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# CHARACTERISTIC HOMOMORPHISM FOR (F<sub>1</sub>, F<sub>2</sub>)-FOLIATED BUNDLES OVER SUBFOLIATED MANIFOLDS

## by José Manuel CARBALLÉS

#### 1. Introduction.

Let  $(F_1, F_2)$  be a couple of foliations on a differentiable manifold M such that the leaves of  $F_1$  contain those of  $F_2$ ; we shall say such couple  $(F_1, F_2)$  a subfoliation on M. While Moussu [9], Feigin [5], Cordero-Gadea [3] and Cordero-Masa [4] have study the (exotic) characteristic homomorphism of a subfoliation  $(F_1, F_2)$  using the techniques of Bernstein-Rozenfeld, Bott-Haefliger and Lehmann, our aim in this paper is to present the construction of the characteristic homorphism of  $(F_1, F_2)$  using the techniques and language of Kamber-Tondeur for foliated bundles.

Our study is based on the notion of  $(F_1, F_2)$ -foliated principal bundle. This is a principal bundle of the form  $P = P_1 + P_2 \longrightarrow M$  of structure group  $G_1 \times G_2$  endowed with a foliated structure given by a connection of the form  $\omega = \omega_1 + \omega_2$  (called adapted connection sum) and where, for each i = 1, 2,  $P_i \longrightarrow M$  is an  $F_i$ -foliated principal bundle of structure group  $G_i$ , and  $\omega_i$  is an adapted connection in  $P_i$ . The most meaningful example of  $(F_1, F_2)$ -foliated bundle over M is a reduction of the bundle of linear frames of the so called normal bundle of  $(F_1, F_2)$  defined by  $\nu(F_1, F_2) = (F_i/F_2) \oplus \nu F_1$ . This vector bundle  $\nu(F_1, F_2)$  has been used in [4] in order to define the characteristic homomorphism of  $(F_1, F_2)$  adapting the Bott [2] well-known construction of the characteristic

homomorphism of a foliation; our construction of the characteristic homorphism of an  $(F_1, F_2)$ -foliated principal bundle generalizes that of Cordero-Masa in the same way as Kamber-Tondeur theory of characteristic classes of foliated bundles generalizes Bott theory. This approach allows, moreover, to initiate the study of the holonomy homomorphism of a "leaf" of a subfoliation, in the line of Goldman's paper [6] for the leaf of a foliation.

The paper is structured as follows. In § 2, we introduce the basic definitions and deduce the filtration preserving properties of the Weil homomorphism  $k(\omega)$  of an adapted connection sum in an  $(F_1, F_2)$ -foliated bundle. As a particular consequence, the vanishing theorem for the normal bundle of a subfoliation [4], [5] is reobtained. These properties of  $k(\omega)$  are used in order to prove the vanishing of  $k(\omega)$  on a differential ideal I of the product Weil algebra  $W(g_1 \oplus g_2)$  (firstly considered by Feigin [5]) and thus, following Kamber-Tondeur's theory, we introduce the generalized characteristic homomorphism of an  $(F_1, F_2)$ -foliated principal bundle P:

$$\Delta_* = \Delta_{(F_1,F_2)}(P) : H(W(g,H)_I) \longrightarrow H_{DR}(M)$$

where  $H \subseteq G$  is a closed Lie subgroup such that P admits an H-reduction. We show that  $\Delta_*$  does not depend on the connection sum  $\omega$  and that it satisfies the usual functorial properties (i.e. naturality under pull-backs and  $\rho$ -extensions). We also deal with the case where  $\omega_1$  and  $\omega_2$  both are basic connections.

In § 3, we relate the generalized characteristic homomorphism  $\Delta_*$  (P) with the generalized characteristic homomorphism (as defined in [7] of each  $P_i$ , i=1,2. Taking into account that any adapted connection sum in P is  $F_2$ -adapted, we deduce some properties of the characteristic homomorphism as  $F_2$ -foliated bundle of an  $(F_1, F_2)$ -foliated bundle as well as of any  $F_2$ -extension of it. This section ends with the construction of the generalized characteristic homomorphism  $\Delta_*$  (P) when considering a foliation F as a subfoliation in the three possible forms.

In § 4 we apply the general results of Kamber-Tondeur on

the cohomology of g-DG-algebras in order to calculate the cohomology  $H(W(g,H)_I)$ . In particular, this allows to refind the characteristic homomorphism of  $(F_1, F_2)$  as defined in [4]. The algebra of secondary characteristic invariants is constructed and a geometric interpretation of the generalized characteristic homomorphism is also given for the general situation.

Finally, in § 5, we restrict the  $(F_1, F_2)$ -foliated bundle P to the leaves of each foliation  $F_i$ , i = 1, 2; this leads us, on the one hand to a slightly generalization of Goldman's study, and, on the other, to define the holonomy homomorphism of a "leaf" of a subfoliation and to discuss an example of Reinhart [10].

Through all this paper, the manifolds, maps, etc, will be assumed differentiable of class  $C^{\infty}$ . Also, we shall adopt the notation of [7].

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## 2. Characteristic homomorphism of an (F<sub>1</sub>, F<sub>2</sub>)-foliated bundle.

Let M be an n-dimensional differentiable manifold, TM its tangent bundle. Through all this paper, we always assume M endowed with a  $(q_1, q_2)$ -codimensional subfoliation  $(F_1, F_2)$ , that is, of a couple of integrable subbundles  $F_i$  of TM of dimension  $n-q_i$ , i=1,2, and  $F_2$  being a subbundle of  $F_1$ . Therefore, for each i,  $F_i$  defines a  $q_i$ -codimensional foliation on M,  $d=q_2-q_1 \ge 0$  and the leaves of  $F_1$  contain those of  $F_2$ .

Let  $Q_i = TM/F_i$  be the normal bundle of  $F_i$ , i = 1, 2, and  $Q_0$  the quotient bundle  $F_1/F_2$ ; then, there is a short exact sequence of vector bundles, canonically associated to  $(F_1, F_2)$ ,  $0 \longrightarrow Q_0 \xrightarrow{i} Q_2 \xrightarrow{\pi} Q_1 \longrightarrow 0$  and the vector bundle  $\nu(F_1, F_2) = Q_0 \oplus Q_1$  is called the normal bundle of  $(F_1, F_2)$ .

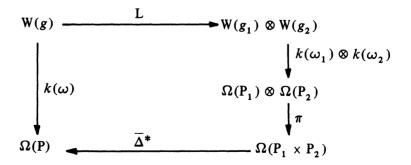
Let  $P_i(M, G_i)$  be an  $F_i$ -foliated principal bundle, i = 1, 2, and let  $\omega_i$  be an adapted connection. Let

$$P(M, G_1 \times G_2) = P_1(M, G_1) + P_2(M, G_2)$$

be the principal bundle sum of  $P_1$  and  $P_2$ ; then  $\omega = \omega_1 + \omega_2$  defines two partial connections in P and  $\omega$  is adapted to both; endowed with these two partial connections, P will be said  $(F_1,F_2)$ -foliated and  $\omega = \omega_1 + \omega_2$  an adapted connection sum. Let us remark that, in particular, P is  $F_2$ -foliated and if both  $\omega_1$  and  $\omega_2$  are basic, then  $\omega = \omega_1 + \omega_2$  is also basic with respect to  $F_2$ .

Let  $L(Q_i)$  be the frame bundle of  $Q_i$ , i=0,1, and  $L(Q_1)+L(Q_0)$  the bundle sum. As it can be easily shown using the results in [4],  $L(Q_1)+L(Q_0)$  is  $(F_1,F_2)$ -foliated and it will be called the bundle of transverse frames of  $(F_1,F_2)$ . Other examples can be obtained as follows; let  $P_i \longrightarrow M$  be a  $G_i$ -principal bundle, i=1,2, endowed with an  $F_i$ -foliated structure,  $F_i$  being the orbit foliation defined on M by a left almost free action of a Lie subgroup  $K_i \subset G_i$  (see 2.4 in [7]); then, if  $K_2 \subset K_1$ ,  $P=P_1+P_2$  is an  $(F_1,F_2)$ -foliated bundle. In particular, if  $P \longrightarrow M$  is a G-principal bundle which is  $F_1$ -foliated by the orbits of the action of a Lie subgroup  $K_1 \subset G$  on M, as above, then for each Lie subgroup  $K_2 \subset K_1$ , the bundle P+P is  $(F_1,F_2)$ -foliated.

Let  $P=P_1+P_2$  be an  $(F_1,F_2)$ -foliated bundle over M,  $\omega=\omega_1+\omega_2$  an adapted connection sum. If we denote  $G=G_1\times G_2$ , its Lie algebra by  $g=g_1\oplus g_2$  and  $k(\omega)$ ,  $k(\omega_1)$ ,  $k(\omega_2)$  the respective Weil homomorphisms, the following commutative diagram allows to write  $k(\omega)=k(\omega_1)\otimes k(\omega_2)$ :



where L denotes the canonical isomorphism,  $\pi$  is defined by  $\pi(\alpha \otimes \beta) = p_1^* \alpha \wedge p_2^* \beta$ ,  $p_i : P_1 \times P_2 \longrightarrow P_i$  the canonical projection, and  $\overline{\Delta}^*$  being induced by the canonical homomorphism  $\overline{\Delta} : P = P_1 + P_2 \longrightarrow P_1 \times P_2$ .

Using  $L: W(g) \cong W(g_1) \otimes W(g_2)$ , the canonical even decreasing filtration of W(g) by G-DG-ideals can be written as

$$F^{2p}W(g) = \bigoplus_{j \geq p} \Lambda^{\bullet} g^* \otimes S^j g^*$$

$$= \bigoplus_{j_1 + j_2 > p} \Lambda^* g^* \otimes S^{j_1} g_1^* \otimes S^{j_2} g_2^*, \quad p \ge 0$$

and we can define a new even decreasing filtration of W(g), also by G-DG-ideals, by

$${}^{\prime}\mathbf{F}^{2p}\mathbf{W}(g) = \bigoplus_{j \geq p} \Lambda^{\bullet} g^* \otimes \mathbf{S}^j g_1^* \otimes \mathbf{S}^{\bullet} g_2^*, p \geq 0.$$

Also,  $\Omega^{\bullet}$  (P) has two decreasing filtrations by G-DG-ideals defined by the sheaves  $Q_i^*$ , i=1,2, of local 1-forms annihilating the foliation  $F_i$  on the base space M; they are given by

$$F^{\rho}\Omega(P) = \Gamma(P, \pi^*\Lambda^{\rho}Q_2^* \cdot \Omega_{p}),$$

$${}'{\rm F}^p\ \Omega({\mathbb P}) = \Gamma({\mathbb P}\,,\pi^*\,\Lambda^p\,\underline{{\mathbb Q}}_1^*\,\boldsymbol{\cdot}\,\Omega_p),\ \ p \geq 0.$$

Then, the Weil homomorphism  $k(\omega)$  of an adapted connection sum  $\omega = \omega_1 + \omega_2$  is filtration-preserving, that is

$$k(\omega) (F^{2p}W(g)) \subset F^p\Omega(P), p \ge 0,$$

and if  $\omega_1$  and  $\omega_2$  are basic, then

$$k(\omega) (F^{2p}W(g)) \subset F^{2p}\Omega(P), p \ge 0.$$

Moreover, one easily proves

PROPOSITION 2.1. – Let  $\omega = \omega_1 + \omega_2$  be an adapted connection sum in  $P = P_1 + P_2$ . Then  $k(\omega) ('F^{2p}W(g)) \subset 'F^p\Omega(P)$ ,  $p \ge 0$ . If  $\omega_1$  and  $\omega_2$  are basic, then  $k(\omega) ('F^{2p}W(g)) \subset 'F^{2p}\Omega(P)$ ,  $p \ge 0$ .

COROLLARY 2.2. – For an adapted connection sum 
$$\omega = \omega_1 + \omega_2$$
,  $k(\omega) F^{2(q_2+1)}W(g) = 0$ ,  $k(\omega) F^{2(q_1+1)}W(g) = 0$ .

If  $\omega_1$  and  $\omega_2$  are basic,

$$k(\omega) \operatorname{F}^{2([q_2/2]+1)} W(g) = 0, \quad k(\omega) \operatorname{F}^{2([q_1/2]+1)} W(g) = 0.$$

If we now consider the algebras of G-basic elements, we obtain similar properties for the Chern-Weil homomorphism  $h(\omega): I(G) = I(G_1 \times G_2) \longrightarrow \Omega(M)$  with respect to the following filtrations of I(G) and  $\Omega(M)$ :

$$F^{2p}I(G) = \bigoplus_{j \ge p} I^{2j}(G), \ 'F^{2p}I(G) = \bigoplus_{j \ge p} I^{2j}(G_1) \otimes I^*(G_2), \ p \ge 0$$

$$F^{p}\Omega(M) = \Gamma(M, \Lambda^{p} Q_{2}^{*} \cdot \Omega_{M}),$$

$$^{\prime}\mathbf{F}^{p}\ \Omega(\mathbf{M}) = \Gamma(\mathbf{M}, \Lambda^{p}\ \mathbf{Q}_{1}^{*}\cdot\Omega_{\mathbf{M}})\,, \ \ p\geqslant 0\,.$$

That is, since  $F^{q_2+1}\Omega(M) = 0$  and  $F^{q_1+1}\Omega(M) = 0$ , we have

COROLLARY 2.3. – Let  $\omega = \omega_1 + \omega_2$  be an adapted connection sum in an  $(F_1, F_2)$ -foliated bundle  $P = P_1 + P_2$ , and let  $h(\omega)$  denote the Chern-Weil homomorphism of P. Then

$$h(\omega) \operatorname{F}^{2(q_2+1)} I(G) = 0, \ h(\omega) \operatorname{F}^{2(q_1+1)} I(G) = 0.$$

If, moreover,  $\omega_1$  and  $\omega_2$  are basic, then

$$h(\omega) F^{2([q_2/2]+1)}I(G) = 0, \ h(\omega)' F^{2([q_1/2]+1)}I(G) = 0.$$

In particular, if P is the bundle of transverse frames of  $(F_1, F_2)$ , then Corollary 2.3 is the Vanishing Theorem for subfoliations stated in [4].

Next, let  $I \subset W(g)$  be the G-DG-ideal given by

$$I = F^{2(q_2+1)}W(g) + 'F^{2(q_1+1)}W(g).$$
 (2.1)

Then, by virtue of Corollary 2.2,  $I \subseteq Ker(k(\omega))$  and there is an induced G-DG-homomorphism  $k(\omega): W(g)_I = W(g)/I \longrightarrow \Omega(P)$ .

For any subgroup  $H \subseteq G$ , there is the relative ideal  $I_H$  of  $W(g,H) = W(g)_H$ , and thus if we construct

$$W(g, H)_{I} = W(g, H)/I_{H} = (W(g)_{I})_{H}$$
,

we can consider the induced DG-homomorphism

$$k(\omega)_{\mathrm{H}}: \mathrm{W}(\mathrm{g}, \mathrm{H})_{\mathrm{I}} \longrightarrow \Omega(\mathrm{P})_{\mathrm{H}}.$$

Now, if we assume H to be closed and P having an H-reduction given by a section  $s: M \longrightarrow P/H$  of the induced map  $\hat{\pi}: P/H \longrightarrow M$ ,

we can construct a DG-homomorphism as the composition

$$\Delta(\omega) = s^* \circ k(\omega)_{\mathsf{H}} : \mathsf{W}(g,\mathsf{H})_{\mathsf{I}} \longrightarrow \Omega(\mathsf{P})_{\mathsf{H}} \cong \Omega(\mathsf{P}/\mathsf{H}) \longrightarrow \Omega(\mathsf{M}).$$

DEFINITION 2.4. – We shall call generalized characteristic homomorphism of the  $(F_1, F_2)$ -foliated bundle P the homomorphism  $\Delta_* = \Delta_{(F_1, F_2)}(P) \colon H(W(g, H)_I) \longrightarrow H_{DR}(M)$  induced by  $\Delta(\omega)$  in cohomology.

Remark. – If both  $\omega_1$  and  $\omega_2$  are basic connections, then  $k(\omega)$  vanishes on the ideal

$$I' = F^{2([q_2/2]+1)}W(g) + F^{2([q_1/2]+1)}W(g)$$

and the generalized characteristic homomorphism of P will be  $\Delta_*: H(W(g,H)_{I'}) \longrightarrow H_{DR}(M)$  because, under these conditions,  $\Delta(\omega)$  factorizes through  $p: W(g,H)_I \longrightarrow W(g,H)_{I'}$ , the canonical projection induced by the injection  $I \subset I'$ .

 $\Delta_* = \Delta_{(F_1, F_2)}(P)$  is independent of the choice of  $\omega = \omega_1 + \omega_2$  in the following sense. Let  $\omega^0 = \omega_1^0 + \omega_2^0$ ,  $\omega^1 = \omega_1^1 + \omega_2^1$  be two adapted connections sum in P. Let an H-reduction of P be given by a section  $s: M \longrightarrow P/H$ , and

$$\Delta_*^i = \Delta(\omega^i)_* : H(W(g, H)_I) \longrightarrow H_{DR}(M)$$

the homomorphism constructed using the connection  $\omega^i$ , i = 0, 1. Then,

Proposition 2.5.  $-\Delta^0_* = \Delta^1_*$ .

Proof. – Let  $f: \mathbf{M} \times [0,1] \longrightarrow \mathbf{M}$  be the canonical projection, and let  $f^{-1}(\mathbf{F}_k)$ , k=1,2, the foliation inverse image of  $\mathbf{F}_k$  via f. If  $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$  is an  $(\mathbf{F}_1, \mathbf{F}_2)$ -foliated bundle over  $\mathbf{M}$  then the inverse image  $\mathbf{P}' = f^*(\mathbf{P}) = f^*(\mathbf{P}_1) + f^*(\mathbf{P}_2)$  of  $\mathbf{P}$  via f is  $f^{-1}(\mathbf{F}_1, \mathbf{F}_2) = (f^{-1}(\mathbf{F}_1), f^{-1}(\mathbf{F}_2))$ -foliated. Moreover, the connection  $\overline{\omega}$  given by

$$\overline{\omega}(\mathbf{X}) = t(f^*\omega^1)(\mathbf{X}) + (1-t)(f^*\omega^0)(\mathbf{X}), \ \mathbf{X} \in \mathbf{T}_{(u,t)}(\mathbf{P}')$$

is obviously an adapted connection sum in P'.

On the other hand, if  $j_t: \mathbf{M} \longrightarrow \mathbf{M} \times [0,1]$  is the canonical injection  $j_t(x) = (x,t)$ , for each  $t \in [0,1]$ , then  $j_t^*(\mathbf{P}') = \mathbf{P}$  for any  $t \in [0,1]$ ,  $\overline{j}_0^* \overline{\omega} = \omega^0$ ,  $\overline{j}_1^* \overline{\omega} = \omega^1$  where  $\overline{j}_t: \mathbf{P} \longrightarrow \mathbf{P}'$ 

denotes the canonical lift of  $j_t$ . Thus, using  $\overline{\omega}$  to construct the generalized characteristic homomorphism of  $P': \overline{\Delta}_* = \Delta_*(\overline{\omega})$ , we have  $\Delta_*^l = (j_i^*)_{DR} \circ \overline{\Delta}_*$ , i = 0, 1. But, since  $(j_0^*)_{DR} = (j_1^*)_{DR}$ , then  $\Delta_*^0 = \Delta_*^1$ .

It is clear from the construction that  $\Delta_*$  depends a priori upon the H-reduction of P given by s. However, this construction is visibly independent of s if the closed subgroup  $H \subset G$  contains a maximal compact subgroup of G.

 $\Delta_*$  has also the following properties of functoriality.

#### (A) $\Delta_*$ is functorial under pullbacks.

This means more precisely the following. Let  $(F_1', F_2')$  and  $(F_1, F_2)$  be  $(q_1, q_2)$ -codimensional subfoliations on M' and M respectively, and let  $f: M' \longrightarrow M$  be a differentiable map such that  $f_*(F_i') \subset F_i$ , i = 1, 2. Let  $P = P_1 + P_2$  be an  $(F_1, F_2)$ -foliated bundle over M, and let

$$P' = f * P = f * P_1 + f * P_2$$

be the inverse image of P via f. Since each  $f^*P_i$  is  $F'_i$ -foliated ([1], Prop. 1.7), then P' is, in fact, an  $(F'_1, F'_2)$ -foliated bundle over M'. Then, if  $H \subseteq G$  is a closed subgroup and  $s: M \longrightarrow P/H$  the section given an H-reduction of P,  $s' = f^*s: M' \longrightarrow P'/H$  gives an H-reduction of P' and we can easily prove

Proposition 2.6. 
$$\Delta_*(P') = f_{DR}^* \circ \Delta_*(P)$$
.

It is clear that this result is applied in the particular case of f being transversal to the subfoliation  $(F_1, F_2)$  on M [4].

#### (B) $\Delta_*$ is functorial under $\rho$ -extensions.

This means more precisely the following. Let

$$\rho = (\rho_1, \rho_2) : G = G_1 \times G_2 \longrightarrow G' = G'_1 \times G'_2$$

a homomorphism of product Lie groups, that is, each  $\rho_i: G_i \longrightarrow G_i'$  a Lie group homomorphism, i=1,2. If P is an  $(F_1,F_2)$ -foliated principal bundle over M and  $\omega$  an adapted connection sum in P, then P', the extension of P by  $\rho$ , is  $(F_1,F_2)$ -foliated and  $\omega'$ , extension of  $\omega$  by  $\rho$ , is an adapted connection sum in P'.

Let H, H' be closed subgroups of G and G', respectively,

such that  $\rho(H) \subset H'$ ; let I' and I be the ideals of W(g') and W(g) given by (2.1). Since  $W(d\rho)$  is graduation-preserving, then  $W(d\rho)$  (I')  $\subset$  I and diagram (4.72) in [7] can be used to state

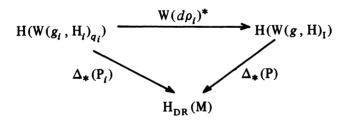
Proposition 2.7.  $\Delta_*(P') = \Delta_*(P) \circ W(d\rho)^*$ .

#### 3. Relation between $\Delta_*(P)$ and $\Delta_*(P_i)$ , i = 1, 2.

Between the generalized characteristic homomorphism  $\Delta_*(P)$  of an  $(F_1, F_2)$ -foliated principal bundle  $P = P_1 + P_2$  and the generalized characteristic homomorphism  $\Delta_*(P_i)$  ([7]) of the  $F_i$ -foliated principal bundle  $P_i$ , i = 1, 2, there exists a canonical relation given as follows.

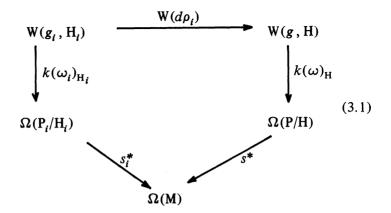
Let  $ho_i\colon G=G_1\times G_2\longrightarrow G_i$  be the canonical projection,  $H_i\subset G_i$  a closed subgroup, i=1,2, and  $H=H_1\times H_2\subset G$ . Let  $s\colon M\longrightarrow P/H$  be a section defining an H-reduction of P and let  $s_i\colon M\longrightarrow P_i/H_i$  be the induced section defining an induced  $H_i$ -reduction of  $P_i$ . Then.

Proposition 3.1. – The diagram



is commutative for each i = 1, 2. In fact, this diagram is also commutative at the cochain level.

**Proof.** – Since  $P_i$  is isomorphic (as  $F_i$ -foliated bundle) to the  $\rho_i$ -extension of P, and because  $\omega_i = (\rho_i)*\omega$  is an adapted connection in  $P_i$ ,  $\omega = \omega_1 + \omega_2$  being an adapted connection sum in P, the following diagram commutes for each i=1,2:



and we are reduced to show that  $W(d\rho_i)$   $(F^{2(q_i+1)}W(g_i)) \subset I$  for each i=1,2.

For i=2, this follows easily because  $W(d\rho_i)$  preserves the bigraduation and then

$$W(d\rho_2) W^{p,2q}(g_2) \subset W^{p,2q}(g).$$

For i = 1, the result follows from the fact that

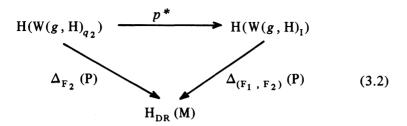
$$\begin{split} \mathbb{W}(d\rho_1) & (\Lambda^u g_1^* \otimes S^v g_1^*) \subset \Lambda^u g^* \otimes S^v g_1^* \otimes S^0 g_2^* \,, \quad u\,, v \geqslant 0 \\ \text{since} & (d\rho_1)^* : Sg_1^* \longrightarrow Sg^* = Sg_1^* \otimes Sg_2^* \quad \text{is given by} \\ & (d\rho_1)^* (\alpha) = \alpha \otimes 1 \,. \end{split}$$

Remarks. -1) Since both  $\omega = \omega_1 + \omega_2$  and  $\omega_i$  are  $F_2$ -adapted connections, we can truncate the Weil algebras in diagram (3.1) at the degree  $q_2$  and thus, going into cohomology, obtain a commutative diagram relating the generalized characteristic homomorphisms of P and  $P_i$  as  $F_2$ -foliated principal bundles.

2) We can use  $\omega = \omega_1 + \omega_2$  to construct the generalized characteristic homomorphism of the  $F_2$ -foliated bundle P:

$$\Delta_{\mathrm{F}_{2}}(\mathrm{P}):\mathrm{H}(\mathrm{W}(g,\mathrm{H})_{q_{2}})\longrightarrow\mathrm{H}_{\mathrm{DR}}(\mathrm{M}).$$

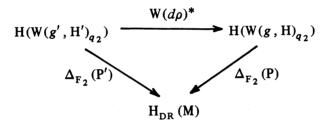
Then, taking into account that the inclusion  $F^{2(q_2+1)}W(g) \subset I$  induces a projection  $p:W(g,H)_{q_2} \longrightarrow W(g,H)_I$ , we obtain a commutative diagram



and, therefore,  $\operatorname{Im} \Delta_{F_2}(P) \subset \operatorname{Im} \Delta_{(F_1,F_2)}(P)$ .

3) Let  $\rho: G = G_1 \times G_2 \longrightarrow G'$  be a homomorphism of Lie groups and consider the structure of  $F_2$ -foliated bundle on the  $\rho$ -extension  $P' = \rho_* P$  induced by the structure of  $F_2$ -foliated bundle underlying the  $(F_1, F_2)$ -foliated structure of  $P = P_1 + P_2$ .

Then, for suitable closed subgroups  $H \subset G$ ,  $H' \subset G'$ , the functoriality under  $\rho$ -extensions of the generalized characteristic homomorphism of foliated bundles ([7]) implies that the following diagram is commutative



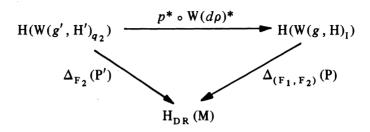
which combined with (3.2) leads to the following

PROPOSITION 3.2.—Let  $P' \longrightarrow M$  be an  $F_2$ -foliated principal bundle with structure group G' and let  $P = P_1 + P_2$  be an  $(F_1, F_2)$ -foliated G-reduction of P. Assume  $i: P \longrightarrow P'$ . be  $F_2$ -foliated compatibly with the homomorphism

$$\rho: G = G_1 \times G_2 \longrightarrow G',$$

and let H, H' be closed subgroups of G, G' respectively, verifying the suitable hypothesis. Then, the generalized characteristic homomorphism  $\Delta_{F_2}(P')$  of P' as  $F_2$ -foliated bundle factorizes through the generalized characteristic homomorphism  $\Delta_{(F_1,F_2)}(P)$ 

of P as  $(F_1, F_2)$ -foliated bundle, that is, the following diagram is commutative:



Example. – Let  $P' = L(Q_2) \cong L(\nu(F_1, F_2))$  be the canonically  $F_2$ -foliated bundle of transverse frames of  $F_2$ , and P the  $(F_1, F_2)$ -foliated bundle of transverse frames of  $(F_1, F_2)$ , which is a (not  $F_2$ -foliated) reduction of P' compatible with the canonical homomorphism  $\rho: Gl(q_1, R) \times Gl(d, R) \longrightarrow Gl(q_2, R)$ .

If we consider in P' the  $\rho$ -extension of the F<sub>2</sub>-foliated structure of P, this is not the canonical F<sub>2</sub>-foliated structure of P'; but, as it can be easily shown using the Lemma 5.3 in [4] (see [8]), both are integrably homotopic. Then, for suitable H, H' the proposition 3.2 provides the corresponding commutative diagram. If, moreover,  $H = O(q_1) \times O(d)$  and  $H' = O(q_2)$ , then  $\Delta_{F_2}(P')$  is just the characteristic homomorphism of the foliation F<sub>2</sub>, whereas  $\Delta_{(F_1, F_2)}(P)$  is the characteristic homomorphism of the subfoliation  $(F_1, F_2)$  [4], as it will be established later.

Now, let us remark that a q-codimensional foliation F on M can be considered as a subfoliation on M in three different ways;  $(C_1): F_1 = F_2 = F$ ,  $q_1 = q_2 = q$ ;  $(C_2): F_1 = TM$ ,  $F_2 = F$ ,  $q_1 = 0$ ,  $q_2 = q$ ;  $(C_3): F_1 = F$ ,  $F_2 = 0$ ,  $q_1 = q$ ,  $q_2 = n$ . Then, all the previous results particularize to these cases as follows:

Case (C<sub>1</sub>). – Here an (F<sub>1</sub>, F<sub>2</sub>)-foliated bundle  $P = P_1 + P_2$  is, in fact, an F-foliated bundle, the ideal I coincides with  $F^{2(q+1)}W(g)$ ,  $p^*$  in diagram (3.2) is an isomorphism and  $\Delta_{(F,F)}(P) = \Delta_F(P)$ .

Case  $(C_2)$ . – Here,  $P = P_1 + P_2$  is the sum of a flat bundle  $P_1$  and an F-foliated bundle  $P_2$ ; since

$$I = F^{2(q+1)} W(g) + 'F^{2}W(g)$$

making calculations we obtain

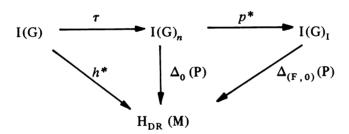
$$W(g)_{I} \cong \bigoplus_{j=0}^{q} \Lambda^{\bullet} g^{*} \otimes S^{0} g_{1}^{*} \otimes S^{j} g_{2}^{*}$$

$$W(g)_{q} \cong W(g)_{I} \oplus \left(\bigoplus_{j=0}^{q-1} \bigoplus_{t=0}^{q-j} \Lambda^{\bullet} g^{*} \otimes S_{1}^{t} g^{*} \otimes S_{2}^{j} g^{*}\right)$$

and then  $p^*$  in diagram (3.2) is surjective. Hence

$$\operatorname{Im}(\Delta_{F}(P)) = \operatorname{Im}(\Delta_{(TM,F)}(P)).$$

Case  $(C_3)$ . — In this case,  $P = P_1 + P_2$  is simply an ordinary bundle (that is, 0-foliated) which is not necessarily F-foliated. Thus, if we take H = G in diagram (3.2) and denote  $\tau: I(G) \longrightarrow I(G)_n$  the canonical projection, we have a commutative diagram



where  $h^*$  denotes the Chern-Weil homomorphism of P. Thus, we can assert the following: if  $P=P_1+P_2$  where  $P_1$  is a foliated bundle, then the Chern-Weil homomorphism of P vanishes on Ker  $(p^*\circ\tau)$ . Again, since any connection in  $P_2$  is basic with respect to the foliation by points on M, if  $P_1$  admits a basic connection then  $h^*$  vanishes on the kernel of the composition

$$I(G) \xrightarrow{r'} I(G)_{[n/2]} \xrightarrow{p'*} I(G)_{i'}.$$

## 4. Difference construction for $\Delta_{(F_1,F_2)}(P)$ . Secondary invariants.

The computation of  $H(W(g,H)_I)$  can be done from the general results in [7], Chapter 5, from where we shall take the notation.

We assume throughout that G is either connected or  $I(G)\cong I(G_0)\equiv I(g)$  for the connected component  $G_0$  of G; the closed subgroup  $H\subset G$  is assumed to have finitely many connected components.

Then, let us consider in the G-DG-algebra  $W(g)_I$  the canonical connection given by the projection  $k:W(g)\longrightarrow W(g)_I$ .

If the pair (g,h) is reductive (h=Lie algebra of H), in accordance with Theorem 5.82 in [7] there exists a homomorphism  $\zeta(W(g)_I,H):A(W(g)_I,H)\longrightarrow (W(g)_I)_H=W(g,H)_I$  which induces an isomorphism in cohomology. In this way, the generalized characteristic homomorphism  $\Delta_{(F_I,F_2)}(P)$  of P will have the same image as the composition

$$H(A(W(g)_{I}, H)) \xrightarrow{\varsigma(W(g)_{I}, H)_{*}} H(W(g, H)_{I}) \xrightarrow{\Delta_{(F_{1}, F_{2})} (P)} H_{DR}(M)$$

induced by the cochain map  $\widetilde{\Delta}(\omega) = \Delta(\omega) \circ \zeta(W(g)_I, H)$ . In fact, the evaluation of  $\widetilde{\Delta}(\omega)$  on the complex

$$A(W(g)_{I}, H) = \Lambda P_{g} \otimes (W(g)_{I})_{g} \otimes I(H) = \Lambda P_{g} \otimes I(G)_{I} \otimes I(H)$$

is equal to that of Theorem 5.95 in [7] for the case of a foliated bundle.

If we now assume the pair (g,h) to be special Cartan (CS), then, by Theorem 5.107 in [7], there is an isomorphism

$$\overline{\beta} : H(\hat{A}(W(g)_I)) \otimes_{I(g)} I(H) \xrightarrow{\cong} H(A(W(g)_I, H))$$

where  $\hat{A}(W(g)_I) = \Lambda \hat{P} \otimes (W(g)_I)_g$ . Thus  $\Delta_{(F_1, F_2)}(P)$  has the same image as the composition  $\Delta_{(F_1, F_2)}(P) \circ \zeta(W(g)_I, H) \circ \overline{\beta}$ . Then taking into account that  $\hat{A}(W(g)_I) \subset A(W(g)_I, H)$ , we consider the composition

$$\hat{\Delta}(\omega): \hat{A}(W(g)_{I}) \longrightarrow A(W(g)_{I}, H) \xrightarrow{\widetilde{\Delta}(\omega)} \Omega(M)$$

and, thus, the characteristic homomorphism  $\Delta_{(F_1,F_2)}(P)$  will be realized by  $\widehat{\Delta}_* \otimes h'_* : H(\widehat{A}(W(g)_I)) \otimes_{I(G)} I(H) \xrightarrow{} H_{DR}(M), h'_*$  being the characteristic homomorphism of the H-reduction P' of P. See 5.112 in [7] for more details.

• In particular, let us assume that  $P = P_1 + P_2$  is the bundle of transverse frames of  $(F_1, F_2)$ , and take

$$H = O(q_1) \times O(d) \subset Gl(q_1, \mathbf{R}) \times Gl(d, \mathbf{R}) = G.$$

Since  $\operatorname{gl}(q_1, \mathbf{R})$  and  $\operatorname{gl}(d, \mathbf{R})$  are reductive Lie algebras and  $(\operatorname{gl}(q_1, \mathbf{R}) \times \operatorname{gl}(d, \mathbf{R}), \operatorname{o}(q_1) \times \operatorname{o}(d))$  is symmetric, this pair will be special Cartan and the previous construction can be used. Then,  $\Delta_{(F_1, F_2)}$  (P) can be considered as defined on  $\operatorname{H}(\hat{A}_I) \otimes_{\operatorname{I}(G)} \operatorname{I}(H)$ , where  $\hat{A}_I = \hat{A}(W(g)_I) = \Lambda \hat{P} \otimes \operatorname{I}(\operatorname{gl}(q_1, \mathbf{R}) \times \operatorname{gl}(d, \mathbf{R}))_I$ . But, as it happens in the case of the bundle of transverse frames of a foliation [7],  $\operatorname{H}(\hat{A}_I) \otimes_{\operatorname{I}(G)} \operatorname{I}(H) \cong \operatorname{H}(\hat{A}_I)$ , and then

$$\hat{\Delta}_{(F_1, F_2)}(P) = \hat{\Delta}_* : H(\hat{A}_I) \longrightarrow H_{DR}(M).$$

On the other hand,  $I(gl(q_1, \mathbf{R})) = \mathbf{R}[c_1, \ldots, c_{q_1}]$ ,  $I(gl(d, \mathbf{R})) = \mathbf{R}[c_1', \ldots, c_d']$  and  $\Lambda \hat{\mathbf{P}} = \Lambda \hat{\mathbf{P}}_1 \otimes \Lambda \hat{\mathbf{P}}_2$ ,  $\hat{\mathbf{P}}_i$  being the Samelson subspace of the pair  $(g_i, h_i)$ , i = 1, 2; since both pairs are special Cartan,

$$\Lambda \hat{P}_1 = \Lambda(y_1, y_2, \dots, y_{q'_1}), \quad \Lambda \hat{P}_2 = \Lambda(y'_1, y'_3, \dots, y'_{q'})$$

where  $y_i = \sigma c_i$ ,  $y_i' = \sigma' c_i'$  and  $q_1' = 2[(q_1 + 1)/2] - 1$ , d' = 2[(d+1)/2] - 1,  $\sigma$  and  $\sigma'$  being the suspension maps. Therefore

$$\hat{A}_{I} = \Lambda(y_{1}, y_{3}, \dots, y_{d'}) \otimes \Lambda(y'_{1}, y'_{3}, \dots, y'_{d'})$$

$$\otimes \ \frac{\mathsf{R}\left[c_{1}\,,\ldots\,,c_{q_{1}}\right] \otimes \ \mathsf{R}\left[c_{1}'\,,\ldots\,,c_{d}'\right]}{\mathsf{I}_{g}}$$

where

$$I_g = I \cap I(g) = \langle \{\alpha \otimes \beta \in I^{j_1}(g_1) \otimes I^{j_2}(g_2)/j_1 > q_1 \text{ or } j_1 + j_2 > q_2 \} \rangle.$$

That is,  $\hat{A}_I = WO_I$ , the graded differential algebra defined in [4]. Therefore, the generalized characteristic homomorphism of the bundle of transverse frames of the subfoliation  $(F_1, F_2)$  coincides with the characteristic homomorphism of  $(F_1, F_2)$  as defined in [4]

$$\lambda_{(F_1, F_2)}^* : H(WO_I) \longrightarrow H_{DR}(M).$$

From this point of view, the generalized characteristic homomorphism  $\Delta_{(F_1, F_2)}(P)$  of an  $(F_1, F_2)$ -foliated bundle  $P = P_1 + P_2$  generalizes the characteristic homomorphism of the subfoliation  $(F_1, F_2)$  in the same way as Kamber-Tondeur's characteristic homomorphism of a foliated bundle generalizes Bott's characteristic homomorphism of a foliation ([7], [2]).

In order to construct the algebra of secondary characteristic invariants, from now on, we shall consider an  $(F_1, F_2)$ -foliated bundle  $P = P_1 + P_2$ ,  $H \subset G$  a closed subgroup with finitely many connected components and such that the pair of Lie algebras (g,h) be reductive. Let us denote P' the H-reduction of P used to define the characteristic homomorphism  $\Delta_{(F_1,F_2)}$  (P) of P and, to simplify the notation, put  $A_I = A(W(g)_I, H)$ .

Let  $p: A_I \longrightarrow I(G)_I \otimes_{I(G)} I(H)$  the composition of the canonical projection along  $\Lambda P_g$ ,  $\lambda: A_I \longrightarrow I(G)_I \otimes I(H)$  with the canonical map.

DEFINITION 4.1. –  $H(K_I)$ , where  $K_I = \text{Ker } p$ , is called the algebra of secondary characteristic invariants of P.

Proposition 4.2. — There is a short exact sequence of algebras

$$0 \longrightarrow H(K_{I}) \longrightarrow H(W(g,H)_{I}) \longrightarrow I(G)_{I} \otimes_{I(G)} I(H) \longrightarrow 0. \quad (4.1)$$

**Proof.** - Consider the short exact sequence of complexes

$$0 \, \longrightarrow \, K_{I} \, \longrightarrow \, A_{I} \, \longrightarrow \, I(G)_{I} \otimes_{I(G)} I(H) \, \longrightarrow \, 0 \, .$$

Then (4.1) appears by writing up the associated long exact sequence of homology whose connecting homomorphism is null, and because  $H(A_I) \cong H(W(g, H)_I)$ .

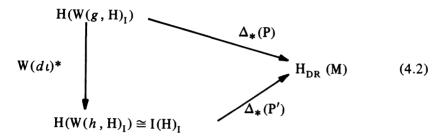
The non-triviality of  $\Delta_{(F_1,F_2)}(P)/_{H(K_1)}$  is a measure for the incompatibility of the  $(F_1,F_2)$ -foliated structure of  $P=P_1+P_2$  with its H-reduction P'; that is,

PROPOSITION 4.3. – Let  $P = P_1 + P_2$  be an  $(F_1, F_2)$ -foliated bundle,  $H = H_1 \times H_2 \subset G$  a closed subgroup and P' an H-reduction of P which is  $(F_1, F_2)$ -foliated and such that, if

 $\iota \colon H \longrightarrow G$  is the injection, then the  $(F_1, F_2)$ -foliated structure of P is, in fact, the  $\iota$ -extension of that of P'. Then

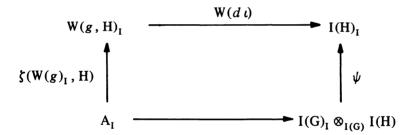
$$\Delta_{(F_1, F_2)} (P)/_{H(K_1)} = 0$$
.

*Proof.* – Applying Proposition 2.7 to the homomorphism  $\iota: (H, H) \longrightarrow (G, H)$  we obtain the commutative diagram



and hence  $\Delta_*(P)/_{Ker(W(d,i)^*)} = 0$ .

Moreover, there is a commutative diagram



where  $\psi$  is the canonical projection of

$$I(G)_{I} \otimes_{I(G)} I(H) \cong I(H)/I \cdot I(H)$$

onto  $I(H)_I = I(H)/I$ . Thus, going into cohomology, we obtain a factorization  $H(W(g,H)_I) \longrightarrow I(G)_I \otimes_{I(G)} I(H) \longrightarrow I(H)_I$  of the vertical homomorphism in (4.2). Then, because

$$H(K_I) = Ker \{H(W(g,H)_I) \longrightarrow I(G) \otimes_{I(G)} I(H)\}$$

by virtue of Proposition 4.2, we have  $H(K_1) \subset Ker(W(d\iota)^*)$ .

Moreover, as in the usual case of foliated bundles [7], we have

PROPOSITION 4.4. – There is a splitting homomorphism

$$\kappa: \mathrm{I}(\mathrm{G})_{\mathrm{I}} \otimes_{\mathrm{I}(\mathrm{G})} \mathrm{I}(\mathrm{H}) \longrightarrow \mathrm{H}(\mathrm{W}(g,\mathrm{H})_{\mathrm{I}})$$

of the short exact sequence (4.1) and the composition  $\Delta_*(P) \circ \kappa$  is induced by the characteristic homomorphism of P':

$$h_*(\mathbf{P'}): \mathbf{I}(\mathbf{H}) \longrightarrow \mathbf{H}_{\mathbf{DP}}(\mathbf{M}).$$

#### 5. Restriction to the leaves.

In this section we shall discuss the restriction of an  $(F_1, F_2)$ -foliated bundle  $P = P_1 + P_2$  to the leaves of each foliation  $F_i$ , i = 1, 2. In order to do that, let us previously discuss the restriction to the leaves of an  $F_2$ -foliated bundle.

So, let  $(F_1, F_2)$  be a  $(q_1, q_2)$ -codimensional subfoliation on M, L a leaf of  $F_1$  and  $j: L \longrightarrow M$  the canonical immersion. Since  $F_2 \subset F_1$ ,  $F_2$  induces on L a foliation which will be denoted by  $F_L$ ; note that codim  $(F_L) = d = q_2 - q_1$  while codim  $(F_2) = q_2$ . Obviously, j maps the leaves of  $F_L$  into leaves of  $F_2$ .

Now, let  $\pi: P \longrightarrow M$  be a G-principal fibre bundle and denote  $P' = j^*P$  the inverse image of P via j. Then  $\pi': P' \longrightarrow L$ , the restriction of P to L, is a G-principal fibre bundle and we shall denote  $\overline{j}: P' \longrightarrow P$  the canonical injection. The following result is known [1]:

Proposition 5.1.—If P is  $F_2$ -foliated then P' is  $F_L$ -foliated. Moreover, if  $\omega$  is an adapted connection in P then  $\overline{j}^*\omega$  is an adapted connection in P'.

Precisely the latter condition allows to consider, using connections  $\omega$  and  $\overline{j}^*\omega$ , a commutative diagram

$$H(W(g, H)_{q_2}) \xrightarrow{\Delta_*(P)} H_{DR}(M)$$

$$p^* \downarrow \qquad \qquad \downarrow j^* \qquad (5.1)$$

$$H(W(g, H)_d) \xrightarrow{\Delta_*(P')} H_{DR}(L)$$

where  $p: W(g, H)_{q_2} \longrightarrow W(g, H)_d$  is the canonical projection  $(d \leq q)$ ,  $H \subset G$  is a subgroup satisfying the usual hypothesis and  $\Delta_*(P)$ ,  $\Delta_*(P')$  are the generalized characteristic homomorphisms of P and P'.

For example,  $Q_0 = F_1/F_2$  (the normal bundle of  $F_2$  relative to  $F_1$ ) is an  $F_2$ -foliated vector bundle on account of the existence on it of the so-called Bott connection [4], [1]. Moreover,  $Q_L = TL/F_L$ , the normal bundle of  $F_L$ , is canonically isomorphic to  $j^*Q_0$  [1] in such way that the Bott connection in  $Q_0$  pulls back via j to the Bott connection in  $Q_L$ . Therefore, the frame bundle of  $Q_0$ , P, is an  $F_2$ -foliated  $G1(d,\mathbf{R})$ -principal bundle, and  $P'=j^*P$  is precisely the bundle of transverse frames of  $F_L$ . Thus, through the corresponding isomorphisms, diagram (5.1) becomes:

$$H(W(gl(d, \mathbf{R}), O(d))_{q_2}) \xrightarrow{\Delta_*} H_{DR} (M)$$

$$p^* \downarrow \qquad \qquad \downarrow j^*$$

$$H(W(gl(d, \mathbf{R}), O(d))_d) \cong H(WO_d) \xrightarrow{\Delta'_*} H_{DR} (L)$$

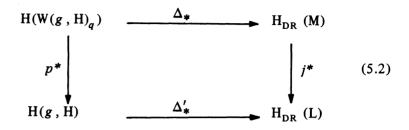
where  $\Delta_{*}^{\prime}$  is just the usual characteristic homomorphism of foliation  $F_{L}$  on  $L\,.$ 

Next, we shall discuss the restriction to a leaf of  $F_2$ . Thus, provided that we do not need to use the foliation  $F_1$ , we shall assume only one foliation F on M, L a leaf of F and  $j:L\longrightarrow M$  the canonical immersion. Now if  $\pi:P\longrightarrow M$  is a G-principal bundle and  $P'=j^*P$  is the inverse image of P via j, we have

Proposition 5.2. — Each F-foliated bundle structure on P determines a flat bundle structure on P' in such way that if  $\omega$  is an adapted connection in P, then  $\omega' = \bar{j}^* \omega$  is a flat connection in P'.

Therefore, if we consider on M the subfoliation (F,F) then the foliation  $F_L$  induced on L is trivial, that is,  $F_L = TL$ ,

and taking into account that  $W(g, H)_0 \cong (\Lambda g^*)_H \cong \Lambda(g/h)^{*H}$ , diagram (5.1) becomes



 $\Delta'_*$  being the generalized characteristic homomorphism of P' as flat bundle [7].

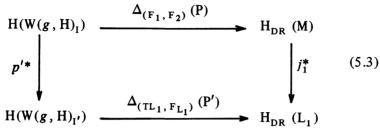
Example. – Let P be the bundle of transverse frames of F. Then, if  $\nu F = TM/F$  is the normal bundle of F,  $\nu L = \nu F/L$  is the normal bundle of the leaf L of F and P' = j\*P is just the bundle of frames of  $\nu L$ . Following Goldman [6], any connection in P adapted to its canonical structure of F-foliated bundle will be said a foliation connection, and a connection in P' obtained as inverse image of a foliation connection will be said a leaf connection. In fact, Goldman showed that there is an unique leaf connection which is flat, and one easily checks that  $\Delta'_*$  in diagram (5.2) is nothing but the so-called holonomy homomorphism of the leaf L [6].

Again, let  $(F_1, F_2)$  be a  $(q_1, q_2)$ -codimensional subfoliation on M,  $L_1$  a leaf of  $F_1$ ,  $j_1: L_1 \longrightarrow M$  the canonical immersion,  $F_{L_1}$  the foliation on  $L_1$  induced by  $F_2$ ,  $P = P_1 + P_2$  an  $(F_1, F_2)$ -foliated bundle on M and  $P' = j_1^*P$  its inverse image via  $j_1$ . Then, since P is also  $F_2$ -foliated we can apply to it all previous results; so, in particular, we can construct a diagram (5.1) for this  $P = P_1 + P_2$ . On the other hand,

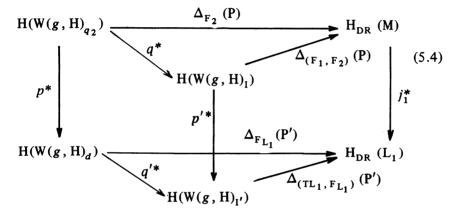
$$P' = j_1^* P_1 + j_1^* P_2 ;$$

then, applying the previous results to each  $j_1^*P_i$ , i=1,2, it follows that P' is  $(TL_1, F_{L_1})$ -foliated over  $L_1$ . Moreover, if  $\omega$  is an adapted connection sum in P then  $\omega' = j_1^*\omega$  is an adapted connection sum in P'. If I and I' are the ideals given by (2.1) for the pairs  $(q_1, q_2)$  and (0, d), respectively, then

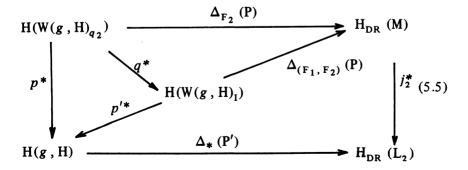
 $I \subset I'$  and, for an appropiate subgroup H, we obtain a commutative diagram



where  $p'^*$  is induced by the canonical projection. If we now combine (5.3) with (5.1) through (3.2), we obtain



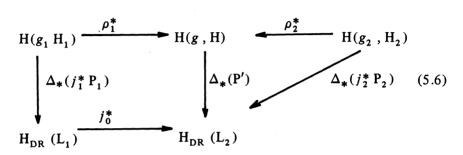
Now, if  $L_2$  is a leaf of  $F_2$  and  $j_2:L_2 \longrightarrow M$  is its canonical immersion, then  $(F_1,F_2)$  induces on  $L_2$  the trivial subfoliation  $(TL_2,TL_2)$ . Therefore, the restriction to  $L_2$  of an  $(F_1,F_2)$ -foliated bundle  $P=P_1+P_2$  is a flat bundle, and hence we obtain a commutative diagram similar to (5.4):



If we now assume that  $L_1$  contains  $L_2$ ,  $j_0:L_2\longrightarrow L_1$  being the canonical immersion with  $j_2=j_1\circ j_0$ , then, using Proposition 3.1 and taking the closed subgroups

$$H_1 \subset G_1$$
,  $H_2 \subset G_2$ ,  $H = H_1 \times H_2$ 

and  $\rho_i\colon G=G_1\times G_2\longrightarrow G_i$  , i=1,2 , the canonical projections, there is a commutative diagram



All these results, when particularized in certain examples, provide a starting point for a study of the holonomy of the leaves of a subfoliation similar to that of Goldman [6] for the leaves of a foliation.

Example. – With the previous notations, let

$$P = L(Q_1) + L(Q_0)$$

be the bundle of transverse frames of  $(F_1,F_2)$  and let  $(L_1,L_2)$  be a "leaf" of  $(F_1,F_2)$  (that is,  $L_i$  leaf of  $F_i$  and  $L_2 \subset L_1$ ); then  $P'=j_2^*P=L(j_0^*(\nu L_1))+L(j_2^*(Q_0))$  is a reduction of the bundle of frames of the "normal bundle of  $(L_1,L_2)$ " defined as  $\nu(L_1,L_2)=j_0^*(\nu L_1)\oplus j_2^*(Q_0)$ ,  $\nu L_1$  being the normal bundle of the leaf  $L_1$  [6], that is,  $\nu L_1=Q_1/L_1$ . With a terminology analogous to that of Goldman, we call leaf connection any connection in P' obtained by pull-back of any adapted connection sum in P. Then, the following proposition can be easily proved:

PROPOSITION 5.3. – There exists a unique leaf connection in P'. Moreover, this connection is flat.

Through this result, we can state easily vanishing and obstruction theorems for the leaves of a subfoliation similar

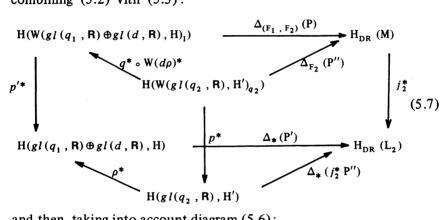
to those in [6] for the leaves of a foliation, because the real Pontrjagin ring of  $\nu(L_1, L_2)$  is trivial and this fact provides a necessary condition for a pair (N<sub>1</sub>, N<sub>2</sub>) of connected manifolds, with injective immersions  $j_i: N_i \longrightarrow M$ ,  $i = 1, 2, j_0: N_2 \longrightarrow N_1$ and  $j_2 = j_1 \circ j_0$ , to be a leaf of a subfoliation on M.

Now, if 
$$P'' = L(Q_2)$$
 and  $\rho: G1(q_1, \mathbb{R}) \times G1(d, \mathbb{R}) \longrightarrow G1(q_2, \mathbb{R})$ 

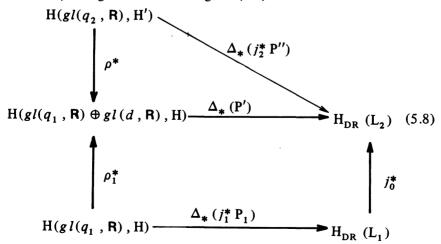
is the canonical homomorphism, then, using the example that follows Proposition 3.2 and taking a closed subgroup

$$H' \subset G1(q_2, \mathbf{R})$$

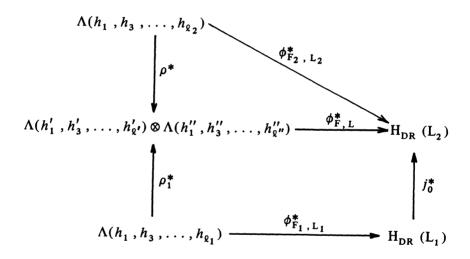
such that  $\rho(H) \subset H'$ , we first obtain a commutative diagram combining (5.2) vith (5.5):



and then, taking into account diagram (5.6):

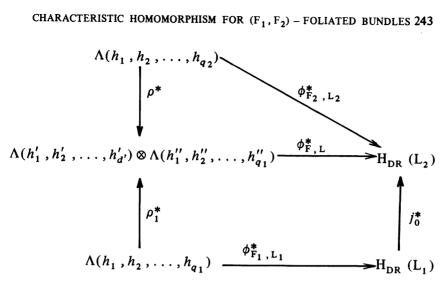


Now, we assume  $H=O(q_1)\times O(d)$  and  $H'=O(q_2)$ . In this case Goldman shows that  $p^*$  in diagram (5.7) is the zero homomorphism and concludes that the secondary foliation classes of  $F_2$  vanish in the leaves  $L_2$ . Essentially with the same arguments, one can prove that the homomorphism  $p'^*$  in diagram (5.7) is also zero and assert that the restriction to  $L_2$  of every secondary subfoliation class of  $(F_1,F_2)$  vanishes. Moreover, the homomorphism  $\Delta_*(P')$  is similar to the holonomy homomorphism defined by Goldman, and hence it can be called the holonomy homomorphism of the leaf  $(L_1,L_2)$  and denoted by  $\phi_{F,L}^*$ . Then, diagram (5.8) relate the holonomy homomorphism of  $(L_1,L_2)$  with that of each  $L_l$ , i=1,2. Through the canonical isomorphisms we obtain the following commutative diagram:



where 
$$\ell_i = 2[(q_i + 1)/2] - 1$$
,  $i = 1, 2$ ;  $\ell' = 2[(d + 1)/2] - 1$ ;  $\ell'' = \ell_1$ .

Obviously, the case of a subfoliation with trivialized normal bundle can be also discussed; to do that, it suffices to take H' as the trivial subgroup, and the diagram (5.8) becomes



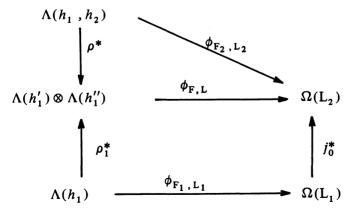
This result may be used in order to obtain topological obstructions to the existence of subfoliations. Reinhart [10] exhibits a first example of these obstructions which can be expressed in our language as follows.

Let  $(F_1, F_2)$  be a (1,2)-codimensional subfoliation on a manifold M with trivialized normal bundle; suppose  $F_1$  defined by the global 1-form  $\alpha_2$  and  $\,F_2\,$  defined by the global 1-forms  $\alpha_1$  ,  $\alpha_2$  . Hence there exist 1-forms  $au_{11}$  ,  $au_{21}$  ,  $au_{22}$  on  $ext{M}$  such that  $d\alpha_1 = \alpha_1 \wedge \tau_{11} + \alpha_2 \wedge \tau_{21}$ ,  $d\alpha_2 = \alpha_2 \wedge \tau_{22}$ .

If  $(L_1, L_2)$  is a leaf of  $(F_1, F_2)$ , let us consider the 1-forms on L<sub>2</sub> given by

$$\tau_{11}^{L} = j_{2}^{*}(\tau_{11}), \quad \tau_{21}^{L} = j_{2}^{*}(\tau_{21}), \quad \tau_{22}^{L} = j_{2}^{*}(\tau_{22}).$$

In this case, the previous diagram writes, at the cochain level, as



and, from it, we obtain the following holonomy classes:

a) for  $L_1$  as leaf of  $F_1$ :

$$\phi_{F_1,L_1}^*(h_1) = [\tau_{22/L_1}] \in H_{DR}(L_1)$$

b) for  $L_2$  as leaf of  $F_2$ :

$$\phi_{\text{F}_2,\text{L}_2}^*(h_1) = [\tau_{11}^{\text{L}} + \tau_{22}^{\text{L}}] \in \mathcal{H}_{\text{DR}}(\mathcal{L}_2), \quad \phi_{\text{F}_2,\text{L}_2}^*(h_2) = 0$$

since  $h_2 \in \text{Ker } \rho^*$ . In fact, Reinhart shows the vanishing of  $\phi_{F_2,L_2}^*(h_2)$  through a direct computation.

c) for  $(L_1, L_2)$  as leaf of  $(F_1, F_2)$ :

$$\begin{split} \phi_{\mathrm{F},\,\mathrm{L}}^{*}(h_{1}') &= [\tau_{11}^{\mathrm{L}}] \!\in\! \mathrm{H}_{\mathrm{DR}}^{}\left(\mathrm{L}_{2}\right), \quad \phi_{\mathrm{F},\,\mathrm{L}}^{*}(h_{1}'') = [\tau_{22}^{\mathrm{L}}] \!\in\! \mathrm{H}_{\mathrm{DR}}^{}\left(\mathrm{L}_{2}\right) \\ \phi_{\mathrm{F},\,\mathrm{L}}^{*}(h_{1}' + h_{1}'') &= [\tau_{11}^{\mathrm{L}} + \tau_{22}^{\mathrm{L}}] \!\in\! \mathrm{H}_{\mathrm{DR}}^{}\left(\mathrm{L}_{2}\right). \end{split}$$

Now, by comparing with Reinhart results one can deduce:

- 1) the vanishing of certain holonomy classes of  $L_2$  follows from the fact that they are obtained from elements of  $Ker \rho^*$ .
- 2) the image of  $\Lambda(h'_1)$  by  $\phi_{F,L}^*$  gives holonomy classes which cannot be obtained if we consider each leaf separately.

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