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NON-DEGENERESCENCE
OF SOME SPECTRAL SEQUENCES

by K.S. SARKARIA

1. Introduction.

Given a Lie algebra $\mathcal{F}$ of vector fields (smooth sections of the complexified tangent bundle) of a smooth manifold $M^m$ one has a decreasing filtered complex

$$A(M) = A_0(\mathcal{F}) \supset A_1(\mathcal{F}) \supset \ldots \supset A_m(\mathcal{F}) \supset A_{m+1}(\mathcal{F}) = 0; \quad (1)$$

here $A(M)$ is the differential graded algebra of all smooth complex valued forms on $M$ (the de Rham complex of $M$) and, for each $i > 0$, $A_i(\mathcal{F})$ denotes the $i$th power of the ideal generated by 1-forms which vanish on $\mathcal{F}$. Like any filtered complex, (1) has an associated spectral sequence which will be denoted by $E^k(S?)$. (We will follow Griffiths and Harris [2] regarding terminology for spectral sequences.) Two special cases of this construction will be important for us. (a) If $M^{2n}$ is a complex manifold and $\mathcal{F}$ consists of vector fields which in local complex coordinates $z_1, z_2, \ldots, z_n$ are of the form $\sum_j \varphi_j \frac{\partial}{\partial z_j}$, then $E_k(\mathcal{F})$ is the well known Fröhlicher spectral sequence [1] of the complex manifold $M$. The usual notation for $E^p,q_1(\mathcal{F})$ is $H^q(M, \Omega^p)$: the $q$th cohomology of $M$ with coefficients in the sheaf $\Omega^p$ of germs of holomorphic $p$-forms on $M$. (b) If $M$ carries a smooth foliation and $\mathcal{F}$ consists of all vector fields tangent to the leaves, then $E_k(\mathcal{F})$ will be called the spectral sequence of the foliated manifold $M$. It has been considered e.g. in [3].
THEOREM. — (A) For all integers \( m \geq 1 \), \( 0 < c < m \), the \( m \)-torus \( T^m \) admits a codimension \( c \) real analytic foliation such that \( E^\mu,0(\mathcal{F}) \) is infinite dimensional; here \( \mu = \min(c, m - c + 1) \).

(B) For all integers \( n \geq 1 \), \( T^{2n-1} \times \mathbb{R} \) admits a complex structure such that \( E^n,0(\mathcal{F}) \) is infinite dimensional. (C) For all integers \( m \geq 1 \), \( \mathbb{R}^m \) admits a compactly supported Lie algebra \( \mathcal{F} \) of vector fields such that \( E^m,0(\mathcal{F}) \) is infinite dimensional.

This theorem (which will be proved in § 2) is best possible in the sense that dimension considerations show that \( E^k(\mathcal{F}) \) is isomorphic to the de Rham cohomology for \( k > \mu + 1 \) (resp. \( k > n + 1 \), resp. \( k > m + 1 \)) and so is finite dimensional for these values of \( k \). We note that the cohomology considered by Schwarz [4] is precisely the \( E^2,0(\mathcal{F}) \) of a foliated manifold. In response to a question posed by Bott, he constructed smooth (non-analytic) foliations of compact manifolds for which \( E^2,0(\mathcal{F}) \) is infinite dimensional. Thus (A) can be considered as an improvement on [4]; besides our construction is different and simpler and leads to foliations which are of a natural and non-pathological kind. Regarding (B) we remark that the degenerescence problem for compact complex manifolds is much more delicate; we hope to give some results about it in a subsequent paper.

2.

In each of the examples (A)–(C) we will check, for \( k = \mu, n \) and \( m \) respectively, that \( \text{Im} \{ H^k(A_k(\mathcal{F})) \rightarrow H^k(A_1(\mathcal{F})) \} \) is infinite dimensional. The infinite dimensionality of \( E^k,0(\mathcal{F}) \) (see [2], p. 441 for definition) would follow as an immediate consequence.

Proof of (A) (We ignore the trivial cases \( c = 0, m \).) — The torus \( T^m \) will be considered as the quotient \( \mathbb{R}^m/(2\pi \mathbb{Z})^m \). Furthermore we put \( \nu = m - (2\mu - 1) \) and identify \( \mathbb{R}^m \) with \( \mathbb{R}^{\mu-1} \times \mathbb{R}^{\mu-1} \times \mathbb{R} \times \mathbb{R}^\nu \); thus a point of \( T^m \) will have coordinates \((\theta_{1,0}, \ldots, \theta_{\mu-1,0}; \theta_{1,1}, \ldots, \theta_{\mu-1,1}; r; t_1, \ldots, t_\nu)\). Note that \( \mu = \ell + 1 \) (here \( \ell = m - c \)) or \( \mu = c \); correspondingly \( \mu - 1 = \ell \) or \( \mu - 1 + \nu = \ell \). We define \( \mu - 1 \) real analytic vector fields on \( T^m \) by

\[
X_i = \frac{\partial}{\partial \theta_{i,0}} + \sin r \cdot \frac{\partial}{\partial \theta_{i,1}}, \quad 1 \leq i \leq \mu - 1.
\]
In case $\ell > \mu - 1$ we also take the vector fields $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_\nu}$. Taken together these $\ell$ vector fields span an Abelian $\ell$ dimensional Lie algebra $\mathcal{A}$ of analytic vector fields. At each point of $T^m$ these vector fields are linearly independent; so they determine a real analytic $\ell$ dimensional tangent plane field on $T^m$. By Frobenius theorem this involutive plane field is tangent to a real analytic $\ell$ dimensional foliation of $T^m$. We will prove that $E_\mu^0(\mathcal{F})$ is infinite dimensional for this foliation. (Note that $\mathcal{A}$ and $\mathcal{F}$ have the same filtration (1) and so have the same spectral sequence.)

For each smooth function $\varphi(r)$ of period $2\pi$ we define a closed form $\Omega_\varphi$ of degree $\mu$ on $T^m$ by

$$\Omega_\varphi = \sum_{(\alpha_1, \ldots, \alpha_{\mu-1})} \varphi(r)(-\sin r)^{\mu-1-\alpha} dr \wedge d\theta_{1, \alpha_1} \wedge \ldots \wedge d\theta_{\mu-1, \alpha_{\mu-1}};$$

where the summation is over all multi-indices $(\alpha_1, \ldots, \alpha_{\mu-1})$ with entries 0 or 1, and $\alpha = \alpha_1 + \ldots + \alpha_{\mu-1}$. Using (2) it follows that the interior products $\iota_{X_i}(\Omega_\varphi)$ and $\iota_{\theta_k}(\Omega_\varphi)$ vanish for $1 \leq i \leq \mu - 1$ and $1 \leq k \leq \nu$. Thus the forms $\Omega_\varphi$ constitute an infinite dimensional subspace of closed degree $\mu$ forms of $\mathcal{A}_\mu(\mathcal{F})$. We assert that if $\Omega_\varphi = d\omega$ where $\omega \in \mathcal{A}_1(\mathcal{F})$, then the form $\Omega_\varphi$ is the zero form (i.e. $\varphi \equiv 0$). This will suffice to prove the infinite dimensionality of $\text{Im } \{H^\mu(\mathcal{A}_\mu(\mathcal{F})) \to H^\mu(\mathcal{A}_1(\mathcal{F}))\}$.

Let $T^{2\mu-2}$ denote the subtorus of $T^m$ obtained by putting the last $\nu + 1$ coordinates equal to zero. For each $\theta \in T^{2\mu-2}$ we have the translation $L_\theta : T^m \to T^m$ given by $L_\theta(u) = \theta + u$. From (3) we see that $\Omega_\varphi$ is preserved by this action of $T^{2\mu-2}$ on $T^m$. Since the infinitesimal generators of this action are $\frac{\partial}{\partial \theta_{i, \alpha_i}}$ and $\left[ \frac{\partial}{\partial \theta_{i, \alpha_i}}, X_k \right] = 0 = \left[ \frac{\partial}{\partial \theta_{i, \alpha_i}}, \frac{\partial}{\partial t_k} \right]$, we conclude also that our Lie algebra $\mathcal{A}$ is preserved by this action. Hence if $\omega \in \mathcal{A}_1(\mathcal{F})$ is such that $d\omega = \Omega_\varphi$ we can assume in addition that $\omega$ is also preserved by this action; for, otherwise, we can replace $\omega$ by the form $\int_{T^{2\mu-2}} L_\theta^*(\omega) \, d\theta$ obtained by averaging with respect to the normalized Haar measure $d\theta$ of $T^{2\mu-2}$. Let us now write
the expression of $\omega$ in the chosen coordinate system. We have

$$\omega = \sum_{(a_1, \ldots, a_{\mu-1})} \omega_{a_1} \ldots a_{\mu-1} \ d\theta_1, a_1 \wedge \ldots \wedge d\theta_{\mu-1}, a_{\mu-1} + \tilde{\omega}$$  \hspace{1cm} (4)

where $\tilde{\omega}$ denotes all terms other than those written out in the beginning. Since $\omega$ is preserved by the action of $T^{2\mu-2}$ every coefficient of $\omega$ in (4) is a function only of $r, t_1, \ldots, t_{\mu}$. But $d\omega = \Omega_\varphi$ where $\Omega_\varphi$ is given by (3). This shows that $d\tilde{\omega} = 0$ and that the coefficients $\omega_{a_1} \ldots a_{\mu-1}$ are functions only of $r$.

Since $\omega$ is a degree $\mu - 1$ form in $A_1(\mathbb{S})$ we should get zero when the operator $\ ^\omega X_{\mu-1} \circ \ldots \circ X_1$ is applied to (4). It is clear that on applying this to $\tilde{\omega}$ we get zero. On applying it to the first part we get the condition

$$\sum_{(a_1, \ldots, a_{\mu-1})} (\sin r)^a \omega_{a_1} \ldots a_{\mu-1} = 0.$$  \hspace{1cm} (5)

Next we use the conditions $\ ^\omega X_i (d\omega) = 0$, $1 \leq i \leq \mu - 1$ on (4). We see that for each multi-index $(a_1, \ldots, a_{\mu-1})_k$ of length $\mu - 2$ obtained by omitting the $k$th entry we have

$$\omega'_{a_1} a_2 \ldots 0 \ldots a_{\mu-1} + \sin r \cdot \omega'_{a_1} a_2 \ldots 1 \ldots a_{\mu-1} = 0$$  \hspace{1cm} (6)

where primes denote differentiation with respect to $r$ and 0 and 1 are placed at the $k$th place. This in turn implies the following identities

$$\sum_{s \geq 2} \frac{\alpha!}{(\alpha - s)!} (\sin r)^{\alpha-s} \omega'_{a_1} a_2 \ldots a_{\mu-1} = 0, \ 0 \leq s \leq \mu - 2.$$  \hspace{1cm} (7)_s

(To check (7), use (6) and the binomial identities

$$\sum_{\alpha=2}^{\mu-1} \frac{\alpha!}{(\alpha - s)!} \left( \frac{\mu - 1}{\alpha} \right) (-1)^{\alpha} = 0.)$$

We now differentiate (5) with respect to $r$ and use (7) to get

$$\cos r \sum_{(a_1, \ldots, a_{\mu-1}) \ s.t. \alpha \geq 1} \alpha (\sin r)^{\alpha-1} \cdot \omega_{a_1} a_2 \ldots a_{\mu-1} = 0$$

which implies —since $\cos r$ is non-zero on an open dense subset of $T^m$— that

$$\sum_{(a_1, \ldots, a_{\mu-1}) \ s.t. \alpha \geq 1} \alpha (\sin r)^{\alpha-1} \cdot \omega_{a_1} a_2 \ldots a_{\mu-1} = 0.$$
Next we differentiate this and use (7), etc., etc.; finally we get $\omega_{11\ldots1} = 0$. Now by using (6) it follows that $\omega_{a_1 a_2 \ldots a_{n-1}} = 0$ for all multi-indices $(\alpha_1, \ldots, \alpha_{n-1})$ which means that $d\omega = \Omega_\varphi = 0$.

**Proof of (B).** We identify $T^{2n-1} \times \mathbb{R}$ with the quotient $(\mathbb{C}^{n-1} \times \mathbb{C})/\Gamma$; here $\Gamma$ is the subgroup of $\mathbb{C}^{n-1} \times \mathbb{C}$ consisting of elements $(w_1, \ldots, w_{n-1}; u)$ with $u$ and each $\text{Re } w_k, \text{Im } w_k$ an integral multiple of $2\pi$. Now let $\mathcal{E}$ be the Abelian $n$ dimensional Lie algebra of vector fields of $T^{2n-1} \times \mathbb{R}$ spanned by $\frac{\partial}{\partial w_k}$ and

$$X_k = \frac{\partial}{\partial w_k} + 2e^{iu} \frac{\partial}{\partial w_k}, \quad 1 \leq k \leq n-1. \quad (8)$$

We note that at each point $x$ of $T^{2n-1} \times \mathbb{R}$ these $n$ vector fields are linearly independent; so they span an $n$ dimensional subspace $\mathcal{F}(x)$ of the complexified tangent space $T(x)$. Furthermore $\mathcal{F}(x) + \overline{\mathcal{F}(x)} = T(x)$ (because $\frac{\partial}{\partial w_k} + 2e^{iu} \frac{\partial}{\partial w_k}$ is always linearly independent from $\frac{\partial}{\partial w_k} + 2e^{-iu} \frac{\partial}{\partial w_k}$). By the well known complex Frobenius theorem (of Newlander and Nirenberg) this involutive almost complex structure is integrable i.e. we can choose local complex coordinates $z_1, z_2, \ldots, z_n$ so that the smooth sections of $\mathcal{F}$ are precisely those vector fields which are of the type $\sum \varphi_j \frac{\partial}{\partial z_j}$. We assert that $E_n^0(\mathcal{F})$ is infinite dimensional for this complex structure.

For each entire function $\varphi(u)$ of period $2\pi$ we define a closed form of degree $n$ on $T^{2n-1} \times \mathbb{R}$ by

$$\Omega_\varphi = \sum_{(\alpha_1, \ldots, \alpha_{n-1})} \varphi(u) (-2e^{iu})^{n-\alpha} \ du \wedge dw_1, \alpha_1 \wedge \ldots \wedge dw_{n-1}, \alpha_{n-1}. \quad (3')$$

Here the summation is over all multi-indices $(\alpha_1, \ldots, \alpha_{n-1})$ with entries 0, 1 and $\alpha = \alpha_1 + \ldots + \alpha_{n-1}$; furthermore $dw_k, \alpha_k$ denotes $dw_k$ if $\alpha_k = 0$ and $d\bar{w}_k$ if $\alpha_k = 1$. We can now prove, by a method entirely analogous to that in part (A), that these $\Omega_\varphi$ constitute an infinite dimensional subspace of closed forms in $A_n(\mathcal{F})$ and that no non-zero $\Omega_\varphi$ can be the boundary of a form $\omega \in A_1(\mathcal{F})$. 

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Proof of (C). – Case \( m = 1 \) is trivial (take \( \mathcal{F} = 0 \)); so we assume \( m \geq 2 \). Let us take the closed disk \( D \) in \( \mathbb{R}^2 \) with origin as centre and radius 1. In its interior we choose a countably infinite number of disjoint closed disks \( D_\beta \). Let us choose polar coordinates \((r_\beta, \phi_\beta)\) for each \( D_\beta \); so \( D_\beta \) is given by

\[
0 \leq \phi_\beta < 2\pi, \quad 0 \leq r_\beta \leq \rho_\beta
\]

where \( \rho_\beta \) is the radius of \( D_\beta \). On the torus \( \mathbb{T}^{m-2} = \mathbb{R}^{m-2}/\mathbb{Z}^{m-2} \) we choose coordinates \( \theta_1, \theta_2, \ldots, \theta_{m-2} \). Now we define \( m-1 \) smooth vector fields on \( \mathbb{T}^{m-2} \times \mathbb{D}^2 \) by

\[
X_i = \sum_\beta c_\beta f_\beta(r_\beta) \frac{\partial}{\partial \theta_i} \quad \text{if } 1 \leq i < m-1
\]

\[
= \sum_\beta c_\beta f_\beta(r_\beta) \frac{\partial}{\partial \phi_\beta} \quad \text{if } i = m-1.
\]

(9)

Here \( f_\beta(r_\beta) \) is a smooth function on \( \mathbb{R}^+ \), positive on \( (\frac{\rho_\beta}{2}, \rho_\beta) \) and zero outside this interval, while the positive constants \( c_\beta \) are so chosen that the sums in (9) converge in the \( C^\infty \) topology. One notes that \( [X_i, X_j] = 0 \) \( \forall i,j \) and that each vector field \( X_i \) vanishes on the boundary \( \mathbb{T}^{m-2} \times \partial \mathbb{D}^2 \) of our solid torus \( \mathbb{T}^{m-2} \times \mathbb{D}^2 \). The latter can be smoothly imbedded in \( \mathbb{R}^m \) and we can extend these vector fields to all of \( \mathbb{R}^m \) by defining them to be zero outside the solid torus. This gives us an Abelian \( m-1 \) dimensional Lie algebra \( \mathcal{F} \) of compactly supported vector fields on \( \mathbb{R}^m \); we will prove that \( E^{m,\mathcal{F}}_m(\mathcal{F}) \) is infinite dimensional.

For each \( \beta \), let us choose a smooth \( m \)-form \( \Omega_\beta \) supported inside the region of \( \mathbb{T}^{m-2} \times D_\beta \) given by \( r_\beta < \frac{\rho_\beta}{2} \) and such that

\[
\int_{\mathbb{T}^{m-2} \times D_\beta} \Omega_\beta \neq 0.
\]

(10)

Note that each \( \Omega_\beta \) lies in \( A_\mathcal{F}_m(\mathcal{F}) \). If possible let us suppose that \( \Omega \) is a non-trivial finite linear combination of the \( \Omega_\beta \) such that \( \Omega = d\omega \) where \( \omega \) is a smooth \( m-1 \) form lying in \( A_{m-1}(\mathcal{F}) \). This last condition implies that \( \omega(X_1, \ldots, X_{m-1}) = 0 \) and so in \( \mathbb{T}^{m-2} \times D_\beta \) we have

\[
(c_\beta f_\beta(r_\beta))^{m-1} \omega \left( \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{m-2}}, \frac{\partial}{\partial \phi_\beta} \right) = 0.
\]

(11)
In the region $\frac{\rho_\beta}{2} < r_\beta < \rho_\beta$ the coefficient of (11) is non-zero; so in this domain we must have
\[
\omega\left(\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{m-2}}, \frac{\partial}{\partial \phi_{\beta}}\right) = 0
\]
which remains valid on $r_\beta = \rho_\beta$ by continuity. Hence $\omega$ induces, on the boundary of each $T^{m-2} \times D_\beta$, the zero form. By Stoke's theorem this contradicts (10). Thus we have proved that
\[
\text{Im} \{H^m (A_1(\mathcal{F})) \longrightarrow H^m (A_1(\mathcal{F}))\}
\]
has infinite dimension.

3. Remarks.

The proof of (C) given above is inspired by the work of Schwarz [4]. We note that it is possible to use analogous ideas to construct some more smooth (non-analytic) codimension $c$ foliations on $T^m$ with $E_{\mu,\phi}(\mathcal{F})$ infinite dimensional. One can also ensure that the singular set of such foliations is nowhere dense. (The singular set is made up of those points where the infinitesimal transformations of the foliation fail to span the tangent space. In [3] it is proved that for a compact foliated manifold with empty singular set, $E_2(\mathcal{F})$ is finite dimensional.) In another direction we can prove that if a smooth manifold $M^m$ admits one smooth codimension $c$ foliation, then it admits another with $E_{\mu,\phi}(\mathcal{F})$ infinite dimensional.

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