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Estimates of one-dimensional oscillatory integrals


<http://www.numdam.org/item?id=AIF_1983__33_4_189_0>
1. Introduction.

If $U$ is an open domain in $\mathbb{R}^k$ and if $f$ is a smooth, real valued function on $U$, one may define the associated oscillatory integral as

$$E_f(\vartheta) = \int_U \vartheta(x)e^{2\pi i f(x)} \, dx,$$

where $\vartheta$ belongs to $\mathcal{D}(U)$, the space of test functions on $U$.

When $f$ has the form $f = \sum_{j=1}^{n} \eta_j \psi_j$, where the $\psi_j \in C^\infty(U)$ are real-valued functions and $\eta_j$ are real parameters, one is interested in the asymptotic behaviour of $E_{\Sigma \eta \psi}(\vartheta)$ as $(\eta_1, \ldots, \eta_n)$ tends to infinity, for several reasons.

For example, if $\mu$ is a smooth measure on a smooth submanifold of $\mathbb{R}^m$, and if the support of $\mu$ is sufficiently small, then the Fourier-Stieltjes transform $\hat{\mu}(\eta_1, \ldots, \eta_n)$ may always be written as $E_{\Sigma \eta \psi}(\vartheta)$ for certain functions $\psi_j$ and $\vartheta$.

Good information about the asymptotic behaviour of such Fourier-Stieltjes transforms is needed to solve the synthesis problem for smooth submanifolds of $\mathbb{R}^m$ (see e.g. [7]). And, as Professor Y. Domar has pointed out to me, such knowledge would also yield information about the decay at infinity of solutions of partial differential equations (see e.g. [5]).

As far as I know, satisfactory answers to the above problem have only been given for oscillatory integrals $E_{\Sigma \eta \psi}(\vartheta)$ with

$$\Sigma \eta_j \psi_j(x_1, \ldots, x_k) = \sum_{j=1}^{k} \eta_j x_j + \eta_{k+1} \psi_{k+1}(x_1, \ldots, x_k),$$
which correspond to surface carried measures (see [2], [4], [6]). In some sense, the other extreme is the case where $\sum \eta_j \psi_j$ is a function of only one real variable, which corresponds to measures on curves. For this case, we will prove some quite general results.

2.

Let $\psi \in C^\infty(I,\mathbb{R}^n)$, $\psi = (\psi_1, \ldots, \psi_n)$, where $I \neq \emptyset$ is some bounded open interval in $\mathbb{R}$. For $\xi, \eta \in \mathbb{R}^n$ let $\xi \cdot \eta$ denote the Euclidean inner product on $\mathbb{R}^n$, and correspondingly let

$$\eta \cdot \psi(x) = \sum_{j=1}^n \eta_j \psi_j(x).$$

Further let

$$|\eta| := \max_j |\eta_j| \quad \text{for} \quad \eta \in \mathbb{R}^n.$$ 

Define the torsion $\tau$ of $\psi$ by

$$\tau(x) = \det (\psi_j^{(i+1)}(x))_{i,j=1,\ldots,n} = \det (\psi''(x)\psi'''(x)\ldots\psi^{(n+1)}(x)),$$

where $\psi$ is regarded as a column vector and $\psi^{(k)}$ denotes the $k$-th derivative of $\psi$. At least for $n = 2$ we have $\tau(x) = k(x)|\psi''(x)|^2$, where $k$ is the torsion of the curve $\gamma = \{(x,\psi(x)) : x \in I\}$ in $\mathbb{R}^{n+1}$. Let

$$e(t) = e^{2\pi it} \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and} \quad e(g) = e \circ g$$

for $g \in C^\infty(I,\mathbb{R})$. If $\psi_0(x) = x$ for $x \in \mathbb{R}$, then for $\theta \in \mathcal{D}(I)$, $\eta_0 \in \mathbb{R}$ and $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$, we have

$$E_n \sum_{\theta} \eta \psi_j = (\theta e(\eta \cdot \psi))(-\eta_0).$$

So it will be slightly more general to study the behaviour of $|\theta e(\eta \cdot \psi)|_{PM}$ as $|\eta| \to \infty$, where

$$|\varphi|_{PM} = \sup_{t \in \mathbb{R}} |\hat{\varphi}(t)|$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$. 
For certain reasons (see [3]; [7], Th. 4.1), we will also study $|3e(\eta \cdot \psi)|_\Lambda$, where

$$|\psi|_\Lambda = \int |\hat{\psi}(t)| \, dt$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$.

We will first state our main results and prove some corollaries:

**Theorem 1.** — Let $\vartheta \in \mathcal{D}(I)$. Then

(i) $|\vartheta(\eta \cdot \psi)|_\Lambda = O(|\eta|^{\frac{1}{2}})$, as $|\eta| \to \infty$.

(ii) If for some subinterval $J$ of $I$ and some $\sigma > 0$

$$|\vartheta(x)| \geq \sigma \quad \text{and} \quad |\vartheta(x) - \vartheta(y)| < \sigma/2 \quad \text{for all} \quad x, y \in J,$$

and if $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine linear functions, then there is a constant $C > 0$, such that

$$|\vartheta(\eta \cdot \psi)|_\Lambda \geq C(1 + |\eta|)^{\frac{1}{2}}$$

for all $\eta \in \mathbb{R}^n$.

**Corollary 1.** — The following two conditions are equivalent:

(i) For each $\vartheta \in \mathcal{D}(\mathbb{R})$, $\vartheta \neq 0$, there are constants $c > 0$, $C > 0$, such that for all $\eta \in \mathbb{R}^n$

$$c(1 + |\eta|)^{\frac{1}{2}} \leq |\vartheta(\eta \cdot \psi)|_\Lambda \leq C(1 + |\eta|)^{\frac{1}{2}}.$$

(ii) $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine linear functions on every non empty open subinterval of $I$.

**Proof of Corollary 1.** — (i) follows directly from (ii) by Theorem 1. Now suppose that there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$, such that $v \cdot \psi$ is affine linear on some open subinterval $J \neq \emptyset$ of $I$. Then we have for any non-trivial $\vartheta \in \mathcal{D}(J)$

$$|\vartheta(sv \cdot \psi)|_\Lambda = |\vartheta|_\Lambda \neq 0 \quad \text{for all} \quad s \in \mathbb{R},$$

since $e(sv \cdot \psi)$ is the product of a unimodular complex number and a unitary character of $\mathbb{R}$.

Thus (i) is not fulfilled, q.e.d.
Remark. — Condition (ii) of Corollary 1 is clearly satisfied if \( \tau^{-1}(\{0\}) \) has empty interior. As will be shown later (Lemma 3), this is always the case if \( \psi_1, \ldots, \psi_n \) are real analytic and linearly independent modulo affine mappings. However one should notice that global linear independence does not in general imply local linear independence.

**Theorem 2.**

(i) If \( \tau^{-1}(\{0\}) = \emptyset \), then for \( \theta \in \mathcal{D}(I) \)

\[
|\theta e(\eta \cdot \psi)|_{PM} = 0(|\eta|^{-1/(n+1)}) \quad \text{as} \quad |\eta| \to \infty.
\]

(ii) If \( \theta \in \mathcal{D}(I) \), and if there exists an \( x_0 \in I \) with \( \theta(x_0) \neq 0 \) and \( \tau(x_0) \neq 0 \), then there exists an \( \varepsilon > 0 \) and a function \( \xi \in C^\infty((-\varepsilon, \varepsilon), \mathbb{R}^n) \) with

\[
\det (\xi(y) \xi'(y) \ldots \xi^{(n-1)}(y)) \neq 0 \quad \text{for all} \quad y \in (-\varepsilon, \varepsilon),
\]

such that, for some \( C > 0 \),

\[
|\theta e(s \xi(y) \cdot \psi)|_{PM} \geq C(1 + |s|^{-1/(n+1)})
\]

for all \( s \in \mathbb{R} \) and \( y \in (-\varepsilon, \varepsilon) \).

Assume that \( \tau^{-1}(\{0\}) \) has empty interior. Then we have

**Corollary 2.** — There exists a \( \theta \in \mathcal{D}(I) \), \( \theta \neq 0 \), such that for all positive \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) with \( \sum_1^n \alpha_j \leq (n+1)^{-1} \), there exists a constant \( C = C(\alpha_1, \ldots, \alpha_n) > 0 \) such that

\[
|\theta e(\eta \cdot \psi)|_{PM} \leq C \prod_{j=1}^{n} |\eta_j|^{-\alpha_j}.
\]

Conversely, if \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) are positive, and if there exists a \( \theta \in \mathcal{D}(I) \), \( \theta \neq 0 \), and a \( C > 0 \) such that (2.1) holds, then

\[
\sum_{j=1}^{n} \alpha_j \leq (n+1)^{-1}.
\]

**Proof of Corollary 2.** — If \( \tau^{-1}(\{0\}) \) has empty interior, then there is of course an \( x_0 \in I \) with \( \tau(x_0) \neq 0 \), and so, for \( \theta \in \mathcal{D}(I) \) with sufficiently small support near \( x_0 \),

\[
|\theta e(\eta \cdot \psi)|_{PM} \leq C(1 + |\eta|)^{-1/(n+1)}
\]

by Theorem 2, (i).
If \( \alpha_1, \ldots, \alpha_n \) are positive and \( \Sigma \alpha_j \leq (n+1)^{-1} \), then
\[
\prod_j |\eta_j|^2 \leq |\eta|^{1/(n+1)} \quad \text{for} \quad |\eta| \geq 1,
\]
hence
\[
|\vartheta(\eta \cdot \psi)|_{PM} \leq C \prod_j |\eta_j|^{-2} \quad \text{for} \quad |\eta| \geq 1,
\]
and the same estimate holds for all \( \eta \) if one replaces \( C \) by \( C + |\vartheta|_{L^1} \).

Conversely, let now \( \vartheta \in \mathcal{D}(I), \vartheta \neq 0 \), such that (2.1) holds for some \( \alpha_j \geq 0 \), and assume
\[
\Sigma \alpha_j = (n+1)^{-1} + \delta, \quad \delta > 0.
\]
Since \( \tau^{-1}(\{0\}) \) has empty interior, there is an \( x_0 \in I \) with \( \vartheta(x_0) \neq 0 \) and \( \tau(x_0) \neq 0 \). Choose \( \varepsilon > 0 \) and \( \xi \in C^\infty((-\varepsilon, \varepsilon), \mathbb{R}^n) \) as in Theorem 2 (ii). Since \( \det (\xi(y)\xi'(y) \ldots \xi^{(n-1)}(y)) \neq 0 \) for all \( y \in (-\varepsilon, \varepsilon) \), there exists a \( y_0 \in (-\varepsilon, \varepsilon) \) with
\[
\xi_j(y_0) \neq 0 \quad \text{for} \quad j = 1, \ldots, n.
\]
It follows
\[
|\vartheta(s\xi_j(y_0) \cdot \psi)|_{PM} \geq C'(1 + |s|)^{-1/(n+1)}.
\]
On the other hand, (2.1) yields
\[
|\vartheta(s\xi_j(y_0) \cdot \psi)|_{PM} \leq C \prod_j |s\xi_j(y_0)|^{-2j} \leq \left( C \prod_j |\xi_j(y_0)|^{-2j} \right) |s|^{-1/(n+1)} |s|^{-\delta}.
\]
For \( |s| \) sufficiently large this leads to a contradiction to (2.2), q.e.d.

Corollary 2 demonstrates that the result in Theorem 2 is in some sense best possible.

3.

Before we start to prove the theorems above we will state some lemmas. The first one is due to J.-E. Björk and is cited in [3], Lemma 1.6:

**Lemma 1.** Let \( I \neq \emptyset \) be a bounded, open interval in \( \mathbb{R} \), and let \( \varphi \in \mathcal{D}(I), \, g \in C^p(I) \) with
\[
0 < C_1 \leq |g'(x)| + |g''(x)| + \cdots + |g^{(p)}(x)| \leq C_2
\]
if $x \in I$, where $C_1$ and $C_2$ are constants and $p$ is a positive integer. Then there exists a constant $C$ not depending on $g$, such that

$$\left| \int \varphi(x)e^{2\pi ig(x)} \, dx \right| \leq C(1+|t|)^{-1/p}$$

for every $t \in \mathbb{R}$.

The second lemma will be used to prove the remark following Corollary 1. I would like to thank Professor H. Leptin for pointing out to me a shorter proof than my original one. By $\langle \wedge \rangle$ we denote the exterior product in the Grassmann algebra $\Lambda(\mathbb{R}^n)$.

**Lemma 2.** – Let $\psi \in C^\infty(I, \mathbb{R}^n)$. Then

$$\psi(x) \wedge \psi'(x) \ldots \wedge \psi^{(n-1)}(x) = 0$$

for all $x \in I$ implies

$$\psi^{(k_1)}(x) \wedge \psi^{(k_2)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = 0$$

for all $x \in I$ and $k_1, \ldots, k_n \in \mathbb{N}_0$.

**Proof.** – Fix $x_0 \in I$, and assume first $\psi(x_0) \neq 0$. If $u \in C^\infty(I, \mathbb{R})$, then

$$(u\psi)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} u^{(k-j)} \psi^{(j)},$$

so $\psi \wedge \psi' \wedge \ldots \wedge \psi^{(n-1)} \equiv 0$ implies

$$(u\psi) \wedge (u\psi') \wedge \ldots \wedge (u\psi)^{(n-1)} \equiv 0.$$ 

So, it is no loss of generality to assume

$$\psi_n(x) = 1 \quad \text{for} \quad x \in I.$$

If $\{e_j\}_j$ denotes the canonical basis of $\mathbb{R}^n$, we may thus write

$$\psi(x) = \sum_{j=1}^{n-1} \psi_j(x)e_j + e_n = \rho(x) + e_n,$$  where $\rho(x) \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$.

This yields

$$0 = \psi(x) \wedge \psi'(x) \wedge \ldots \wedge \psi^{(n-1)}(x) = \rho(x) \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x) + e_n \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x),$$
and since $\rho(x), \rho'(x), \ldots, \rho^{(n-1)}(x)$ are clearly linearly dependent, we get

$$0 = \rho'(x) \wedge \rho''(x) \wedge \ldots \wedge \rho^{(n-1)}(x).$$

By induction over $n$, we now may assume

$$0 = \rho^{(k_2)}(x) \wedge \rho^{(k_2)}(x) \wedge \ldots \wedge \rho^{(k_n)}(x)$$

for $x \in I$ and $k_j \geq 1$.

This implies

$$\psi^{(k_1)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = e^{(k_1)}(x) \wedge \rho^{(k_2)}(x) \wedge \ldots \wedge \rho^{(k_n)}(x) = 0$$

for $0 \leq k_1 < k_2 < \cdots < k_n$, where we considered $e_n$ as the function $e_n(x) = e_n$.

Thus we have proved

$$\psi^{(k_1)}(x_0) \wedge \psi^{(k_2)}(x_0) \wedge \ldots \wedge \psi^{(k_n)}(x_0) = 0$$

for all $x_0 \in I_0 = \{x \in I : \psi(x) \neq 0\}$ and $k_j \geq 0$. By continuity, the same holds true for $x_0 \in I_0 \wedge I$, hence for all $x_0 \in I$, since for $y \in I \setminus I_0$ clearly $\psi^{(k)}(y) = 0$ for every $k \in \mathbb{N}_0$.

**Lemma 3.** If $\psi = (\psi_1, \ldots, \psi_n) \in C^\infty(I, \mathbb{R}^n)$ is real analytic, and if $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine mappings, then $\tau^{-1}(\{0\})$ has empty interior, where $\tau$ denotes the torsion of $\psi$.

**Proof.** Assume $\tau(x) = 0$ for every $x$ in some nonempty open interval $J \subset I$. Fix $x_0 \in J$. Then, passing to a possibly smaller interval, we may assume that $\psi_j$ has an absolute convergent series expansion

$$\psi_j(x) = \sum_{k=0}^{\infty} a_k^j (x-x_0)^k, \quad j = 1, \ldots, n, \quad x \in J.$$

Define vectors

$$a_k = (a_k^j)_{j=1, \ldots, n} \in \mathbb{R}^n$$

and

$$a^j = (a_k^j)_{k=2, \ldots, \infty} \in \mathbb{R}^n, \quad N_1 = \mathbb{N} \setminus \{0, 1\}.$$

By Lemma 2, $\psi^{(k_1)}(x_0), \ldots, \psi^{(k_n)}(x_0)$ are linearly dependent for any $k_j \in \mathbb{N}$ with $2 \leq k_1 < \cdots < k_n$, i.e. $a_1, \ldots, a_n$ are linearly dependent for $2 \leq k_1 < \cdots < k_n$. But this implies that $a^1, \ldots, a^n$ are linearly
dependent, i.e. there exist $v_1, \ldots, v_n \in \mathbb{R}$, not all zero, with
\[ 0 = \sum_j v_j a^j, \quad \text{i.e.} \]
\[ \sum_j v_j \psi_j(x) = \sum_j v_j a^j_0 + v_j a^j_1 (x - x_0) \quad \text{for} \quad x \in J. \]

But, since $\psi$ is real analytic, this equation holds for all $x \in I$, i.e. $\sum_j v_j \psi_j$ is affine linear.

4.

Proof of Theorem 1. — It is well-known (see e.g. [1], [7]) that for $\varphi \in \mathcal{D}(\mathbb{R})$ one has the estimate

\[ |\varphi|_A \leq \{ 2 |\text{supp } \varphi| |\varphi|_\infty |\varphi'|_\infty \}^{1/2}, \]

where $|\text{supp } \varphi|$ denotes the Lebesgue measure of the support of $\varphi$. From (4.1) one immediately gets (i) of Theorem 1.

Now, suppose there exists a subinterval $J$ in $I$ and a $\sigma > 0$ such that $|\varphi(x)| \geq \sigma$ and $|\varphi(x) - \varphi(y)| < \sigma/2$ for $x, y \in J$, and such that $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine mappings on $J$. Then a simple compactness argument yields:

There are constants $\varepsilon > 0$, $\delta > 0$, such that for every $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ there is an interval $J_\eta$ of length $2\varepsilon$ in $J$ with

\[ |\eta \cdot \psi(x)| \geq \delta \quad \text{for all} \quad x \in J_\eta. \]

Now choose $\varphi \in \mathcal{D}(-\varepsilon, \varepsilon)$, $\varphi \geq 0$, with $\int \varphi(x) \, dx = 1$. For fixed $\eta \in \mathbb{R}^n$, $\eta \neq 0$, set $\eta' = |\eta|^{-1} \eta$, and choose $J_{\eta'}$ as in (4.2). Let $\tilde{\varphi}$ be a suitable translate of $\varphi$ such that $\text{supp } \tilde{\varphi} \subset J_{\eta'}$. Then we get

\[ 0 < \sigma/2 \leq \left| \int g(x) \hat{\psi}(x) \, dx \right| \]
\[ = \left| \int g(x) e(\eta \cdot \psi)(x) \hat{\varphi}(x) e(-\eta \cdot \psi)(x) \, dx \right| \]
\[ \leq |g e(\eta \cdot \psi)|_A |\hat{\varphi} e(-\eta \cdot \psi)|_{PM}, \]

since $J_{\eta'} \subset J$. 
For $\xi \in \mathbb{R}$ one has
\[
\{\phi e(\eta \cdot \psi)\}(-\xi) = \int \phi(x)e(-\xi x - \eta \cdot \psi(x)) \, dx
\]
\[
= \int \varphi(x)e(-|\eta|g(x)) \, dx,
\]
where $g$ is a function on $[-\varepsilon, \varepsilon]$ which is a certain translate of the function
\[x \mapsto \xi'x + \eta' \cdot \psi(x) \quad \text{on} \quad J_n',\]
where $\xi' = |\eta|^{-1}\xi$.

But (4.2) implies
\[\delta \leq |g''(x)| \quad \text{for every} \quad x \in [-\varepsilon, \varepsilon].\]

Moreover, if we set $A = 2 \sup_{x \in J} |\psi'(x)|$, $B = \sup_{x \in J} |\psi''(x)|$, then for $|\xi| \leq A|\eta|$:
\[|g'(x)| + |g''(x)| \leq |\xi'| + |\eta'|(A + B) \leq 2A + B\]
for every $x \in [-\varepsilon, \varepsilon]$.

Thus, by Lemma 1, there exists a $C > 0$, such that for $|\xi| \leq A|\eta|$:
\[(4.4) \quad \left| \int \phi(x)e(-\xi x - \eta \cdot \psi(x)) \, dx \right| \leq C(1 + |\eta|)^{-1/2}.
\]

And, if $|\xi| > A|\eta|$, then integration by parts yields
\[(4.5) \quad \left| \int \phi(x)e(-\xi x - \eta \cdot \psi(x)) \, dx \right|
\]
\[= \left| \int e(-|\eta|g(x)) \left( \frac{\varphi}{2\pi i|\eta|g'}(x) \right) \, dx \right|
\]
\[\leq (2\pi|\eta|)^{-1} \int \left\{ \frac{|\varphi'(x)|}{|g'(x)|} + \frac{|\varphi(x)||g''(x)|}{|g'(x)|^2} \right\} \, dx
\]
\[\leq C'|\eta|^{-1},\]

where $C'$ is some constant depending on $\varphi$, $\psi$ and $A$ only, since for $x \in [-\varepsilon, \varepsilon]$ we have $|g''(x)| \leq B$ and $|g'(x)| = |\xi' + \eta' \psi'(y)| \geq A - A/2$ for some $y \in J$. 

Now, by (4.4), (4.5),
\[ |\Phi e(-\eta \cdot \psi)_{PM}| \leq (C+C')|\eta|^{-1/2} \quad \text{if} \quad |\eta| \geq 1, \]
which together with (4.3) proves Theorem 1 (ii).

**Proof of Theorem 2.** Assume \( \tau(x) \neq 0 \) for every \( x \in I \), and let \( \vartheta \in \mathcal{D}(I) \), \( \vartheta \neq 0 \). Passing to a smaller interval, we may even assume that \( I \) is closed.

Set \( A = \sup_{x \in I} |\psi'(x)| \), and for \( \xi' \in \mathbb{R}, \ |\xi'| \leq A, \ \eta' \in \mathbb{R}^n, \ |\eta'| = 1, \ x \in I \) let
\[ Q_{\xi,\eta}(x) = \sum_{j=1}^{n+1} |(\xi' x + \eta' \cdot \psi(x))^{(j)}(x)|. \]
Since \( \tau^{-1}([0]) = \emptyset \), we have \( Q_{\xi,\eta}(x) \neq 0 \) for every \( x \in I \), and since \( Q_{\xi,\eta}(x) \) is continuous in \( \xi', \eta' \) and \( x \) on the compact space \([-A,A] \times \{ \eta' \in \mathbb{R}^n: |\eta'| = 1 \} \times I \), there exist constants \( C_1 > 0, \ C_2 > 0 \), such that
\[(4.6) \quad C_1 \leq Q_{\xi,\eta}(x) \leq C_2 \]
for all \( x \in I, \ \xi', \eta' \) with \( |\xi'| \leq A, |\eta'| = 1 \).

So, using quite the same arguments as in the proof of Theorem 1 (ii), we can deduce from (4.6) by Lemma 1:
\[ |\delta e(\eta \cdot \psi)|_{PM} \leq C(1 + |\eta|)^{-1/(n+1)} \]
for some constant \( C > 0 \), which proves (i).

To prove (ii), we will assume, for convenience, \( x_0 = 0 \), i.e. \( 0 \in I \), and \( \vartheta(0) \neq 0, \ \tau(0) \neq 0 \).

Let \( \varepsilon > 0 \) such that \( \tau(x) \neq 0 \) for \( x \in [-\varepsilon,\varepsilon] \).

Since \( \psi'(x), \ \psi''(x), \ldots, \psi^{(n+1)}(x) \) are linearly independent for \( x \in [-\varepsilon,\varepsilon] \), there exists a function \( \xi \in C^\infty([-\varepsilon,\varepsilon], \mathbb{R}^n) \), such that for every \( x \in [-\varepsilon,\varepsilon] \)
\[(4.7) \quad \xi(x) \cdot \psi^{(j)}(x) = 0, \quad j = 2, \ldots, n, \]
and
\[(4.8) \quad \xi(x) \cdot \overline{\psi}^{(n+1)}(x) = 1. \]
Differentiating (4.7) and inserting (4.8), we get

\[ \xi'(x) \cdot \psi'(x) = 0 \quad \text{for} \quad j = 2, \ldots, n - 1, \]

and

\[ \xi'(x) \psi^{(n)}(x) = -1. \]

Repeating this process, one inductively obtains for \( k = 0, \ldots, n - 1 \)

\[
\begin{cases}
\xi^{(k)}(x) \cdot \psi^{(j)}(x) = 0 & \text{for} \quad j = 2, \ldots, n - k, \\
\xi^{(k)}(x) \cdot \psi^{(n+1-k)}(x) = (-1)^{k}.
\end{cases}
\]

(4.9)

So, if we define matrices

\[
S(x) = (\xi^{(j-i)}(x))_{i,j=1,\ldots,n}, \quad T(x) = (\psi^{(j+1)}(x))_{i,j=1,\ldots,n},
\]

then (4.9) means that \( S(x) T(x) \) is an upper triangular matrix with diagonal elements 1 or \(-1\), which yields

\[
(4.10) \quad \det (\xi(x) \xi'(x) \ldots \xi^{(n-1)}(x)) = \det S(x) = |\tau(x)|^{-1} \neq 0
\]

for all \( x \in [-\varepsilon, \varepsilon] \).

We now claim:

There is a constant \( C > 0 \), such that for all \( y \in (-\varepsilon, \varepsilon) \) and \( s \in \mathbb{R} \)

\[
(4.11) \quad |\mathfrak{A}e(s \xi(y) \cdot \psi)|_{PM} \geq C(1 + |s|)^{-1/(n+1)}.
\]

Choose \( y \in (-\varepsilon, \varepsilon) \). Then by (4.7), \( (\xi(y) \cdot \psi)^{(j)}(y) = \delta_{j,n+1} \) for \( j = 2, \ldots, n + 1 \), and so a Taylor expansion of \( \xi(y) \cdot \psi \) yields (for \( \varepsilon \) small enough)

\[
(4.12) \quad (\xi(y) \cdot \psi)(x) = \alpha + \beta x + (x-y)^{n+1} g(x) \quad \text{for} \quad x \in (-2\varepsilon, 2\varepsilon),
\]

where \( g \) is some smooth function on \( (-2\varepsilon, 2\varepsilon) \) which depends on \( y \), and where \( \alpha \) and \( \beta \) are some real numbers.

Let us remark here that although \( g = g_y \) depends on \( y \), \( \sup_{|x| < 2\varepsilon} |g_y'(x)| \) is uniformly bounded for \( y \in (-\varepsilon, \varepsilon) \).

Now take \( \rho \in \mathcal{D}(\mathbb{R}) \) with \( \text{supp} \rho \subset (-\varepsilon, \varepsilon) \), \( \rho \geq 0 \) and

\[
\int \rho(x) \, dx = 1, \quad \text{and set} \quad \tilde{\rho}(x) = \rho(|s|^{1/(n+1)}(x-y)).
\]
If we choose $\varepsilon$ small enough such that

$$|\vartheta(0) - \vartheta(x)| < \frac{1}{2} |\vartheta(0)|$$

for $x \in (-2\varepsilon, 2\varepsilon)$, then we get

$$\left| \int \vartheta(x) \varphi (x) \, dx \right| = \left| \int \vartheta(|s|^{-1/(n+1)} x + y) \varphi (x) \, dx \right| |s|^{-1/(n+1)}$$

$$\geq \frac{1}{2} |\vartheta(0)| |s|^{-1/(n+1)}, \quad \text{if } |s| \geq 1;$$

and since

$$\left| \int \vartheta(x) \varphi (x) \, dx \right| = \left| \int \vartheta(|s|^{-1/(n+1)} x + y) \varphi (x) e(-s\xi(y) \cdot \psi) \, dx \right|$$

$$\leq |\vartheta e(s\xi(y) \cdot \psi)|_{L^1} |\varphi (x) e(-s\xi(y) \cdot \psi)|_{A},$$

(4.11) will follow if we can show that $|\varphi e(-s\xi(y) \cdot \psi)|_{A}$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1$.

Now, regular affine mappings of $\mathbb{R}$ induce isometries of the Fourier algebra $A = A(\mathbb{R})$, thus

$$|\varphi e(-s\xi(y) \cdot \psi)|_{A} = |\varphi e(-s\xi(y) \cdot \bar{\psi})|_{A},$$

where $\bar{\psi}(x) = \psi(|s|^{-1/(n+1)} x + y)$.

Since for $x \in \text{supp } \varphi$ and $|s| \geq 1$,

$$|s|^{-1/(n+1)} x + y \in (-2\varepsilon, 2\varepsilon),$$

(4.12) yields

$$\xi(y) \cdot \bar{\psi}(x) = \alpha + \beta y + |s|^{-1/(n+1)} x + |s|^{-1} x^{n+1} g(|s|^{-1/(n+1)} x + y).$$

Thus

$$|\varphi e(-s\xi(y) \cdot \psi)|_{A} = |\varphi e(h)|_{A},$$

where $h(x) = -s|s|^{-1} x^{n+1} g(|s|^{-1/(n+1)} x + y)$. If we again apply estimate (4.1), we easily see that $|\varphi e(h)|_{A}$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1, \text{ q.e.d.}$
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Manuscrit reçu le 27 juillet 1982.

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