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A RESULT ON EXTENSION OF C. R. FUNCTIONS

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1. Introduction.

We prove here a result on extension of C.R. functions, given locally on the boundary of an open set of $\mathbf{C}^n$, of the form $\{\varphi < c\}$, where the Hessian of φ satisfies some conditions. This result will be established, by reducing the problem to one consisting of solving a $\bar{\partial}$-problem with compact support (see [6], see also [2] [9]) in intersections of domains defined by functions the Hessian of which may degenerate in some sense.

Of course, it is essential to note that one has not necessarily one positive eigenvalue for the Levi form of the domain defined by $\{\varphi < c\}$, near the point considered; for this case is solved by H. Lewy a long time ago [8], and generalized by R.O. Wells [10]. Let us mention also in this area, far from being complete, the following works [1] [3] [9]). We should mention also even in higher codimension, the method of construction of discs the boundary of which lie in $\partial\Omega$, or in the given manifold M ([5], [7], and the use of a recent result of Baouendi-Trèves [3], on the approximate Poincaré Lemma. But this construction seems to be very hard in the case where the Levi form degenerates.

2. Notations and definitions.

Let us recall some notations and definitions used in [6]. Let φ be a real, C^2 function in $\mathbf{C}^n$. For every multi-index J, with

$|\mathrm{J}| = q+1$, $q \leqslant n-1$, consider the matrix $\left(\frac{\partial^2\varphi}{\partial z_i\,\partial \bar z_j}\right)_{i,j\in \mathrm{J}}$. Let $\lambda_{i,\mathrm{J}}$ be continuous functions for $i \in \mathrm{J}$, such that

$$\sum_{i,j\in \mathrm{J}} \frac{\partial^2\varphi}{\partial z_i\,\partial \bar z_j}\, a_i \bar a_j \geqslant \sum_{i\in \mathrm{J}} \lambda_{i,\mathrm{J}}(\varphi)\, |a_i|^2$$

for any vector $a = (a_i, i \in \mathrm{J})$ (observe that $\lambda_{i,\mathrm{J}}$ is not necessarily non negative).

Now, for every multi-index I, with $|\mathrm{I}| = q$, define

$$\lambda_{\mathrm{I}}(\varphi) = \sum_{i\notin \mathrm{I}} \lambda_{i(i\mathrm{I})}(\varphi)$$

where $(i\mathrm{I})$ is the ordered multi-index obtained by adding i to I.

$$\lambda_q(\varphi) = \inf_{|\mathrm{I}|=q} \lambda_{\mathrm{I}}(\varphi)\,.$$

3. Extension of C.R. functions.

Now, using a result in [6], we have the following:

THEOREM 3.1. – *Let* ω *be defined near* $z_0 \in \partial\omega$, *by* $\omega = \{\varphi < c\}$ *with* φ *a* C^2 *function. Assume the following:*

a) $\lambda_{q+1}(\varphi) \geqslant |h|$, *where* h *is a holomorphic function near* z_0, $h \not\equiv 0$.

b) *For every neighborhood* V *of* z_0, *there exists a positive function* φ_{V}, *of class* C^2 *such that:*

$$\omega \cap \{\varphi_{\mathrm{V}} > 1\} \subset \mathrm{V}\,, \quad \varphi_{\mathrm{V}}(z_0) > 1\,, \quad \lambda_{q+1}(\varphi_{\mathrm{V}}) \geqslant |h|\,.$$

Let f *be a* $(0,q)$ *form which has the form* $f = h^2 g$ *with* $\mathrm{supp}(g) \subset \mathrm{U} \cap \bar\omega$, $\bar\partial g = 0$ *in* U, $g \in L^2(\mathrm{U})$, U *being a neighborhood of* z_0.

Then there exist a neighborhood V *of* z_0 *and* $u \in L^2_{(0,q-1)}$ *such that* $\bar\partial u = f$ *in* V, $\mathrm{supp}(u) \subset \bar\omega \cap \mathrm{V}$.

Proof. – *The theorem 4.1 in* [6] *works if we prove the following*

LEMMA 3.2. – *If* $0 \leqslant q \leqslant n-2$, *then*
$$\lambda_q \geqslant [(n-q)/(n-q-1)]\,.\,\lambda_{q+1}\,.$$

More precisely, for every choice of λ_{q+1} *depending on the* $\lambda_{i,J}$*'s,* $|J| = q+2$, *there exists such a* λ_q.

Proof. – Let $\lambda^{q+1}_{i,J}$, λ^{q+1}_{I}, λ_{q+1} be as above in the definition of λ_{q+1}, choose a multi-index J, $|J| = q+1$. Let S_J denote all multi-indices K, $|K| = q+2$, so that $J \subset K$ S_J contains of course $(n-q-1)$ elements.

If $j \in J$, let $\lambda_{j,J}$ be defined by $\lambda_{j,J} = \dfrac{1}{n-q-1} \sum_{K \in S_J} \lambda^{q+1}_{j,K}$.

Moreover λ_I, λ_q are defined as in the above procedure. We must show that

$$\sum_{i,j\in J} \frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j} t_i \bar{t}_j \geqslant \sum_{j \in J} \lambda_{j,J} \, |t_{j,J}| \, |t_j|^2 . \tag{*}$$

For every $K \in S_J$, $K = (kJ)$, choosing $t_k = 0$, one always has

$$\sum_{i,j\in J} \frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j} t_i \bar{t}_j \geqslant \sum_{j \in J} \lambda^{q+1}_{j,K} \, |t_j|^2 .$$

Adding up these inequalities over $K \in S_J$, one obtains

$$(n-q-1) \sum_{i,j\in K} \frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j} t_i \bar{t}_j \geqslant \sum_{j \in J} \Big(\sum_{K \in S_j} \lambda^{q+1}_{j,K} \Big) \, |t_j|^2$$

and $(*)$ follows.

To show that $\lambda_q \geqslant [(n-q)/(n-q-1)] \lambda_{q+1}$ it suffices to show for each I, $|I| = q$, that $\lambda_I \geqslant \lambda_{q+1} \cdot \dfrac{n-q}{n-q-1}$.

Computing, one obtains

$$\begin{aligned}
\lambda_I &= \sum_{i \notin I} \lambda_{i(iI)} = \frac{1}{n-q-1} \sum_{i \notin I} \sum_{K \in S_{(iI)}} \lambda^{q+1}_{i,K} \\
&= \frac{1}{n-q-1} \sum_{\substack{|K| = q+2 \\ K \supset I \\ i \in K - I}} \lambda^{q+1}_{i,K} = \frac{1}{n-q-1} \sum_{\substack{J \supset I \\ J = q+1 \\ i \notin J}} \lambda^{q+1}_{i,(iJ)} \\
&= \frac{1}{n-q-1} \sum_{J \supset I} \lambda^{q+1}_{J} \geqslant \frac{1}{n-q-1} \sum_{J \supset I} \lambda_{q+1} \\
&= \frac{n-q}{n-q-1} \lambda_{q+1} .
\end{aligned}$$

COROLLARY. – *If* $\mathrm{Hess}(\varphi) \geqslant |h|$, *then* $\lambda_q \geqslant |h|$, *for* $q \leqslant n-1$.

Now the lemma shows that if $\lambda_{q+1} \geqslant |h|$, then $\lambda_q \geqslant |h|$. Then the hypothesis of theorem 4.1 in [6] are fulfilled and the theorem follows.

Now we wish to establish a result on extension of C.R. functions given locally, near $z_0 \in \partial\omega$ on $\partial\omega$, with $\omega = \{\varphi < c\}$.

THEOREM 3.3. – *Assume the hypothesis of theorem 3.1 with* $q = 1$. *Assume moreover that, for every neighborhood* U *of* z_0, *one has* $\mathrm{U} \cap \{h_i = 0\} \cap \complement\, \bar{\omega} \neq \phi$, *for every component* h_i *of* h.

Let f *be a C.R. function on* $\partial\omega$, *near* z_0, *of class* C^4. *Assume also that* $\partial\omega$ *is* C^4. *Then there is a holomorphic function* $\tilde{f}$ *in* $\mathrm{V} \cap \omega$, *continuous in* $\mathrm{V} \cap \bar{\omega}$ *such that* $\tilde{f}|_{\partial\omega \cap \mathrm{V}} = f$, *where* V *is a neighborhood of* z_0.

Before the proof, we need some lemmas.

LEMMA 3.4. – *Let* $g = (z-1)h$ *be two holomorphic functions on* $\Delta = \{|z| < 1\} \subset \mathbf{C}$. *Assume* g *is bounded and that* $g(e^{i\theta})/e^{i\theta} - 1$ *is uniformly bounded a.e. Then* h *is bounded.*

Proof. – Assume, say $|g(e^{i\theta})| < 1$ and $|g(e^{i\theta})|/|e^{i\theta} - 1| < 1$. For $0 < r < 1$, $\theta \in \mathbf{R}$, let $e^{i\psi} = \dfrac{1 - re^{i\theta}}{1 - re^{-i\theta}}$

and
$$\lambda_{re^{i\theta}}(z) = (z-1)\, h\left(\frac{z - re^{i\theta}}{1 - zre^{-i\theta}} \cdot e^{-i\psi}\right).$$

Since we only have scaled the variable near $z = 1$, λ remains bounded. Therefore, its maximum is given by the maximum of its radial limits (a.e.). Except at $z = 1$, the radial limit of the second factor is at most 1. So $|\lambda| < 2$.

Now, the subgroup of the automorphisms of the unit disc fixing 1 acts transitively, hence $\{-re^{i\theta - i\psi}\} = \Delta$ and the lemma is proved.

LEMMA 3.5. – *Let* $\mathrm{D} \subset\subset \mathbf{C}$ *be a domain with* C^2 *boundary,* $\lambda : \mathrm{D}' \longrightarrow \mathbf{C}$ *a holomorphic function defined on* $\mathrm{D}' \supset \bar{\mathrm{D}}$, $\lambda \not\equiv 0$. *Assume* h *is holomorphic in* D, *that* h *extends continuously to* $\bar{\mathrm{D}} \backslash \{\lambda = 0\}$ *and that* $h|_{\partial\mathrm{D} \backslash \{\lambda = 0\}}$ *extends continuously to* $\partial\mathrm{D}$.

Then, if λh *is bounded on* D, h *is also bounded in* D.

Proof. – We may assume $\lambda = \Pi(z-a_j)^{m_j}$, where $\{a_j\}$ is a finite set of points on ∂D. We may also assume $D = \Delta$ since the result is local and biholomorphic maps between C^2 domains are Lip 1. Then $h \cdot \left(\prod_{j=1}^{N} (z-a_j)^{m_j}\right)$ is bounded on Δ, and h extends continuously to $\bar{D} - \{a_j\}$, while $h|_{\partial D - \{a_j\}}$ extends continuously to ∂D.

We may finally assume as well that there is only one $a_{j_0} = 1$, say $(z-1)^{m_{j_0}}\left(\prod_{j \neq j_0} (z-a_j)^{m_j} h\right)$ is bounded. The above lemma gives $(z-1)^{m_{j_0}-1}\left(\prod_{j \neq j_0} (z-a_j)^{m_j} h\right)$ bounded. Now we apply successively the same lemma to get that h is bounded.

Now we have the following.

PROPOSITION 3.6. – *Let* $\omega \subset\subset \mathbf{C}^n$, *with* C^2 *boundary,* $z_0 \in \partial\omega$, $\lambda \not\equiv 0$, *a holomorphic function in a neighborhood* $U = U(z_0)$. *Assume* $h \in \mathcal{O}(\omega \cap U)$, $\lambda h \in C(\bar{\omega} \cap U)$ $h|_{\partial\omega \cap U - \{\lambda = 0\}} \in C(\partial\omega \cap U)$. *Then* $h \in C(\bar{\omega} \cap U)$.

Proof. – It suffices to show that h is continuous at z_0, assuming that $\lambda(z_0) = 0$. Let $\tilde{h} \in C(\partial\omega \cap U)$ the continuous extension of $h|_{\partial\Omega \cap U - \{\lambda=0\}}$ we may assume $\tilde{h}(z_0) = 0$. Hence we need to show that $\overline{\lim}_{\substack{z \in \omega \\ z \to z_0}} |h| = 0$.

Let L be a complex line intersecting $\{\lambda = 0\}$ at z_0. We may assume $L \cap \{\lambda = 0\} = \{z_0\}$, except for points far away. Also we may assume $L \cap \omega$ is a nice C^2 domain in L (near z_0). Applying the above lemma to $L' \cap \omega$, for small translates L' of L, it follows that $|_{L' \cap \omega}$ is bounded for all L'. Since the above values are uniformly bounded in L', h is uniformly bounded in ω, near z_0.

Consider now a boundary point p of $L' \cap \omega = V \subset L'$ near z_0 in some linear coordinates in L', we may assume $p = 0$ and $T(\partial V)_0 = \{\operatorname{Re} \tau = 0\}$.

Now consider functions of the form $e^{-K\sqrt{\tau}} h$ in V for K large. This bounded holomorphic function in V is very small on

∂V, if K large enough. Hence $e^{-K\sqrt{\tau}} h$ is small everywhere. So $|h|$ is small near p. This is sufficiently uniform in L′ and p to prove that $\lim\limits_{\substack{z \in \omega \\ z \to z_0}} |h| < \epsilon$, for every $\epsilon > 0$.

PROPOSITION 3.7. – *Assume f is continuous on $\bar{\omega} \cap U(z_0)$ and holomorphic on $\omega \cap U(p)$. Assume furthermore that there exists a continuous function g on $\partial\omega \cap U(z_0)$ such that $f|_{\partial\omega \cap U(z_0)} = \lambda g$, where λ is a holomorphic function in $U(z_0)$ satisfying the following:*

a) $\{\lambda = 0\} \cap U(z_0)$ *is irreducible, and* λ *is irreducible on* $U(z_0)$

b) $\{\lambda = 0\} \cap U(z_0) \cap \complement\, \bar{\omega} \neq \emptyset$.

Then there exists a continuous function f' on $\bar{\omega} \cap U(z_0)$, holomorphic on $\omega \cap U(z_0)$ such that $f = \lambda f'$ on $\bar{\omega} \cap U(z_0)$.

Proof. – Define $\varphi : \{\lambda = 0\} \cap U(z_0) \longrightarrow \mathbf{C}$ by

$$\varphi = f \quad \text{on} \quad \{\lambda = 0\} \cap U(z_0) \cap \bar{\omega}$$
$$\varphi = 0 \quad \text{on} \quad \{\lambda = 0\} \cap U(z_0) \cap \complement\, \bar{\omega}\,.$$

Applying Rado's theorem, it follows that φ is holomorphic on the regular points of $\{\lambda = 0\} \cap U(z_0)$. Since $\varphi \equiv 0$ on a non empty, relatively open set, it follows moreover that $\varphi \equiv 0$ on the regular points of $\{\lambda = 0\} \cap U(z_0)$. This implies that f vanishes on $\{\lambda = 0\} \cap \omega \cap U(z_0)$. So we can write $f = \lambda f'$, for some holomorphic function f', in $\omega \cap U(z_0)$.

So we obtain that f' is a holomorphic function with the following properties:

a) f' is holomorphic in $\omega \cap U(z_0)$

b) $\lambda f'$ is continuous in $\bar{\omega} \cap U(z_0)$ and $f'|_{\partial\omega \cap U - \{\lambda = 0\}}$ which is equal to g extends to continuous function on $\partial\omega$. So by the preceding proposition f' is continuous in $\bar{\omega} \cap U(z_0)$.

Proof of the theorem 3.3. – Let f be a C^4 C.R. function on the C^4 boundary $\partial\omega$, near z_0. One can extend f to a C^1 function F whose $\bar{\partial}$ vanishes to order 1 on $\partial\omega$.

Now consider $\bar{\partial} F$, and $g = \begin{cases} h^2\, \bar{\partial} F = \bar{\partial}(h^2 F) & \text{on } \bar{\omega} \cap U \\ 0 & \text{on } U - \bar{\omega} \end{cases}$

we have

a) g is $\bar{\partial}$ closed in $U(z_0)$, $g = h^2 \tilde{g}$, $\tilde{g} \in C^0$

b) $\mathrm{supp}(g) \subset \bar{\omega} \cap U(z_0)$.

So, applying theorem 3.1, there exists a function u in a neighborhood V of z_0, such that

$$(*) \qquad \begin{cases} \bar{\partial} u = g & \text{in} \quad V \\ \mathrm{supp}(u) \subset \bar{\omega} \cap V \\ u \in L^2(V). \end{cases}$$

From the hypoellipticity of $\bar{\partial}$, the function u is, in fact, C^0. So, we obtain from (*)

$$\begin{cases} h^2 F - u \text{ is holomorphic in } \omega \cap V, \text{ continuous in } \bar{\omega} \cap V \\ \text{and } \; h^2 F - u|_{\partial\omega \cap V} = h^2 f. \end{cases}$$

Now, if we write $h^2 = \Pi h_i^{\alpha_i}$, h_i irreducible, we are using the proposition inductively: $\Pi h_i^{\alpha_i} F - u = \Pi h_i^{\alpha_i} \tilde{f}$, for some $\tilde{f}$, holomorphic in $\omega \cap V$, continuous in $\bar{\omega} \cap V$. Now it is easy to see that $\tilde{f}$ is an extension of f, which ends the proof of the theorem.

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