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UNFOLDINGS OF FOLIATIONS WITH MULTIFORM FIRST INTEGRALS

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In this note we study unfoldings of codim 1 local foliations $F = (\omega)$ generated by germs ω of the form

$$\omega = f_1 \dots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$$

for some germs f_i of holomorphic functions and complex numbers λ_i, generalizing the situation considered in [10].

For such a foliation F satisfying some side conditions, we determine the set U(F) of equivalence classes of first order unfoldings ((1.7) Proposition) and give explicitly a universal unfolding of F ((1.11) Theorem) as an application of the versality theorem in [7]. In section 2, it is shown that the unfolding theory for $F = (\omega)$, $\omega = f_1 \dots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$ is equivalent to the unfolding theory for the "multiform function" $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$. In section 3, we consider foliations with holomorphic or meromorphic first integrals. In either case, it turns out that the given generator ω is of the form considered in section 1. Thus, under the conditions of (1.11) Theorem, such a foliation has a universal unfolding (Theorems (3.4) and (3.10)). If the conditions are not satisfied, then the space U(F) may have obstructed elements ((3.6) Example).

This work was inspired by the extension theory of Cerveau and Moussu for forms with holomorphic integrating factors [1,4]. An unfolding is certainly an extension and, by the implicit function

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theorem, an extension can be thought of as an unfolding. Also a morphism in the unfolding theory is a morphism in the extension theory. However, the converse is not true in general. Thus a versal unfolding is a versal extension but not vice versa. In [1] and [4], it is proved that a germ ω of the form in section 1 of this note (or more generally, ω with holomorphic integrating factor f, i.e., $d\left(\frac{\omega}{f}\right) = 0$ for some f in $\mathcal{O}$) has a mini-versal extension.

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1. Unfoldings of $\omega = f_1 \dots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$.

Let ${}_n\mathcal{O}$ (or simply $\mathcal{O}$) denote the ring of germs of holomorphic functions at the origin 0 in $\mathbf{C}^n = \{(z_1, \dots, z_n)\}$ and let ${}_n\Omega$ (or simply Ω) denote the $\mathcal{O}$-module of germs of holomorphic 1-forms at 0. For an element ω in Ω, we denote by $S(\omega)$ (the germ at 0 of) the set of zeros of ω and call it the singular set of ω.

Let ω be an element in ${}_n\Omega$ of the form

$$\omega = f_1 \dots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i},$$

where f_i are germs in $\mathcal{O}$ and λ_i are complex numbers. If we set $F_i = f_1 \dots \hat{f}_i \dots f_p$ (omit f_i) for each $i = 1, \dots, p$, we may write $\omega = \sum_{i=1}^{p} \lambda_i F_i \, df_i$. Note that ω is integrable; $d\omega \wedge \omega = 0$. By regrouping the f_i's, if necessary, we may always assume that

(1.1) $\qquad \lambda_i \neq \lambda_j \ (\neq 0), \quad \text{if } i \neq j.$

In what follows we also assume that $\operatorname{codim} S(\omega) \geqslant 2$, which implies that

(1.2) each f_i is reduced, i.e., for any non-unit g in $\mathcal{O}$, f_i is not divisible by g^2,

and that

(1.3) f_i and f_j are relatively prime, if $i \neq j$.

Let F be the codim 1 local foliation at 0 in $\mathbf{C}^n$ generated by ω as above ([6] 4, [7] 1, [8]). The set U(F) of equivalence classes of first order unfoldings of F is given by ([6] 6, [7] 1).

$$\mathrm{U(F)} = \mathrm{I}(\omega)\Big/\Big(\sum_{i=1}^{p} \lambda_i \mathrm{F}_i \partial f_i\Big),$$

where $\mathrm{I}(\omega)$ is an ideal in $\mathcal{O}$ defined by

$$\mathrm{I}(\omega) = \{h \in \mathcal{O} \mid h d\omega = \eta \wedge \omega \quad \text{for some} \quad \eta \in \Omega\}$$

and $\Big(\sum_{i=1}^{p} \lambda_i \mathrm{F}_i \partial f_i\Big)$ is the ideal generated by

$$\sum_{i=1}^{p} \lambda_i \mathrm{F}_i \frac{\partial f_i}{\partial z_1}, \ldots, \sum_{i=1}^{p} \lambda_i \mathrm{F}_i \frac{\partial f_i}{\partial z_n}.$$

For a q-tuple of integers $i_1, \ldots, i_q$ with $1 \leqslant i_1 < \ldots < i_q \leqslant p$, let $\mathrm{I}(i_1, \ldots, i_q)$ denote the ideal in $\mathcal{O}$ generated by

$$f_{i_2} \ldots f_{i_q}, \ldots, f_{i_1} \ldots \hat{f}_{i_j} \ldots f_{i_q} \text{ (omit } f_{i_j}), \ldots, f_{i_1} \ldots f_{i_{q-1}}.$$

Note that $\mathrm{I}(1, \ldots, p) = (\mathrm{F}_1, \ldots, \mathrm{F}_p)$ (the ideal generated by $\mathrm{F}_1, \ldots, \mathrm{F}_p$). We denote by $ht\,\mathrm{I}$ the height of an ideal I in $\mathcal{O}$.

(1.4) LEMMA. – *Suppose* $ht(f_i, f_j, f_k) = 3$ *if* i, j, k *are distinct and* f_i, f_j, f_k *are non-units. Then we have*

$$\mathrm{I}(i_1, \ldots, i_q) = \bigcap_{\{j_1, \ldots, j_{q-1}\} \subset \{i_1, \ldots, i_q\}} \mathrm{I}(j_1, \ldots, j_{q-1})$$

for $q \geqslant 3$.

Proof. – Without loss of generality, we may assume that $(i_1, \ldots, i_q) = (1, \ldots, q)$. Obviously, the left hand side in the above equality is in the right hand side. Take any element h in the right hand side. We set $\mathrm{F}'_{ij} = f_1 \ldots \hat{f}_i \ldots \hat{f}_j \ldots f_q$ (omit f_i and f_j) for each pair of distinct indexes i, j and

$$\mathrm{F}'_{ijk} = f_1 \ldots \hat{f}_i \ldots \hat{f}_j \ldots \hat{f}_k \ldots f_q$$

for each triple of distinct indexes i, j, k. Then we may write

$$(1.5) \qquad h = \sum_{i \neq j} a_{ij} \mathrm{F}'_{ij}, \quad a_{ij} \in \mathcal{O},$$

for each $j = 1, \ldots, q$. Now we show that a_{ij} is in the ideal (f_i, f_j) for each i, j with $i \neq j$, which would imply that h is in $\mathrm{I}(1, \ldots, q)$.

This is obviously true if f_i or f_j is a unit. Thus we assume that f_i and f_j are non-units. If k is an index different from i or j, we have, from (1.5),

$$F'_{ijk}(a_{ij} f_k - a_{ik} f_j) = \Big(\sum_{\ell \neq i,k} a_{\ell k} F'_{i\ell k} - \sum_{m \neq i,j} a_{mj} F'_{imj} \Big) f_i .$$

By our assumption, f_i and F'_{ijk} are relatively prime. Hence

$$a_{ij} f_k - a_{ik} f_j = af_i$$

for some a in $\mathcal{O}$. Thus $a_{ij} f_k$ is in (f_i, f_j). If f_k is a unit, then a_{ij} is in (f_i, f_j). If f_k is a non-unit, then by our assumption $ht(f_i, f_j, f_k) = 3$. Hence a_{ij} is in (f_i, f_j). Q.E.D.

(1.6) COROLLARY. – *Under the assumption of (1.4) Lemma,*

$$(F_1, \ldots, F_p) = \bigcap_{i \neq j} (f_i, f_j) .$$

(1.7) PROPOSITION. – *If the assumption of (1.4) Lemma is satisfied and if* $df_1 \wedge \ldots \wedge df_p \neq 0$, *then we have* $I(\omega) = (F_1, \ldots, F_p)$, *thus*

$$U(F) = (F_1, \ldots, F_p) \Big/ \Big(\sum_{i=1}^{p} \lambda_i F_i \partial f_i \Big) .$$

Proof. – If we set $F_{ij} = f_1 \ldots \hat{f}_i \ldots \hat{f}_j \ldots f_p$ for $i \neq j$, we have

$$d\omega = \sum_{1 \leqslant i < j \leqslant p} (\lambda_j - \lambda_i) F_{ij} df_i \wedge df_j .$$

From this we see easily that

$$\lambda_i F_i d\omega = \sum_{i \neq j} (\lambda_i - \lambda_j) F_{ij} df_j \wedge \omega ,$$

which shows that $(F_1, \ldots, F_p) \subset I(\omega)$. Conversely, take any element h in $I(\omega)$. Thus

$$hd\omega = \eta \wedge \omega \qquad (1.8)$$

for some η in Ω. Let U be a small neighborhood of 0 on which the germs $f_1, \ldots, f_p$, h and η have representatives and let S be the set of zeros of $df_1 \wedge \ldots \wedge df_p$ in U. By our assumption, the set S is an analytic set of codim $\geqslant 1$. As in the proof of [10] (2.1) Lemma, from (1.8), we may write

(1.9) $$\eta = \sum_{i=1}^{p} \phi_i \, df_i ,$$

(1.10) $$(\lambda_j - \lambda_i)\, h = \lambda_j \phi_i f_i - \lambda_i \phi_j f_j$$

for some holomorphic functions $\phi_1, \ldots, \phi_p$ on $U - S$. Now we show that ϕ_i can be extended to holomorphic functions on U. From (1.9) and (1.10), we have

$$\phi_i \omega = \lambda_i F_i \eta + h \sum_{j \neq i} (\lambda_j - \lambda_i)\, F_{ij}\, df_j$$

for each $i = 1, \ldots, p$. Since the right hand side is holomorphic in U, this shows that ϕ_i is holomorphic in $U - S(\omega)$. Therefore, by the assumption that $\operatorname{codim} S(\omega) \geqslant 2$, ϕ_i can be extended to a holomorphic function on U. Thus from (1.10) and (1.6) Corollary, we see that h is in $(F_1, \ldots, F_p)$. Q.E.D.

For an element h in $\mathcal{O}$, we denote the corresponding element in $\mathcal{O}\Big/\Big(\sum_{i=1}^{p} \lambda_i F_i \partial f_i\Big)$ by $[h]$. The following result follows from (1.7) Proposition and the versality theorem in [7] (cf. the proof of [10] (2.4) Theorem).

(1.11) THEOREM. – *Let* $F = (\omega)$ *be a* codim 1 *local foliation at* 0 *in* $\mathbf{C}^n$ *generated by a germ* ω *of the form*

$$\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$$

for some f_i *in* $\mathcal{O}$ *and* λ_i *in* $\mathbf{C}$. *Suppose the conditions (a)* $\lambda_i \neq \lambda_j$ $(\neq 0)$ *for* $i \neq j$, *(b)* $\operatorname{codim} S(\omega) \geqslant 2$, *(c)* $ht(f_i, f_j, f_k) = 3$ *for* $i \neq j \neq k \neq i$ *such that* f_i, f_j, f_k *are non-units, and (d)* $df_1 \wedge \ldots \wedge df_p \neq 0$ *are satisfied. If the dimension of the* **C***-vector space* $(F_1, \ldots, F_p)\Big/\Big(\sum_{i=1}^{p} \lambda_i F_i \partial f_i\Big)$, $F_i = f_1 \ldots \hat{f_i} \ldots f_p$, *is finite, then* F *has a universal unfolding. In fact, if*

$$\Big[\sum_{i=1}^{p} \lambda_i u_i^{(1)} F_i\Big], \ldots, \Big[\sum_{i=1}^{p} \lambda_i u_i^{(m)} F_i\Big], \quad u_i^{(j)} \in \mathcal{O},$$

is a **C***-basis of* $(F_1, \ldots, F_p)\Big/\Big(\sum_{i=1}^{p} \lambda_i F_i \partial f_i\Big)$, *then the unfolding*

$\mathcal{F} = (\tilde{\omega})$ of F *with parameter space* $\mathbf{C}^m = \{(t_1, \ldots, t_m)\}$ *generated by* $\tilde{\omega} = \tilde{f}_1 \ldots \tilde{f}_p \sum_{i=1}^{p} \lambda_i \dfrac{d\tilde{f}_i}{\tilde{f}_i}$, *where* $\tilde{f}_i$ *are germs in* ${}_{n+m}\mathcal{O}$ *given by*

$\tilde{f}_i = f_i + \sum_{k=1}^{m} u_i^{(k)} t_k$, *is universal.*

(1.12) Corollary (Cerveau-Lins Neto [1] Th. E_5, [2] Prop. 6, see also [9] (3.2) Th.). – *If* $F = (\omega)$ *is the* codim 1 *local foliation at* 0 *in* $\mathbf{C}^n = \{(z_1, \ldots, z_n)\}$ *generated by* $\omega = z_1 \ldots z_n \sum_{i=1}^{n} \lambda_i \dfrac{dz_i}{z_i}$ *for some* λ_i *in* $\mathbf{C}$ *with* $\lambda_i \neq \lambda_j \neq 0$ $(i \neq j)$, *then every unfolding of* F *is trivial, in fact* $U(F) = 0$.

Proof. – We have

$$(F_1, \ldots, F_n) = \Big(\sum_{i=1}^{n} \lambda_i F_i \partial f_i\Big) = (z_1 \ldots \hat{z}_i \ldots z_n).$$

Hence $U(F) = 0$.

(1.13) *Remark.* – The universal unfolding given in (1.11) Theorem is infinitesimally versal. However, if the conditions in (1.11) are not satisfied, U(F) may have obstructed elements (see (3.6) Example).

(1.14) *Remark.* – Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbf{C}^n$ generated by a germ ω of the form

$$\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}, \quad \lambda_i \neq \lambda_j \neq 0 \ (i \neq j),$$

with $\operatorname{codim} S(\omega) \geqslant 2$ and let $\mathcal{F}$ be an unfolding of F with parameter space $\mathbf{C}^\ell$. Then by a result of Cerveau and Moussu ([1] 4e Partie, Th. C_4, [4]), we have that

(1.15) $\mathcal{F}$ has a generator $\tilde{\omega}$ of the form

$$\tilde{\omega} = \tilde{f}_1 \ldots \tilde{f}_p \sum_{i=1}^{p} \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}, \quad \tilde{f}_i \in {}_{n+\ell}\mathcal{O}.$$

Moreover, if ω has no meromorphic first integrals (Sec. 3), then we may assume that ([1] 2e Partie, Ch. I, Prop. 1.5, [3])

(1.16) $\tilde{f}_i(z, 0) = f_i(z), \quad i = 1, \ldots, p.$

The facts (1.15) and (1.16) also follow from (1.11) Theorem in case the conditions in (1.11) are satisfied.

(1.17) *Remark.* – If a foliation F is generated by a germ ω of the form $\omega = f_1 \dots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$, then F has a generator of a similar form such that each function germ involved in the expression is a non-unit.

2. Multiform functions.

A germ of multiform function at 0 in $\mathbf{C}^n$ is an expression $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ for some germs f_i in ${}_n\mathcal{O}$ and non-zero complex numbers λ_i. Two multiform functions $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ and $g_1^{\mu_1} \dots g_q^{\mu_q}$ are equal if they are equal as germs of multivalued functions, i.e., $f_1^{\lambda_1} \dots f_p^{\lambda_p} g_1^{-\mu_1} \dots g_q^{-\mu_q} = 1$. Let $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ be a multiform function. By regrouping the factors of the f_i's, if necessary, we may always assume that the conditions (1.1), (1.2) and (1.3) are satisfied. Then the expression $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ is uniquely determined up to the order of the f_i's and units of $\mathcal{O}$. The critical set $\mathrm{C}(f)$ of $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ is defined to be the singular set $\mathrm{S}(\omega)$ of the 1-form $\omega = f_1 \dots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$. In this section, we consider only multiform functions f with $\operatorname{codim} \mathrm{C}(f) \geqslant 2$.

An *unfolding* of $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ is a germ $\tilde{f}$ of multiform function at 0 in $\mathbf{C}^n \times \mathbf{C}^m = \{(z, t)\}$ which can be written as $\tilde{f} = \tilde{f}_1^{\lambda_1} \dots \tilde{f}_p^{\lambda_p}$ for $\tilde{f}_i$ in ${}_{n+m}\mathcal{O}$ with $\tilde{f}_i(z, 0) = f_i(z)$, $i = 1, \dots p$. We call $\mathbf{C}^m$ the parameter space of $\tilde{f}$.

(2.1) DEFINITION. – *Let* $\tilde{f} = \tilde{f}_1^{\lambda_1} \dots \tilde{f}_p^{\lambda_p}$ *and* $g = g_1^{\lambda_1} \dots g_p^{\lambda_p}$ *be two unfoldings of* $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ *with parameter spaces* $\mathbf{C}^m$ *and* $\mathbf{C}^\ell$, *respectively. A morphism from* g *to* $\tilde{f}$ *consists of germs of holomorphic maps* $\Phi : (\mathbf{C}^n \times \mathbf{C}^\ell, 0) \longrightarrow (\mathbf{C}^n \times \mathbf{C}^m, 0)$ and $\phi : (\mathbf{C}^\ell, 0) \longrightarrow (\mathbf{C}^m, 0)$ *such that*

(a) the diagram

$$\begin{array}{ccc} (\mathbf{C}^n \times \mathbf{C}^\ell, 0) & \xrightarrow{\Phi} & (\mathbf{C}^n \times \mathbf{C}^m, 0) \\ \downarrow & & \downarrow \\ (\mathbf{C}^\ell, 0) & \xrightarrow{\phi} & (\mathbf{C}^m, 0) \end{array}$$

is commutative, where the vertical maps are the projections,

(b) $\Phi(z, 0) = (z, 0)$ *and*

(c) $g = \Phi^* \tilde{f}$, *i.e.*, $g_1^{\lambda_1} \ldots g_p^{\lambda_p} = (\Phi^* \tilde{f}_1)^{\lambda_1} \ldots (\Phi^* \tilde{f}_p)^{\lambda_p}$.

(2.3) DEFINITION. – *An unfolding* $\tilde{f}$ *of* f *is versal if for any unfolding* g *of* f, *there is a morphism from* g *to* $\tilde{f}$.

Note that if $\tilde{f} = \tilde{f}_1^{\lambda_1} \ldots \tilde{f}_p^{\lambda_p}$ is an unfolding of $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$, then $\tilde{\mathcal{F}} = (\tilde{\omega})$, $\tilde{\omega} = \tilde{f}_1 \ldots \tilde{f}_p \sum_{i=1}^{p} \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$, is an unfolding of $\mathrm{F} = (\omega)$, $\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$, with the same parameter space as that of $\tilde{f}$. For the definition of morphisms for unfoldings of foliations, see [10] (1.2) Definition.

(2.4) LEMMA. – *Let* $\tilde{f} = \tilde{f}_1^{\lambda_1} \ldots \tilde{f}_p^{\lambda_p}$ *and* $g = g_1^{\lambda_1} \ldots g_p^{\lambda_p}$ *be two unfoldings of* $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ *with parameter spaces* $\mathbf{C}^m$ *and* $\mathbf{C}^\ell$, *respectively. A pair* (Φ, ϕ) *of germs of holomorphic maps* $\Phi : (\mathbf{C}^n \times \mathbf{C}^\ell, 0) \longrightarrow (\mathbf{C}^n \times \mathbf{C}^m, 0)$ *and* $\phi : (\mathbf{C}^\ell, 0) \longrightarrow (\mathbf{C}^m, 0)$ *is a morphism from* g *to* $\tilde{f}$ *if and only if it is a morphism from*

$$\mathcal{F}' = (\theta), \quad \theta = g_1 \ldots g_p \sum_{i=1}^{p} \lambda_i \frac{dg_i}{g_i},$$

to

$$\tilde{\mathcal{F}} = (\tilde{\omega}), \quad \tilde{\omega} = \tilde{f}_1 \ldots \tilde{f}_p \sum_{i=1}^{p} \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i} .$$

Proof. – We first note that if $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ and

$$\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i},$$

we may write $d \log f = \frac{df}{f} = \frac{1}{f_1 \ldots f_p}\, \omega$. Suppose (Φ, ϕ) is a morphism from g to $\tilde{f}$. Then we have

$$\chi \cdot \theta = \Phi^* \tilde{\omega}, \tag{2.5}$$

where $\chi = \dfrac{\Phi^* \tilde{f}_1 \ldots \Phi^* \tilde{f}_p}{g_1 \ldots g_p}$. Since the right hand side of (2.5) is holomorphic and $\operatorname{codim} \mathrm{S}(\theta) \geqslant 2$, we see that χ is in ${}_{n+\ell}\mathcal{O}$.

Moreover, since $\widetilde{f}_i(z, 0) = g_i(z, 0) = f_i(z)$ and $\Phi(z, 0) = (z, 0)$, we have $\chi(z, 0) = 1$. Hence (Φ, ϕ) is a morphism from $\mathfrak{F}'$ to $\mathfrak{F}$.

Conversely, suppose (Φ, ϕ) is a morphism from $\mathfrak{F}'$ to $\mathfrak{F}$. Then there is a germ χ in ${}_{n+\ell}\mathcal{O}$ with $\chi(z, 0) = 1$ satisfying $\chi \cdot \theta = \Phi^* \widetilde{\omega}$. Now we prove that χ is equal to $\dfrac{\Phi^* \widetilde{f}_1 \dots \Phi^* \widetilde{f}_p}{g_1 \dots g_p}$. Once this is done, we have $d \log g = d \log \Phi^* f$. Since the restrictions of g and $\Phi^* \widetilde{f}$ to $\mathbf{C}^n \times \{0\}$ are both equal to f, we get $g = \Phi^* \widetilde{f}$, which shows that (Φ, ϕ) is a morphism from g to $\widetilde{f}$. Let $s = (s_1, \dots, s_\ell)$ be coordinates on $\mathbf{C}^\ell$. In general, for an element $\widetilde{h}$ in ${}_{n+\ell}\mathcal{O}$, consider the power series expansion of $\widetilde{h}$ in s; $\widetilde{h}(z, s) = \sum_{|\nu| \geq 0} h^{(\nu)}(z) s^\nu$, where ν denotes an ℓ-tuple $(\nu_1, \dots, \nu_\ell)$ of non-negative integers, $|\nu| = \nu_1 + \cdots + \nu_\ell$, $s^\nu = s_1^{\nu_1} \dots s_\ell^{\nu_\ell}$ and $h^{(\nu)}$ are germs in ${}_n\mathcal{O}$. If $h^{(0)} \neq 0$, $(0) = (0, \dots, 0)$, then for each ν, there is a germ $\phi^{(\nu)}$ of meromorphic function at 0 in $\mathbf{C}^n$ such that

$$\sum_{\lambda + \mu = \nu} h^{(\lambda)} \phi^{(\mu)} = \begin{cases} 1 \dots |\lambda| = 0, \\ 0 \dots |\lambda| > 0. \end{cases}$$

Thus we have an expression $\dfrac{1}{\widetilde{h}} = \sum_{|\nu| \geq 0} \phi^{(\nu)} s^\nu$. If we set

$$\rho = \chi \cdot \frac{g_1 \dots g_p}{\Phi^* \widetilde{f}_1 \dots \Phi^* \widetilde{f}_p},$$

we may write $$\rho(z, s) = \sum_{|\nu| \geq 0} \rho^{(\nu)}(z) s^\nu,$$

where $\rho^{(\nu)}$ are germs of meromorphic functions at 0 in $\mathbf{C}^n$ with $\rho^{(0)} = 1$. For our purpose, it suffices to show that $\rho^{(\nu)} = 0$ if $|\nu| > 0$. We may also write

$$d \log \Phi^* \widetilde{f} = \sum_{|\nu| \geq 0} \alpha^{(\nu)} s^\nu + \sum_{k=1}^{\ell} \sum_{|\nu| \geq 0} \nu_k \mathrm{F}^{(\nu)}(z) s^{\nu - 1_k} ds_k,$$

$$d \log g = \sum_{|\nu| \geq 0} \beta^{(\nu)} s^\nu + \sum_{k=1}^{\ell} \sum_{|\nu| \geq 0} \nu_k \mathrm{G}^{(\nu)}(z) s^{\nu - 1_k} ds_k,$$

where 1_k denotes the ℓ-tuple with 1 in the k-th component and 0 in the others, the addition and substraction of two ℓ-tuples are done componentwise, $\alpha^{(\nu)}$ and $\beta^{(\nu)}$ are germs of meromorphic 1-forms and $F^{(\nu)}$ and $G^{(\nu)}$ are germs of meromorphic functions at 0 in $\mathbf{C}^n$. Note that $\alpha^{(0)} = \beta^{(0)}$. Since $d \log \Phi^* \tilde{f}$ and $d \log g$ are both closed forms, we have

$$(2.6) \qquad dF^{(\nu)} = \alpha^{(\nu)} \quad \text{and} \quad dG^{(\nu)} = \beta^{(\nu)} .$$

On the other hand, from $\rho\, d \log g = d \log \Phi^* \tilde{f}$, we have

$$(2.7) \ \alpha^{(\nu)} = \sum_{\lambda+\mu=\nu} \rho^{(\lambda)} \beta^{(\mu)} \quad \text{and} \quad \nu_k F^{(\nu)} = \sum_{\lambda+\mu=\nu} \mu_k \rho^{(\lambda)} G^{(\mu)}$$

for all ν. From (2.6) and (2.7), it is not difficult to show that $\rho^{(\nu)} = 0$ for $|\nu| > 0$. Q.E.D.

In view of (1.14) Remark and (2.4) Lemma, the unfolding theory for multiform functions $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ satisfying (1.1), (1.2), (1.3) and $\operatorname{codim} C(f) \geqslant 2$ (as well as other conditions described in (1.14)) is equivalent to the unfolding theory for foliations $F = (\omega)$ with $\operatorname{codim} S(F) \geqslant 2$ generated by germs ω of the form $\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$, $\lambda_i \neq \lambda_j \neq 0$ $(i \neq j)$. In particular, from (1.11) Theorem, we have the following

(2.8) THEOREM. – *Let $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ be a germ of multiform function at 0 in $\mathbf{C}^n$ satisfying (1.1), (1.2), (1.3),* $\operatorname{codim} C(f) \geqslant 2$ *and the conditions (c) and (d) in (1.11) Theorem. If*

$$\dim_{\mathbf{C}} (F_1, \ldots, F_p) \Big/ \Big(\sum_{i=1}^{p} \lambda_i F_i \partial f_i \Big), \quad F_i = f_1 \ldots \hat{f_i} \ldots f_p ,$$

is finite, then f has a versal unfolding. In fact if $\tilde{f}_i$ are the germs in (1.11), then the unfolding $\tilde{f} = \tilde{f}_1^{\lambda_1} \ldots \tilde{f}_p^{\lambda_p}$ of f is versal.

3. Foliations with holomorphic or meromorphic first integrals.

The following application of the results in section 1 was pointed out by K. Saito. First we observe the following

(3.1) LEMMA. – *Let* f *be a germ in* $\mathcal{O}$ *with* $f(0) = 0$ *and let* g *be a reduced germ in* $\mathcal{O}$. *If* $df = g\theta$ *for some* θ *in* Ω, *then* f *is divisible by* g^2.

Proof. – From the condition, we see that f vanishes on the zero set of g. Hence g divides f; $f = f'g$ for some f' in $\mathcal{O}$. Then we have $df = gdf' + f'dg$. Thus f' must be also divisible by g. Q.E.D.

Similarly we have

(3.2) LEMMA. – *Let* f *be a germ in* $\mathcal{O}$ *with* $f(0) = 0$ *and let* g *be a germ in* $\mathcal{O}$ *of the form* $g = f_1^{k_1} \dots f_r^{k_r}$ *for some germs* f_i *in* $\mathcal{O}$ *and positive integers* k_i *such that (a)* f_i *are reduced, and (b)* f_i *and* f_j *are relatively prime if* $i \neq j$. *If* $df = g\theta$ *for some* θ *in* Ω, *then* f *is divisible by* $f_1^{k_1+1} \dots f_r^{k_r+1}$.

Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbf{C}^n$ with codim $S(\omega) \geqslant 2$. Suppose ω has a holomorphic first integral f, i.e., $\omega \wedge df = 0$ for some f in $\mathcal{O}$ ([5] p. 470). Without loss of generality, we may always assume that $f(0) = 0$. Since codim $S(\omega) \geqslant 2$, we may write $df = g\omega$ for some g in $\mathcal{O}$. If g is a unit in $\mathcal{O}$, $F = (\omega) = (df)$ is a Haefliger foliation and unfoldings of F are well understood [7,10]. We may write $g = f_1^{k_1} \dots f_r^{k_r}$, where k_i are positive integers with $k_i \neq k_j$ for $i \neq j$ and f_i are (non-constant) germs in $\mathcal{O}$ satisfying the conditions (a) and (b) in (3.2) Lemma. Then, from (3.2) Lemma, we have $f = f_1^{k_1+1} \dots f_r^{k_r+1} f_{r+1}$ for some f_{r+1} in $\mathcal{O}$. By computing df, we have

$$(3.3)\quad \omega = f_1 \dots f_{r+1} \sum_{i=1}^{r+1} \lambda_i \frac{df_i}{f_i}, \quad \lambda_i = \begin{cases} k_i + 1 \dots 1 \leqslant i \leqslant r, \\ 1 \dots i = r+1. \end{cases}$$

Note that, since codim $S(\omega) \geqslant 2$, f_{r+1} is reduced and that f_{r+1} and f_i are relatively prime for $i = 1, \dots, r$. Let $p = r$ and replace λ_i by $f_{r+1}\lambda_i$ if f_{r+1} is a constant and let $p = r+1$ otherwise. Then from (1.11) Theorem, we have

(3.4) THEOREM. – *Let* $F = (\omega)$ *be a* codim 1 *local foliation at* 0 *in* $\mathbf{C}^n$ *with* codim $S(F) \geqslant 2$. *If* $\omega \wedge df = 0$ *for some* f *in* $\mathcal{O}$, *then* ω *can be written as (3.3). Moreover, if (a)* $ht(f_i, f_j, f_k) = 3$

for distinct indexes $i, j, k = 1, \ldots, p$ *such that* f_i, f_j, f_k *are non-units, (b)* $df_1 \wedge \ldots \wedge df_p \neq 0$ *and (c)*

$$\dim_{\mathbf{C}}(F_1, \ldots, F_p) \Big/ \Big(\sum_{i=1}^{p} \lambda_i F_i \partial f_i \Big), \quad F_i = f_1 \ldots \hat{f}_i \ldots f_p ,$$

is finite, then F *has a universal unfolding. In fact, a universal unfolding is constructed explicitly as in (1.11) Theorem.*

(3.5) *Example.* – Let $F = (\omega)$ be the foliation at 0 in $\mathbf{C}^2 = \{(x, y)\}$ generated by

$$\omega = y(3x + 2y^2)\, dx + 2x(x + 2y^2)\, dy .$$

For $f = x^2 y^2 (x + y^2)$ and $g = xy$, we have $df = g\omega$. Letting $f_1 = F_2 = xy$, $f_2 = F_1 = x + y^2$, $\lambda_1 = 2$ and $\lambda_2 = 1$, we see that the complex vector space

$$(F_1, F_2) \Big/ \Big(\sum_{i=1}^{2} \lambda_i F_i \partial f_i \Big) = (x + y^2, xy)/(y(3x + 2y^2), x(x + 2y^2))$$

is three dimensional and we may choose $[x + y^2] = \left[\frac{1}{2} \lambda_1 F_1 \right]$, $[xy] = [\lambda_2 F_2]$ and $[x^2] = \left[\frac{1}{2} \lambda_1 x F_1 - \lambda_2 y F_2 \right]$ as its basis. Thus by (3.4) Theorem, we see that the unfolding $\mathcal{F} = (\tilde{\omega})$ of F with parameter space $\mathbf{C}^3 = \{(t_1, t_2, t_3)\}$ given by

$$\tilde{\omega} = 2\tilde{f}_2\, d\tilde{f}_1 + \tilde{f}_1\, d\tilde{f}_2 ,$$

$$\tilde{f}_1 = xy + \frac{1}{2} t_1 + \frac{1}{2} x t_3 , \quad \tilde{f}_2 = x + y^2 + t_2 - y t_3$$

is universal. Note that $d\tilde{f} = \tilde{g}\tilde{\omega}$ for $\tilde{f} = \tilde{f}_1^2 \tilde{f}_2$ and $\tilde{g} = \tilde{f}_1$.

Here is an example of $F = (\omega)$ with a holomorphic first integral which has obstructed elements in U(F).

(3.6) *Example.* – Let $F = (\omega)$ be the foliation at 0 in $\mathbf{C}^2 = \{(x, y)\}$ generated by

$$\omega = y(3x + 2y)\, dx + x(3x + 4y)\, dy .$$

For $f = x^2 y^3 (x + y)$ and $g = x^2 y^3$, we have $df = g\omega$. Thus in the previous situation, we have $f_1 = x$, $f_2 = y$, $f_3 = x + y$, $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 1$. Note that $ht(f_1, f_2, f_3) = 2$. If we set $h = 3x + 4y$, then $h\,d\omega = \eta \wedge \omega$ for $\eta = 3dx$. Hence $[h]$ is in U(F) and $\mathcal{F}^{(1)} = (\tilde{\omega})$,

$$\tilde{\omega} = y(3x + 2y)\,dx + (3x^2 + 4xy + t)\,dy + (3x + 4y)\,dt$$

is a first order unfolding of F corresponding to $[h]$. However, it is not difficult to show that there is no unfolding corresponding to $[h]$.

Next we consider a foliation $F = (\omega)$ (codim $S(\omega) \geqslant 2$) with a meromorphic first integral, i.e., we suppose that $\omega \wedge d\left(\frac{f}{g}\right) = 0$ for some relatively prime germs f and g in $\mathcal{O}$. In what follows we assume that g is reduced. Since codim $S(\omega) \geqslant 2$, we may write

$$gdf - fdg = h\omega \tag{3.7}$$

or

$$d\left(\frac{f}{g}\right) = \frac{h}{g^2}\,\omega \tag{3.8}$$

for some h in $\mathcal{O}$. Note that if h is a unit, F is generated by $gdf - fdg$ and unfoldings of such an F are well understood [10]. Since f and g are relatively prime and g is reduced, from (3.7), we see that g and h are relatively prime. Thus by (3.8), $\frac{f}{g} = c$ is a constant on the zero set of h. If we write $h = f_1^{k_1} \dots f_r^{k_r}$, where k_i are positive integers with $k_i \neq k_j$ for $i \neq j$ and f_i are non-constant germs in $\mathcal{O}$ satisfying the conditions (a) and (b) in (3.2) Lemma, then we have $f - gc = f_1^{k_1+1} \dots f_r^{k_r+1} f_{r+2}$ for some f_{r+2} in $\mathcal{O}$. We set $f_{r+1} = g$. By computing $d\left(\frac{f}{g}\right)$, we have

$$\omega = f_1 \dots f_{r+2} \sum_{i=1}^{r+2} \lambda_i \frac{df_i}{f_i}, \quad \lambda_i = \begin{cases} k_i + 1 \dots 1 \leqslant i \leqslant r, \\ -1 \dots i = r+1, \\ 1 \dots i = r+2. \end{cases} \tag{3.9}$$

Note that, since codim $S(\omega) \geqslant 2$, f_{r+2} is also reduced and that f_i and f_j are relatively prime for distinct indexes i, j with $1 \leqslant i, j \leqslant r+2$. Let $p = r+1$ and replace λ_i by $f_{r+2}\lambda_i$ if f_{r+2} is a constant and let $p = r+2$ otherwise. Then from (1.11) Theorem, we have

(3.10) THEOREM. – *Let* $F = (\omega)$ *be a* codim 1 *local foliation at* 0 *in* $\mathbf{C}^n$ *with* codim $S(F) \geqslant 2$. *Suppose* $\omega \wedge d\left(\frac{f}{g}\right) = 0$ *for some* f *and* g *in* $\mathcal{O}$ *such that* f *and* g *are relatively prime and that* g *is reduced.*

Then ω *can be written as (3.9). If (a)* $ht(f_i, f_j, f_k) = 3$ *for distinct indexes* $i, j, k = 1, \ldots, p$ *such that* f_i, f_j, f_k *are non-units, (b)* $df_1 \wedge \ldots \wedge df_p \neq 0$ *and (c)* $\dim_{\mathbf{C}}(F_1, \ldots, F_p)\Big/\Big(\sum_{i=1}^{p} \lambda_i F_i \partial f_i\Big)$, $F_i = f_1 \ldots \hat{f_i} \ldots f_p$, *is finite, then* F *has a universal unfolding.*

In fact, a universal unfolding is constructed as in (1.11) Theorem.

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