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Finitely generated ideals in $A(\omega)$


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FINITELY GENERATED IDEALS IN $A(\Omega)$

by J. E. FORNÆSS and N. ØVRELID

1. Let $\Omega \subset \subset C^2(z,w)$ be a bounded pseudoconvex domain with smooth boundary containing the origin and let $A(\Omega)$ denote the set of continuous functions on $\bar{\Omega}$ which are holomorphic in $\Omega$. In the special case when $\Omega$ is the unit ball, A. Gleason [4] asked the following:

The Gleason Problem: If $f \in A(\Omega)$ and $f(0,0) = 0$, does there exist $g$, $h \in A(\Omega)$ such that $f = zg + wh$?

This was solved affirmatively by Leibenzon, see [5], in the ball case and by Henkin [5], Kerzman-Nagel [6], Lieb [9] and Øvrelid [12] in the strongly pseudoconvex case. Beatrous [1] solved the problem for weakly pseudoconvex domains under the extra hypothesis that there exists a complex line through 0 which intersects the boundary of $\Omega$ only in strongly pseudoconvex points. In this paper we discuss the real analytic case.

Main theorem. — Let $0 \in \Omega \subset \subset C^2(z,w)$ be a pseudoconvex domain with real analytic boundary. If $f \in A(\Omega)$ and $f(0) = 0$, then there exist $g$, $h \in A(\Omega)$ such that $f = zg + wh$.

The main difficulty is that the Levi flat boundary points, $w(\partial \Omega)$, can be two-dimensional. This means that the projection of $w(\partial \Omega)$ into the space of complex lines through 0 (a $\mathbb{P}^1$) can be onto. Thus no such complex line avoids $w(\partial \Omega)$ and therefore Beatrous' theorem does not apply. (On the other hand, if $w(\partial \Omega)$ is one-dimensional, then of course the Main Theorem is a direct consequence of Beatrous' result.)

To handle this difficulty we study the structure of $w(\partial \Omega)$. We show (Proposition 3) that except for a one-dimensional subset, $w(\partial \Omega)$ consists
of R-points. The R-points were first studied by Range [11] who proved sup norm estimates for \(\partial\) at such points. We give a precise definition of R-points in the next section. Their main property is that they allow holomorphic separating functions. In particular we thus show in this paper that the Kohn-Nirenberg points [8] constitute an at most one-dimensional subset of \(\partial \Omega\). Next we choose a complex line through 0 intersecting \(w(\partial \Omega)\) only in R-points. Then, one has good enough \(\partial\)-results to complete the proof along the same line as Beatrous.

The Main Theorem can still be proved if we replace \(A(\Omega)\) by various holomorphic Hölder- and Lipschitz-spaces and if we replace \(z\) and \(w\) by arbitrary generators of the maximal ideal at 0 in these spaces. This requires several hard \(\partial\)-estimates. Therefore, in order to keep the length of this paper down, the authors have decided to postpone these generalizations to a later paper. We will then also show how these techniques can be used to prove that bounded pseudoconvex domains with real analytic boundary in \(C^2\) have the Mergelyan property (see [3]).

2. We will make a detailed discussion of the weakly pseudoconvex boundary points \(W = w(\partial \Omega)\) of a bounded pseudoconvex domain \(\Omega\) with smooth real analytic boundary in \(C^2\). First we need a stratification of \(W\) into totally real manifolds.

Lemma 1. — There exist pairwise disjoint real analytic manifolds \(S_0, S_1, S_2 \subset \partial \Omega\) with the following properties:

(i) Each \(S_j\) consists of finitely many \(j\)-dimensional totally real real analytic manifolds,

(ii) \(W = S_0 \cup S_1 \cup S_2\),

(iii) \(S_1\) is closed in \(\partial \Omega - S_0\); \(S_2\) is closed in \(\partial \Omega - (S_0 \cup S_1)\) and

(iv) Each connected component of \(S_2\) consists of points of the same finite type only.

Here finite type is in the sense of Kohn [7].

The sets \(S_0\), \(S_1\) and \(S_2\) are actually semi-analytic. During the proof we will use repeatedly standard facts about semi-analytic sets. The reader can consult Łojasiewicz [10] for details.

Proof. — Let \(r\) be a real analytic defining function for \(\Omega\). (For example, one can choose \(r\) to be the Euclidean distance to \(\partial \Omega\) outside \(\Omega\),
but close to $\partial \Omega$, and the negative of the Euclidean distance in $\Omega$ close to $\partial \Omega$.) Also let $s$ be a real valued real analytic function defined on a neighbourhood of $\partial \Omega$ vanishing at a $p \in \partial \Omega$ if and only if $p$ is a weakly pseudoconvex boundary point. (One can for example let

$$s(z,w) = \frac{\partial^2 r/\partial z \partial \bar{z} \cdot |\partial r/\partial w|^2 - 2\Re \partial^2 r/\partial z \partial \bar{w} \cdot \partial r/\partial w \cdot \partial r/\partial \bar{z}}{\partial^2 r/\partial w \partial \bar{w} \cdot |\partial r/\partial z|^2}.$$}

Hence the weakly pseudoconvex boundary points, $W$, is the common zero set $\{r=s=0\}$ of global real analytic functions.

Using real coordinates, $x + iy = z$, $u + iv = w$, we can identify as usual $C^2(z,w)$ with $R^4(x,y,u,v)$ with complex coordinates

$$X = x + ix', \quad Y = y + iy', \quad U = u + iu', \quad V = v + iv'.$$

Then $r$, $s$ have unique extensions to holomorphic functions $R(X,Y,U,V)$ and $S(X,Y,U,V)$ respectively. The complexification $M$ of $\partial \Omega$ is then given by $\{R=0\}$ which is a complex manifold since $dr \neq 0$. From now on we will consider only points of $M$. In $M$, $\Sigma := \{S=0\} \cap M$ is a complex hypersurface, hence has (complex) dimension 2.

Let $p$ be any point in $W \subset \Sigma$. Since $\Sigma$ and $M$ are closed under complex conjugation, there exists a holomorphic function $h = h_p(X,Y,U,V)$ defined in a neighbourhood of $p$ in $C^4$ which, when restricted to $M$, generates the ideal of $\Sigma$ at every point of $\Sigma$ in that neighbourhood, and such that $h$ is real valued at points in $C^2 = R^4$. The function $h$ has a nonvanishing gradient (on $M$) at regular points of $\Sigma$. Since $\Im h \equiv 0$ on $\partial \Omega$ it follows that $W$ is given by $\{r = \Re h \equiv 0\}$ near such regular points of $\Sigma$ and that $\partial \Omega \cap \text{reg } \Sigma$ is a pure 2-dimensional real analytic manifold. By Diederich-Fornæss [2] $\partial \Omega$ cannot contain a complex manifold. This implies that $\partial \Omega \cap \text{reg } \Sigma$ is totally real at a (relatively) dense set of points. A point in $\partial \Omega \cap \text{reg } \Sigma$ is totally real if and only if $\lambda := (\partial \bar{r})_{(z,w)} \wedge \partial (\Re h_p)_{(z,w)} \neq 0$ there. Here derivatives are taken in $C^2$. This condition does not depend on $p$ since different $(\Re h_p)$'s only differ by real multiples on $\partial \Omega$.

Let $S' \subset W$ be the (at most) one-dimensional closed real analytic set consisting of $\partial \Omega \cap \text{sing } \Sigma$ and the zeroes in $W$ of the coefficient of $\lambda$. By Lojasiewicz [10], $W - S'$ consists of finitely many connected, pairwise disjoint semi-analytic sets, $C_1, \ldots, C_r$. Each $C_j$ is a two dimensional
totally real real analytic manifold whose closure $\mathcal{C}_j$ is also a semi analytic set, and $\mathcal{C}_j - C_j \subset S'$.

Locally, there exists a holomorphic vector field

$$L = a \frac{\partial}{\partial z} + b \frac{\partial}{\partial w} \neq 0$$

with real analytic coefficients tangent to the boundary, i.e. $L(r) = 0$ on $\partial \Omega$. The type of a point $p \in \partial \Omega$ is then given as the smallest integer $2k$ for which $(\partial r, L^{k-1}[L, L](r))(p) \neq 0$. This number is independent of the choices of $r$ and $L$. Let $n_j$ be the maximum type of points in $C_j$, and let $T_j$ consist of all boundary points of type $> n_j$. Then $T_j$ is a real analytic set. In particular, $\mathcal{C}_j \cap T_j$ is a semi analytic set of dimension at most one. Then $S_2 := \mathcal{C}_j - T_j$ is a pure 2-dimensional totally real real analytic manifold with finitely many connected components on each of which the type is constant. Also, $W - S_2$ is a closed semi analytic set in $C^2$ of dimension at most one, and can hence be written as $S_0 \cup S_1$ where $S_0$ is a finite set of points and $S_1$ is a relatively closed 1-dimensional real analytic manifold in $W - S_0$ with finitely many connected components. This completes the proof of Lemma 1.

Range [11] introduced a convexity condition which is satisfied by many weakly pseudoconvex boundary points.

**Definition 2.** Let $D = \{\rho < 0\} \subset \subset C^n$ be a domain with $C^\infty$ boundary. A point $p \in \partial D$ is an R-point (of order $m$) if there exists a neighbourhood $U$ of $p$ and a $C^\infty$ function

$$F(\zeta, z) : (\partial D \cap U)(\zeta) \times U(z) \to C$$

such that

(i) $F$ is holomorphic in $z$,

(ii) $F(\zeta, \zeta) \equiv 0$ and $d_\zeta F \neq 0$ and

(iii) $\rho(z) \geq \varepsilon |z - \zeta|^m$ whenever $F(\zeta, z) = 0$, $\varepsilon > 0$ some constant.

Using the Levi polynomial

$$F(\zeta, z) = \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{i, j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(\zeta_i - z_i)(\zeta_j - z_j)$$

one immediately obtains that strongly pseudoconvex boundary points are R-points of order 2.
PROPOSITION 3. — Every point in $S_2$ is an R-point.

In the proof of the proposition we will need two elementary inequalities.

LEMMA 4. — Let $p_k(s,t) := (s+t)^{2k} - s^{2k} - 2kts^{2k-1}$ for $s, t \in \mathbb{R}$, $k \in \{1,2,\ldots\}$. Then there exists a constant $c_k > 0$ such that

$$p_k(s,t) \geq c_k(s^{2k-2}t^2 + t^{2k})$$

for all $s, t$.

**Proof.** — For each fixed $s$, $q_s(t) = (s+t)^{2k}$ is a convex function of $t$ and $T_s(t) = s^{2k} + 2kts^{2k-1}t$ is an equation for the tangentline through $(0,s^{2k})$. Hence,

$$p_k(s,t) = q_s(t) - T_s(t) > 0$$

whenever $t \neq 0$. Since

$$p_k(s,t) = t^2 \left[ \frac{(2k)}{2}s^{2k-2} + O(t) \right]$$

and $s^{2k-2}t^2 + t^{2k} = t^2[s^{2k} + O(t)]$

it follows that there exists a $c_k > 0$ such that

$$p_k(s,t) \geq c_k(s^{2k}t^2 + t^{2k})$$

for all $(s,t)$ on the unit circle and hence by homogeneity for all $(s,t)$.

LEMMA 5. — Let $k \in \{1,2,\ldots\}$ and $\delta > 0$, $\delta < 4^{-k^2}$ be given. Then $y^{2k} + \delta \Re(z^{2k}) \geq 2^{-k}\delta|z|^{2k}$ for every complex number $z = x + iy$.

**Proof.** — Expanding $\Re z^{2k}$, we get

$$y^{2k} + \delta \Re(z^{2k}) \geq y^{2k} + \delta x^{2k} - R(z)$$

with $R(z) = 2^{2k-1} \delta y^2 \max(|x|,|y|)^{2k-2}$. Elementary computation gives $y^{2k} \geq 2R(z)$ when $|x| < 2^k|y|$, while $\delta x^{2k} \geq 2R(z)$ otherwise. In any case,

$$y^{2k} + \delta \Re(z^{2k}) \geq \frac{\delta}{2}(x^{2k} + y^{2k}) \geq 2^{-k}\delta(x^2 + y^2)^k,$$

so the lemma follows.

To simplify our computations it is convenient to change coordinates locally so that $S_2$ becomes a plane.
LEMMA 6. — Let $p_0 \in S_2$. There exist local holomorphic coordinates $z = x + iy, w = u + iv$ in a neighbourhood $U$ of $p_0$, such that in $U$,

(i) $S_2$ is given by $y = v = 0$, and

(ii) $\partial \Omega$ is tangent to the plane $v = 0$ along $S_2$.

As a consequence $T_p \partial \Omega$ is given by $w = 0$ along $S_2$.

Proof. — Local coordinates satisfying (i) are constructed by choosing a real analytic parametrization $F : W \to S_2$ near $p_0$, with $W$ open in $\mathbb{R}^2$. Since $S_2$ is totally real, the prolongation $\tilde{F}$ of $F$ to complex arguments is invertible near $p_0$, and we set $(z(p), w(p)) = \tilde{F}^{-1}(p)$. Then (ii) means that the vector field $\frac{\partial}{\partial y} = J \frac{\partial}{\partial x}$ is tangential to $\partial \Omega$ on $S_2$, i.e. $\left(\frac{\partial}{\partial x}\right)_p \in T_p \partial \Omega$ when $p \in S_2$. Now $L = T_{S_2} \cap T^\ast \partial \Omega$ is a real analytic line field on $S_2$, and we just have to choose a parametrization $F$ where the curves $u = \text{const.}$ are integral curves of $L$ to complete the proof.

When $v = -V(x, y, u)$ is a local parametrization of $\partial \Omega$, $\Omega$ is given near $p_0$ by $\rho = v + V(x, y, u) < 0$, provided $\partial / \partial v$ points out of $\partial \Omega$. We may write

$$\rho = v + g(x, y, u) = v + \sum_{\ell = 2k} a_{\ell}(x, u) y'$$

for some $k > 1$ and $a_{2k} > 0$, since $\Omega$ is weakly pseudoconvex of constant type on $S_2$.

After these preliminary remarks we can prove Proposition 3. To show that $p_0 \in S_2$ is an $R$-point, choose at first a neighbourhood $U = U(p_0)$ of $p_0$ on which $a_{2k}(x, u) > a > 0$. We will shrink $U$ whenever necessary without saying so each time.

For $\zeta = (z_0, w_0) \in U \cap \partial \Omega$, we write $z = z_0 + z'$, $w = w_0 + w'$, $w' = u' + iv'$ etc., and Taylor-expand $\rho$ around $\zeta$. Since $\rho(\zeta) = 0$ we get

$$\rho = v' + g_x(\zeta)x' + g_y(\zeta)y' + g_u(\zeta)u' + a_{2k}(x_0, u_0) p_k(y_0, y') + R$$

where the remainder $R$ satisfies an estimate

$$|R| \leq C (|z'| + |w'|)^2 (|y_0| + |z'| + |w'|)^{2k - 1}$$

in $U$ with $C$ independent of $\zeta$. 
The linear function \( \tilde{w} = (g_x(\zeta) + ig_y(\zeta))z' + (1 + ig_w(\zeta))w' \) has imaginary part equal to the linear part of \( \rho \), so by Lemma 4 \( \rho \geq \tilde{\nu} + a_k(y_0^{2k+2}y^2 + y'^{2k}) - |R| \) in \( U \).

Set \( F_\zeta(z,w) = i\tilde{w} + \varepsilon(y_0^{2k-2}z'^2 + z'^{2k}) \), with \( 0 < \varepsilon < 4^{-k}c_\varepsilon \). On the zero set of \( F_\zeta \)

\[
\tilde{w} = i\varepsilon(y_0^{2k-2}z'^2 + z'^{2k}), \text{ and in particular } \\
\tilde{\nu} = \varepsilon(y_0^{2k-2}\text{Re}(z'^2) + \text{Re}(z'^{2k})).
\]

Applying Lemma 5 this gives \( \rho \geq 2^{-k}\varepsilon(y_0^{2k-2}|y|^2 + |z'|^{2k}) - |R| \).

Since \( g_x, g_y \) and \( g_w \) are small near the origin, it follows from (1) and the definition of \( \tilde{w} \) that \( |w'| < |z'| \) on \( \{F_\zeta = 0\} \cap U \) whenever \( \zeta \in U \). Thus

\[
\rho \geq 2^{-k}\varepsilon(y_0^{2k-2}|z'|^2 + |z'|^{2k}) - c'|z'|^2(|y_0| + |z'|)^{2k-1} \\
\geq \varepsilon(y_0^{2k-2}|z'|^2 + |z'|^{2k}) \\
\geq 2^{-k}\varepsilon l(z,w - \zeta)^{2k}.
\]

It follows that \( F(\zeta(z,w)) = F_\zeta(z,w) \) satisfies Range's condition in Definition 2 with order \( m = 2k \). This completes the proof of Proposition 3.

3. We can now prove the Main Theorem. Let \( \Omega \) be a bounded pseudoconvex domain in \( C^2 \) with real analytic boundary: By Lemma 1 the weakly pseudoconvex points \( w(\partial \Omega) \) can be stratified by real analytic sets \( S_0, S_1 \) and \( S_2 \) where \( S_j \) has dimension \( j \), \( j = 0,1,2 \). Proposition 3 gives that \( S_2 \) consists only of \( R \)-points. We need the following \( \bar{\partial} \)-result by Range [11].

**Theorem 7.** — Let \( D \subset C^2 \) be a pseudoconvex domain with \( C^\infty \) boundary. Assume that \( D \) has a Stein neighbourhood basis. If \( \lambda \) is a \( \bar{\partial} \)-closed \((0,1)\)-form with uniformly bounded coefficients on \( D \) whose support clusters on \( \partial D \) only at \( R \)-points, then there exists a continuous function \( g \) on \( D \) with \( \bar{\partial}g = \lambda \) on \( D \).

This theorem applies as it is shown in [2] that \( \Omega \) has a Stein neighbourhood basis.

By rotation of the axis we may assume that the \( z \)-axis does not intersect \( S_0 \cup S_1 \). In particular, if \( \varepsilon > 0 \) is small enough, \( F_\varepsilon := \{(z,w) \in \partial \Omega; \varepsilon/2 \leq |w| \leq \varepsilon\} \) consists only of \( R \)-points.
Following Beatrous [1], if \( f \in A(\Omega) \) and \( f(0) = 0 \), we can write
\[
\frac{f(z)}{z} = g^1 + w h^1
\]
in a small neighbourhood of 0. On the set \( \{(z,w) \in \Omega_1; |z| > \varepsilon\} \) we can write
\[
f = zg^2 + wh^2 \quad \text{with} \quad g^2 = f/z \quad \text{and} \quad h = 0, \quad \varepsilon \ \text{arbitrarily small}.
\]
Solving an additive Cousin problem we obtain the decomposition
\[
f = zg^3 + wh^3
\]
on the set
\[
\Omega_1 = \{(z,w) \in \Omega; |w| < \varepsilon\},
\]
with \( g^3, h^3 \) holomorphic and continuous up to the boundary. On the set
\[
\Omega_2 = \{(z,w) \in \Omega; |w| > \varepsilon/2\}
\]
we have the decomposition
\[
f = zg^4 + wh^4 \quad \text{where} \quad g^4 = 0 \quad \text{and} \quad h^4 = f/w.
\]
Where the two sets overlap, we get the equation
\[
G := \frac{g^3 - g^4}{w} = \frac{h^4 - h^3}{z}.
\]
We need holomorphic functions \( G_1, G_2 \) with continuous boundary values on \( \Omega_1, \Omega_2 \) respectively so that \( G = G_1 - G_2 \) on the intersection. This reduces in a standard way to solving a \( \partial \)-problem for a form with support in \( \Omega_1 \cap \Omega_2 \). Hence Theorem 7 shows that such \( G_1, G_2 \) exist.

We then obtain the decomposition
\[
f = zg + wh, \quad g, h \in A(\Omega)
\]
by letting
\[
g = \begin{cases} g^3 - w G_1 & \text{on } \Omega_1, \\ g^4 - w G_2 & \text{on } \Omega_2 \end{cases}, \quad h = \begin{cases} h^3 + z G_1 & \text{on } \Omega_1, \\ h^4 - z G_2 & \text{on } \Omega_2 \end{cases}.
\]
This completes the proof of the Main Theorem.

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