ALEXANDRU BUIUM

Degree of the fibres of an elliptic fibration


<http://www.numdam.org/item?id=AIF_1983__33_1_269_0>
DEGREE OF THE FIBRES OF AN ELLIPTIC FIBRATION

by Alexandru BUIUM

1. Statement of the results.

Let \( f : X \rightarrow B \) be an elliptic fibration over the complex field i.e. a morphism from a smooth complex projective surface \( X \) to a smooth curve \( B \) such that the general fibre \( F \) of \( f \) is a smooth elliptic curve and no fibre contains exceptional curves of the first kind. Consider the following subsets of \( \text{Pic}(X) \):

\[
\begin{align*}
N_e &= \{ \mathcal{L} \in \text{Pic}(X), \mathcal{L} = \mathcal{O}_X(D) \text{ for some effective } D \} \\
N_s &= \{ \mathcal{L} \in \text{Pic}(X), \mathcal{L} \text{ is spanned by global sections} \} \\
N_a &= \{ \mathcal{L} \in \text{Pic}(X), \mathcal{L} \text{ is ample} \} \\
N_v &= \{ \mathcal{L} \in \text{Pic}(X), \mathcal{L} \text{ is very ample} \}
\end{align*}
\]

and let \( n_e, n_s, n_a, n_v \) be the minima of the non-zero intersection numbers \( (\mathcal{L}, F) \) when \( \mathcal{L} \) runs through \( N_e, N_s, N_a \) and \( N_v \) respectively. In [3] p. 259, Enriques investigates the possibility of finding a birational model of \( X \) in the projective space \( \mathbb{P}^3 \) such that the fibres of \( f \) have degree \( n_e \). His analysis suggests the following problem: find the minimum possible degree of the fibres of \( f \) in an embedding of \( X \) in a projective space. In other words: find \( n_v \).

There obviously exist inequalities: \( n_e \leq n_s \leq n_v \) and \( n_a \leq n_v \).

Let \( m \) denote the maximum of the multiplicities of the fibres of \( f \). The aim of this paper is to prove the following propositions:

PROPOSITION 1. — Equality \( n_e = n_s \) holds if and only if \( n_e \geq 2m \).
Proposition 2. — Equality $n_a = n_v$ holds if and only if $n_a \geqslant 3m$.

The statements above are consequences of the following more precise results:

Theorem 1. — There exists a constant $C_1$ depending only of the fibration such that for any effective divisor $D$ on $X$ which does not contain in its support any component of any reducible fibre and such that $D$ is either reduced dominating $B$, or ample, the following conditions are equivalent:

1) $(D.F) \geqslant 2m$.

2) $\mathcal{O}_X(D) \otimes f^*L$ is spanned by global sections for any $L \in \text{Pic}(B)$ with $\deg(L) \geqslant C_1$.

3) $\mathcal{O}_X(D) \otimes f^*L$ is spanned by global sections for some $L \in \text{Pic}(B)$.

Theorem 2. — There exists a constant $C_2$ depending only on the fibration such that for any ample sheaf $\mathcal{E} \in \text{Pic}(X)$ the following conditions are equivalent:

1) $(\mathcal{E}.F) \geqslant 3m$.

2) $\mathcal{E} \otimes f^*L$ is very ample for any $L \in \text{Pic}(B)$ with $\deg(L) \geqslant C_2$.

3) $\mathcal{E} \otimes f^*L$ is very ample for some $L \in \text{Pic}(B)$.

Our proofs are based on Bombieri's technique from [2]. Therefore the main point will be to prove that certain divisors on $X$ are numerically connected.

2. Two lemmas.

Lemma 1. — Let $D$ be an effective divisor on $X$ which does not contain in its support any component of any reducible fibre. Suppose $D$ is either reduced or ample and put $T = D + a_1 F_1 + \ldots + a_p F_p$ where $F_i$ are distinct fibres and $a_i \in \mathbb{Q}$, $a_i > 0$ for $1 \leqslant i \leqslant p$. Suppose furthermore that $a_1 + \ldots + a_p \geqslant 2$. Then we have:

1) If $(D.F) \geqslant 2m$ then $T$ is 2-connected.

2) If $(D.F) \geqslant 3m$ and $D$ is integral and ample then $T$ is 3-connected.
Proof. Suppose $T = T_1 + T_2$ where $T_k > 0$ and

$$T_k = D_k + A_k$$

$$D_1 + D_2 = D$$

$$A_1 + A_2 = A = a_1 F_1 + \ldots + a_p F_p .$$

We get

$$(T_1, T_2) = (D_1, D_2) + (D_1, A_2) + (D_2, A_1) + (A_1, A_2).$$

If in addition $D$ is integral we may suppose $D_2 = 0$. Since by [6] ample divisors are 1-connected it follows that in any case $(D_1, D_2) \geq 0$. On the other hand we have $(D_1, A_2) \geq 0$ and $(D_2, A_1) \geq 0$ because any common component of $D$ and $A$ must be a rational multiple of a fibre. We may write $A_2 = Z_1 + \ldots + Z_p$ where $Z_i \leq a_i F_i$ for $1 \leq i \leq p$. We get

$$(A_1, A_2) = (A - A_2, A_2) = -(A_2^2) = -(Z_1^2) - \ldots - (Z_p^2).$$

By [1] p. 123 we have $(Z_i^2) \leq 0$ for any $i$. Suppose first that there exists an index $i$ such that $(Z_i^2) < 0$. By [5], $(Z_i^2) = -2$, consequently $(T_1, T_2) \geq 2$. If an addition $D$ is integral and ample then $A_2 \neq 0$ (because otherwise $T_2 = 0$) hence $(D_1, A_2) \geq 1$ and we get $(T_1, T_2) \geq 3$.

Now suppose $(Z_i^2) = 0$ for any $i$. Then by [1] p. 123, we must have $Z_i = c_i F_i$ where $c_i \in \mathbb{Q}$, $0 \leq c_{i2} \leq a_i$, hence

$$A_1 = c_{i1} F_1 + \ldots + c_{p1} F_p$$

where $c_{i1} + c_{i2} = a_i$. If both $D_1$ and $D_2$ dominate $B$ we get $(D_k, F) \geq 1$ for $k = 1, 2$ hence

$$(T_1, T_2) \geq (D_1, A_2) + (D_2, A_1) \geq c_{i2} + \ldots + c_{p2} + c_{i1} + \ldots + c_{p1}$$

$$= a_1 + \ldots + a_p \geq 2$$

and we are done. If $D_k = 0$ for $k = 1$ or $k = 2$ then $A_k \neq 0$ hence there exists an index $i_0$ such that $c_{i_0 k} > 0$. Now if $m_0$ denotes the multiplicity of $F_{i_0}$ we have $c_{i_0 k} \geq 1/m_0 \geq 1/m$. Consequently we get $(T_1, T_2) = (A_k, D) \geq c_{i_0 k} (D, F) \geq (D, F)/m$ and we are done again. Finally if $D_k \neq 0$ and $D_k$ does not dominate $B$ we get $(T_1, T_2) \geq (D_1, D_2) = (D, D_k) \geq (D, F)/m$ and the lemma is proved.

Lemma 2. Let $m_1, \ldots, m_p$ denote the multiplicities of the multiple fibres of $f$. Then for any reduced effective divisor $D$ not
containing in its support any component of any reducible fibre we have \((D^2) \geq - (D \cdot F) (\chi(\Theta_X) + \sum_{j=1}^{r} (m_j - 1)/m_j)\).

**Proof.** — We may suppose \( D = D_1 + \ldots + D_t \) where \( D_i \) are integral, distinct, dominating \( B \). For any \( i = 1, \ldots, t \) let \( E_i \) be the normalization of \( D_i \). By adjunction formula and by Hurwitz formula we get:

\[
(D_i^2) + (D_i \cdot K) = 2p_a(D_i) - 2 \geq 2p_a(E_i) - 2 \geq [E_i: B] (2p_a(B) - 2).
\]

Consequently:

\[
(D^2) \geq \sum_{i=1}^{t} (D_i^2) \geq \left( \sum_{i=1}^{t} [E_i: B] \right) (2p_a(B) - 2) - (D \cdot K)
= (D \cdot F) (2p_a(B) - 2) - (D \cdot F) (2p_a(B) - 2 + \chi(\Theta_X))
+ \sum_{j=1}^{r} (m_j - 1)/m_j
\]

because of the formula for the canonical divisor \( K \) (see [4] p. 572) and we are done.

3. Proofs of Theorems 1 and 2.

Suppose \( m_1 Y_1, \ldots, m_r Y_r \) are all the multiple fibres of \( f \) each having multiplicity \( m_j \), \( 1 \leq j \leq r \) and take \( b_j \in B \) such that \( m_j Y_j = f^*(b_j) \). By the formula for the canonical divisor \( K \) we may write

\[
\Theta_X(K) = f^* M \otimes \Theta_X \left( \sum_{j=1}^{r} (m_j - 1) Y_j \right)
\]

where \( M \in \text{Pic}(B) \), \( \deg(M) = 2p_a(B) - 2 + \chi(\Theta_X) \).

Furthermore for any points \( x, x_1, x_2 \) on \( X \) denote by \( p: \widetilde{X} \longrightarrow X \) and \( q: \widetilde{X} \longrightarrow X \) the blowing ups of \( X \) at \( x \) and \( \{x_1, x_2\} \) respectively and let \( W, W_1, W_2 \) be the corresponding exceptional curves. Put \( y = f(x), y_1 = f(x_1), y_2 = f(x_2) \).

**Proof of Theorem 1.** — To prove 1) \( \Longrightarrow \) 2) it is sufficient by [2] to prove that \( H^1(\widetilde{X}, p^* \Theta_X(D) \otimes p^* f^* L \otimes \Theta_{\widetilde{X}}(- W)) = 0 \) for any \( x \in X \) hence by Bombieri-Ramanujam vanishing theorem [2] to prove that the linear system
contains an 1-connected divisor with self-intersection $> 0$. Now by Lemma 2 the self-intersection of $\Lambda$ is

$$(D^2) - 2(D.K) + 2(D.F) \deg(L) - 4 > 0$$

provided $\deg(L) \geq \alpha_1$ where $\alpha_1$ is a constant depending only on the fibration. Now by Riemann-Roch on $B$ we get that

$$|L \otimes M^{-1} \otimes \Theta_B(-b_1 - \ldots - b_r - 2y)| \neq \emptyset$$

provided $\deg(L) - \deg(M) - r - 2 \geq p_a(B)$. Hence there exists a constant $\alpha_2$ depending only on $f$ such that for $\deg(L) \geq \alpha_2$ we may find a divisor $b \in |L \otimes M^{-1}|$ with $b_1 + \ldots + b_r + 2y \leq b$. It follows that

$$G = p^*(D + f^*b - \sum_{j=1}^r (m_j - 1)Y_j) - 2W \in \Lambda.$$

Now for $\deg(L) - \deg(M) - \sum_{j=1}^r (m_j - 1)/m_j \geq 2$ the divisor $D + f^*b - \sum_{j=1}^r (m_j - 1)/m_j Y_j$ must be 2-connected by Lemma 1. It follows by a standard computation that in this case $G$ is 1-connected. Hence we may choose $C_1 = \max \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_3 = \deg(M) + \sum_{j=1}^r (m_j - 1)/m_j + 2$ and we are done.

2) $\implies$ 3) is obvious.

To prove 3) $\implies$ 1) we may suppose that $L$ is trivial and that $D$ has no common components with $Y$, where $mY$ is some fibre of multiplicity $m$. We only have to prove that $(D.Y) \geq 2$. Suppose $(D.Y) = 1$. By Riemann-Roch on the (possibly singular) curve $Y$ we get

$$h^0(\Theta_Y(D)) = h^0(\omega_Y(-D)) + \deg(\Theta_Y(D)) + \chi(\Theta_Y) = h^0(\Theta_Y(-D)) + 1$$

because the dualizing sheaf $\omega_Y$ is trivial. Now since $\Theta_Y(-D) \subset \Theta_Y$ we get $H^0(\Theta_Y(-D)) \subset H^0(\Theta_Y)$. Since by [5], $H^0(\Theta_Y)$ consists only of constants and since $\Theta_Y(-D)$ is not trivial we get $h^0(\Theta_Y(-D)) = 0$ hence $h^0(\Theta_Y(D)) = 1$. Since $\Theta_Y(D)$ is not trivial, it follows that $\Theta_Y(D)$ cannot be spanned by global sections, contradiction.
Proof of Theorem 2. — Note that 2) \( \implies \) 3) is obvious and that 3) \( \implies \) 1) follows easily considering as above a multiple fibre of the form \( mY \) and noting that \( Y \) must have degree at least 3 with respect to any very ample divisor because \( p_a(Y) = 1 \).

Let us prove 1) \( \implies \) 2). Start with an ample \( \mathcal{E} \in \text{Pic}(X) \) with \( (\mathcal{E}, F) \geq 3m \), put \( \mathcal{N} = \mathcal{E} \otimes f^*L \) for \( L \in \text{Pic}(B) \) and let us prove first that \( |\mathcal{N}| \) has no fixed components among the components of the reducible fibres of \( f \) provided \( \deg(L) \geq \beta_1 \) for some constant \( \beta_1 \). Let \( Z_1 \) be a component of a reducible fibre \( F \) and look for a divisor in \( |\mathcal{N}| \) not containing \( Z_1 \) in its support. Note that by [5], \( Z_1 \) is smooth rational with selfintersection \( (Z_1^2) = -2 \). According to [5] there are two cases which may occur: either \( (Z_1, Z_2) \leq 1 \) for any other component \( Z_2 \) of \( F \), or \( F = b(Z_1 + Z_2) \) for some natural \( b \) where \( Z_2 \) is smooth rational with \( (Z_2^2) = -2 \) and \( (Z_1, Z_2) = 2 \). In the first case put \( Z = Z_1 \) and choose a point \( p \in Z \). In the second case, since \( b(\mathcal{E}, Z_1) + b(\mathcal{E}, Z_2) = (\mathcal{E}, F) \geq 3m \geq 3b \) we must have \( (\mathcal{E}, Z_k) \geq 2 \) for \( k = 0 \) or \( k = 1 \). Put in this case \( Z = Z_1 + Z_2 - Z_k \) and take \( p \in Z_1 \cap Z_2 \). It will be sufficient to find a divisor in \( |\mathcal{N}| \) not passing through \( p \). We have the following exact sequence:

\[
0 \rightarrow H^0(\mathcal{N}(-Z)) \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_{p_1}(c)) \rightarrow H^1(\mathcal{N}(-Z))
\]

where \( c = (\mathcal{E}, Z) \geq 1 \). It is sufficient to prove that \( H^1(\mathcal{N}(-Z)) = 0 \).

We use Ramanujam's vanishing theorem [6]. By Serre duality it is sufficient to prove that

\[
(\mathcal{N}(-Z - K)^2) > 0 \quad \text{and} \quad (\mathcal{N}(-Z - K).R) \geq 0
\]

for any integral curve \( R \). Now

\[
(\mathcal{N}(-Z - K)^2) = (\mathcal{E}^2) + 2(\mathcal{E}, F) \deg(L) - 2 - 2(\mathcal{E}, Z) - 2(\mathcal{E}, K) \>
2(\mathcal{E}, F) (\deg(L) - 1 - d) - 2
\]

where \( d \in \mathbb{Q}, K \equiv dF \). Consequently the selfintersection is \( > 0 \) for \( \deg(L) \geq d + 2 \).

To check the second inequality suppose first that \( R \) is contained in a fibre \( F \). We get \( (\mathcal{N}(-Z - K).R) = (\mathcal{E}, R) - (Z, R) \geq 0 \) because the only case when \( (Z, R) = 2 \) is \( F = b(Z_1 + Z_2) \) and \( R = Z_k \). Now if \( R \) dominates \( B \) we get

\[
(\mathcal{N}(-Z - K).R) = (\mathcal{E}, R) + (F, R) \deg(L) - (Z, R) - (K, R) 
\>
(F, R) \deg(L) - (F, R) - (F, R) \geq 0
\]
for $\deg(L) \geq d + 1$, and we are done. Now if $\beta_1$ is chosen also such that $\beta_1 \geq 2p_d(B)$ it follows that $\mathcal{M}$ is still ample hence by Theorem 1 the linear system $|\mathcal{E} \otimes f^*L|$ is ample and base point free provided $\deg(L) \geq \beta_2 = \beta_1 + C_1$. By Bertini's theorem the above system contains an integral member $D$. To prove 1) $\implies$ 2) it is sufficient by [2] to prove that

$$H^1(\tilde{X}, p^*\mathcal{O}_X(D) \otimes p^*f^*L \otimes \mathcal{O}_X(-2W)) = 0$$
$$H^1(\tilde{X}, q^*\mathcal{O}_X(D) \otimes q^*f^*L \otimes \mathcal{O}_X(-W_1 - W_2)) = 0$$

for any $x, x_1, x_2 \in X$, provided $\deg(L) \geq \beta_3$ for some constant $\beta_3$; in this case the constant $C_2 = \beta_2 + \beta_3$ will be convenient for our purpose.

Now exactly as in the proof of the Theorem 1 we may find a constant $\beta_3$ such that for $\deg(L) \geq \beta_3$ the linear systems

$$|p^*\mathcal{O}_X(D - K) \otimes p^*f^*L \otimes \mathcal{O}_X(-3W)|$$
$$|q^*\mathcal{O}_X(D - K) \otimes q^*f^*L \otimes \mathcal{O}_X(-2W_1 - 2W_2)|$$

have strictly positive selfintersections and contain divisors of the form

$$G_1 = p^*\left(D + \sum a_iF_i\right) - 3W$$
$$G_2 = q^*\left(D + \sum b_iF_i\right) - 2W_1 - 2W_2$$

with $a_i, b_i \in \mathbb{Q}, a_i \geq 0, b_i \geq 0, \sum a_i \geq 2, \sum b_i \geq 2$ and where $F_i$ are fibres. Then by Lemma 1 the divisors $D + \sum a_iF_i$ and $D + \sum b_iF_i$ are 3-connected hence by a standard computation, $G_1$ and $G_2$ are 1-connected and the Theorem is proved.

**BIBLIOGRAPHY**


Manuscrit reçu le 20 avril 1982.

Alexandru Buium,
Department of Mathematics
National Institute for Scientific
and Technical Creation
Bd. Pacii 220
79622 Bucarest (Romania).