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A G-MINIMAL MODEL FOR PRINCIPAL G-BUNDLES

by Shrawan KUMAR

Introduction.

Sullivan built a minimal model theory for simplicial complexes. He showed that given a simply connected simplicial complex X with all its Betti numbers being finite, there is associated to it a certain uniquely determined (up to DGA isomorphism) DGA over Q (called minimal model for the space X) which contains exactly the rational homotopy information of the space X. Actually large part of this theory goes through for nilpotent simplicial complexes as well. For a quick exposition of this theory, see [3; Sections 1 to 3], [4] or [7].

Suppose $E \xrightarrow{p} B$ is a principal G-bundle, then the C^∞ de-Rham complex $\Omega(E)$ of E acquires additional structures due to the action of G on E. $\Omega(E)$ becomes a $\mathfrak{G}$(= Lie-algebra of G) algebra [see section 1]. In this paper we formulate a certain « natural » model $\mu_G[E]$ (which we call the G-minimal model) for the space E which is a collection of mutually « $\mathfrak{G}$-homotopic » $\mathfrak{G}$-algebras $\{A_\theta\}$, such that the DGA of basic elements in A_θ is the minimal model for B and any A_θ has the complete rational homotopy information of the space E (and B) (see theorem (2.2)).

In general (probably) we don't get a $\mathfrak{G}$-morphism from any A_θ to $\Omega(E)$ inducing isomorphism in cohomology. We analyze a more general question in theorem (2.3). It turns out that it is equivalent to the existence of a « special » connection in the bundle E. The nature of « special » connection seems interesting. For example, such a connection Φ_0 (if it exists) in a principal G-bundle E with highly connected base space B, would have the property that the corresponding (to Φ_0) lower characteristic forms themselves vanish. This actual vanishing of characteristic forms figures in the definition of secondary characteristic classes by Chern-Simons [2].

Section 1 contains the various definitions and some examples. The main theorems of the paper (Theorems 2.2 and 2.3) are formulated in section 2. Section 3 contains the proofs and examples of some G-bundles which admit « special » connections. We add an appendix to give a spectral sequence which converges to the cohomology of B and which has $H(E) \otimes H(BG)$ as its E_1 term.

We intend to take up the question « which principal G-bundles admit a « special » connection » in a separate paper.

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Throughout G will denote a compact connected real Lie group and $\mathfrak{G}$ its real Lie-algebra. All the G-bundles will be principal and in the smooth ($=C^\infty$) category with simply-connected base space B. Further we assume that all the Betti numbers of B are finite. Vector spaces will be over reals and linear maps would mean **R**-linear maps. Isomorphism would always mean surjective isomorphism.

1. Definitions.

(1.1) Definitions. — (a) *A Differential Graded Algebra (abbreviated as DGA) is an associative graded algebra* $A = \bigoplus_{k \geqslant 0} A^k$ *with unity and a differential* $d : A \to A$ *of degree* $+1$ *satisfying*

1) A *is graded commutative i.e.* $x.y = (-1)^{k\ell} y.x$ *for* $x \in A^k$ *and* $y \in A^\ell$.

2) *d is derivation i.e.*

$$d(x.y) = (dx).y + (-1)^k x.dy \quad \text{for} \quad x \in A^k$$

and

3) $d^2 = 0$.

A *is connected if* $H^0(A)$ *is the ground field.* A *is simply connected if in addition* $H^1(A) = 0$.

(*b*). — *Let* A *and* B *be two DGA with morphisms* $f, g : A \to B$. f *and* g *are said to be homotopic, if there exists a morphism* $H : A \to B \otimes_{\mathbf{R}} \mathbf{R}(t,dt)$ *such that* $\varepsilon_0 \circ H = f$ *and* $\varepsilon_1 \circ H = g$, *where* $\varepsilon_0, \varepsilon_1 : B \otimes \mathbf{R}(t,dt) \to B$ *are evaluations at* 0 *and* 1 *respectively.*

(*c*) [1(*a*), section 4]. — *By a* $\mathfrak{G}$*-algebra we mean a DGA* A *with two linear maps* $L : \mathfrak{G} \to \mathrm{Der}_0 A$ *and* $i : \mathfrak{G} \to \mathrm{Der}_{-1} A$ ($\mathrm{Der}_\ell A$ *denotes the set of all derivations of degree* ℓ *i.e. linear maps* $\theta : A^k \to A^{k+\ell}$ *satisfying* $\theta(ab) = \theta(a)b + (-1)^{\ell k} a\theta(b)$ *for* $a \in A^k$) *satisfying*

1) $i(X) \circ i(X) = 0$

2) $L(X)i(Y) = i(Y)L(X) + i[X,Y]$

3) $L(X) = di(X) + i(X)d$

for all $X, Y \in \mathfrak{G}$.

Remarks. — 1) i and L correspond to inner and Lie derivatives respectively.

2) As a consequence of (2) and (3) above, L is a Lie algebra homomorphism.

Notation. — We denote by $A^{\mathfrak{G}} = \{a \in A : L(X)a = 0 = i(X)a \text{ for all } X \in \mathfrak{G}\}$ and call them *basic elements* and by

$$I(A) = \{a \in A : L(X)a = 0 \text{ for all } X \in \mathfrak{G}\}$$

and call them *invariant elements*. In the example (1) of (1.2) below, the basic elements correspond exactly to the forms on the base.

(*d*) Let A_1 and A_2 be two $\mathfrak{G}$-algebras. A $\mathfrak{G}$*-morphism* $\varphi : A_1 \to A_2$ is a DGA homomorphism commuting with L and i actions.

(1.2) *Examples of* $\mathfrak{G}$*-algebras.* — (1) The main motivating example is the smooth de Rham complex $\Omega(E)$ of the total space E of a G-bundle.

(2) *Weil algebra of* $\mathfrak{G}$, which is defined to be the algebra $S(\mathfrak{G}^*) \otimes \Lambda(\mathfrak{G}^*)$ where $S(\mathfrak{G}^*)$ (respectively $\Lambda(\mathfrak{G}^*)$) denotes the total symmetric (respectively exterior) algebra of $\mathfrak{G}^*$ (= the dual of $\mathfrak{G}$).

For details of the operators d, L and i on $W(\mathfrak{G})$, see [1(*a*), section 6].

(3) $\Lambda(\mathfrak{G}^*)$ considered as a DGA with the operators

$$i(X)\omega = \omega(X) \qquad \text{and} \qquad [L(X)\omega]Y = -\omega[X, Y]$$

for $\omega \in \mathfrak{G}^*$ and $X, Y \in \mathfrak{G}$. Extend $i(X)$ and $L(X)$ as derivations on the whole of $\Lambda(\mathfrak{G}^*)$. We denote $I(\Lambda(\mathfrak{G}^*))$ by $I_\Lambda(\mathfrak{G})$.

(1.3) DEFINITIONS. — (1) *A connection in a* $\mathfrak{G}$*-algebra* A *is, by definition, a* $\mathfrak{G}$*-morphism from* $W(\mathfrak{G})$ *to* A.

It is not difficult to see that a connection in $\Omega(E)$ *in this sense gives rise to a connection in the* G*-bundle* E *in the usual geometric sense and vice-versa. See* [1(*a*); *sections* 5 *and* 6].

(2) *We call a* $\mathfrak{G}$*-algebra* A *with connection to be irreducible if there does not exist a* $\mathfrak{G}$*-subalgebra* B (*of* A) *admitting a connection such that* $A \supsetneqq B \supset A^{\mathfrak{G}}$.

2. Formulations of the main results.

Let $E \xrightarrow{p} B$ be a principal G-bundle. We are tacitly assuming that the base space B is simply connected although this restriction is more of a convenience than necessity. One can have suitable formulations for non simply-connected B as well by taking ℓ-stage minimal model for the space B, which always exists for finite ℓ. See [3; theorem (1.1)]. We associate a « G-model » as below.

(2.1) *A « G-model » associated to* E. — Let us fix a minimal model $\rho : \mu \to \Omega(B)$ in the sense of Sullivan [3; section 1]. Let $S_E = \{\theta : \theta$ is a DGA morphism from $I = I_S(\mathfrak{G})$ to μ such that the map induced in cohomology : $I \to H^*(\mu) \xrightarrow{\approx} H^*(B)$ is the characteristic cohomology homomorphism induced from some (and hence any) connection in E$\}$.

$I_S(\mathfrak{G}) \subset W(\mathfrak{G})$ denotes the algebra of all the invariant polynomials on $\mathfrak{G}$. Since I is a polynomial algebra, any two morphisms in S_E are homotopic. (Actually, $\theta \in S_E$ is nothing but an induced map at the minimal model level corresponding to the unique homotopy class of maps : $B \to B(G)$ determined by E).

Given a $\theta \in S_E$, we associate a $\mathfrak{G}$-algebra $A_\theta = W(\mathfrak{G}) \underset{I}{\otimes} \mu$, where μ is considered as an I-module via θ. The operators i_X and L_X, for all $X \in \mathfrak{G}$, are defined to be 0 on μ. i_X, L_X and d, being I-linear on both $W(\mathfrak{G})$ and μ, extend to operators on $W(\mathfrak{G}) \underset{I}{\otimes} \mu$. It is easy to see that A_θ becomes a $\mathfrak{G}$-algebra.

Let $\Phi_{res.} : I \to \Omega(B)$ be the characteristic homomorphism (i.e. the evaluation of the invariant polynomial after substituting the curvature) corresponding to a smooth connection Φ on the bundle E.

As the maps $\gamma = \rho \circ \theta$, $\Phi_{res.}$ are homotopic, there is a diagram of $\mathfrak{G}$-algebras (and $\mathfrak{G}$-morphisms)

$$\begin{tikzcd}
A_\theta = W(\mathfrak{G}) \underset{I}{\otimes} \mu \arrow[r, "\mathrm{Id.} \otimes \rho"] & W(\mathfrak{G}) \overset{\gamma}{\underset{I}{\otimes}} \Omega(B) & \\
\text{(D)} & & W(\mathfrak{G}) \underset{I}{\otimes} [\Omega(B) \underset{\mathbf{R}}{\otimes} \mathbf{R}(t,dt)] \arrow[ul, "\varepsilon_0"'] \arrow[dl, "\varepsilon_1"] \\
\Omega(E) & W(\mathfrak{G}) \overset{\Phi_{res.}}{\underset{I}{\otimes}} \Omega(B) \arrow[l, "\tilde{\Phi}"'] &
\end{tikzcd}$$

$W(\mathfrak{G}) \overset{\gamma}{\underset{I}{\otimes}} \Omega(B)$ denotes the tensor product, where $\Omega(B)$ is considered as an I-module via γ. The map $\tilde{\Phi}$ is extension of the connection $\Phi : W(\mathfrak{G}) \to \Omega(E)$ and the canonical inclusion $\Omega(B) \hookrightarrow \Omega(E)$.

In view of the lemma (3.3) of this paper, all the maps in diagram (D) induce isomorphism in cohomology. Since E is a nilpotent space (B being simply connected, by assumption), for any $\theta \in S_E$, the DGA A_θ contains all the rational homotopy information of the space E. (Of course, the minimal model μ, of the base space, sits inside A_θ as exactly the set of its basic elements and hence the $\mathfrak{G}$-algebra A_θ contains the complete rational homotopy information of the base space as well).

If we choose another $\theta_1 \in S_E$ then, θ, θ_1 being homotopic, A_θ and A_{θ_1} are « $\mathfrak{G}$-homotopic » in the following sense.

$$\begin{tikzcd} & W(\mathfrak{G}) \overset{\theta}{\underset{I}{\otimes}} \mu = A_\theta \\ W(\mathfrak{G}) \underset{I}{\otimes} [\mu \underset{R}{\otimes} \mathbf{R}(t,dt)] \arrow[ur, "\varepsilon_0"] \arrow[dr, "\varepsilon_1"'] & \\ & W(\mathfrak{G}) \overset{\theta_1}{\underset{I}{\otimes}} \mu = A_{\theta_1} \end{tikzcd}$$

Now let $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B'$ be two bundles with a G-morphism $f : E \to E'$. This induces, of course, a morphism : $\Omega(B') \to \Omega(B)$ and hence a map $\tilde{f} : \mu' \to \mu$ at the minimal model level. It is easy to see that, for any $\theta' \in S_{E'}$, $\tilde{f}\theta' \in S_E$. There exists a canonical $\mathfrak{G}$-morphism $\mu_G[f]$:

$$A_{\theta'} = W(\mathfrak{G}) \underset{I}{\otimes} \mu' \xrightarrow{\text{Id.} \otimes \tilde{f}} A_{\tilde{f} \circ \theta'} = W(\mathfrak{G}) \underset{I}{\otimes} \mu .$$

We summarize all this in the following.

(2.2) THEOREM. — *Let* $E \xrightarrow{p} B$ *be a G-bundle* (B *being simply connected and having all its betti nos. finite). There is associated a collection* $\mu_G[E] = \{A_\theta\}_{\theta \in S_E}$ *of mutually « $\mathfrak{G}$-homotopic » $\mathfrak{G}$-algebras admitting connections, as defined above. Moreover, for any* $\theta \in S_E$, $H^*(A_\theta)$ *is isomorphic with* $H^*(E)$ *and* $A_\theta^{\mathfrak{G}}$ *is a minimal model for* B. *In fact,* A_θ *contains the complete rational homotopy information of the space* E (*and* B).

Further, given two G-bundles E, E' *and a G-morphism* $f : E \to E'$, *there exists a « natural » $\mathfrak{G}$-morphism* $\mu_G[f]$ *from* $\mu_G[E']$ *to* $\mu_G[E]$ (*that is, for any* $\theta' \in S_{E'}$ *there exists a* $\theta \in S_E$ *and a « natural » $\mathfrak{G}$-morphism* : $A_{\theta'} \to A_\theta$) *as defined above.*

We call $\mu_G[E]$ the G-*minimal model* associated to the bundle E.

Remark. — Similarly, we can associate a $\mathfrak{G}$-minimal model to any $\mathfrak{G}$-algebra A which is finite dimensional in each degree, admits a connection and such that $A^{\mathfrak{G}}$ is simply-connected.

For this, we choose a connection $\Phi : W(\mathfrak{G}) \to A$. This gives a $\mathfrak{G}$-morphism : $W(\mathfrak{G}) \overset{\Phi_{\text{res.}}}{\underset{I}{\otimes}} A^{\mathfrak{G}} \to A$, which induces isomorphism in cohomology by lemma (3.3). Now we choose a minimal model $\rho : \mu \to A^{\mathfrak{G}}$ and take various homotopy lifts θ to make the construction of $A_\theta = W(\mathfrak{G}) \overset{\theta}{\underset{I}{\otimes}} \mu$.

$$\begin{tikzcd} & \mu \arrow[d, "\mathfrak{G}"] \\ I \arrow[ur, dashed, "\theta"] \arrow[r, "\Phi_{res.}"'] & A \end{tikzcd}$$

Observe that, homotopy class of the map $\Phi_{res.}$ does not depend upon the particular choice of connection in A. □

Now we study the existence of a $\mathfrak{G}$-morphism : $A_\theta \to \Omega(E)$.

There exists a $\mathfrak{G}$-morphism $\varphi : A_\theta \to \Omega(E)$ inducing the map ρ at the base if and only if there exists a connection Φ in the bundle E such that the following diagram is (actually) commutative

$$\begin{tikzcd} I \arrow[r, "\Phi_{res.}"] \arrow[dr, "\theta"'] & \Omega(B) \\ & \mu \arrow[u, "\rho"'] \end{tikzcd}$$

More generally, we have the following result.

(2.3) THEOREM. — *Let* $E \xrightarrow{p} B$ *be a G-bundle. There exists a* $\mathfrak{G}$*-algebra* $A = \bigoplus_{k \geqslant 0} A^k$ *and a* $\mathfrak{G}$*-morphism* $\varphi : A \to \Omega(E)$ *satisfying*

1) A^k *is finite dimensional for all* $k \geqslant 0$ *and* A^0 *is the ground field.*

2) φ *induces isomorphism in cohomology.*

3) $\varphi|_{A^{\mathfrak{G}}} : A^{\mathfrak{G}} \to \Omega(B)$ *is a minimal model in the sense of Sullivan if and only if there exist a connection* $\Phi : W(\mathfrak{G}) \to \Omega(E)$, *a minimal model* $\rho : \mu \to \Omega(B)$ *and a DGA morphism* $\theta : I = I_s(\mathfrak{G}) \to \mu$ *making the following diagram actually (not merely homotopically, which always exists) commutative.*

$$\begin{tikzcd} I \arrow[r, "\Phi_{res.}"] \arrow[dr, "\theta"'] & \Omega(B) \\ & \mu \arrow[u, "\rho"'] \end{tikzcd}$$

(D′)...

Notes. – (1) We call any such connection a « special » connection. (2) Given such a diagram (D′), there is a « canonical » $\mathfrak{G}$-morphism $\varphi_{\Phi,D'} : A_\theta \to \Omega(E)$, induced from the connection Φ and the map $\rho : \mu \to \Omega(B) \hookrightarrow \Omega(E)$, satisfying (1), (2) and (3) above.

(2.4) Corollary. – *If* A *is any* $\mathfrak{G}$*-algebra with a* $\mathfrak{G}$*-morphism* $\varphi : A \to \Omega(E)$ *satisfying* (1), (2) *and* (3) *above then there exist a « special » connection* Φ *in the algebra* $\Omega(E)$ *and a commutative diagram* (D′) *with the property that there exists a* $\mathfrak{G}$*-morphism* $\alpha : A_\theta \to A$ *satisfying* $\varphi \circ \alpha = \varphi_{\Phi,D'}$.

So, if A *is irreducible,* α *is a surjective morphism. We prove theorem* (2.3) *and its corollary in the next section.*

(2.5) *Remark.* – The following result due to Kostant [6; Theorem 0.2. and lemma 1] gives that, as a graded vector space over $\mathbf{R}$, A_θ can be identified with $\Lambda(\mathfrak{G}^*) \underset{\mathbf{R}}{\otimes} H \underset{\mathbf{R}}{\otimes} \mu$.

« Let H be any graded $\mathfrak{G}$-submodule of $S(\mathfrak{G}^*)$ satisfying $I_S(\mathfrak{G})^+ S(\mathfrak{G}^*) \oplus H = S(\mathfrak{G}^*)$ ($I_S(\mathfrak{G})^+$ denotes the set of all the $\mathfrak{G}$-invariant polynomials on $\mathfrak{G}$ with zero constant term). Then, the canonical map from $H \otimes I_S(\mathfrak{G})$ to $S(\mathfrak{G}^*)$, given by $f \otimes g \mapsto fg$, is a $\mathfrak{G}$-module isomorphism.

H can be taken to be, for example, the set of all G-harmonic polynomials on $\mathfrak{G}$ where G is the adjoint group of $\mathfrak{G}$. »

3. Proofs and some examples.

First we prove the following lemmas.

(3.1) Lemma. – *Any* $\mathfrak{G}$*-algebra* A, *with a* $\mathfrak{G}$*-morphism* $\varphi : A \to \Omega(E)$ *satisfying* (1), (2) *and* (3) *of theorem* (2.3) *admits a connection. In fact* (3) *can be replaced by a weaker assumption that* $A^{\mathfrak{G}} \to \Omega(B)$ *induces isomorphism in cohomology.*

Proof. – We show that there exists a linear map $\xi : \mathfrak{G}^* \to A^1$ commuting with the actions i and L.

Let us fix a point $e_0 \in E$. Consider the map $\varepsilon : G \to E$ defined by $\varepsilon(g) = e_0 g$. ε gives rise to a map $\varepsilon^* : \Omega(E) \to \Omega(G)$. We claim that $\varepsilon^* \varphi(A^k) \hookrightarrow \Lambda^k(\mathfrak{G}^*)$ (i.e. the left invariant k-forms on G). This is because, for $X_1, \ldots, X_k \in \mathfrak{G}$ and $a \in A^k$,

$$i(X_1) \circ \ldots \circ i(X_k) \circ \varepsilon^*\varphi(a) = \varepsilon^*\varphi \circ i(X_1) \circ \ldots \circ i(X_k)(a)$$

which is a constant function on G (since $A^0 \simeq \mathbf{R}$).

We further assert that $\varepsilon^*\varphi(A^1) = \mathfrak{G}^*$. Assuming this for a moment, let K be the kernel of the map $\varepsilon^*\varphi : A^1 \to \mathfrak{G}^*$ and $K^\perp$ be a $\mathfrak{G}$-submodule (under the L action) of A^1 such that $K \oplus K^\perp = A^1$. $\varepsilon^*\varphi|_{K\perp}$ is an isomorphism. Taking $(\varepsilon^*\varphi|_{K\perp})^{-1} : \mathfrak{G}^* \to K^\perp \hookrightarrow A^1$ gives a desired map ξ.

Extend this map to an algebra morphism $\tilde{\xi} : \Lambda(\mathfrak{G}^*) \to A$. We define the curvature from $\mathfrak{G}^* \to A^2$ by $\omega \mapsto d(\xi(\omega)) - \tilde{\xi}(d_{\mathfrak{G}}(\omega))$, where $d_{\mathfrak{G}}$ denotes the differential in the complex $\Lambda(\mathfrak{G}^*)$, and extend this to $S(\mathfrak{G}^*)$. These two maps together give a unique algebra map (again denoted by) $\tilde{\xi} : W(\mathfrak{G}) \to A$. It is a routine checking that the map $\tilde{\xi}$ is a connection in the $\mathfrak{G}$-algebra A.

We return to prove that $\varepsilon^*\varphi(A^1) = \mathfrak{G}^*$. Let ω be a primitive element in $I^k_A(\mathfrak{G})$. As ω is universally transgressive, there exists a form $\tilde{\omega} \in \Omega^k(E)$ such that $\varepsilon^*\tilde{\omega} = \omega$ and $d\tilde{\omega} \in p^*(\Omega^{k+1}(B))$. We can further assume that $\tilde{\omega} \in I(\Omega^k(E))$, i.e. $L(X)\tilde{\omega} = 0$ for all $X \in \mathfrak{G}$. Since $H^{k+1}(A^{\mathfrak{G}}) \xrightarrow{\sim} H^{k+1}(B)$, there exists an element $y \in A^{\mathfrak{G}}$ such that $dy = 0$ and $\varphi(y) = d\tilde{\omega} + p^*(d\theta)$ for some $\theta \in \Omega^k(B)$. But then by taking $\tilde{\omega} + p^*(\theta)$ in place of $\tilde{\omega}$, we can assume that $\varphi(y) = d\tilde{\omega}$. By assumption $H(A) \xrightarrow{\sim} H(E)$, so that $y = dx$ for some $x \in A^k$. Since $H(I(A)) \xrightarrow{\sim} H(A)$ (as can be easily seen from the relation $L(X) = di(X) + i(X)d$), we can choose $x \in I(A^k)$.

Now $d(\varphi(x) - \tilde{\omega}) = 0$ and hence $\varphi(x) - \tilde{\omega} = \varphi(y') + d\theta'$ for some form $\theta' \in I(\Omega^{k-1}(E))$ and $y' \in I(A^k)$ (We are using $H(I(A)) \xrightarrow{\sim} H(I(\Omega(E)))$). This gives $\varepsilon^*\varphi(x) - \varepsilon^*\tilde{\omega} = \varepsilon^*\varphi(y') + d\varepsilon^*(\theta')$. Since $d\varepsilon^*(\theta')$ is a bi-invariant form on G which is a coboundary and hence is 0. So $\varepsilon^*\tilde{\omega} = \omega \in \varepsilon^*\varphi(A)$ and hence $\varepsilon^*\varphi(A)$ contains all the bi-invariant forms on G. But the image $\varepsilon^*\varphi(A)$ is closed under the actions of $i(X)$ and $L(X)$ which would imply that $\varepsilon^*\varphi(A^1) = \mathfrak{G}^*$, proving the lemma.

(3.2) LEMMA. — *Let A be a $\mathfrak{G}$-algebra admitting a connection Φ. Let Z denote the subalgebra of horizontal elements i.e.*

$$Z = \{a \in A : i(X)a = 0 \text{ for all } X \in \mathfrak{G}\}.$$

Then the map $\beta : \Lambda(\mathfrak{G}^*) \otimes Z \to A$, *defined by* $\beta|_{\Lambda(\mathfrak{G}^*)} = \Phi|_{\Lambda(\mathfrak{G}^*)}$ *and* $\beta|_Z$ *is the inclusion, is a graded algebra (but not* DGA *in general) isomorphism commuting with the natural* i *and* L *actions.*

Proof. — Let us choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{G}$ and let $\{X_1^*, \ldots, X_n^*\}$ be the dual basis (of $\mathfrak{G}^*$).

(*a*) β is injective. — For let

$$\beta\left(\sum_{\substack{0 \leq k \leq \ell \\ i_1 < \ldots < i_k}} X_{i_1}^* \wedge \ldots \wedge X_{i_k}^* \otimes h_{i_1, \ldots, i_k}\right) = 0.$$

By operating $i(X_{j_\ell}) \circ \cdots \circ i(X_{j_1})$ on both the sides, we get $h_{j_1, \ldots, j_\ell} = 0$ and hence β is injective.

(*b*) β is surjective. — Let A_ℓ denote the set

$$\{a \in A : i(Y_1) \circ \cdots \circ i(Y_\ell)a = 0 \quad \text{for all} \quad Y_1, \ldots, Y_\ell \in \mathfrak{G}\}.$$

Clearly $A = A_{n+1} \supset A_n \cdots \supset A_1 = Z$. Assume, by induction, that A_ℓ is in the image of β (of course A_1 is in the image of β) and let $a \in A_{\ell+1}$. Consider the element

$$b = \sum_{i_1 < \ldots < i_\ell} \beta(X_{i_1}^* \wedge \ldots \wedge X_{i_\ell}^*) . i(X_{i_\ell}) \circ \cdots \circ i(X_{i_1})a.$$

By operating $i(X_{j_\ell}) \circ \cdots \circ i(X_{j_1})$ on both the sides, we get

$$i(X_{j_\ell}) \circ \cdots \circ i(X_{j_1})b = i(X_{j_\ell}) \circ \cdots \circ i(X_{j_1})a.$$

This implies that $b - a \in A_\ell$ and hence, by induction hypothesis, $b - a \in \text{Image}\,\beta$, but $b \in \text{Image}\,\beta$ and hence a also is in the image.

We prove the following lemma which is analogue of Leray-Serre spectral sequence for fibrations.

(3.3) Lemma. — *Let* A *be a* $\mathfrak{G}$*-algebra admitting a connection which is finite dimensional in each degree. Then there exists a convergent spectral sequence with* $E_2^{p,q} \simeq H^q(\mathfrak{G}) \otimes H^p(A^{\mathfrak{G}})$ *and converging to the cohomology of* A.

Remarks. — (1) Observe that a principal G-bundle (for G a connected group, which we are always assuming) is always orientable.

(2) The hypothesis that A admits a connection is necessary. For, take a $\mathfrak{G}$-algebra A with connection and then define

$$B = \sum_{\ell \geqslant 1} \Lambda(\mathfrak{G}^*) \otimes Z^\ell \oplus A^0 .$$

For « appropriate » A, B will provide a counter example.

Proof (of the lemma). – Let Φ be a connection in A. By the previous lemma (3.2), this induces an isomorphism $\Lambda(\mathfrak{G}^*) \otimes Z \xrightarrow{\sim} A$. Consider the filtration $A = A_0 \supset A_1 \supset \cdots \supset A_p \supset \cdots$ where $A_p = \sum_{\ell \geqslant p} \Lambda(\mathfrak{G}^*) \otimes Z^\ell$. This is of course a convergent filtration bounded above. We compute $E_r^{p,q}$ for $r = 0, 1, 2$.

Clearly $E_0^{p,q} \simeq \Lambda^q(\mathfrak{G}^*) \otimes Z^p$. Further $E_1^{p,q} \simeq H^q(\mathfrak{G}, Z^p) \simeq H^q(\mathfrak{G}, (A^{\mathfrak{G}})^p)$. We are using the fact that the Lie-algebra cohomology of a reductive Lie-algebra $\mathfrak{G}$, with coefficients in a nontrivial finite dimensional irreducible $\mathfrak{G}$-module V_ρ, vanishes i.e. $H(\mathfrak{G}, V_\rho) = 0$. See [5; Section 5-theorem 10]. Lastly $E_2^{p,q} \simeq H^q(\mathfrak{G}) \otimes H^p(A^{\mathfrak{G}})$.

Note. – The above given filtration does not depend upon the choice of the connection in A.

Now the proofs of the theorem (2.3) and its corollary are immediate.

(3.4) *Proof* (*of theorem* (2.3)). – The existence of a « special » connection is necessary, for take any connection Φ' in A (which exists by the Lemma 3.1) and compose this with the $\mathfrak{G}$-morphism $\varphi : A \to \Omega(E)$ to get a connection $\Phi = \varphi \circ \Phi'$ in the bundle E. It is easy to see that Φ is a « special » connection.

Conversely, we fix a « special » connection Φ in E and *a* commutative diagram (D′) as stated in the theorem. We have a $\mathfrak{G}$-morphism $\varphi_{\Phi, D'} : A_\theta \to \Omega(E)$ as defined in Note (2) of the theorem. Since the map $\varphi_{\Phi, D'} : A_\theta \to \Omega(E)$ preserves the filtrations (given in the proof of lemma 3.3) of A_θ and $\Omega(E)$, it induces maps

$$\varphi^*_{\Phi, D'} : E_r^{p,q}(A_\theta) \to E_r^{p,q}(\Omega(E)) .$$

Moreover $\varphi^*_{\Phi, D'} : E_2^{p,q}(A_\theta) \to E_2^{p,q}(\Omega(E))$ is an isomorphism for all p and q (lemma 3.3) and hence $\varphi_{\Phi, D'}$ induces isomorphism in cohomology. This proves the theorem.

(3.5) *Proof of the corollary* (2.4). – Let us fix a connection Φ' in A (exists by lemma 3.1). Then $\Phi = \varphi \circ \Phi'$ is a special connection in the bundle E. Consider the commutative diagram

$$\begin{tikzcd} & I \arrow[dl, "\Phi'_{\text{res.}} = \theta"'] \arrow[dr, "\Phi_{\text{res.}}"] & \\ \mu = A^{\mathfrak{G}} \arrow[rr, "\varphi_{\text{res.}} = \rho"'] & & \Omega(B) \end{tikzcd}$$

It is easily seen that the map $\alpha : A_\theta \to A$, defined by $\alpha|_{W(\mathfrak{G})} = \Phi'$ and $\alpha|_\mu$ is the inclusion, is a $\mathfrak{G}$-morphism satisfying $\varphi \circ \alpha = \varphi_{\Phi, D'}$. $\square$

Let $\mathscr{A}(E)$ denote the set of $\mathfrak{G}$-isomorphism classes of all the irreducible $\mathfrak{G}$-algebras A with a $\mathfrak{G}$-morphism : $A \to \Omega(E)$ satisfying (1), (2) and (3) of theorem (2.3). The following remark describes $\mathscr{A}(E)$, in fact it gives slightly sharper result.

(3.6) *Remark.* – Let J and J' be graded ideals in A_θ and $A_{\theta'}$ respectively which are closed under d, i and L, so that A_θ/J (respectively $A_{\theta'}/J'$) itself is a $\mathfrak{G}$-algebra. Assume further that $J \cap A_\theta^{\mathfrak{G}} = 0 = J' \cap A_{\theta'}^{\mathfrak{G}}$ (and hence $(A_\theta/J)^{\mathfrak{G}} \approx A_\theta^{\mathfrak{G}}$). If there exists a $\mathfrak{G}$-morphism $f : A_\theta/J \to A_{\theta'}/J'$ inducing isomorphism in cohomology, then there exists a DGA isomorphism $\tilde{f} : \mu \to \mu$ making the following diagram commutative.

$$\begin{tikzcd} & I_S(\mathfrak{G}) \arrow[dl, "\theta"'] \arrow[dr, "\theta'"] & \\ \mu \arrow[rr, "\tilde{f}"] & & \mu \end{tikzcd}$$

and hence A_θ is $\mathfrak{G}$-isomorphic with $A_{\theta'}$. To prove this, observe the following

(1) A_θ admits a unique connection.

(2) Let A, A' be two $\mathfrak{G}$-algebras with connection which are finite dimensional in each degree and f a $\mathfrak{G}$-morphism from A to A' which induces isomorphism in cohomology, then the map $f_{\text{res.}} : A^{\mathfrak{G}} \to A'^{\mathfrak{G}}$ also induces isomorphism in cohomology. This follows from the spectral sequence given in the appendix.

(3) A morphism of minimal differential algebras inducing an isomorphism in cohomology is itself an isomorphism, see [4; lecture 12].

(3.7) *Examples.* – We give below some examples of G-bundles which admit special connections.

(1) If G is abelian (i.e. G is a torus) then any G-bundle admits a special connection.

Since $\mathfrak{G}$ acts trivially on $S(\mathfrak{G}^*)$, the characteristic ring is the total algebra $S(\mathfrak{G}^*)$. Choose a basis $C = \{C_1, \ldots, C_n\}$ of $\mathfrak{G}^*$. Let Φ_0 be a connection in E and let $\{\beta_1, \ldots, \beta_n\}$ be the corresponding characteristic forms with respect to the basis $\underset{\sim}{C}$ (i.e. $\beta_i = \Phi_0(C_i)$). Let $\{\alpha_1, \ldots, \alpha_n\}$ be arbitrary elements in $\Omega^1(B)$. It can be easily seen that there exists a connection Φ in the bundle E such that the characteristic forms, with respect to the connection Φ, are $\{\beta_i + d\alpha_i\}_{1 \leqslant i \leqslant n}$. This ensures that E admits special connections. Moreover, it can be seen that the $\mathfrak{G}$-algebra A_θ does not depend (upto $\mathfrak{G}$-isomorphism) on θ.

(2) Let $E(G) \xrightarrow{p} B(G)$ be a universal G-bundle. Let Φ be a connection in $E(G)$. As is well known, the homomorphism $\Phi_{\text{res.}} : I_S(\mathfrak{G}) \to \Omega(B(G))$ induces isomorphism in cohomology (this follows easily from the spectral sequence given in the appendix) and $I_S(\mathfrak{G})$ is a polynomial algebra. Hence $\Phi_{\text{res.}}$ is a minimal model for the base space $B(G)$. This implies that the bundle $E(G)$ admits special connections. Moreover, it can be easily seen that any A_θ is $\mathfrak{G}$-isomorphic with $W(\mathfrak{G})$.

Note. – This bundle is not in the finite dimensional smooth category, but the underlying difficulty is not serious and we omit the precise formulation.

(3) Let $E \xrightarrow{p} B$ be a G-bundle which admits a special connection and let $f : B' \to B$ be a map inducing isomorphism at de-Rham cohomology level, then $f^*(E)$ (the pull-back bundle) also admits a special connection.

(4) Let $E_i \xrightarrow{p_i} B_i$ be G_i bundles which admit special connections for $i = 1, 2$. Then the $G_1 \times G_2$ bundle $E_1 \times E_2 \xrightarrow{p_1 \times p_2} B_1 \times B_2$ also admits a special connection.

(5) Let $E \xrightarrow{p} B$ be a G-bundle admitting a special connection and let $\rho : G \to H$ be a Lie-group homomorphism. Let E_ρ denote the associated principal H-bundle, then E_ρ also admits a special connection. In particular

a G-bundle, which admits a reduction of its structural group to a maximal torus of G, has a special connection.

(6) Let $E \xrightarrow{p} B$ be a G-bundle. Suppose that a compact connected Lie-group H operates on E by bundle morphisms and hence H acts on the base B. Let $I_H(\Omega(B))$ denote the set of H-invariant forms on B. Then, of course, $I_H(\Omega(B)) \hookrightarrow \Omega(B)$ induces isomorphism in cohomology. If we can choose a minimal model $\rho : \mu_B \to I_H(\Omega(B))$ for the algebra $I_H(\Omega(B))$ so that ρ is surjective (e.g. if B is a symmetric space under the action of H) then E admits a special connection, because an H invariant connection in E can be checked to be « special ».

Appendix.

THEOREM. — *Let* A *be a* $\mathfrak{G}$*-algebra, which is finite dimensional in each degree and which admits a connection. Then, there is a « natural » spectral sequence with* $E_1^{p,q} \simeq H^{q-p}(A) \otimes I_S^p(\mathfrak{G})$ *and converging to the cohomology of* $A^{\mathfrak{G}}$.

$I_S^p(\mathfrak{G})$ *denotes the set of all the invariant homogeneous polynomials on* $\mathfrak{G}$ *of degree* p *(and hence grade degree* $2p$*).*

Proof. — We sketch the derivation of this spectral sequence. Consider the tensor product of two $\mathfrak{G}$-algebras $A \otimes W(\mathfrak{G})$. There is a canonical inclusion $A \to A \otimes W(\mathfrak{G})$. Restriction of this map from $A^{\mathfrak{G}} \to [A \otimes W(\mathfrak{G})]^{\mathfrak{G}}$ induces isomorphism in cohomology, see [1(*b*); Theorem 3]. The projection

$$A \otimes W(\mathfrak{G}) = A \otimes \Lambda(\mathfrak{G}^*) \otimes S(\mathfrak{G}^*) \to A \otimes S(\mathfrak{G}^*)$$

induces bijection of $[A \otimes W(\mathfrak{G})]^{\mathfrak{G}}$ onto $I(A \otimes S(\mathfrak{G}^*))$ (i.e. the set of invariants). So, by transporting, we get a differential D in the algebra $I(A \otimes S(\mathfrak{G}^*))$ to make it a DGA. Explicity, this differential D is given by

$$D(a \otimes b) = (da) \otimes b - \sum_{j=1}^{n} i(X_j)a \otimes X_j^* b$$

for $a \in A$ and $b \in S(\mathfrak{G}^*)$, where $\{X_j\}_{1 \leqslant j \leqslant n}$ is a basis of $\mathfrak{G}$ and $\{X_j^*\}$ is the dual basis. (Although D is defined