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THE WIENER TEST FOR DEGENERATE ELLIPTIC EQUATIONS

by E. FABES(*), D. JERISON(*), C. KENIG(*)

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Introduction.

This is the second in a series of three articles examining solutions to degenerate elliptic equations in divergence form

$$Lu = -\partial_j(a_{ij}(x)\,\partial_i u(x)) = 0 \ (^{**}).$$

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(**) Here and elsewhere ∂_j denotes $\partial/\partial x_j$, $j = 1, \ldots, n$ and repeated indices are summed.

In the first article, two of the present authors and R. Serapioni established a Harnack inequality, Hölder continuity of solutions and certain other basic estimates that we will use here. Our main purpose in this article is to prove a Wiener test for regular points in the Dirichlet problem for L. We will suppose that the coefficients $a_{ij}(x)$ are real-valued, measurable, symmetric, and satisfy

$$c^{-1}\,|\xi|^2\,w(x) \leqslant a_{ij}(x)\,\xi_i\,\xi_j \leqslant C\,|\xi|^2\,w(x)$$

for all x and ξ in $\mathbf{R}^n$ and some constant $c \geqslant 1$. The weight $w(x)$ will be a non-negative, measurable function satisfying either Muckenhoupt's condition (A_2) or the condition (QC). These conditions are defined as follows:

$$(\mathrm{A}_2) \qquad \sup_{\mathrm{B}} \left(\frac{1}{|\mathrm{B}|}\int_{\mathrm{B}} w(x)\,dx\right)\left(\frac{1}{|\mathrm{B}|}\int_{\mathrm{B}} w(x)^{-1}\,dx\right) \leqslant \mathrm{C}$$

where the supremum is taken over all Euclidean balls B and $|\mathrm{B}| = \int_{\mathrm{B}} dx$.

$$(\mathrm{QC}) \qquad w(x) = |f'(x)|^{1-2/n},$$

where f is a global quasiconformal mapping $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ and $|f'(x)|$ denotes the absolute value of the Jacobian determinant of f. For example, all functions $w(x)$ of the form $|x|^{\alpha}, \alpha > -n$ satisfy either (A_2) or (QC). For more details on the nature of these conditions see [2, 5], and [6].

Denote $w(\mathrm{E}) = \int_{\mathrm{E}} w(x)\,dx$ and $\mathrm{B}(x,r) = \{y \in \mathbf{R}^n : |x-y| \leqslant r\}$. Fix a large ball Σ of radius R. The first main result is an approximate formula for the Green function in Σ,

$$g(x,y) \simeq \int_{|x-y|}^{\mathrm{R}} \frac{s^2}{w(\mathrm{B}(x,s))}\,\frac{ds}{s}, \quad \text{for } x, y \in \frac{1}{4}\,\Sigma.$$

(For a precise statement, see Theorem 3.3.) This formula shows that locally the Green function for L exhibits essentially the same simple radial behavior as the classical Green function. (The third article is devoted in part to estimates of the Green function near the boundary.)

The formula reveals an amusing difference from classical Green functions. The limit on $y \longrightarrow x$ of $g(x,y)$ need not be infinite. The following properties are equivalent:

(i) $\limsup_{y \to x} g(x,y) < \infty$. (We will see that the limit exists.)

(ii) the punctured ball $\Sigma \setminus \{x\}$ is regular for the Dirichlet problem.

(iii) The capacity (1.19) of $\{x\}$ is positive.

(iv) $\displaystyle\int_0^R \frac{s^2}{w(B(x, s))} \frac{ds}{s} < \infty .$

At first glance property (ii) seems to contradict the maximum principle, since we can assign boundary values 1 at x and 0 on $\partial\Sigma$. However, as property (iii) indicates, the set $\{x\}$ is not removable in any appropriate sense, so the maximum principle remains intact. The extra phenomenon of (i)-(iv) is reflected in the Wiener test (Theorem 5.1(a)). A corollary of the Wiener test is that regular points depend only on $w(x)$ and not on the particular operator L. Another by-product of the argument is that the capacity we are considering has the usual equivalent descriptions (Theorem 4.7, 4.10). These descriptions coupled with the formula for the Green function above give a convenient way to calculate capacities and hence the Wiener criterion 5.1 (b).

If $w(x)$ satisfies (QC), then a change of variable by the quasiconformal map f transforms the problem into one for $w \equiv 1$, that is a uniformly elliptic equation with bounded, measurable coefficients such as was treated by Littman, Stampacchia, and Weinberger [8]. In that case our results follow from theirs. The point is to prove the results directly so that they apply to weights that satisfy conditions (like A_2) that are more easily verified. In fact, our proof will apply to a wider class of weights satisfying six properties listed in [5]. The single most important of these is

$$(*) \int_\Omega |\varphi(x)|^2 \varphi(x)\,dx \leqslant C \int_\Omega |\nabla\varphi(x)|^2 \, w(x)dx\,, \text{ all } \varphi \in C_0^\infty(\Omega)\,,$$

where C depends only on w and Ω.

We will follow the outline of the paper of Littman et al. [8]. The main differences are in Section 3.

1. Preliminaries.

Recall that $w(x)$ is a non-negative function satisfying either (A_2) or (QC). Two well-known facts are

(1.1) $w(x)\,dx$ and dx are mutually absolutely continuous

(1.2) $w(B(x, 2r)) \leqslant C w(B(x, r))$. *(Doubling condition.)*

Ω will always denote a bounded, open connected subset of $\mathbf{R}^n$.

Function Spaces. Denote by $L^p(\Omega, w)$ the Lebesgue class with norm $\|f\|_p^p = \int_\Omega |f(x)|^p\, w(x)\,dx$. $\mathrm{Lip}(\bar\Omega)$ is the restriction to $\bar\Omega$ of functions Ψ on $\mathbf{R}^n$ satisfying the Lipschitz condition $|\Psi(x) - \Psi(y)| \leqslant M|x - y|$ for some M. $\mathrm{Lip}_0(\Omega)$ denotes the class of functions of $\mathrm{Lip}(\bar\Omega)$ with compact support in Ω. (All functions are real valued.) Consider the inclusion

$$\mathrm{Lip}(\bar\Omega) \longrightarrow [L^p(\Omega, w)]^{n+1}$$

given by $\varphi \longrightarrow (\varphi, \nabla\varphi) = (\varphi, \partial_1\varphi, \ldots, \partial_n\varphi)$. $H^{1,p}(\Omega)$ denotes the closure of the image of $\mathrm{Lip}(\bar\Omega)$ in $[L^p(\Omega, w]^{n+1}$. Similarly, $H_0^{1,p}(\Omega)$ denotes the closure of the image of $\mathrm{Lip}_0(\Omega)$ in $[L^p(\Omega, w)]^{n+1}$. When $p \geqslant 2$, an $(n+1)$-tuple $\underline{u} = (u_0, u_1, \ldots, u_n)$ in $H^{1,p}(\Omega)$ is uniquely determined by its first component u_0 (see [5] 2.1). If $w^{-1} \in L^1(dx)$ and $\underline{u} \in H^{1,2}(\Omega)$, then u_0 is a distribution and $(u_1, \ldots, u_n) = \nabla u_0$ in the sense of distributions, but this is not true in general. But since $(u_1, \ldots, u_n)$ are determined by u_0, we can use the symbol ∇u_0 for $(u_1, \ldots, u_n)$. We will also shift notation and refer to u_0 as an element of $H^{1,2}(\Omega)$. The *Dirichlet form* $D : H^{1,2}(\Omega) \times H^{1,2}(\Omega) \longrightarrow \mathbf{R}$ is defined by $D(u, v) = \int_\Omega a_{ij}(x)\, u_i(x)\, v_j(x)\,dx$, where $(u_1, \ldots, u_n) = \nabla u$ and $(v_1, \ldots, v_n) = \nabla v$. (We will use this notation consistently).

Let $\frac{1}{p} + \frac{1}{p'} = 1$. The dual space of $H_0^{1,p'}(\Omega)$ for $p' < \infty$ is the space

$$H^{-1,p}(\Omega) = \{f_0 - \operatorname{div}\vec f : \vec f = (f_1, \ldots, f_n),\ f_j/w \in L^p(\Omega, w)\ j = 0, 1, \ldots, n\}.$$

To see this, observe first that since $w \in L^1(\Omega, dx)$, a function f satisfying $f/w \in L^p(\Omega, w)$ belongs to $L^1(\Omega, dx)$. Hence, an element $T = f_0 - \operatorname{div}\vec f$ of $H^{-1,p}(\Omega)$ is a distribution and acts on $\mathrm{Lip}_0(\Omega)$ by

$$\langle T, \varphi\rangle = \int_\Omega f_0(x)\,dx + \int_\Omega \vec f \cdot \nabla\varphi\,dx.$$

This action extends in a unique way to all u in $H_0^{1,p'}(\Omega)$

$$\langle T, \underline{u} \rangle = \sum_{j=0}^{n} \int_\Omega f_j(x)\, u_j(x)\, dx .$$

DEFINITION 1.3. – *Let* $T \in H^{-1,2}(\Omega)$. *We say that* $Lu = T$ *in the* $H^{1,2}(\Omega)$ *sense if* $u \in H^{1,2}(\Omega)$ *and* $D(u, v) = \langle T, v \rangle$ *for every* $v \in H_0^{1,2}(\Omega)$.

THEOREM 1.4. – *For every* T *in* $H^{-1,2}(\Omega)$ *and every* h *in* $H^{1,2}(\Omega)$ *there is a unique* u *in* $H^{1,2}(\Omega)$ *satisfying* $Lu = T$ *in the* $H^{1,2}(\Omega)$ *sense and* $u - h \in H_0^{1,2}(\Omega)$.

Property (*) of the introduction says that the inner product $D(u, v)$ is non-degenerate when restricted to $H_0^{1,2}(\Omega)$. Thus the proof of 1.4 consists of the usual Hilbert space argument. (See, for example, [7].)

Fundamental Inequalities. We recall now the results from [5] needed. The constants C, $k > 1$, $\alpha > 0$, and $p_0 < \infty$ below depend only on the (A_2) or (QC) constants of $w(x)$. In particular, they are independent of r and p. Denote

$$B = B_1 = \{y : |x - y| < r\} \quad \text{and} \quad B_2 = \{y : |x - y| < 2r\} .$$

The basic inequality (*) of the introduction is the consequence of a stronger inequality ([5], 2.3, 4))

$$\left(\frac{1}{w(B)} \int_B |\varphi|^{2k} w\right)^{1/2k} \leqslant Cr\left(\frac{1}{w(B)} \int_B |\nabla \varphi|^2 w\right)^{1/2} \quad \text{for all} \quad \varphi \in H_0^{1,2}(B) . \tag{1.5}$$

A slight variant ([5], 2.3, 5)) is

$$\left(\frac{1}{w(B)} \int_B |\varphi - \varphi_B|^{2k} w \leqslant Cr\left(\frac{1}{w(B)} \int_B |\nabla \varphi|^2 w\right)^{1/2}, \quad \text{for all} \quad \varphi \in H^{1,2}(B) , \quad \varphi_B = \frac{1}{w(B)} \int_B \varphi w . \tag{1.6}$$

Let u satisfy $Lu = 0$ in the $H^{1,2}(B_2)$ sense. Then u is Hölder continuous and ([5], 2.3.1, 2.3.12)

$$\max_{B_1} |u| \leqslant C\left(\frac{1}{w(B_2)} \int_{B_2} u^2 w\right)^{1/2} , \tag{1.7}$$

$$(1.8) \quad \sup_{|z-x|<\rho} |u(z) - u(x)| \leqslant C\left(\frac{1}{w(B_2)} \int_{B_2} u^2 w\right)^{1/2} (\rho/r)^\alpha \text{, for } \rho < r.$$

If u is also non-negative, then Harnack's inequality says ([5], 2.3.8).

$$(1.9) \qquad \max_{y\in B_1} u(y) \leqslant C \min_{y\in B_1} u(y)$$

(1.10) *Notations.* – Let $\Sigma = \{y : |y| < R\}$ be fixed from now on. The mapping $G : H^{-1,2}(\Sigma) \longrightarrow H_0^{1,2}(\Sigma)$ is uniquely defined by the conditions $u = G(T) \in H_0^{1,2}(\Sigma)$ and $Lu = T$ in the $H^{1,2}(\Sigma)$ sense. (See 1.4).

The proof of 1.4 shows that

(1.11) $\quad G : H^{-1,2}(\Sigma) \longrightarrow H_0^{1,2}(\Sigma)$ is an isomorphism.

If $T \in H^{-1,p}(\Sigma)$ for $p \geqslant p_0$, then $u = G(T)$ is Hölder continuous in $\bar{\Sigma}$ and ([5], 2.4.8).

$$(1.12) \qquad \sup_{\Sigma} |u| \leqslant C\, \|T\|_{H^{-1,p}(\Sigma)}$$

$$(1.13) \qquad \sup_{\substack{x,y\in\Sigma \\ |x-y|<\rho}} |u(x) - u(y)| \leqslant C\rho^\alpha\, \|T\|_{H^{-1,p}(\Sigma)} .$$

Note that since $u \in H_0^{1,2}(\Sigma)$, u vanishes on $\partial\Sigma$.

Finally, we have the standard lemma ([8], 2.1): If $Lu = 0$ in the $H^{1,2}(B_2)$ sense, then

$$(1.14) \qquad \int_{B_1} |\nabla u|^2 w \leqslant C r^{-2} \int_{B_2} |u|^2 w .$$

The boundary variant says ([5], 2.4.2) that if $Lu = 0$ in the $H^{1,2}(\Sigma \cap B_2)$ sense and $u = 0$ on $\partial\Sigma \cap B_2$ in the $H^{1,2}(\Sigma \cap B_2)$ sense, then

$$(1.14)' \qquad \int_{B_1} |\nabla u|^2 w \leqslant C r^{-2} \int_{B_2} |u|^2 w .$$

Truncation. Let $\underline{u} = (u, u_1, \dots, u_n)$ be an element of $H_0^{1,2}(\Omega)$. For $\mathcal{E} \geqslant 0$, denote $u^{(\mathcal{E})}(x) = \min\{u(x), \mathcal{E}\}$. Let $\varphi_j \in \mathrm{Lip}_0(\Omega)$ be a sequence tending to $\underline{u}$ in $H_0^{1,2}(\Omega)$ norm. Then $\varphi_j^{(\mathcal{E})}$ tends to $u^{(\mathcal{E})}$ in $L^2(\Omega, w)$ norm and $\varphi_j^{(\mathcal{E})}$ tends weakly to some $\underline{v} = (v, v_1, \dots, v_n)$ in $H_0^{1,2}(\Omega)$. But then $u^{(\mathcal{E})} = v$, and as remarked above, v uniquely determines $\underline{v}$. Hence $\underline{v}$ is unique and we have proved

LEMMA 1.15. – *If* $\underline{u}$ *belongs to* $H_0^{1,2}(\Omega)$ $\mathcal{E} \geqslant 0$, *then there is a unique* $\underline{u}^{(\mathcal{E})}$ *such that for every sequence* $\varphi_j \in \mathrm{Lip}_0(\Omega)$ *with* $\varphi_j \longrightarrow \underline{u}$ *in* $H_0^{1,2}(\Omega)$, $\varphi_j^{(\mathcal{E})}$ *tends weakly to* $\underline{u}^{(\mathcal{E})}$ *in* $H_0^{1,2}(\Omega)$. *Moreover, the first component of* $\underline{u}^{(\mathcal{E})}$ *is* $u^{(\mathcal{E})}$.

Notice also that replacing $\varphi_j^{(\mathcal{E})}$ by the arithmetic means of a subsequence, we can suppose (by the theorem of Banach and Saks) that $\varphi_j^{(\mathcal{E})} \longrightarrow u^{(\mathcal{E})}$ in $H_0^{1,2}(\Omega)$ norm.

We will have no further need in the remainder of the section to distinguish between $\underline{u}$ and its first component. Similar considerations to 1.15 yield

PROPOSITION 1.16. – *If* u *belongs to* $H_0^{1,2}(\Omega)$, *then* $|u|$, $u^+ = \max(u, 0)$ *and* $u^{(\mathcal{E})}$ $\mathcal{E} \geqslant 0$ *belong to* $H_0^{1,2}(\Omega)$. *Furthermore,* $\|u^{(\mathcal{E})}\|_{H_0^{1,2}(\Omega)} \leqslant \|u\|_{H_0^{1,2}(\Omega)}$ *and other analogous norm inequalities hold.*

DEFINITION 1.17. – *Let* $K \subset \overline{\Omega}$. *We say that* $u \geqslant c$ *on* K *in the* $H^{1,2}(\Omega)$ *sense if there exist* $\varphi_j \in \mathrm{Lip}(\overline{\Omega})$ *such that* $\varphi_j(x) \geqslant c$ *for all* $x \in K$ *and* $\varphi_j \longrightarrow u$ *in* $H^{1,2}(\Omega)$. *(There is a similar definition for* $u \leqslant c$ *on* K, *and* $u = c$ *on* K *means* $u \leqslant c$ *and* $u \geqslant c$ *on* K.)

The weak maximum principle of Stampacchia says

THEOREM 1.18. – *If* $u \in H^{1,2}(\Omega)$, $u \geqslant 0$ *on* $\partial\Omega$ *in the* $H^{1,2}(\Omega)$ *sense, and* $D(u, v) \geqslant 0$ *for every* v *in* $H_0^{1,2}(\Omega)$ *such that* $v \geqslant 0$ *on* Ω *in the* $H^{1,2}(\Omega)$ *sense, then* $u(x) \geqslant 0$ a.e. x *in* Ω.

The proof is well-known and uses truncation. (See [7] or [8]).

Capacity. – Let K be a compact subset of Σ. (1.10).

DEFINITION 1.19. – *The capacity of* K *in* Σ *is*

$$\mathrm{cap}(K) = \inf \{D(u, u) : u \in H_0^{1,2}(\Sigma)$$

and $u \geqslant 1$ *on* K *in the* $H^{1,2}(\Sigma)$ *sense*$\}$.

THEOREM 1.20. – *There exist a unique* u *in* $H_0^{1,2}(\Sigma)$ *satisfying* $D(u, u) = \mathrm{cap}(K)$ *and* $u \geqslant 1$ *on* K *in the* $H^{1,2}(\Sigma)$ *sense. Moreover,* $u = 1$ *on* K *in the* $H^{1,2}(\Sigma)$ *sense and* $D(u, v) \geqslant 0$ *for every* $v \in H_0^{1,2}(\Sigma)$ *such that* $v \geqslant 0$ *on* K *in the* $H^{1,2}(\Sigma)$ *sense.*

Proof. – One can easily check that the infimum is taken over a closed convex set in $H_0^{1,2}(\Sigma)$. As we observed in 1.4, $D(u, v)$ is an inner product for the Hilbert space, so the extremal function u exists and is unique. A limiting argument using truncation shows that $u = u^{(1)} = 1$ on K in the $H^{1,2}(\Sigma)$ sense. Finally, if $v \in H_0^{1,2}(\Sigma)$ and $v \geqslant 0$ on K in the $H_0^{1,2}(\Sigma)$ sense, then

$$D(u + \mathcal{E}v, u + \mathcal{E}v) \geqslant D(u, u)$$

for all $\mathcal{E} > 0$. Hence, $2\mathcal{E}D(u, v) + \mathcal{E}^2 D(v, v) \geqslant 0$, which implies $D(u, v) \geqslant 0$.

The function u is the *capacitary potential* of K in Σ. It follows from 1.20 that $Lu = 0$ in the $H^{1,2}(\Sigma \setminus K)$ sense. Also, $u = 1$ on K and $u = 0$ on $\partial\Sigma$ in the $H^{1,2}(\Sigma)$. Therefore, by 1.18,

COROLLARY 1.21. – *A capacitary potential* u *satisfies* $0 \leqslant u(x) \leqslant 1$ a.e. x *in* Σ.

Next, for any $\varphi \in C_0^\infty(\Sigma)$ satisfying $\varphi \geqslant 0$ on K, we have $D(u, \varphi) \geqslant 0$. By L. Schwartz' theorem, there exists a positive measure μ supported on K such that $D(u, \varphi) = \int \varphi d\mu$ for all $\varphi \in C_0^\infty(\Sigma)$. It is easy to see that the previous equality also holds for all $\varphi \in \mathrm{Lip}_0(\Sigma)$. The measure μ is known as the *capacitary distribution of* K in Σ.

PROPOSITION 1.22. – *The capacitary distribution* μ *defined above is supported on* ∂K *and* $\mu(K) = \mathrm{cap}(K)$.

Proof. – We can arrange using truncation that u is the limit of $\Psi_j \in \mathrm{Lip}_0(\Sigma)$ such that $\Psi_j = 1$ on K. If $\varphi \in C_0^\infty(\Sigma)$ is supported in the interior of K, then $D(u, \varphi) = \lim_{j\to\infty} D(\Psi_j, \varphi) = 0$, because $\nabla\Psi_j = 0$ in the interior of K. Hence μ is supported on ∂K. Also, $\mu(K) = \lim_{j\to\infty} \int \Psi_j \, d\mu = \lim_{j\to\infty} D(\Psi_j, u) = D(u, u) = \mathrm{cap}(K)$.

DEFINITION 1.23. – *A measure* μ *is said to belong to* $H^{-1,2}(\Sigma)$ *if* $|\int \varphi \, d\mu| \leqslant C \|\varphi\|_{H_0^{1,2}(\Sigma)}$ *for all* $\varphi \in C_0^\infty(\Sigma)$. *It is then clear that there exists a unique* $T \in H^{-1,2}(\Sigma)$ *so that* $\int \varphi d\mu = \langle T, \varphi\rangle$ *for all* $\varphi \in C_0^\infty(\Sigma)$. *This equality immediately extends to all* $\varphi \in \mathrm{Lip}_0(\Sigma)$.

Remark 1.24. – If μ is the capacitary distribution of $K \subset \Sigma$, then $\mu \in H^{-1,2}(\Sigma)$, and if u is the corresponding capacitary potential, $Lu = \mu$ in the $H^{1,2}(\Sigma)$ sense.

Before proceeding with our development, we need to take a closer look at capacity and at continuity properties of elements of $H_0^{1,2}(\Sigma)$. The results will be applied in the forthcoming sections. Most of the material that follows is known in one form or another. Unfortunately, we have been unable to find any reference in the literature where these results are stated in the precise form we need them.

For an open set $\mathcal{O}$ in Σ, and an arbitrary set E in Σ, denote

$$\text{cap}(\mathcal{O}) = \sup\{\text{cap}(K) : K \text{ compact}, \ K \subset \mathcal{O}\}$$

$$\text{cap}^*(E) = \inf\{\text{cap}(U) : U \text{ open}, \ U \supset E\}.$$

We will say that an equality holds *quasi-everywhere* (abbreviated *q.e.*) on a set $S \subset \Sigma$ if it holds on $S \backslash E$, where $\text{cap}^*(E) = 0$.

PROPOSITION 1.25. – *If the non-negative measure* μ *belongs to* $H^{-1,2}(\Sigma)$ *then, if* E *is a Borel set and* $\text{cap}^*(E) = 0$, $\mu(E) = 0$.

Proof. – Given $\mathcal{E} > 0$, there exists an open set U, $E \subset U$ such that $\text{cap}(U) \leqslant \mathcal{E}$. Let K be any compact subset of U. Then, $\text{cap}(K) \leqslant \mathcal{E}$. It is easy to see that $\text{cap}(K) = \inf\{D(\varphi, \varphi) : \varphi \geqslant 1$ on K, $\varphi \in \text{Lip}_0(\Sigma)\}$. Replacing φ by φ^+, we can choose $\varphi \geqslant 0$ in Σ, $\varphi \geqslant 1$ on K, $\varphi \in \text{Lip}_0(\Sigma)$ such that $D(\varphi, \varphi) \leqslant 2\mathcal{E}$. Then, $\mu(K) \leqslant \int \varphi \, d\mu = \langle T, \varphi \rangle \leqslant CD(\varphi, \varphi) \leqslant C\mathcal{E}$. Thus, $\mu(U) \leqslant C\mathcal{E}$, and as $\mathcal{E} > 0$ is arbitrary, $\mu(E) = 0$.

DEFINITION 1.26. – *A function* u *defined q.e. in* Σ *is called quasi-continuous, if given* $\mathcal{E} > 0$, *there exists an open set* $U \subset \Sigma$, *with* $\text{cap}(U) < \mathcal{E}$ *so that* u *is continuous on* $\Sigma \backslash U$. *Our main goal is to prove the following two propositions.*

PROPOSITION 1.27. – *Given* $u \in H_0^{1,2}(\Sigma)$, *there exists a sequence* $\{\varphi_j\} \in \text{Lip}_0(\Sigma)$, *and a sequence of open sets* $\mathcal{O}_k$, *such that* $\text{cap}(\mathcal{O}_k) \longrightarrow 0$, $\varphi_j \longrightarrow u$ *in* $H_0^{1,2}(\Sigma)$, *and* $\{\varphi_j\}$ *converges uniformly in* $\Sigma \backslash \mathcal{O}_k$ *for each* k. *Moreover, if* u *is bounded, the* φ_j *can be taken to be uniformly bounded, and if* $u = 1$ *on* K *in the* $H^{1,2}(\Sigma)$ *sense, the* φ_j *can be taken to be* $\equiv 1$ *on* K.

As a consequence of the proposition we see that given $u \in H_0^{1,2}(\Sigma)$, there exists $\widetilde{u}$ in $H_0^{1,2}(\Sigma)$ with $u = \widetilde{u}$ a.e., and $\widetilde{u}$ quasi continuous.

PROPOSITION 1.28. – *If* $\widetilde{u}_1$ *and* $\widetilde{u}_2$ *belong to* $H_0^{1,2}(\Sigma)$, *are quasi-continuous, and agree almost everywhere, they agree quasi-everywhere.*

The following corollary is the main application of 1.27 and 1.28 that will be needed in the sequel.

COROLLARY 1.29. – *Let* μ *be a positive measure in* $H^{-1,2}(\Sigma)$. *Then, if* $u \in H_0^{1,2}(\Sigma)$, *is Borel measurable, bounded and quasi-continuous we have* $\int u\, d\mu = \langle T, u\rangle$, *where* T *is as in definition 1.23.*

Proof of corollary 1.29. – Pick a sequence $\{\varphi_j\}$ of $\mathrm{Lip}_0(\Sigma)$ functions and a sequence of open sets $\{\mathcal{O}_k\}$ as in 1.27. Let $E = \cap\, \mathcal{O}_k$. Then E is a Borel set, and $\mathrm{cap}(E) = 0$. Let $\widetilde{u} = \lim \varphi_j$, where the limit is taken in the pointwise sense. Clearly $\widetilde{u}$ is defined everywhere in $\Sigma \setminus E$, and as $\mu(E) = 0$, it is μ-measurable. Also, as $|E| = 0$, $\widetilde{u}$ is in $H_0^{1,2}(\Sigma)$, and $\widetilde{u} = u$ almost everywhere. Because of the uniform convergence of $\{\varphi_j\}$ in $\Sigma \setminus \mathcal{O}_k$, we see that $\widetilde{u}$ is quasi-continuous. But then, $\widetilde{u} = u$ quasi-everywhere by 1.28. Thus, there exists a G_δ set $\widetilde{E}$ so that $\widetilde{u} = u$ for every point in $\Sigma \setminus \widetilde{E}$. By 1.25, $\mu(E \cup \widetilde{E}) = 0$. Thus,

$$\int u d\mu = \int_{\Sigma\setminus E\cup\widetilde{E}} u d\mu = \int_{\Sigma\setminus E\cup\widetilde{E}} \widetilde{u} d\mu = \int_{\Sigma\setminus E\cup\widetilde{E}} \lim \varphi_j\, d\mu$$

$$= \lim_j \int_{\Sigma\setminus E\cup\widetilde{E}} \varphi_j\, d\mu = \lim_j \int_\Sigma \varphi_j\, d\mu = \lim_j \langle T, \varphi_j\rangle = \langle T, u\rangle .$$

The interchange of lim and integration is justified by the uniform boundedness of $\{\varphi_j\}$.

We now turn to the proof of propositions 1.27 and 1.28.

Proof of proposition 1.27. – We first note that

$$\mathrm{cap}(K_1 \cup K_2) \leqslant \mathrm{cap}(K_1) + \mathrm{cap}(K_2)$$

for any two compact sets K_1 and K_2. This follows easily by considering the test function $u = \max(u_1, u_2)$ where the u_i are the capacitary potentials of K_i. From this it easily follows that $\mathrm{cap}\left(\bigcup_i \mathcal{O}_i\right) \leqslant \sum_i \mathrm{cap}(\mathcal{O}_i)$ for any sequence of open sets $\mathcal{O}_i$. Pick now a sequence $\varphi_j \in C_0^\infty(\Sigma)$, $\varphi_j \longrightarrow u$ in $H_0^{1,2}(\Sigma)$ so fast that

$\sum_j 4^j \|\varphi_{j+1} - \varphi_j\|^2_{H_0^{1,2}(\Sigma)} < \infty$. Let now

$$U_j = \left\{ x \in \Sigma : |\varphi_{j+1}(x) - \varphi_j(x)| > \frac{1}{2^j} \right\},$$

and $\mathcal{O}_k = \bigcup_{j=k}^{\infty} U_j$. We know that $\operatorname{cap}(\mathcal{O}_k) \leqslant \sum_{j=k}^{\infty} \operatorname{cap}(U_j)$. On the other hand, $2^j |\varphi_{j+1}(x) - \varphi_j(x)| \geqslant 1$ on U_j, and so

$$\begin{aligned} \operatorname{cap}(U_j) &\leqslant D(2^j |\varphi_{j+1} - \varphi_j|, 2^j |\varphi_{j+1} - \varphi_j|) \\ &\leqslant 4^j D(\varphi_{j+1} - \varphi_j, \varphi_{j+1} - \varphi_j). \end{aligned}$$

Thus, $\operatorname{cap}(\mathcal{O}_k) \longrightarrow 0$ as $k \longrightarrow \infty$, and the proposition follows. Proposition 1.28 follows immediately from Theorem 5 in [3], once we show that our definition of capacity of an open set $\mathcal{O}$ coincides with the encombrement of an open set $\mathcal{O}$ as defined in [3]. We recall the definition of $\operatorname{enc}(\mathcal{O})$. For an open set $\mathcal{O} \subset \Sigma$, let $U_{\mathcal{O}} = \{u \in H_0^{1,2}(\Sigma) : u \geqslant 1$ almost everywhere on $\mathcal{O}\}$. Then,

$$\operatorname{enc}(\mathcal{O}) = \begin{cases} +\infty & \text{if } U_{\mathcal{O}} = \phi \\ \inf_{u \in U_{\mathcal{O}}} D(u, u) & \text{if } U_{\mathcal{O}} \neq 0 \end{cases}$$

We then have

PROPOSITION 1.30. – *For any open set* $\mathcal{O} \subset \Sigma$, $\operatorname{enc}(\mathcal{O}) = \operatorname{cap}(\mathcal{O})$.

Proof. – We first claim that if $K \subset \Sigma$,

$$K_\rho = \{x \in \Sigma : \operatorname{dist}(x, K) \leqslant \rho\},$$

and $u \in H_0^{1,2}(\Sigma)$ is non-negative a.e. in K_ρ, then $u \geqslant 0$ in K in the $H^{1,2}(\Sigma)$ sense. To see this, pick $\varphi \equiv 1$ in K, $\operatorname{supp} \varphi \subset K_\rho$, $\varphi \in C_0^\infty(\Sigma)$. Then, $\varphi u \in H_0^{1,2}(\Sigma)$, $\varphi u \geqslant 0$ in Σ. Thus, using truncation we can find a sequence g_j, $g_j \geqslant 0$ in Σ, $g_j \in \operatorname{Lip}_0(\Sigma)$, such that $g_j \longrightarrow \varphi u$ in $H_0^{1,2}(\Sigma)$. Pick $h_j \in C_0^\infty(\Sigma)$, $h_j \longrightarrow u$ in $H_0^{1,2}(\Sigma)$. Then, $(1 - \varphi) h_j + g_j \longrightarrow u$ in $H_0^{1,2}(\Sigma)$, and on K, $(1 - \varphi) h_j + g_j = g_j \geqslant 0$, and the claim follows. Now, assume $\operatorname{enc}(\mathcal{O}) < +\infty$, $\operatorname{cap}(\mathcal{O}) < +\infty$. Then, there exists a $u \in U_{\mathcal{O}}$. Let $K \subset\subset \mathcal{O}$. By the claim, $u \geqslant 1$ on K in the $H_0^{1,2}(\Sigma)$ sense. Thus, $\operatorname{cap}(\mathcal{O}) \leqslant D(u, u)$, and so $\operatorname{cap}(\mathcal{O}) \leqslant \operatorname{enc}(\mathcal{O})$. Pick now a sequence of compact sets $\{K_j\}$, $K_j \subset \mathring{K}_{j+1}$, $K_j \subset \mathcal{O}$, $K_j \nearrow \mathcal{O}$. Let u_j be the capacitary potential of K_j. Since $\operatorname{cap}(\mathcal{O}) < +\infty$, $D(u_j, u_j) \leqslant C$.

Thus, there exists a subsequence u_{j_k} and $u \in \mathrm{H}_0^{1,2}(\Sigma)$ so that $u_{j_k} \longrightarrow u$ weakly. Because of the Banach Saks theorem, it is easy to see that $u \geqslant 1$ a.e. on $\mathcal{O}$. But, then,

$$\begin{aligned} \mathrm{D}(u\,,u) = \lim_{k\to\infty} \mathrm{D}(u\,,u_{j_k}) &\leqslant \overline{\lim_k}\, \mathrm{D}(u_{j_k}\,,u_{j_k})^{1/2}\,.\,\mathrm{D}(u\,,u)^{1/2} \\ &\leqslant \overline{\lim_k}\, \mathrm{cap}(\mathrm{K}_{j_k})^{1/2}\,.\,\mathrm{D}(u\,,u)^{1/2} \\ &\leqslant \mathrm{cap}(\mathcal{O})^{1/2}\,\mathrm{D}(u\,,u)^{1/2}\,, \end{aligned}$$

and so, $\mathrm{D}(u\,,u) \leqslant \mathrm{cap}(\mathcal{O})$. Hence, $\mathrm{enc}(\mathcal{O}) \leqslant \mathrm{cap}(\mathcal{O})$. As it is easy to see that $\mathrm{enc}(\mathcal{O}) = +\infty$ iff $\mathrm{cap}(\mathcal{O}) = +\infty$, the proposition follows.

As mentioned before, proposition 1.28 follows from 1.30 by the results in [3].

2. Weak solutions and the Green function.

Recall from 1.12 and 1.13 that if $p \geqslant p_0$, G maps $\mathrm{H}^{-1,p}(\Sigma)$ into $\mathrm{C}_0(\Sigma)$, the class of continuous functions in $\overline{\Sigma}$ that vanish on $\partial\Sigma$. Denote by $\mathrm{M}(\Sigma)$ the class of finite measures supported in Σ. A function u in $\mathrm{L}^1(\Sigma\,,w)$ is called a *weak solution vanishing on* $\partial\Sigma$ *to* $\mathrm{L}u = \mu$ provided

$$\int_\Sigma u(x)\,\Psi(x)\,w(x)\,dx = \int_\Sigma \mathrm{G}(\Psi w)\,d\mu\,,$$

for every $\Psi \in \mathrm{L}^\infty(\Sigma\,,w)$. Notice that $\Psi w \in \mathrm{H}^{-1,p}(\Sigma)$ for all p, since $\Psi w = f_0 - \operatorname{div}\vec{f}$, where $\vec{f} = 0$ and $f_0/w = \Psi \in \mathrm{L}^p(\Sigma\,,w)$ for all p. Consequently, $\mathrm{G}(\Psi w)$ is continuous in $\overline{\Sigma}$ and the right hand integral makes sense.

Proposition 2.1. – *For every* $\mu \in \mathrm{M}(\Sigma)$, *there exists a unique weak solution* u *to* $\mathrm{L}u = \mu$. *Moreover, there exists* $\underline{u} \in \mathrm{H}_0^{1,p'}(\Sigma)$ *so that* $\underline{u} = (u\,,u_1\,,\ldots,u_n)$ *and* $\|\underline{u}\|_{\mathrm{H}_0^{1,p'}(\Sigma)} \leqslant \mathrm{C}\,\|\mu\|_{\mathrm{M}(\Sigma)}$ *for* $1 \leqslant p' \leqslant p_0'$.

Proof. – The existence of u follows from the fact that the adjoint of G, G*, is bounded from $\mathrm{M}(\Sigma)$ to $\mathrm{H}_0^{1,p'}(\Sigma)$. Put $\underline{u} = \mathrm{G}^*(\mu)$, then by definition (using the representation $\Psi w = f_0 - \operatorname{div}\vec{f}$ with $\vec{f} = 0$ above)

$$\int_\Sigma u\Psi w = \langle \underline{u}\,,\Psi w\rangle = \langle \mathrm{G}^*(\mu)\,,\Psi w\rangle = \langle \mu\,,\mathrm{G}(\Psi w)\rangle = \int_\Sigma \mathrm{G}(\Psi w)\,d\mu\,.$$

The function u is unique in $L^1(\Sigma, w)$ because it is determined by $\int_\Sigma u\Psi w$ for all $\Psi \in L^\infty(\Sigma, w)$.

PROPOSITION 2.2. – *If* $\mu \geqslant 0$, *then the weak solution* u *to* $Lu = \mu$ *is non-negative* a.e. *in* Σ.

Proof. – It is enough to show that $\int_\Sigma u\Psi w \geqslant 0$ for all non-negative $\Psi \in L^\infty(\Sigma, w)$. But $\Psi w \geqslant 0$ implies $G(\Psi w) \geqslant 0$ by the weak maximum principle 1.18. Hence,

$$\int_\Sigma u\Psi w = \int_\Sigma G(\Psi w)\, d\mu \geqslant 0 .$$

PROPOSITION 2.3. – *Assume that* $\mu \geqslant 0$, *and* $\mu \in H^{-1,2}(\Sigma)$. *Then, the weak solution* u *of* $Lu = \mu$ *belongs to* $H_0^{1,2}(\Sigma)$. *Moreover,* $Lu = \mu$ *in the* $H_0^{1,2}(\Sigma)$ *sense.*

Proof. – Since $\mu \in H^{-1,2}(\Sigma)$, by 1.4 and 1.29 there exists a $v \in H_0^{1,2}(\Sigma)$ such that $D(v,\varphi) = \langle \mu, \varphi\rangle$ for all $\varphi \in C(\overline{\Sigma}) \cap H_0^{1,2}(\Sigma)$. For $\Psi \in L^\infty(\Sigma, w)$, $G(\Psi w) \in C(\overline{\Sigma}) \cap H_0^{1,2}(\Sigma)$, and thus by definition of G, $\langle v, \Psi w\rangle = D(G(\Psi w), v) = D(v, G(\Psi w)) = \langle \mu, G(\Psi w)\rangle$. Hence, v is the weak solution to $Lv = \mu$, and the proposition follows.

Note that because of remark 1.24 and 2.3, if μ is a capacitary distribution and u is the corresponding capacitary potential, then u is the weak solution to $Lu = \mu$.

Fix $y \in \Sigma$. Denote by $g(x,y)$ the weak solution of $Lg = \delta_y$ as a function of x. (δ_y is the unit mass at y.) By Proposition 2.2 $g(x,y) \geqslant 0$ for a.e. x in Σ.

PROPOSITION 2.4. – $g(.,y) \in H^{1,2}(\Sigma \backslash B(y,r))$ *for any* $r > 0$. *Moreover,* $g(.,y)$ *can be modified on a set of measure zero so that it is Hölder continuous in* $\Sigma \backslash \{y\}$ *and vanishes on* $\partial\Sigma$.

Proof. – Define a measure $d\mu_j = \Psi_j w dx$, where

$$\Psi_j(x) = \begin{cases} 1/w(B(y, j^{-1})) & x \in B(y, j^{-1}) \\ 0 & \text{elsewhere}. \end{cases}$$

Then μ_j tends to δ_y weakly, and $\mu_j \in H^{-1,p}(\Sigma)$ for every p. Let u_j be the weak solution to $Lu_j = \mu_j$. Since G^* is bounded from

$M(\Sigma)$ to $L^{p'}(\Sigma, w)$, $p \geqslant p_0$, $\|u_j\|_{L^{p'}(\Sigma,w)} \leqslant C$ is independent of j and u_j tends to $g(.\,, y)$ weakly in $L^{p'}(\Sigma, w)$.

By 2.3, we see that $u_j \in H_0^{1,2}(\Sigma)$. Moreover, $Lu_j = 0$ in the $H^{1,2}(\Sigma \setminus B(y, j^{-1}))$ sense, Thus we can choose the Hölder continuous representative of u_j(1.8). Also by 2.2, u_j is non-negative, so Harnack's principle applies (1.9). Thus

$$u_j(x) \leqslant C\left(\frac{1}{w(B)} \int_B u_j(z)\, w(z)\, dz\right) \leqslant \frac{C}{w(B)} \|u_j\|_{L^{p'}(\Sigma, w)},$$

where $x \in \partial B(y, r)$, $B = B(x, r/2)$, $j^{-1} < r/4$. Thus by the maximum principle, $0 \leqslant u_j(x) \leqslant C_r$ for all $x \in \Sigma \setminus B(y, r)$, $j^{-1} < r/4$. Next, by 1.14 and 1.14′ we also have

$$\int_{\Sigma \setminus B(y, r)} |\nabla u_j(z)|^2\, w(z)\, dz \leqslant C_r,$$

and thus the sequence u_j is uniformly bounded in $H^{1,2}(\Sigma \setminus B(y, r))$ norm. Thus a subsequence converges weakly and it follows that $g(\cdot, y) \in H^{1,2}(\Sigma \setminus B(y, r))$. Finally, it also follows that $Lg(.\,, y) = 0$ in the $H^{1,2}(\Sigma \setminus B(y, r))$ sense, and $g(\cdot, y)$ vanishes on $\partial\Sigma$ in the $H^{1,2}(\Sigma \setminus B(y, r))$ sense. Therefore $g(\cdot, y)$ is Hölder continuous in $\overline{\Sigma} \setminus \{y\}$ and vanishes on $\partial\Sigma$. (See 1.12 and 1.13).

LEMMA 2.5. – *Let* $\Psi \in L^\infty(\Sigma, w)$. *Then*

$$G(\Psi w)(y) = \int_\Sigma g(x, y)\, \Psi(x)\, w(x)\, dx.$$

Proof. –

$$\begin{aligned}\int_\Sigma g(x, y)\, \Psi(x)\, w(x)\, dx &= \langle g(\cdot, y), \Psi w\rangle \\ &= \langle \delta_y, G(\Psi w)\rangle = G(\Psi w)(y).\end{aligned}$$

From now on we will only use the representative of $g(x, y)$ that is continuous in x for $x \in \overline{\Sigma} \setminus \{y\}$.

PROPOSITION 2.6. – $g(x, y)$ *is jointly continuous in* $\Sigma \times \Sigma \setminus \Delta$, *where* $\Delta = \{(y, y) : y \in \Sigma\}$.

Proof. – Fix x and let

$$\Psi_j^{(x)}(z) = \begin{cases} 1/w(B(x, j^{-1})) & z \in B(x, j^{-1}) \\ 0 & \text{elsewhere}.\end{cases}$$

From 2.5, the solution $\Phi_j^{(x)} \in H_0^{1,2}(\Sigma) \cap C(\overline{\Sigma})$ of $L\Phi_j^{(x)} = \Psi_j^{(x)} w$ is given by $\Phi_j^{(x)}(y) = \int_\Sigma g(z, y)\, \Psi_j^{(x)}(z)\, w(z)\, dz$. If $x \neq y$, then $\Phi_j^{(x)}(y)$ tends to $g(x, y)$. Even better, if J and K are disjoint compact sets in Σ, then $\Phi_j^{(x)}(y) \longrightarrow g(x, y)$ uniformly for $x \in J$, $y \in K$. In fact, 1.8 implies for $z \in J^* = \{z : \operatorname{dist}(z, J) < j^{-1}\}$ and j large,

$$|g(z, y) - g(x, y)| \leqslant C|z - x|^\alpha \left(\int_{J^{**}} g(z', y)^2 w(z')\, dz'\right)^{1/2}.$$

By the proof of Proposition 2.4, the integral is bounded independent of $y \in K$. Hence, for sufficiently large j,

$$|\Phi_j^{(x)}(y) - g(x, y)| = \left|\int_\Sigma (g(z, y) - g(x, y))\, \Psi_j^{(x)}(z)\, w(z)\, dz\right| \leqslant Cj^{-\alpha}.$$

On the other hand, $\Phi_j^{(x)}(y)$ is continuous in $J \times K$ for large j. This is because (1.12) implies

$$\sup_{y \in K} |\Phi_j^{(x)}(y) - \Phi_j^{(z)}(y)| \leqslant C\, \|\Psi_j^{(x)} - \Psi_j^{(z)}\|_{L^p(\Sigma, w)}$$

for $p \geqslant p_0$. Clearly the right hand side tends to zero as $z \longrightarrow x$. Finally, for $x \in J$,

$$|\Phi_j^{(x)}(y) - \Phi_j^{(x)}(y')| \leqslant C\, |y - y'|^\alpha\, \|\Psi_j^{(x)}\|_{L^p(\Sigma, w)}.$$

LEMMA 2.7. – *For every* $\mu \in M(\Sigma)$, $u(x) = \int g(x, y)\, d\mu(y)$ *exists for a.e.* x *and* u *is the weak solution to* $Lu = \mu$.

Proof. – Assume that $\mu \geqslant 0$. Since $g(x, y)$ is continuous in $\Sigma \times \Sigma \setminus \Delta$ and $dx \times d\mu(\Delta) = 0$, $g(x, y)$ is $dx \times d\mu(y)$ measurable on $\Sigma \times \Sigma$. Let $\Psi \geqslant 0$, $\Psi \in L^\infty(\Sigma, w)$. Then by Fubini's theorem, and 2.5,

$$\langle G(\Psi w), \mu\rangle = \iint g(x, y)\, \Psi(x)\, w(x)\, dx\, d\mu(y) = \int \Psi(x)\, w(x)\, u(x)\, dx$$

because $g(x, y) \geqslant 0$ for $(x, y) \notin \Delta$. These integrals are always finite because $G(\Psi w)$ is continuous. Thus $u(x)$ exists a.e. x, and we can drop the restrictions $\Psi \geqslant 0$ and $\mu \geqslant 0$.

PROPOSITION 2.8. – $g(x, y) = g(y, x)$.

Proof. – Fix $x_0 \neq y_0$. Let $\Psi \in L^\infty(\Sigma, w)$ be supported near

y_0 and disjoint from x_0. By 2.7 and 2.3,

$$\Phi_1(x) = \int g(x, y)\, \Psi(y)\, w(y)\, dy$$

represents the $H_0^{1,2}(\Sigma)$ solution to $L\Phi = \Psi w$. By 2.5

$$\Phi_2(x) = \int g(y, x)\, \Psi(y)\, w(y)\, dy$$

also represents this solution. But since $g(x, y)$ is continuous near (x_0, y_0), both Φ_1 and Φ_2 are continuous near x_0. Therefore, $\Phi_1(x_0) = \Phi_2(x_0)$. Now using arbitrary Ψ and continuity of g we see that $g(x_0, y_0) = g(y_0, x_0)$.

3. The size of the Green function.

We will say that $A_1(x) \simeq A_2(x)$ if there exist positive constants c_1 and c_2 such that $c_1 < A_1(x)/A_2(x) < c_2$. The constants depend only on the (A_2) or (QC) constants of w and not on x, whose range will be specified.

LEMMA 3.1. – *If* $B(x, 2r) \subset \Sigma$ *and* $y \in \partial B(x, r)$, *then* $g(x, y) \cong 1/\mathrm{cap}(B(x, r))$.

Proof. – Let μ be the capacitary distribution of $B(x, r)$ and $\hat{u}(z) = \int g(z, y)\, d\mu(y)$ a representative of its capacitary potential (see 2.3 and 2.7). Since μ is supported on $\partial B(x, r)$ and $\hat{u}(z) = 1$ on $B(x, r)$ in the $H^{1,2}(\Sigma)$ sense, $\hat{u}$ is continuous in the interior of $B(x, r)$ and $1 = \hat{u}(x) = \int_{\partial B(x,r)} g(x, y)\, d\mu(y)$ (see 1.20 and 1.22). Therefore,

$$\min_{y \in \partial B(x,r)} g(x, y)\ \mathrm{cap}(B(x, r)) \leqslant 1 \leqslant \max_{y \in \partial B(x,r)} g(x, y)\ \mathrm{cap}(B(x, r).$$

By 2.4 and 2.8, $g(x, \cdot) \in H^{1,2}\ \Sigma \setminus \left(B\left(x, \frac{r}{2}\right)\right)$ and $g(x, \cdot) \geqslant 0$ there. Thus Harnack's principle applies, giving

$$\max_{y \in \partial B(x,r)} g(x, y) \simeq \min_{y \in \partial B(x,r)} g(x, y),$$

and 3.1 is proved.

LEMMA 3.2. – *If* $x \in \Sigma$ *and* $\frac{3}{2}\, r \leqslant \mathrm{dist}\ (x, \partial\Sigma) \leqslant 8r$, *then* $\mathrm{cap}(B(x, r)) \simeq w(B(x, r))/r^2$.

Proof. – Choose $\Psi \in \mathrm{Lip}_0(\Sigma)$ with $\Psi = 1$ on

$$B(x, r), \ \mathrm{supp}\, \Psi \subset B(x, 2r), \ 0 \leqslant \Psi \leqslant 1, \ \text{and} \ |\nabla \Psi| \leqslant Cr^{-1}.$$

Then $\mathrm{cap}(B(x, r)) \leqslant D(\Psi, \Psi) \leqslant C\int |\nabla\Psi|^2 w \cong w(B(x, r))/r^2$, by (1.2). Conversely, let u be the capacitary potential of $B(x, r)$. Let $\bar{x}$ be the point of $\partial\Sigma$ closest to x. By Hölder continuity at the boundary (1.13), for $\rho < r$

$$\max_{z \in B(\bar{x}, \rho) \cap \Sigma} u(z) \leqslant C(\rho/r)^\alpha \left(\frac{1}{w(B(\bar{x}, r))} \int_{B(\bar{x}, r) \cap \Sigma} u(z)^2 w(z)\, dz\right)^{1/2} \leqslant C(\rho/r)^\alpha .$$

The last expression is less than $\frac{1}{2}$ if $\rho = \delta r$ for some $\delta > 0$ (independent of r). Now applying Harnack's principle on a chain of balls connecting the point of $\partial B(\bar{x}, \delta r) \cap \Sigma$ on the ray between $\bar{x}$ and x to $\partial B(x, (1 + \delta) r)$ to the (non-negative) function $1 - u(z)$, we find that $1 - u(z) \geqslant c > 0$ for all

$$z \in B(x, (1 + 2\delta) r) \backslash B(x, (1 + \delta) r) = A .$$

By the doubling condition (1.2), $w(A) \simeq w(B(x, 2r))$. Also, $u \leqslant 1$ almost everywhere. Hence,

$$\bar{u} = \frac{1}{w(B(x, (1 + 2\delta) r))} \int_{B(x,(1+2\delta)r)} u(z) w(z)\, dz \leqslant 1 - \delta' ,$$

where $\delta' > 0$, and δ' depends only on c, δ and the constant in (1.2). Define $\varphi(z) = \Psi(z)(u(z) - \bar{u})$. Then $\varphi \geqslant \delta'$ in $B(x, r)$ and by 1.6,

$$(\delta')^2 w(B(x, r)) \leqslant c\int_{B(x,r)} |\varphi|^2 w \leqslant c\int_{B(x,2r)} \Psi^2 |u - \bar{u}|^2 w \leqslant cr^2 \int_\Sigma |\nabla u|^2 w \simeq r^2 D(u, u) = r^2 \mathrm{cap}(B(x, r)) .$$

THEOREM 3.3. – *Let x and y belong to* $\Sigma' = \left\{ z : |z| < \frac{1}{4} R \right\}$. *Denote* $r = |x - y|$. *Then*

$$g(x, y) \simeq \int_r^R \frac{s^2}{w(B(x, s))} \frac{ds}{s} .$$

Proof. – Denote by $g_j(x, y)$ the Green function for $B(x, 2^j r)$, $j = 0, 1, \ldots, N$, with $2^{N+1} r \leqslant R < 2^{N+2} r$. Lemmas 3.1 and 3.2 show that $g_j(x, y) \simeq (2^j r)^2/w(B(x, 2^j r))$ for $y \in \partial B(x, 2^{j-1} r)$.

Denote $u_j(y) = g_j(x, y) - g_{j-1}(x, y)$ for $y \in B(x, 2^{j-1} r)$. A limiting procedure like the one in the proof of 2.4 shows that $u_j(y)$ solves $Lu_j = 0$ in the $H^{1,2}(\mathring{B}(x, 2^{j-1} r))$ sense. Also, by (2.4) u_j is continuous in the closed ball $B(x, 2^{j-1} r)$ with $u_j(y) = g_j(x, y)$ on $\partial B(x, 2^{j-1} r)$ continuously and in the $H^{1,2}(\mathring{B}(x, 2^{j-1} r))$ sense. Thus by the maximum principle $u_j(y) \simeq (2^j r)^2 / w(B(x, 2^{j-1} r))$ for all $y \in B(x, 2^{j-1} r)$. Let $u(y) = g(x, y) - g_N(x, y)$. A similar argument shows that $u(y) \cong (2^N r)^2 / w(B(x, 2^N r))$ for $y \in B(x, 2^N r)$. In all, by 1.2

$$g(x, y) = u(y) + \sum_{j=1}^{N} u_j(y) \simeq \sum_{j=1}^{N+1} (2^j r)^2 / w(B(x, 2^j r))$$
$$\simeq \int_r^R \frac{s^2}{w(B(x, s))} \frac{ds}{s} .$$

We will now define $g(y, y)$ as follows. If $\mathrm{cap}(\{y\}) = 0$, then let $g(y, y) = \infty$. If $\mathrm{cap}(\{y\}) > 0$, then let $g(y, y) = 1/\mathrm{cap}(\{y\})$. This definition is justified by

PROPOSITION 3.4. – *If* $\mathrm{cap}(\{y\}) = 0$, *then* $\lim_{x \to y} g(x, y) = \infty$. *If* $\mathrm{cap}(\{y\}) > 0$, *then* $\lim_{x \to y} g(x, y) = 1/\mathrm{cap}(\{y\})$.

Proof. – We first claim that $\mathrm{cap}(\{y\}) = \lim_{r \to 0} \mathrm{cap}(B(y, r))$. Clearly $\mathrm{cap}(B(y, r))$ decreases as r decreases and

$$\lim_{r \to 0} \mathrm{cap}(B(y, r)) \geq \mathrm{cap}(\{y\}) .$$

Let u_r denote the capacitary potential of $B(y, r)$. Then for $r \leq r_0$, $D(u_r, u_r) = \mathrm{cap}(B(y, r)) \leq \mathrm{cap}(B(y, r_0))$, so that u_r is uniformly bounded in $H_0^{1,2}(\Sigma)$ norm as $r \to 0$. Choose a sequence $r_j \downarrow 0$ so that u_{r_j} converges weakly to u in $H_0^{1,2}(\Sigma)$. Then u is the capacitary potential of $\{y\}$. By the Banach-Saks theorem we can pass to a sub-sequence (still denoted r_j) such that the means $\Psi_j = j^{-1}(u_{r_1} + \cdots + u_{r_j})$ converge to u in $H_0^{1,2}(\Sigma)$ norm. Thus $\mathrm{cap}(\{y\}) = D(u, u) = \lim_{j \to \infty} D(\Psi_j, \Psi_j)$. But $\Psi_j = 1$ in the $H^{1,2}(\Sigma)$ sense on $B(y, r_j)$, so $D(\Psi_j, \Psi_j) \geq \mathrm{cap}(B(y, r_j))$ and the claim follows. Lemma 3.1 and the claim imply that if $\mathrm{cap}(\{y\}) = 0$, then $\lim_{x \to y} g(x, y) = \infty$.

Now suppose that $\mathrm{cap}(\{y\}) > 0$. Recall from the proof of 3.1 that

$$\min_{x\in\partial B(y,r)} g(x,y)\,\mathrm{cap}(B(y,r)) \leqslant 1 \leqslant \max_{x\in\partial B(y,r)} g(x,y)\,\mathrm{cap}(B(y,r)) .$$

Thus it suffices to show that

$$\lim_{r\to 0} \sup_{\substack{|x-y|=r\\|x'-y|=r}} |g(x,y) - g(x',y)| = 0 .$$

Since $\mathrm{cap}(\{y\}) > 0$, the claim and 3.1 imply that $g(x,y) \leqslant c$ for all $x \neq y$. Hence by Theorem 3.3, $\int_0^R \frac{s^2}{w(B(y,s))}\frac{ds}{s} < \infty$. Recalling 1.2, we have in particular, $\lim_{s\to 0}\frac{s^2}{w(B(y,s))} = 0$. The capacitary potentiel u for $\{y\}$ mentioned above satisfies $Lu = \mu$ for a measure μ supported on $\{y\}$. Also, $\mu(\{y\}) = \mathrm{cap}(\{y\})$. Thus

$$g(x,y) = u(x)/\mathrm{cap}(\{y\}) .$$

There is a dimensional constant N such that every two points x and x' of $\partial B(y,r)$ can be connected by points

$$x = x_1, x_2, \dots, x_{N-1},$$

$x_N = x'$ such that $x_j \in \partial B(y,r)$ and $|x_{j+1} - x_j| < \frac{1}{100} r$. Then 1.6 and 1.7 imply

$$|u(x_{j+1}) - u(x_j)| \leqslant Cr\left(\frac{1}{w(B(x_j, r/2))}\int_{B(x_j,r/2)} |\nabla u|^2 w\right)^{1/2}$$

$$\leqslant C\left(\frac{r^2}{w(B(x,r))}\right)^{1/2} \|u\|_{H_0^{1,2}(\Sigma)},$$

which tends to zero as $r \longrightarrow 0$.

From 3.4, and 2.6 it follows that

COROLLARY 3.5. – $g(x,y)$ is Borel measurable.

Another consequence of 3.4 is

LEMMA 3.6. – *If μ is a positive measure, then*

$$\hat{u}(x) = \int g(x,y)\, d\mu(y)$$

is lower semicontinuous, that is, $\liminf_{y\to x} \hat{u}(y) \geqslant \hat{u}(x)$.

Henceforth we will always use the lower semicontinuous representative given above of the weak solution to $Lu = \mu$.

Proof. – Fix a point $x_0 \in \Sigma$, and write $\mu = \mu_1 + \mu_2$, where $\mu_1(\{x_0\}) = 0$, and $\mu_2 = \mu(\{x_0\})\, \delta_{x_0}$. Then,

$$\hat{u}(x) = \int g(x, y)\, d\mu_1(y) + \mu(\{x_0\})\, g(x, x_0).$$

(We use the convention $0 . \infty = 0$). Hence,

$$\liminf_{x \to x_0} \hat{u}(x) \geqslant \liminf_{x \to x_0} \int g(x, y)\, d\mu_1(y) + \mu\{x_0\}\, g(x_0, x_0),$$

by 3.4. Pick now a sequence of functions $\varphi_j \in \mathrm{Lip}(\mathbf{R})$, $\varphi_j \leqslant \varphi_{j+1}$, $\varphi_j \equiv 0$ near 0, $\varphi_j(t) \equiv 1$ for $t \geqslant \frac{1}{j}$, so that $\varphi_j(t) \longrightarrow \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$. Let $g_j(x, y) = g(x, y)\, \varphi_j(|x - y|)$. Then $g_j(x, y) \leqslant g_{j+1}(x, y)$, $g_j(x, y) \nearrow g(x, y)$ except at $x = y$. Since $\mu_1(\{x_0\}) = 0$, $\int g(x_0, y)\, d\mu_1(y) = \lim_{j \to \infty} \int g_j(x_0, y)\, d\mu_1(y)$. But, $\int g_j(x, y)\, d\mu(y)$ is a continuous function by 2.8, and so,

$$\int g_j(x_0, y)\, d\mu_1(y) = \liminf_{x \to x_0} \int g_j(x, y)\, d\mu_1(y) \leqslant \liminf_{x \to x_0} \int g(x, y)\, d\mu_1(y).$$

Thus, $\int g(x_0, y)\, d\mu_1(y) \leqslant \liminf_{x \to x_0} \int g(x, y)\, d\mu_1(y)$, and the proposition follows.

4. Capacitary potentials and distributions.

In this section we will prove basic results on capacitary potentials needed in the proof of the Wiener test. These results are easily deduced from the properties of the Green function of Section 3.

LEMMA 4.1. – *Suppose that* $\mathrm{cap}(\{y\}) = 0$ *and* μ *is a capacitary distribution. Then* $\lim_{r \to 0} \int_{|x - y| < r} g(y, x)\, d\mu(x) = 0$.

Proof. – By 1.24 and 1.25, $\mu(\{y\}) = 0$. Also,

$$\int g(y, x)\, d\mu(x) \leqslant 1$$

because the capacitary potential of a set K, $u(y) = \int g(y, x)\, d\mu(x)$, is continuous in $\Sigma \backslash K$ and $\mathring{K}$, $\leqslant 1$ a.e., (and hence everywhere on $\Sigma \backslash K$ and $\mathring{K}$) and is lower semicontinuous. The result now follows from the dominated convergence theorem.

LEMMA 4.2. – *Let* μ_1 *and* μ_2 *be positive measures, and* $u_j(x) = \int g(x,y)\,d\mu_j(y)$, $j = 1, 2$. *Then,*

$$\int u_1(y)\,d\mu_2(y) = \int u_2(x)\,d\mu_1(x) = \iint g(x,y)\,d\mu_1(x)\,d\mu_2(y).$$

The lemma follows from 2.8, 3.5, 3.6 and Fubini's Theorem.

LEMMA 4.3. – *Let* $K \subset \Sigma$ *be a compact set. Let*

$$u(y) = \int g(x,y)\,d\mu(x)$$

be the lower semicontinuous representative of its capacitary potential. Then u *is quasi-continuous in* Σ.

Proof. – The proof follows very closely the one of Lemma 6, section III of [1]. Some modifications are needed to take care of the points y such that $\operatorname{cap}\{y\} > 0$.

Let $E_1 = \{x \in \partial K : \operatorname{cap}\{x\} = 0\}$, $E_2 = \{x \in \partial K : \operatorname{cap}\{x\} > 0\}$. By 3.4 and 3.5 E_1 and E_2 are Borel sets. We first claim that given $\mathcal{E} > 0$, there exists a closed set $F_2 \subset E_2$, with $\mu(E_2/F_2) < \mathcal{E}$ such that if $\mu_2(E) = \mu(E \cap F_2)$, and $u_2(x) = \int g(x,y)\,d\mu_2(y)$, then u_2 is continuous in Σ. (Note that as $\mu_2(E) \leqslant \mu(E)$, $\mu_2 \in H^{-1,2}(\Sigma)$, and therefore by 2.7 and 2.3, $u_2 \in H_0^{1,2}(\Sigma)$). Given $y \in E_2$, define $\omega(m,y) = \sup\limits_{\substack{x_0 \in \bar{\Sigma} \\ x \in \bar{\Sigma}, |x - x_0| \leqslant 1/m}} |g(x,y) - g(x_0,y)|$. By the definition of E_2, and 2.6 and 3.4, $g(-,y)$ is continuous in $\bar{\Sigma}$. Thus, $\omega(m,y)$ is a Borel function of y. Let $E_{m,n} = \left\{y \in E_2 : \omega(m,y) < \frac{1}{n}\right\}$. E_{mn} are Borel sets, and for every fixed n, $E_{m,n} \nearrow E_2$. Thus, given $\mathcal{E} > 0$, and n, we can find a closed set F_n, $F_n \subset E_{m_n,n}$, so that $\mu(E_2 \backslash F_n) < \frac{\mathcal{E}}{2^n}$. Let now $F_2 = \bigcap\limits_n F_n$, so that $\mu(E_2 \backslash F_2) \leqslant \mathcal{E}$. Also, given $\eta > 0$, we can choose $\delta = \delta(\eta)$ so that, for any $y \in F_2$ and, for all $x_0 \in \bar{\Sigma}$, $x \in \bar{\Sigma}$, $|x - x_0| \leqslant \delta$ implies that

$$|g(x,y) - g(x_0,y)| \leqslant \eta.$$

Thus, $u_2(x)$ is continuous in Σ. For further reference, define $\nu_2(E) = \mu(E \cap (E_2 \backslash F_2))$, so that $\nu_2(\Sigma) \leqslant \mathcal{E}$, and

$$v_2(x) = \int g(x,y)\,d\nu_2(y).$$

By lemma 4.1, we know that for each $x \in E_1$, we have $\lim_{r \to 0} \int_{|x-y|<r} g(x, y)\, d\mu(y) = 0$. By standard measure theoretic arguments, given $\mathcal{E} > 0$, we can find a closed set $F_1 \subset E_1$, with $\mu(E_1 \setminus F_1) \leqslant \mathcal{E}$, so that given $\delta > 0$, there exists $\eta = \eta(\delta)$ so that for all $x \in F_1$, $\int_{|x-y|<\eta} g(x, y)\, d\mu(y) < \delta$. Let

$$\mu_1(E) = \mu(E \cap F_1), \text{ and } u_1(x) = \int g(x, y)\, d\mu_1(y).$$

We claim that u_1 is continuous in Σ. It is enough to check it for $x \in F_1$. Let $\{x_n\}$ be any sequence, $x_n \longrightarrow x_0 \in F_1$. Then,

$$\overline{\lim_n}\, u_1(x_n) = \overline{\lim_n} \int g(x_n, y)\, d\mu_1(y) \leqslant \int_{|x_0 - y| \geqslant \eta} g(x_0, y)\, d\mu_1(y) + \overline{\lim_n} \int_{|x_n - y|<\eta} g(x_n, y)\, d\mu_1(y).$$

To analyze the last term, note that there exists a number N, depending only on the dimension of the space, such that for any x there exist N overlapping closed cases Q_v, with vertices at x, such that if ξ_v is the point of $Q_v \cap F_1$ which is closest to x, any other point $y \in F_1$ is closer to some ξ_v then to x. Because of 3.3, if the ξ_v are chosen for $x = x_n$, $g(x_n, y) \leqslant C\left(\sum_{v=1}^{N} g(\xi_v, y)\right)$ for all $y \in F_1$. Thus,

$$\overline{\lim_n}\, u_1(x_n) \leqslant \int_{|y-x_0| \geqslant \eta} g(x_0, y)\, d\mu_1(y) + C\, \overline{\lim_n} \sum_{1}^{N} \int_{|\xi_v - y|<\eta} g(\xi_v, y)\, d\mu_1(y) \leqslant \int_{|y-x_0| \geqslant \eta} g(x_0, y)\, d\mu_1(y) + CN\delta.$$

Hence, $\overline{\lim_n}\, u_1(x_n) \leqslant u_1(x_0)$, and so from 3.6 we see that u_1 is continuous at x_0.

Let $\nu_1(E) = \mu(E \cap (E_1 \setminus F_1))$ so that

$$\nu_1(\Sigma) \leqslant \mathcal{E}, \text{ and } v_1(x) = \int g(x, y)\, d\nu_1(y).$$

Then, $u(x) = u_1(x) + v_1(x) + u_2(x) + v_2(x)$, and the u_i are continuous. Let $S_n^1 = \{v_1 > 1/n\}$, $S_n^2 = \{v_2 > 1/n\}$. By 3.6 these sets are open. Let $K \subset S_n^1$, K compact, and u_K its lower-semi-continuous capacitary potential. $\frac{1}{n}\mu_K(K) \leqslant \int v_1\, d\mu_K = \int u_K\, d\nu_1$ by 4.2. By the remarks in the proof of 4.1, $u_K \leqslant 1$, and so $\operatorname{cap}(S_n^1) \leqslant n\mathcal{E}$. Similarly, $\operatorname{cap}(S_n^2) \leqslant n\mathcal{E}$. Let $\eta > 0$ be given, and choose $n_i \nearrow \infty$, $\mathcal{E}_i \searrow 0$ so that $\sum_i n_i \mathcal{E}_i \leqslant \eta/2$. Let $\mathcal{O} = \bigcup_i S^1_{n_i} \cup \bigcup_i S^2_{n_i}$. Then, $\mathcal{O}$ is open, and $\operatorname{cap}(\mathcal{O}) \leqslant \eta$. Also, it is easy to see that u is continuous in $\Sigma \setminus \mathcal{O}$, and thus the lemma follows.

We now turn to an alternative definition of capacity.

Let K be a compact subset of Σ. Then define

$$\operatorname{cap}_1(K) = \sup\left\{\nu(K) : \int g(x, y)\, d\nu(y) \leqslant 1 \quad \text{for all } x \in K, \; \nu \text{ is a positive measure}\right\}.$$

Let μ be the capacitary distribution of K and u the (lower semicontinuous representative of the) capacitary potential. By lower semicontinuity $u(x) \leqslant 1$ for all x. Moreover, $\mu(K) = \operatorname{cap}(K)$. Therefore,

$$\operatorname{cap}_1(K) \geqslant \operatorname{cap}(K). \tag{4.4}$$

We will say that an equality or inequality holds *p.p.* if it holds except on a set of cap_1 size zero. The capacity cap_1 is treated for instance in Carleson's book [1]. Although the hypotheses are slightly more restricted there, the same proofs hold with some modifications to take care of y with $g(y, y) < +\infty$, as in 3.6 and 4.3. In particular, we have ([1] Theorems 4 and 7, Chapter III.)

THEOREM 4.5. – *All analytic sets are capacitable for* cap_1.

THEOREM 4.6. – *For any compact set* $K \subset \Sigma$, *thee exists a positive measure* ν *supported on* K *such that*

$v(x) = \int g(x, y)\, d\nu(y) \leqslant 1$ *everywhere* $v(x) = 1$ p.p. *on* K *and* $\nu(K) = \operatorname{cap}_1(K)$.

Our goal is to prove

THEOREM 4.7. – *With the notations above,* $\operatorname{cap}_1(K) = \operatorname{cap}(K)$, $v = u$, $\nu = \mu$, *and* p.p. *and* q.e. *are equivalent.*

THEOREM 4.8. – ν belongs to $H^{-1,2}(\Sigma)$.

By 4.2, $\iint g(x, y)\, d\nu(x)\, d\nu(y) = \int v(x)\, d\nu(x) \leqslant \nu(K) < \infty$. Recall that G is an isomorphism $G : H^{-1,2}(\Sigma) \rightleftarrows H_0^{1,2}(\Sigma)$ and the norm on $H_0^{1,2}(\Sigma)$ can be taken to be $\|u\|_{H_0^{1,2}(\Sigma)} = D(u, u)^{1/2}$. Let $\Psi \in L^\infty(\Sigma, w)$, then

$$\| \Psi w \|^2_{H^{-1,2}(\Sigma)} \cong D(G(\Psi w), G(\Psi w)) = \langle G(\Psi w), \Psi w \rangle$$
$$= \iint g(x, y)\, \Psi(x)\, w(x)\, dx \Psi(y)\, w(y)\, dy .$$

Let $Q_{j,k}$ be a grid of non-overlapping cubes of side 2^{-k} covering Σ. Let $c_{j,k} = \nu(Q_{j,k})/w(Q_{j,k})$ and $d\nu_k(x) = \Sigma_j\, c_{j,k}\, w(x)\, \chi_{Q_{j,k}}(x)\, dx$. The measures ν_k tend to ν weakly (and hence in the sense of distributions). It therefore suffices to show that $\|\nu_k\|_{H^{-1,2}(\Sigma)}$ is uniformly bounded as $k \longrightarrow \infty$. Since $\operatorname{supp} \nu = K \subset\subset \Sigma$, we can assume k is so small that all the cubes Q_{jk} for which $c_{jk} \neq 0$, have the property that their doubles are still contained in Σ.

$$\iint g(x, y)\, d\nu_k(x)\, d\nu_k(y) = \sum_{i,j} \iint g(x, y)\, w(x)\, dx w(y)\, dy c_{j,k}\, c_{i,k} .$$

By Harnack's inequality, $g(x, y)$ is essentially constant for (x, y) in $Q_{i,k} \times Q_{j,k}$ provided $Q_{i,k}$ and $Q_{j,k}$ are neither adjacent nor equal. Thus it is easy to dominate this part of the sum by the corresponding integral for ν. To handle the case of pairs of nearby cubes, consider a ball $B = B(x_0, 2^{-k+2})$. Let $\lambda = g(x_0, y)$ for some $y \in \partial B$. By Harnack's principle, if x, $y \in B$ and $|x - y| > \frac{1}{100} 2^{-k}$, then $g(x, y) \simeq \lambda$. Furthermore, by 3.3, if x and y are any points of B, then $Cg(x, y) \geqslant \lambda$. Therefore,

$$C \int_B \int_B g(x, y)\, d\nu(x)\, d\nu(y) \geqslant \lambda\, \nu(B)^2 .$$

Suppose that we can show for any $x \in B$

$$\int_B g(x, y)\, w(y)\, dy \leqslant C \int_{\substack{2^{-k+1} < |x-y| < 2^{-k+2} \\ y \in B}} g(x, y)\, w(y)\, dy . \tag{4.9}$$

Then the proof of 4.8 is concluded as follows. If Q_1 and Q_2 are cubes of side 2^{-k} in B, then (recalling 1.2),

$$\int_{Q_1} \int_{Q_2} g(x, y)\, w(x)\, dx\, w(y)\, dy\, \frac{\nu(Q_1)}{w(Q_1)} \frac{\nu(Q_2)}{w(Q_2)}$$
$$\leqslant C \int_B \int_B g(x, y)\, w(x)\, dx\, w(y)\, dy\, \frac{\nu(Q_1)}{w(Q_1)} \frac{\nu(Q_2)}{w(Q_2)}$$
$$\leqslant C \lambda\, w(B)^2\, \frac{\nu(Q_1)}{w(Q_1)} \frac{\nu(Q_2)}{w(Q_2)}$$
$$\leqslant C \lambda\, \nu(B)^2$$

$$\leqslant C \int_B \int_B g(x, y)\, d\nu(x)\, d\nu(y)\,.$$

This shows that $\iint g(x, y)\, d\nu_k(x)\, d\nu_k(y)$ is uniformly bounded.

To prove 4.9, note first that we can assume (for simplicity) that $x = x_0$. Denote $w_i = w(B(x_0, 2^{-i}))$, $L_i = \int_{2^{-i}}^{R} \frac{s^2}{w(B(x_0, s))} \frac{ds}{s}$. Then the left hand side of 4.9 is equivalent to

$$\sum_{i=k}^{\infty} (w_i - w_{i+1})\, L_i = w_k L_k + \sum_{i=k+1}^{\infty} w_i (L_i - L_{i-1})\,.$$

But $w_i(L_i - L_{i-1}) = w(B(x_0, 2^{-i})) \int_{2^{-i}}^{2^{-i+1}} \frac{s^2}{w(B(x_0, s))} \frac{ds}{s} \simeq 2^{-2i}$. In particular, $w_i L_i \geqslant c\, 2^{-2i}$. Hence

$$w_k L_k + \sum_{i=k+1}^{\infty} w_i (L_i - L_{i-1}) \simeq w_k L_k + 2^{-2k} \simeq w_k L_k\,,$$

and $w_k L_k$ is comparable to the right hand side of 4.9.

Proof of 4.7. – By 4.4 and 4.6, $v = 1$ q.e. on K. Hence, by 1.24, 1.25 and 4.2, $\mu(K) = \int v\, d\mu = \int u\, d\nu$. By the proof of 4.1, $u \leqslant 1$. Recall that $u = 1$ on K in the $H_0^{1,2}(\Sigma)$ sense. Thus, there exists a sequence $\varphi_j \in \mathrm{Lip}_0(\Sigma)$ such that $\varphi_j \equiv 1$ on K, and $\varphi_j \longrightarrow u$ in $H_0^{1,2}(\Sigma)$ norm. By 4.8, 4.3 and 1.29,

$$\int u\, d\nu = \lim_{j \to \infty} \int \varphi_j\, d\nu = \nu(K)\,.$$

Thus, $\mathrm{cap}_1(K) = \mathrm{cap}(K)$ and q.e. and p.p. are equivalent. By the proof of 4.3, v is quasi-continuous in Σ, and is also 1 q.e. on K. By 4.8, 1.29 and 2.3, $D(v, v) = \int v\, d\nu = \nu(K) = \mathrm{cap}(K)$, so that $D(v, v) = D(u, u)$. By a similar argument,

$$D(v, u) = \int v\, d\mu = \mu(K) = \mathrm{cap}(K)\,.$$

Hence, $u = v$ as elements of $H_0^{1,2}(\Sigma)$ and by 2.3, $\mu = \nu$. But then, $u = v$ pointwise.

THEOREM 4.10. – *The extremal problems*

$$A^{-1} = \inf \left\{ \iint g(x, y)\, d\nu(x)\, d\nu(y) : \nu(K) = 1, \nu \text{ a positive measure} \right\}$$

$$\mathrm{B} = \inf \{\nu(\mathrm{K}) : \int g(x, y)\, d\nu(y) \geqslant 1 \text{ p.p. on } \mathrm{K}\}$$

$$\mathrm{C} = \sup \{\nu(\mathrm{K}) : \int g(x, y)\, d\nu(y) \leqslant 1 \text{ p.p. on } \mathrm{K}\}$$

are equivalent so that $\mathrm{A} = \mathrm{B} = \mathrm{C} = \mathrm{cap}(\mathrm{K})$.

This is an easy consequence of 4.5, 4.6, and 4.7. (See [1], Theorem 5, Chapter III.) The pair of characterizations B and C and the formula for the Green function 3.3 make calculation of capacity up to a bounded factor as easy as in the classical case.

5. Regular points.

Let $\Omega \subset \Sigma' = \left\{ z : |z| < \frac{1}{4} \mathrm{R} \right\}$. Denote by H the quotient space $\mathrm{H}^{1,2}(\Omega)/\mathrm{H}_0^{1,2}(\Omega)$. By Theorem 1.4, there is a bounded linear map $\mathrm{B} : \mathrm{H} \longrightarrow \mathrm{H}^{1,2}(\Omega)$ such that if $\bar{h} \in \mathrm{H}^{1,2}(\Omega)$ represents an element h of H, $u = \mathrm{B}h$ satisfies $\mathrm{L}u = 0$ in the $\mathrm{H}^{1,2}(\Omega)$ sense and $u - \bar{h} \in \mathrm{H}_0^{1,2}(\Omega)$. Notice that by 1.8, $\mathrm{B}h$ is Hölder continuous in Ω. If $\bar{h}$ is bounded on $\partial\Omega$ in the $\mathrm{H}^{1,2}(\Omega)$ sense, then

$$\sup_{\Omega} |\mathrm{B}h| \leqslant \max_{\partial\Omega} |h|,$$

where $\max_{\partial\Omega} |h|$ means the smallest number c such that $\bar{h} \leqslant c$ and $-\bar{h} \leqslant c$ on $\partial\Omega$ in the $\mathrm{H}^{1,2}(\Omega)$ sense. In fact, $|||\mathrm{B}h||| \leqslant \mathrm{C} \max_{\partial\Omega} |h|$, where

$$|||g||| = \sup_{\Omega} |g| + \sup \left\{ r^{\mathrm{M}} \frac{|g(x) - g(y)|}{|x - y|} : |x - y| < \frac{1}{2} r,\ \mathrm{dist}(x, \partial\Omega) = r \right\}$$

for some large value of M. Since the restriction of $\mathrm{C}^\infty(\mathbf{R}^n)$ to $\partial\Omega$ is dense in the space of continuous functions with the supremum norm, B extends uniquely to a mapping from continuous functions h on $\partial\Omega$ to functions $\mathrm{B}h$ that are Hölder continuous in Ω. (It is also easy to see that $\mathrm{LB}h = 0$ in the $\mathrm{H}^{1,2}(\Omega')$ sense for any $\Omega' \subset\subset \Omega$).

A point $y \in \partial\Omega$ is *regular* if for every continuous function h on $\partial\Omega$, the solution $u = \mathrm{B}h$ to the Dirichlet problem satisfies $\lim_{\substack{x \to y \\ x \in \Omega}} u(x) = h(y)$.

THEOREM 5.1 (Wiener test). – $y \in \partial\Omega$ *is regular if and only if*

a) $\displaystyle\int_0^R \frac{s^2}{w(B(y,s))} \frac{ds}{s} < \infty$, *or*

b) $\displaystyle\int_0^R \operatorname{cap}(K_\rho) \frac{\rho^2}{w(B(y,\rho))} \frac{d\rho}{\rho} = \infty$,

where $K_\rho = (\Sigma \setminus \Omega) \cap B(y,\rho)$.

COROLLARY 5.2. – *The set of regular points of* $\partial\Omega$ *depends only on* $w(x)$ *and not on the particular operator* L.

Conditions a) and b) are mutually exclusive. Condition a) does not depend on Ω, so in case a) y is regular for any domain. (See (i)-(iv) of the introduction.)

Denote the capacitary distribution of K_ρ by μ_ρ. Denote $u_\rho(x) = \int g(x,y)\, d\mu_\rho(y)$ the lower semicontinuous representative of the capacitary potential of K_ρ. The following lemma is proved in the same way as the remark at the end of section 3 of [8].

LEMMA 5.3. – y *is regular if and only if* $\displaystyle\lim_{\substack{x\to y\\ x\in\Sigma\setminus K_\rho}} u_\rho(x) = 1$ *for all* $\rho > 0$.

LEMMA 5.4. – *Suppose that* $\operatorname{cap}(\{y\}) = 0$. *Let* μ *be a positive* $H^{-1,2}$ *measure and* $u(x) = \int g(x,z)\, d\mu(z)$. *Then,*

$$u(y) \geqslant \liminf_{x\to y} u(x).$$

Proof. – The proof follows closely the one of Lemma 8.1 of [8]. Let

$$F_a(t) = \begin{cases} t & t \leqslant a \\ t - \dfrac{1}{4a}(t-a)^2 & a \leqslant t \leqslant 3a. \\ 2a & t \geqslant 3a \end{cases}$$

By monotone convergence, $u(y) = \lim_{a\to\infty} \int F_a(g(y,z)\, d\mu(z)$. Since $g(y,y) = +\infty$, $F_a(g(y,-)) \in H_0^{1,2}(\Sigma) \cap C(\overline{\Sigma})$, and is the weak solution of

$$LF_a(g(y,-)) = \begin{cases} \dfrac{1}{2a} a_{ij}\partial_i g(y,x)\, \partial_j g(y,x) \text{ on } a \leqslant g(y,-) \leqslant 3a \\ 0 \end{cases} = f(x)\, dx,$$

where $f \in L^1(dx)$. By 1.28, $F_a(g(y,-)) = \int g(x,z) f(z)\, dz$ q.e., and so, because of 1.25, and 4.2

$$\int F_a(g(y,z))\, d\mu(z) = \frac{1}{2a} \int u(x)\, a_{ij}\, \partial_i g(y,x)\, \partial_j g(y,x)\, dx .$$

By our assumption on y, $\{x : a \leqslant g(y,x) \leqslant 3a\}$ shrinks to y as $a \longrightarrow \infty$. Also, $y \in \operatorname{int} J_a$, where $J_a = \{x : g(y,x) \geqslant a\}$, by 3.4, and by 2.6. $\partial J_a = \{x : g(y,x) = a\}$. Let ν be the capacitary distribution of J_a, $v(x) = \int g(x,z)\, d\nu(z)$. Then,

$$1 = v(y) = \int_{\partial J_a} g(y,z)\, d\nu(z) = a \operatorname{cap}(J_a) ,$$

so that $\operatorname{cap}(J_a) = \dfrac{1}{a}$. By the remarks prior to 1.21, we see that v equals 1 on J_a, and $\dfrac{g(y,x)}{a}$ on $\Sigma \backslash J_a$, so that

$$\frac{1}{a} = \operatorname{cap}(J_a) = \frac{1}{a^2} \int_{\Sigma \backslash J_a} a_{ij}\, \partial_i g(y,x)\, \partial_j g(y,x)\, dx .$$

(Here we used the fact that $\nabla h = 0$ a.e. on the set where $h = 0$, for $h \in H_0^{1,2}(\Sigma)$. See the remark prior to Lemma 2.1 in [5].) Therefore,

$$\frac{1}{2a} \int_{a < g < 3a} a_{ij}\, \partial_i g(y,x)\, \partial_j g(y,x)\, dx = 1 .$$

Hence,

$$u(y) = \lim_{a \to \infty} \frac{1}{2a} \int u(x)\, a_{ij}\, \partial_i g(y,x)\, \partial_j g(y,x)\, dx \geqslant \liminf_{x \to y} u(x) .$$

LEMMA 5.5. – *Suppose that* $\operatorname{cap}\{y\} = 0$, *and that* u *is the lower semi-continuous capacitary potential of a compact set* K. *Then,*

$$u(y) = \liminf_{\substack{x \to y \\ x \in \Sigma \backslash K}} u(x).$$

Proof. – By 3.6, $u(y) \leqslant \liminf_{\substack{x \to y \\ x \in \Sigma \backslash K}} u(x)$. Let

$$\bar{u}(x) = \begin{cases} u(x) & \text{in } \Sigma \backslash K \\ 1 & \text{in } K \end{cases} .$$

Then, $u(x) = \bar{u}(x)$ a.e., and the proof of lemma 5.4 shows that

$u(y) \geqslant \liminf_{x \to y} \bar{u}(x)$. Now, by the proof of 4.1, $u(x) \leqslant 1$, and so $\liminf_{x \to y} \bar{u}(x) = \liminf_{\substack{x \to y \\ x \in \Sigma \setminus K}} u(x)$, and the lemma follows.

COROLLARY 5.6. – *Suppose that* $\operatorname{cap}\{y\} = 0$. *Then* y *is regular if and only if* $u_\rho(y) = 1$ *for all* $\rho > 0$.

The corollary follows from 5.3 and 5.5.

The following lemma follows closely 9.8 of [8].

LEMMA 5.7. – *Suppose that* $\operatorname{cap}(\{y\}) = 0$. *Then,* y *is not regular if and only if* $\lim_{\rho \to 0} u_\rho(y) = 0$.

Proof. – If $u_\rho(y) \longrightarrow 0$, by 5.6 y is irregular. Assume now that y is irregular. Because $u_\rho(y) \leqslant 1$ (see the proof of 4.1), we must have $u_{\rho_0}(y) < 1$ for some ρ_0, by 5.6. By 4.1, given $\mathcal{E} > 0$, we can find $\sigma < \rho_0$ such that $\int_{|z-y|>\sigma} g(y, z)\, d\mu_{\rho_0}(z) \leqslant \mathcal{E}$. Let $v(x) = \int_{|z-y|>\sigma} g(x, z)\, d\mu_{\rho_0}(z)$, $\quad u(x) = \int_{|z-y|>\sigma} g(x, z)\, d\mu_{\rho_0}(z)$. Then $u_{\rho_0}(x) = v(x) + u(x)$, $v(y) \leqslant \mathcal{E}$, and in view of 2.7 and 2.3, $v, u \in H_0^{1,2}(\Sigma)$. Also u is continuous at y, $u(y) \leqslant u_{\rho_0}(y) < 1$. Hence, there is a τ with $2\tau < \sigma$ such that $u(x) \leqslant \frac{1}{2}(1 + u_{\rho_0}(y))$ on $B(y, 2\tau)$. Therefore, by the claim in the beginning of the proof of 1.30, $v(x) = 1 - u(x) \geqslant \frac{1}{2}(1 - u_{\rho_0}(y))$ on K_τ in the $H^{1,2}(\Sigma)$ sense. Since $u_{\rho_0}(x) \equiv 1$ on K_{ρ_0}, and hence on K_τ in $H^{1,2}(\Sigma)$ sense, $v(x) \geqslant \frac{1}{2}(1 - u_{\rho_0}(y))\, u_{\rho_0}(x)$ on K_τ in the $H^{1,2}(\Sigma)$ sense. By 1.20, 2.3 and 1.18, $v(x) \geqslant \frac{1}{2}(1 - u_{\rho_0}(y))\; u_\tau(x)$ almost everywhere in $\Sigma \setminus K_\tau$. Choose now $\mathcal{E} < \frac{1}{2}(1 - u_{\rho_0}(y))$. Then, as $v(x) \geqslant \frac{1}{2}(1 - u_{\rho_0}(y))$ a.e. on K_τ, and $v(y) \leqslant \mathcal{E}$, $v(x)$ is bounded away from $v(y)$ a.e. on K_τ. Because of 3.6, $v(y) \leqslant \liminf_{\substack{x \to y \\ x \in \Sigma \setminus K_\tau}} v(x)$, and the proof of 5.4 shows that $v(y) \geqslant \liminf_{\substack{x \to y \\ x \in \Sigma \setminus K_\tau}} \bar{v}(x)$, for any $\bar{v}$ which equals v a.e. Thus, $v(y) = \liminf_{\substack{x \to y \\ x \in \Sigma \setminus K_\tau}} v(x)$. On $\Sigma \setminus K_\sigma$, u_τ

and v are continuous, and so, $v(x) \geqslant \frac{1}{2}(1 - u_{\rho_0}(y)\; u_\tau(x)$ everywhere in $\Sigma\backslash K_\sigma$. Moreover, by 5.5, $u_\tau(y) = \liminf_{\substack{x\to y\\ x\in\Sigma\backslash K_\tau}} u_\tau(x)$. Thus,

$$\mathcal{E} \geqslant v(y) = \liminf_{\substack{x\to y\\ x\in\Sigma\backslash K_\tau}} v(x) \geqslant \frac{1}{2}(1 - u_{\rho_0}(y)) \liminf_{\substack{x\to y\\ x\in\Sigma\backslash K_\tau}} u_\tau(x)$$
$$= \frac{1}{2}(1 - u_{\rho_0}(y))\, u_\tau(y)\,.$$

Therefore, the lemma follows.

LEMMA 5.8. – If $\rho > r$, *then*

$$\mu_r(K_r) = \mu_\rho(K_r) + \int_{K_\rho/K_r} u_r\, d\mu_\rho\,.$$

In particular, $\mu_\rho(K_r) \leqslant \mu_r(K_r) = \operatorname{cap}(K_r)$.

Proof. – By 1.25, 4.6, 4.7 and 4.2

$$\mu_r(K_r) = \int u_\rho\, d\mu_r = \int u_r\, d\mu_\rho = \int_{K_r} u_r\, d\mu_\rho + \int_{K_\rho\backslash K_r} u_r\, d\mu_\rho$$
$$= \mu_\rho(K_r) + \int_{K_\rho\backslash K_r} u_r\, d\mu_\rho\,.$$

Proof of 5.1.

Case I $\operatorname{cap}(\{y\}) > 0$.

Let u be the (continuous representative of the) capacitary potential of $\{y\}$. (See the proof of 3.4). By the maximum principle, $u(x) \leqslant u_\rho(x)$ for $x \in \Sigma\backslash K_\rho$. But by 3.4, $\lim_{x\to y} u(x) = 1$, so $\lim_{x\to y} u_\rho(x) = 1$ in $\Sigma\backslash K_\rho$ and y is regular by Lemma 5.3. Moreover, $\operatorname{cap}(\{y\}) > 0$ is equivalent to (a).

Case II $\operatorname{cap}(\{y\}) = 0$.

$$u_\rho(y) = \int_{K_\rho} g(x, y)\, d\mu_\rho(x) \simeq \sum_{j=0}^{\infty} \int_{2^{-j}\rho}^{R} \frac{s^2}{w(B(y, s))} \frac{ds}{s} (\mu_\rho(K_{2^{-j}\rho})$$
$$- \mu_\rho(K_{2^{-j-1}\rho})).$$

Notice that by 4.1,

$$\lim_{r\to 0} u_\rho(K_r) \int_r^R \frac{s^2}{w(B(y, s))} \frac{ds}{s} \leqslant c \lim_{r\to 0} \int_{K_r} g(x, y)\, d\mu_\rho(x) = 0\,.$$

Therefore, we can apply summation by parts and obtain

(5.9)
$$u_\rho(y) \cong \mu_\rho(K_\rho) \int_\rho^R \frac{s^2}{w(B(y, s))} \frac{ds}{s} + \int_0^\rho \frac{r^2}{w(B(y, r))} \mu_\rho(K_r) \frac{dr}{r}.$$

Denote $c(\rho) = \text{cap}(K_\rho)$ and $\theta(s) = s/w(B(y, s))$. Then by 5.8, $u_\rho(y) \leqslant c(\rho) \int_\rho^R \theta(s)\, ds + \int_0^\rho c(s)\,\theta(s)\, ds$. Suppose that (b) fails, then $\int_0^R c(s)\,\theta(s)\, ds = C < \infty$. Evidently, $\int_0^\rho c(s)\,\theta(s)\, ds$ tends to zero as $\rho \longrightarrow 0$. Choose $\delta_2 > 0$ so that $\int_0^{\delta_2} c(s)\,\theta(s)\, ds < \mathscr{E}$. Since, $c(s)$ is increasing and $\lim_{s\to 0} c(s) = 0$ (see the proof of 3.4) we can choose $\delta_1 < \delta_2$ so that for all $\rho < \delta_1$, $c(\rho) < \mathscr{E}\, c(\delta_2)$. Now, for $\rho < \delta_1$

$$c(\rho) \int_\rho^R \theta(s)\, ds \leqslant \int_\rho^{\delta_2} c(s)\,\theta(s)\, ds + \mathscr{E} \int_{\delta_2}^R c(s)\,\theta(s)\, ds \leqslant \mathscr{E} + C\mathscr{E}.$$

In all, $u_\rho(y)$ tends to zero as $\rho \longrightarrow 0$, so that y is not regular (Lemma 5.7).

Conversely, suppose that y is not regular. 5.9 implies that $\int_0^\rho \frac{r^2}{w(B(y, r))} \mu_\rho(K_r) \frac{dr}{r}$ is finite. To prove (b) fails, it suffices to show that $c(r/2) \leqslant 2\,\mu_\rho(K_r)$ for $r \leqslant \rho$ and ρ sufficiently small. In fact,

$$c(r/2) = \mu_{r/2}(K_{r/2}) = \mu_\rho(K_{r/2}) + \int_{K_\rho \setminus K_{r/2}} u_{r/2}\, d\mu_\rho \leqslant \mu_\rho(K_r) + \int_{K_\rho \setminus K_{r/2}} u_{r/2}\, d\mu_\rho .$$

By Harnack's principle and 3.3, if $x \in K_\rho \setminus K_r$ and $z \in K_{r/2}$, then $g(x, z) \leqslant Cg(x, y)$. Hence

$$\int_{K_\rho \setminus K_r} u_{r/2}\, d\mu_\rho = \int_{K_\rho \setminus K_r} \int_{K_{r/2}} g(x, z)\, d\mu_{r/2}(z)\, d\mu_\rho(x) \leqslant C \int_{K_\rho \setminus K_r} \int_{K_{r/2}} g(x, y)\, d\mu_{r/2}(z)\, d\mu_\rho(x) \leqslant Cu_\rho(y)\, c(r/2).$$

Thus, $c(r/2) \leqslant \mu_\rho(K_r) + Cu_\rho(y)\, c(r/2)$. Since $u_\rho(y)$ tends to zero as $\rho \longrightarrow 0$, $c(r/2) \leqslant 2\mu_\rho(K_r)$ for sufficiently small ρ.

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