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On the space of maps inducing isomorphic connections


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ON THE SPACE
OF MAPS INDUCING ISOMORPHIC CONNECTIONS

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1. Introduction.

In this paper we prove the following

**Theorem.** Let \( M \) be a smooth compact manifold, \( P \) a principal bundle on \( M \) with the unitary group \( U(k) \) as structure group, \( A \) a smooth connection on \( P \), and \( \text{Aut} A \) the group of gauge transformations [i.e., automorphisms of \( P \) which act trivially on \( M \)] which leave \( A \) invariant. Let \( B \) be the Grassmanian of \( k \)-planes in a separable Hilbert space \( \mathcal{H} \), \( E \) the Stiefel bundle of orthonormal \( k \) frames in \( \mathcal{H} \), and \( \omega \) the canonical universal connection on \( E \). Denote by \( \Sigma(A) \) the space of maps \( p : M \to B \) such that the pull-back bundle \( p^*(E) \), with the connection \( p^*\omega \), is isomorphic to \( (P, A) \).

Then the space \( \Sigma(A) \), with the \( C^\infty \) topology, has the homotopy type of \( B_{\text{Aut}A} \) where \( B_{\text{Aut}A} \) is the base-space of a universal bundle for \( \text{Aut} A \).

The connectedness of \( \Sigma(A) \) is shown in [6]. We use some ideas from this paper.

To motivate this result, consider the case when \( P \) is a principal \( G \)-bundle with \( G \) a compact Lie group. Let \( \text{Aut} P \) denote the group of gauge transformations of \( P \). Denote by \( \mathcal{E} \) the space of \( C^\infty \) connections on \( P \). The group \( \text{Aut} P \) acts on \( \mathcal{E} \), though not freely in general. Denote by \( \mathcal{Q} \) the quotient.
By [4] there exists a finite dimensional principal G-bundle \( E(G, M) \rightarrow B(G, M) \) with connection such that the following diagram commutes, and the map \( \varphi \) is onto:

\[
\begin{array}{ccc}
\text{Mor}_G(P, E(G, M)) & \xrightarrow{\varphi} & \mathcal{C} \\
\downarrow \text{Aut P} & & \downarrow \mathcal{L} \\
\text{Mor}_P(M, B(G, M)) & \xrightarrow{\mathcal{L}} & \mathcal{C}
\end{array}
\]

Here \( \text{Mor}_G(P, E(G, M)) \) is the space of G-morphisms of P into E and \( \text{Mor}_P(M, B(G, M)) \) is the component of \( C^\infty(M, B(G, M)) \) which induces pull-back bundles isomorphic to P. \( \mathcal{L} \) is the map given by pulling back the universal connection on \( E(G, M) \).

We wish to investigate the fibres of the map \( \mathcal{L} \). It is possible to do so when we consider instead of \( E(G, M) \) a universal bundle \( E_\mathcal{G} \) with connection such that \( E_\mathcal{G} \) is contractible. Suppose then, that in the above diagram we replace \( E(G, M) \) by \( E_\mathcal{G} \) and \( B(G, M) \) by \( B_\mathcal{G} \). Let \( A \in \mathcal{C} \) and \( \mathcal{A} \) its class in \( \mathcal{C} \). We argue heuristically:

The spaces \( \mathcal{C} \) and \( \text{Mor}_G(P, E_\mathcal{G}) \) are both contractible. This would imply that \( \varphi^{-1}(A) \) is contractible (all the mappings being assumed to be good fibrations). The group \( \text{Aut} A \) acts on \( \varphi^{-1}(A) \) to give \( \mathcal{L}^{-1}(\mathcal{A}) \). If all goes well this implies

a) \( \varphi^{-1}(A) \rightarrow \mathcal{L}^{-1}(\mathcal{A}) \) is a universal \( \text{Aut} A \) bundle. The fibre over \( A \) of the map \( \mathcal{L} \) has the same homotopy type as \( B(\text{Aut} A) \).

b) If \( G \) has trivial centre and all connections are generic (i.e. \( \text{Aut} P \) acts freely on \( \mathcal{C} \)) \( \mathcal{L} \) has a section.

The quotient space \( \mathcal{C} \) is relevant in studies of Yang-Mills theories, at present very popular in Physics. It has been pointed out [1] that the Universal Connection Theorem could possibly provide connections between Yang-Mills theories and so-called o-models which concern themselves with the space \( \text{Mor}(M, B) \). Also in the cases when \( \mathcal{L} \) has a section, it could give an alternative to "gauge-fixing" which has been shown to be impossible in general [3, 7, 5].

The paper is organized as follows. In § 2 we imbed E and B as closed submanifolds of Hilbert spaces. In § 3 we describe a one parameter family of isometries \( A_\tau : \mathcal{K} \rightarrow \mathcal{K} \), and also give the
C°° topology to be used on the function spaces Mor_U(k)(P, E) and Mor_p(M, B). In § 4 we prove that \( \varphi^{-1}(A) \) is contractible [Proposition 4.1] using the isometries \( A_y \). Then we prove [Proposition 4.3] that \( \varphi^{-1}(A) \rightarrow \mathcal{L}^{-1}(A) \) is a locally trivial principal fibre space with Aut A as structure group. This involves, among other things, proving that the above projection is closed [Lemma 4.4], which is done by studying a certain differential equation. The completeness of the \( C^\infty \) topology is crucial, and the imbeddings obtained in § 2 simplify proofs throughout.

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2. The bundle of orthonormal \( k \)-frames in a Hilbert space.

Fix an integer \( k > 0 \). Let \( \mathcal{H} \) be an infinite dimensional separable Hilbert space over the complex numbers. Denote by \( E \) the space of orthonormal \( k \)-frames in \( \mathcal{H} \). The group \( U(k) \) acts on \( E \) on the right and the quotient is the Grassmannian \( B \) of \( k \)-dimensional subspaces of \( \mathcal{H} \). In fact \( E \) is a universal principal bundle for \( U(k) \). It also carries a natural connection, which is a universal connection for \( U(k) \).

It will be useful, in the following, to have characterizations of \( E \) and \( B \) as closed submanifolds of Hilbert spaces.

We shall identify a point \( p \) in \( B \) with the orthogonal projector onto the corresponding subspace, denoted by \( H(p) \). Thus \( H(p) = \{ x \in \mathcal{H} \mid px = x \} \). For \( p_0 \in B \), define

\[ \mathcal{R}_0 = \{ p \in B \mid H(p_0) \cap \ker p = \{0\} \}. \]

Then we have a bijection \( L_0 : \mathcal{R}_0 \rightarrow \mathcal{L}(H(p_0), \ker p_0) \) such that for \( p \in \mathcal{R}_0 \) its image \( L \equiv L_0(p) \) has \( H(p) \) as graph.

**Lemma 2.1** [2]. - The charts \( \{ (\mathcal{R}_0, L_0) \} \) give \( B \) the structure of a \( C^\infty \) Hilbert manifold.

Let \( \mathcal{H}_2 \) denote the Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H} \).
PROPOSITION 2.2. — Let \( \psi \) denote the injection \( B \rightarrow \mathcal{S}_2 \) given by associating to each \( k \)-dimensional subspace its orthogonal projector. Then \( \psi \) is a \( C^\infty \) immersion, and a homeomorphism onto its image.

Proof. — Follows from Lemmas 2.3 and 2.4.

Remark. — This shows that \( B \), with the manifold structure given in Lemma 2.1 is a submanifold of \( \mathcal{S}_2 \).

LEMMA 2.3. — On a chart \((\mathcal{S}_0, L_0)\) \( \psi \) is given by \((1 - 3)\). It is a \( C^\infty \) immersion.

Proof. — Let \( L \in \mathcal{S}(H(p_0), \ker p_0) \) and let \( p = \psi L_0^{-1}(L) \). Write

\[
p = A + LA
\]

where \( A : \mathcal{S} \rightarrow H(p_0) \). Then we claim that \( A \) satisfies

\[
A = p_0 + L^+(1 - p_0) - L^+LA
\]

which can be solved to give

\[
A = \frac{1}{1 + L^+L} \left( p_0 + L^+(1 - p_0) \right).
\]

To see that \( p \) given by (2.1)-(2.3) is indeed equal to \( \psi L_0^{-1}(L) \), we verify:

a) Image of \( p = \{ x + Lx \mid x \in H(p_0) \} \). The map is clearly into this set. In fact it is onto since \( A \) is invertible on \( H(p_0) \).

b) \( p^2 = p \). This follows since \( Ap = p \), which in turn is clear because \( Ap \) satisfies the same equation as \( p \).

\[
Ap = p_0p + L^+(1 - p_0)p - L^+LAp = A + L^+LA - L^+LAp = p_0 + L^+(1 - p_0) - L^+LAp.
\]

c) \( p \) is an orthogonal projector, for

\[
\ker p = \{ y - L^+y \mid y \in \ker p_0 \}
\]

which is the orthogonal subspace to \( \operatorname{Im} p \).

(i) \( \psi \) is \( C^\infty \) : To see this split \( \psi \) into the steps:
\[\mathcal{E}(\mathcal{E}, H(p_0)) \to \mathcal{E}(\mathcal{E}, \mathcal{E})\]

\[\{p_0 + L^*(1 - p_0)\} \to \{p_0 + L^*(1 - p_0)\}\]

\[\mathcal{E}(H(p_0), \ker p_0) \to \mathcal{E}(H(p_0), H(p_0)) \to \mathcal{E}(\mathcal{E}, \mathcal{E})\]

\[\{L^*L\} \to \left\{ \frac{1}{1 + L^*L} \right\} \to \left\{ p_0 \left[ \frac{1}{1 + L^*L} \right] p_0 \right\}\]

\(\psi\) is in fact real-analytic.

(ii) It is enough to check the differential at \(L = 0\). Here \(\delta p = \delta L^* (1 - p_0) + p_0 \delta L\) which is clearly injective. Also the image, being defined by \(p_0 \delta pp_0 = (1 - p_0) \delta p (1 - p_0) = 0\) and \(\delta p^* = \delta p\), is closed, and hence admits a supplement.

**Lemma 2.4.** — *The inverse \(\psi^{-1}\) is given by (4) and is continuous.*

*Proof.* — Consider a chart \((\mathcal{R}_0, L_0)\). Let \(p \in \mathcal{R}_0\) and let \(Q = (p_0|_{H(p)})^{-1}\). Then for \(x \in H(p)\), \(Qx = x + (1 - p_0) p Q x\). This gives, for \(L = (1 - p_0) Q\), \(L = (1 - p_0) p (1 + L)\).

This can be solved to give \(p \overset{\psi^{-1}}{\longrightarrow} L\) such that

\[Lx = (1 - p_0) \frac{1}{1 - (1 - p_0) p} x, x \in H(p_0).\]  

(4)

The continuity of \(\psi^{-1}\) follows easily.

We turn now to \(E\). This can be identified with a closed subset of \(\mathcal{E}(\mathbb{C}^k, \mathcal{E})\): \(E = \{U : \mathbb{C}^k \to \mathcal{E} | U^+ U = 1\}\). Standard arguments show:

**Lemma 2.5.** — *\(E\) is a closed submanifold of \(\mathcal{E}(\mathbb{C}^k, \mathcal{E})\). It is a principal \(U(k)\) bundle on \(B\). The \(u(k)\)-valued one-form \(U^+ dU\) is a connection on \(E\).*
LEMMA 2.6. – \( E \) is contractible and hence a universal \( U(k) \) bundle. The connection is a universal \( U(k) \) connection.

Proof. – Both statements are well-known. The first follows also from the remarks after Lemma 4.2. The second is a consequence of the Universal Connection Theorem.

3. Some preliminary remarks and definitions.

(i) A one-parameter-family of isometries on \( \mathcal{H} \).

Following [6], we introduce, on \( \mathcal{H} \), a one-parameter family of isometries which we will use later. Define, for \( t \in [0, 1] \) an isometry \( A_t : \mathcal{H} \to \mathcal{H} \) as follows. Fix an orthonormal basis, so that \( \mathcal{H} \approx \) \{square-summable sequences in \( \mathbb{C} \)\}. Then let \( A_0 = \) Identity

\[
A_t(a_0, a_1, a_2, \ldots) = (a_0, a_1 \ldots a_{n-2}, a_{n-1} \cos \theta_n(t), a_{n-1} \sin \theta_n(t), \ldots)
\]

for \( \frac{1}{n + 1} \leq t \leq \frac{1}{n} \) where \( \theta_n(t) = \frac{\pi}{2^n} n[(n + 1) t - 1] \).

The \( A_t \) are continuous in \( t \) w.r. to the strong operator topology. Note that

\[
A \left( \frac{1}{2} \right) (a_0, a_1, \ldots) = (a_0, 0, a_1 0, \ldots) \in \mathcal{H}_{\text{even}}
\]

\[
A(1) (a_0, a_1, \ldots) = (0, a_0, 0, a_1 \ldots) \in \mathcal{H}_{\text{odd}}
\]

where \( \mathcal{H}_{\text{even}} \) and \( \mathcal{H}_{\text{odd}} \) denote obvious subspaces of \( \mathcal{H} \).

(ii) The topology of the function spaces \( \text{Mor}_{U(k)}(P, E) \) \( \text{Mor}(M, B) \).

We topologize \( \text{Mor}_{U(k)}(P, E) \) as a (closed) subset of \( C^\infty(P, \mathcal{E}(\mathcal{C}^k, \mathcal{H})) \), and \( \text{Mor}(M, B) \) as a (closed) subset of \( C^\infty(M, \mathcal{H}_2) \). The \( C^\infty \) topology is described below:

Let \( X \) be a compact manifold and \( \mathcal{F} \) a Hilbert space. Let \( X_1, \ldots, X_q \) be a set of vector fields on \( X \) which together span the tangent space at each point of \( X \). For a multi index \( \alpha = (\alpha_1, \ldots, \alpha_2) \)
set $D^\alpha = X_1^{\alpha_1}, \ldots, X_q^{\alpha_q}$. We make $C^\omega(X, \mathcal{G})$ a Frechet space w.r.
to the seminorms $\|f\|_{\alpha} = \sup_x \|D^\alpha f\|$ where the heavy bars $\| \|$ denote the Hilbert space norm. The topology is clearly independent
of the choice of $X_1, \ldots, X_q$. If $N \subset \mathcal{G}$ is a closed submanifold then $C^\omega(X, N)$ is a closed subset of $C^\omega(X, \mathcal{G})$ and we give it the relative
topology, which makes it a complete metric space.

We choose now, once and for all, a set of vector fields $X_1, \ldots, X_p$
spanning the tangent space of $M$ at each point. Let $\tilde{X}_1, \ldots, \tilde{X}_p$ be
their lifts to $P$ w.r. to some connection, and let $\tilde{Y}_1, \ldots, \tilde{Y}_{k^2}$ be
vertical vector fields on $P$, the images of a fixed basis $Y_1, \ldots, Y_{k^2}$
in $u(k)$ by the group action. We will use these to determine the
seminorms. Note that $[\tilde{X}_i, \tilde{Y}_\ell] = 0 \ \forall X_i$ and $Y_\ell$. We will let
let $\alpha_L = (\alpha_1, \ldots, \alpha_{k^2})$ and $\alpha = (\alpha_1, \ldots, \alpha_p)$, and write the semi-
norms as $\|f\|_{\alpha_L, \alpha} = \sup_x \|D^\alpha L_i D^\alpha\|$.

When there is no need to distinguish between the vertical and
horizontal vectors we simply denote $(\alpha_L, \alpha)$ by $\gamma$.

**Lemma 3.1.** $\text{Mor}_{U(k)}(P, E)$ and $\text{Mor}(M, B)$ are closed sub-
sets of $C^\omega(P, \mathcal{G}(C^k, \mathcal{U}))$ and $C^\omega(M, \mathcal{U}_2)$ respectively. The map $\text{Mor}_{U(k)}(P, E) \rightarrow \text{Mor}(M, B)$ is continuous.

**Proof.** For $g \in U(k)$ the map $C^\omega(P, E) \rightarrow C^\omega(P, E)$ given
by $f \rightarrow f^g$, $f^g(x) \equiv f(xg)g^{-1}$ $(x \in P)$, is continuous. This follows
since
\[
\|f_1^g - f_2^g\|_{\alpha_L, \alpha} = \sup_{x \in P} \|D^\alpha L_i D^\alpha (f_1(xg)g^{-1} - f_2(xg)g^{-1})\|
\]
\[
= \sup_{x \in P} \|D^\alpha L_i D^\alpha (f_1(xg) - f_2(xg))\|
\]
\[
= \sup_{xg \in P} \|D^{[\alpha_L, \alpha]}_{xg} D^\alpha (f_1(xg) - f_2(xg))\|
\]
\[
= \|f_1 - f_2\|_{[\alpha_2, \alpha], \alpha}
\]
where $D^{[\alpha_L, \alpha]}$ denotes the differential operator
\[
D^{[\alpha_L, \alpha]} = (g^{-1}\tilde{Y}_1g)^{\alpha_1} \ldots (g^{-1}\tilde{Y}_{k^2}g)^{\alpha_{k^2}}.
\]
Here $g^{-1}\tilde{Y}_i g$ is the image of the Lie algebra element $g^{-1}Y_ig$. This
proves the first statement. To prove the second statement, let $f_n \rightarrow f$ in $\text{Mor}_{U(k)}(P, E)$ and let $p_n = f_n f^+_n$. Then
\[ \| p_n - p \|_\alpha = \sup_{x \in B} \| D^\alpha (p_n - p) \| \quad \text{(where } D^\alpha = X^\alpha_1 \ldots X^\alpha_n) \]
\[ = \sup_{x \in P} \| D^\alpha (p_n - p) \| \quad \text{(where } D^\alpha = \hat{X}^\alpha_1 \ldots \hat{X}^\alpha_n) \]
\[ = \sup_{x \in P} \| \sum_{\beta < \alpha} (\alpha \beta) (D^\alpha - \beta f_n D^\beta f^+_n - D^\alpha - \beta f D^\beta f^+) \| \]
\[ \leq \alpha \sum_{\beta < \alpha} \| f_n \|_\beta \| f_n - f \|_{\alpha - \beta} + \| f \|_{\alpha - \beta} \| f_n - f \|_\beta . \]
This proves \( p_n \to p \) in \( \text{Mor}(M, B) \).

4. The topology of the fibres.

We will be interested in the fibres of the map \( \varphi \). Consider first a fibre of \( \varphi \).

**Proposition 4.1.** Let \( A \in \mathcal{C} \). Then \( \varphi^{-1}(A) \) is contractible. In other words the space of morphisms \( P \to E \) which induce a fixed connection on \( P \) is contractible.

**Proof.** The proof proceeds in two steps.

(i) Define a map \( \xi : \varphi^{-1}(A) \times [0, 1/2] \to \varphi^{-1}(A) \) by
\[ \xi_t(f)(x) = A_t \circ f(x) \quad \text{for } f \in \varphi^{-1}(A), \quad x \in P, \quad t \in [0, 1/2]. \]
The map is into \( \varphi^{-1}(A) \) since,

a) \( \xi_t(f)(xg) = A_t \circ f(xg) = A_t \circ f(x) \circ g = \xi_t(f)(x) \circ gU \)

b) \( \xi_t(f)^* d\xi_t(f) = f^* df = A. \)

By lemma 4.2 below \( \xi \) is continuous.

(ii) There exists a \( f_0 \in \varphi^{-1}(A) \) s.t. \( \forall x \in P, f_0(x) \) maps \( \mathcal{C}^k \) into \( \mathcal{C}_{\text{odd}} \) [Apply \( A_1 \) to any \( f \in \varphi^{-1}(A) \) to get such an \( f_0 \)]. Define for \( t \in [1/2, 1] \) a map \( \eta : \varphi^{-1}(A) \times [1/2, 1] \to \varphi^{-1}(A) \) by
\[ \eta_t(f)(x)v = (\sin t \pi) A_{1/2} f(x)v - \cos t \pi f_0(x)v. \]
Again the map is into $\varphi^{-1}(A)$. Note that $A_{1/2}f$ maps into $\mathcal{H}_{\text{even}}$. This means that $\forall (x, t)$, $\eta_t f(x)$ defines an isometry of $\mathcal{C}_k$ into $\mathcal{H}$, for, given $v, v' \in \mathcal{C}_k$,

$$(\eta_t f(x) v, \eta_t f(x) v') = \sin^2 t\pi (A_{1/2} f(x) v, A_{1/2} f(x) v') + (\cos^2 t\pi) (f_0(x) v, f_0(x) v') = (v, v')$$

where $( , )$ denotes the inner product.

The points a), b) above can be checked easily. Lemma 4.2 gives continuity.

(iii) Compose $\xi$ and $\eta$ to get the contraction

$$\psi : \varphi^{-1}(A) \times [0, 1] \rightarrow \varphi^{-1}(A).$$

(See diagram)

LEMMA 4.2. — The maps $\xi, \eta$ constructed in the proof of Proposition 4.1 are continuous (in the product topology).

Proof. — Consider the map $\xi$. Let $(f_n, t_n)$ be a sequence in $\varphi^{-1}(A) \times [0, 1/2]$. Then

$$\|\xi_{t_n}(f_n) - \xi_t(f)\|_\gamma = \sup_{x \in \mathcal{P}} \|A_{t_n} \circ D^\gamma f_n - A_t \circ D^\gamma f\|$$

$$= \sup_{x \in \mathcal{P}} \|A_{t_n} \circ D^\gamma (f_n - f) + (A_{t_n} - A_t) \circ D^\gamma f\|$$

$$\leq \|f_n - f\|_\gamma + \|(A_{t_n} - A_t) f\|_\gamma.$$

This shows continuity of $\xi$. The continuity of $\eta$ follows similarly.

Remark. — The proof of Proposition 4.1 can be extended to prove contractivility of $\text{Mor}_{\Omega(k)}(P, E)$. In particular, taking $P = U(k)$, we see that $E$ itself is contractible.
We turn now to the fibres of the map \( \varphi \). Note that if \( A \in \mathcal{C} \) and \( A \in \mathcal{C} \) is its class, then \( \varphi^{-1}(A) \) projects onto \( \varphi^{-1}(A) \). Also if \( \text{Aut} A \) is the subgroup of Aut that leaves \( A \) fixed \( \text{Aut}(A) \) acts freely on \( \varphi^{-1}(A) \), the quotient being in bijection with \( \varphi^{-1}(A) \).

\( \text{Aut} A \) is the space of maps \( \hat{g} : P \longrightarrow U(k) \) such that

1. \( \hat{g}(xh) = h^{-1}g(x)h \quad x \in P, \ h \in U(k) \)
2. \( A = \hat{g}^{-1}Ag + \hat{g}^{-1}d\hat{g} \).

Since \( \hat{g} \in \text{Aut} A \) is determined by its value at a fixed point in \( P \), we shall, fixing \( y_0 \in P \) (projecting onto \( x_0 \in \mathcal{M} \) identify \( \text{Aut} A \ni \hat{g} \sim \hat{g}(y_0) \in U(k) \).

Thus \( \text{Aut} A \) is a closed subgroup of \( U(k) \) [This is seen either from the equation (ii) above, or noting the fact that under the above identification \( \text{Aut} A \) is the centralizer of the holonomy group at \( y_0 \)] and hence a Lie subgroup.

From now on we assume that the vector fields \( \hat{X}_1 \ldots \hat{X}_p \) have been lifted to \( P \) w.r. to \( A \). Note that then \( \hat{X}_i(\hat{g}) = 0 \) for \( \hat{g} \in \text{Aut} A \).

**Proposition 4.3.** \( \varphi^{-1}(A) \longrightarrow \varphi^{-1}(A) \) is a locally trivial principal fibre space with \( \text{Aut} A \) as structure group.

**Proof.** — The proof proceeds in four steps.

a) \( \text{Aut} (A) \) acts continuously on \( \varphi^{-1}(A) \). For suppose \((f_n, \hat{g}_n) \in \varphi^{-1}(A) \times \text{Aut} A \) and \((f, \hat{g}) \longrightarrow (f, \hat{g}) \). Then for any \( \alpha_L, \alpha \)

\[
\|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} \leq \|(f_n - f) \circ \hat{g}_n\|_{\alpha_L, \alpha} + \|f \circ (\hat{g}_n - \hat{g})\|_{\alpha_L, \alpha}
\]

\[
= \sup_x \|D^\alpha ([D^\alpha (f_n - f)] \hat{g}_n)\| + \sup_x \|D^\alpha ([D^\alpha f] (\hat{g}_n - \hat{g}))\|
\]

(since \( D^\alpha \hat{g} = 0 \))

\[
= \sup_x \left\| \sum_{\beta_L \leq \alpha_L} (\alpha_L) (\beta_L) D^{\alpha_L - \beta_L} D^\alpha (f_n - f) D^\beta_L \hat{g}_n \right\|
\]

\[
+ \sum_{\beta_L \leq \alpha_L} (\alpha_L) (\beta_L) D^{\alpha_L - \beta_L} D^\alpha f D^\beta_L (\hat{g}_n - \hat{g})
\]

\[
\leq \alpha_L! \sum_{\beta_L \leq \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L} + \|f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n - \hat{g}\|_{\beta_L}.
\]
Now, for any \( \hat{Y}_i, \hat{g} \in \text{Aut} A \)

\[
\hat{Y}_i(\hat{g}) = \lim_{t \to 0} \frac{\hat{g}(x \exp t Y_i) - \hat{g}(x)}{t} = [\hat{g}(x), Y_i].
\]

Also, if \( \hat{g}_1, \hat{g}_2 \) are in \( \text{Aut} A \), \( d(\text{Tr}(\hat{g}_1 - \hat{g}_2)^* (\hat{g}_1 - \hat{g}_2)) = 0 \), so that \( \|\hat{g}_1(x) - \hat{g}_2(x)\| = \|\hat{g}_1(y_0) - \hat{g}_2(y_0)\| \).

So, we have

\[
\|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} \leq \alpha_L \sum_{\beta_L < \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L} + \|f\|_{\alpha_L - \beta_L, \alpha} C_{\beta_L} \|\hat{g}_n\|_{\beta_L} \|\hat{g}(p_0) - \hat{g}(p_0)\|
\]

where \( C_{\beta_L} \) is a constant depending on the multiindex \( \beta_L \).

b) Denote by \( G \) the graph of the equivalence relation defined by \( \text{Aut} A \) on \( \varphi^{-1}(A) \). Then the map \( G \to \text{Aut} A \) is continuous. This follows since the map is given by \( (f_1, f_2) \mapsto f_1^*(y_0) f_2(y_0) \) which is clearly continuous.

c) The projection \( \varphi^{-1}(A) \to \mathcal{L}^{-1}(\hat{A}) \) is continuous and closed. Continuity follows from lemma 3.1 and lemma 4.4 shows that it is closed. Thus \( \mathcal{L}^{-1}(\hat{A}) \) has the quotient topology w.r. to the projection.

d) Thus we have shown that \( \varphi^{-1}(A) \to \mathcal{L}^{-1}(\hat{A}) \) is a principal fibre space. Now note that there is a \( \text{Aut} A \)-morphism

\[
\varphi^{-1}(A) \to \mathcal{L}^{-1}(\hat{A}) \to E/\text{Aut} A
\]

given by \( f \mapsto f(y_0) \). Since \( E \to E/\text{Aut} A \) is locally trivial, the proof is complete.

\textbf{Lemma 4.4.} — \textit{The map} \( \varphi^{-1}(A) \to \mathcal{L}^{-1}(\hat{A}) \) \textit{is closed.}

\textit{Proof.} — Let \( f_n \in \varphi^{-1}(A) \) s.t. \( p_n = f_n f_n^* \to p \) \textit{in} \( \mathcal{L}^{-1}(\hat{A}) \).

It is enough to prove that \( \{f_n\} \) contains a convergent subsequence. Since \( p_n(x_0) \to p(x_0) \) and \( E \) has compact fibres one
can assume \( f_n(y_0) \to g_0 \in E \) without loss of generality. Note that the \( f_n \) satisfy
\[
d f_n = f_n A + d p_n f_n.
\] (5)

We now prove that the \( f_n \) are Cauchy in the \( C^0 \) norm so that \( \exists \) a \( C^0 \) function \( f \) such that \( f_n \to f \). Put \( D = f_n - f_m \). Then from (5) we have
\[
d(DD^+) = DD^+ d p_n + d p_n DD^+ + d (p_n - p_m) f_m D^+ + D f_m^+ d (p_n - p_m).
\]
Evaluating on a vector field \( X_t \), taking the trace and then absolute value of both sides we get
\[
|X_t \text{Tr}(DD^+)| \leq |\text{Tr}(DD^+ X_t p_n)| + |\text{Tr}(X_t(p_n) DD^+)|
+ |\text{Tr}(X_t(p_n - p_m) f_m D^+)| + |\text{Tr}(D f_m^+ X_t(p_n - p_m))|
\leq 2 \{ \|D\|^2 \|X_t p_n\| + \|D\| \|X_t(p_n - p_m)\| \}
\]
or,
\[
|X_t \|D\|^2 | \leq 2 \{ \|D\|^2 \|X_t p_n\| + \|X_t(p_n - p_m)\| \}.
\] (6)

Consider now the set \( \{X_t, Y_\ell\} \) which we collectively denote by \( \{Z_\ell\} \). They give a map from \( P \times \mathbb{R}^N \) (where \( N = k^2 + p \)) to the tangent bundle \( TP \) which is onto:
\[
(x, (t_1 \ldots t_N)) \mapsto (x, \sum_i t_i Z_i(x)).
\]
Take the obvious metric on the vector bundle \( P \times \mathbb{R}^n \). This induces a splitting of the above map as well as a Riemannian metric on \( P \). Then we have the following obvious result: if \( X \) is a vector field on \( P \) of norm \( \leq 1 \) and we express \( X = \Sigma a_i Z_i \) with respect to the above splitting then \( |a_i| \leq 1 \forall i \).

Now let \( y \in P \) and let \( \Gamma(y) \) be a minimal geodesic joining \( y_0 \) to \( y \) [such a geodesic exists for \( P \) compact] parametrized with respect to arc-length. Then the length of \( \Gamma(y) < T \) for some constant \( T \) independent of \( y \). Now let \( X_t \) be the tangent vector field to \( \Gamma \) (which is necessarily of norm one). This gives
\[
\|X_t (p_n - p_m)\| = \sum_i \|p_n - p_m\|_i \text{ where } \|p\|_i = \sup_x \|Z_i p\|
= \sum_{|\alpha| = 1} \|p_n - p_m\|_\alpha.
\]
Thus we have, from (6)
\[
|X_t \|D\|^2 | = 2 \{ a \|D\|^2 + b \|D\| \}
\]
with
\[ a = \sum_{|\alpha| = 1} \|p\|_\alpha + c , \ c > 0 \]
and
\[ b = \sum_a \|p_n - p_m\|_\alpha . \]

Consider the ordinary differential equation
\[
\frac{du^2}{dt^2} = 2(au^2 + bu) \]
\[ u(0) = D(y_0) . \]
The solution is clearly:
\[ u(t) = D(y_0) e^{at} + \frac{(e^{at} - 1)}{a} b . \]

Consider the set \( K = \{ t \geq 0 \mid \|D(t)\| > u(t) \} \). \( K \) is open, and hence a union of disjoint open intervals. Let \( t_0 \) be its least boundary point. Clearly \( D(t_0) = u(t_0) \). From the polygonal approximations to \( \|D(t_0)\|^2 \) and \( u^2(t) \) it is clear that in an interval \( (t_0, t_0 + \epsilon) \) we have \( \|D(t)\| \leq u(t) \). Thus \( K = \emptyset \). We have finally,
\[ \|D(y)\| \leq D(y_0) e^{aT} + \frac{(e^{aT} - 1)}{a} b \]
which clearly shows that \( \{f_n\} \) are Cauchy in the \( C^0 \) norm.

Let \( f \) be the \( C^0 \) limit. We now turn back to (5) and ‘bootstrap’ the above result to show that \( f \) is \( C^\infty \) and \( f_n \rightarrow f \) in the \( C^\infty \) topology. Assume, therefore, that \( f \) is \( C^k \) and \( f_n \rightarrow f \) in \( C^k \).

For any multi-index \( \gamma (|\gamma| \geq 1) \) define \( \gamma' \) and \( X^{(\gamma)} \) [here \( X^{(\gamma)} \) is one of the vector fields \( Z_i \)] by \( D^{ \gamma'} = D^{\gamma'} X^{(\gamma)} \) so that \( D^{\gamma'} \) is of order \( |\gamma| - 1 \). Let \( |\gamma| = k + 1 \). Then
\[
D^{\gamma'} f_n = D^{\gamma'} X^{(\gamma)}(f_n) = D^{\gamma'}(f_n A(X^{(\gamma)}) + X^{(\gamma)}(p_n) f_n) \\
= \sum_{\delta < \gamma'} \left( \begin{array}{c} \gamma' \\ \delta \end{array} \right) [D^{\gamma'-\delta} f_n D^\delta A(X^{(\gamma)}) + D^{\gamma'-\delta} X^{(\gamma)}(p_n) D^\delta f_n] .
\]

Then
\[
\|D f_n - \sum_{\delta < \gamma'} \left( \begin{array}{c} \gamma' \\ \delta \end{array} \right) [D^{\gamma'-\delta} f D^\delta A(X^{(\gamma)}) + D^{\gamma'-\delta} X^{(\gamma)}(p) D^\delta f] \| \\
\leq \gamma! \sum_{\delta < \gamma'} \|f_n - f\|_{\gamma'-\delta} \|A(X^{(\gamma)})\|_\delta + \|p_n\|_{\gamma'-\delta, X^{(\gamma)}} \|f_n - f\|_\delta \\
+ \|p_n - p\|_{\gamma'-\delta, X^{(\gamma)}} \|f\|_\delta
\]
where \( \|f\|_{\gamma'-\delta, X^{(\gamma)}} = \sup_x \|D^{\gamma'-\delta} X^{(\gamma)} f\| . \)
This shows $D^\gamma f_n$ tends uniformly to a $C^0$ function, and hence $f$ is $C^{k+1}$. By induction $f$ is $C^\infty$ and $f_n \to f$ in $C^\infty(P,E)$. The proof also shows $df = fA + pf$.

Since $\text{Mor}_{U(k)}(P,E)$ is closed, $f \in \text{Mor}_{U(k)}(P,E)$ and $p = ff^+$ by continuity of the projection $\text{Mor}_{U(k)}(P,E) \to \text{Mor}_p(M,B)$. (One can now easily show that $f^+df = A$, thus showing that the fibre $\mathcal{V}^{-1}(A)$ is closed. This is because we have nowhere in the proof used the fact that $p \in \mathcal{V}^{-1}(A)$).

The Theorem stated in the Introduction now follows.

**BIBLIOGRAPHY**


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