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On the weak $L^1$ space and singular measures


<http://www.numdam.org/item?id=AIF_1982__32_1_119_0>
ON THE WEAK L₁ SPACE AND SINGULAR MEASURES

by Robert KAUFMAN

Introduction.

The class $R$ of finite, complex measures $\mu$ on $(-\infty, \infty)$ such that $\hat{\mu}(\infty) = 0$, has been intensively investigated (since 1916). For this class $o(1)$ is trivial and for absolutely continuous measures, we have the Riemann-Lebesgue Lemma. We investigate the corresponding $o(1)$ condition for the partial-sum operators

$$S_T(x, \mu) \equiv \int D_T(x - t) \mu(dt),$$

$$D_T(t) \equiv (\pi t)^{-1} \sin T t, T > 0.$$

The $o(1)$ condition for $S_T$ depends on the weak $L^1$ norm, defined by

$$\|u\|_1^* \equiv \sup Y \{ m\{ |u| > Y \} \};$$

$$\|S_T(\mu)\|_1^* \leq C \|\mu\|, 0 < T < +\infty.$$

The weak estimate is an easy consequence of Kolmogorov’s estimate for the Hilbert transform [2, Chapter II]. Elementary approximations show that when $\mu = f(x) \, dx$, then $\lim \|S_T(\mu) - f\|_1^* = 0$. When $\mu$ is singular and $\lim \|S_T(\mu) - g\|_1^* = 0$ for a certain measurable $g$, two conclusions can be obtained without great difficulty (see below):

a) $\|S_k(\mu) - S_{k+1}(\mu)\|_1^* \rightarrow 0$ whence $\hat{\mu}(\infty) = 0$;

b) $S_T(\mu) \rightarrow 0$ in measure as $T \rightarrow +\infty$

whence $g = 0$ a.e. This leads us to define:

(*) Presented at the Italian-American Conference on harmonic analysis, Minnesota, 1981.
$W_0$ is the class of measures $\mu$ for which $\|S_T(\mu)\|_1^* \to 0$ as $T \to +\infty$.

We present an elementary structural property of $W_0$, and then show by example that

(A) There exist $M_0$-sets $F$ carrying no measure $\mu \neq 0$ in $W_0$.

The sets $F$ are defined by a purely metrical property, and they need not be especially small. Their construction is based on an idea from the theory of divergent Fourier series [31, Chapter VIII].

(B) The set $F_\theta$ of all sums $\sum_{0}^{\infty} \pm \theta^m \ (0 < \theta < 1/2)$ carries a measure $\lambda \neq 0$ in $W_0$, provided $F_\theta$ is an $M_0$-set.

To elucidate example (B) and the next one we recall that $F_\theta$ fails to be an $M_0$-set (or even an $M$-set) unless $\mu_\theta \in \mathbb{R}$, where $\mu_\theta$ is the Bernoulli convolution carried by $F_\theta$ and that $\mu_\theta \in \mathbb{R}$ except for certain algebraic numbers $\theta$ [311, p. 147-156]. Therefore the next example is somewhat unexpected.

(C) When $0 < \theta < 1/2$, then $\mu_\theta \notin W_0$, in fact

$$\|S_T(\mu_\theta)\|_1^* \geq c(\theta) > 0$$

for large $T > 0$. We observe in passing that $\mu$ is not known to be singular for $1/2 < \theta < 1$ except when $\mu_\theta \notin \mathbb{R}$, e.g., for $\theta^{-1} = (1 + \sqrt{5})/2$.

1.

From the weak estimate for $S_T$ it is clear that $W_0$ is norm-closed in the space of all measures. We shall prove that when $\mu \in W_0$ and $\psi \in C^1 \cap L^\infty$, then $\psi \mu \in W_0$; consequently the same is true if only $\psi \in L^1(\mu)$. We need two lemmas; the first was already used implicitly.

**Lemma 1.** Let $\mu$ be a measure such that $S_k(\mu) - S_{k+1}(\mu) \to 0$ in measure (over finite intervals). Then $\hat{\mu}(\infty) = 0$, i.e., $\mu \in \mathbb{R}$.

**Proof.** $|D_k(t) - D_{k+1}(t)| \leq \min(1, |t|^{-1}) \equiv K(t)$, say, and $K \in L^2(-\infty, \infty)$. Thus the functions $|S_k(\mu) - S_{k+1}(\mu)|$ have a common majorant $\int K(x-t) |\mu|(dt)$ in $L^2$. The hypothesis on
\[ S_k - S_{k+1} \text{ then yields } \| S_k - S_{k+1} \|_2 \to 0. \] 
This means that 
\[ \int_k^{k+1} (|\hat{\mu}(t)|^2 + |\hat{\mu}(-t)|^2) \, dt \to 0 \] 
so \( \hat{\mu}(\infty) = 0 \), because \( \hat{\mu} \) is uniformly continuous.

**Lemma 2.** Let \( \mu \in \mathcal{R} \) and \( \psi \in C^1 \cap L^\infty \). Then as \( T \to +\infty \)
\[ \| S_T(x, \psi \cdot \mu) - \psi(x) S_T(x, \mu) \|_1^* \to 0. \]

**Proof.** Since \( \mu \) can be approximated in norm by measures \( \mu_n \in \mathcal{R} \), each of compact support, we can suppose that \( \mu \) itself has compact support, say \( |t| \leq a \). Now \( S_T(\psi \cdot \mu) - \psi S_T(\mu) \) converges to 0 uniformly on \( [-a - 1, a + 1] \), being equal to 
\[ \pi^{-1} \int \sin T(t - x) \cdot \varphi(x, t) \mu(dt), \]
with \( \varphi(x, t) = (t - x)^{-1}[\psi(t) - \psi(x)] \); \( \varphi(x, t) \) is jointly continuous. This is sufficient to obtain the uniform convergence claimed.

For \( |x| > a + 1 \) we write 
\[ x S_T(x, \mu) = \pi^{-1} \int \sin T(t - x) \cdot \sigma(x, t) \mu(dt) \]
with \( \sigma(x, t) = x(t - x)^{-1} \); now \( |\sigma| \leq a + 1 \) and 
\[ \left| \frac{\partial}{\partial t} \sigma(x, t) \right| \leq a + 1, \]
for \( |t| \leq a \). Therefore \( x S_T(\mu, x) \to 0 \) as \( T \to +\infty \), uniformly for \( |x| \geq a + 1 \). The same applies to \( x S_T(x, \psi \cdot \mu) \), because \( \psi \cdot \mu \in \mathcal{R} \), and these inequalities show that \( \psi S_T(\mu) - S_T(\psi \cdot \mu) \to 0. \)

2. Examples.

I. Let \( F \) be a compact set in \( (-\infty, \infty), \) \( 0 < \alpha < 1, (\varepsilon_j) \) a sequence decreasing to \( 0 \); for each \( j \), let \( F = \bigcup F^j_k \), where 
\[ \text{diam}(F^j_k) \leq \varepsilon_j, \quad d(F^j_k, F^j_{k'}) \geq \varepsilon_j^\alpha, \quad k \neq \ell. \]
Then \( F \) carries no probability measure \( \mu \) in \( W_0 \) (and hence no signed measure \( \mu \neq 0 \) in \( W_0 \)).

We define the following property of a number \( \beta \) in \( [0,1) \), relative to \( \mu \) and the sequence of partitions \( F = \bigcup F^j_k \) :

\[
(\ast\ast) \text{ The total } \mu\text{-measure of the sets } F^j_k, \text{ such that } \mu(F^j_k) \geq \varepsilon_j^\beta, \text{ tends to } 0, \text{ as } j \to +\infty.
\]
Plainly \( \beta = 0 \) has property (**), because \( \mu \), being an element of \( \mathbb{R} \), can have no discontinuities. We shall prove that if \( \beta \) has property (**), and \( 0 \leq \beta < \alpha \), then \( \gamma = \beta + (1 - \alpha)/2 \) has property (**). This leads to a contradiction as soon as \( \gamma > \alpha \), since the number of sets \( F^j_k \neq \emptyset \) is \( O(e^{-\alpha}) \).

Assuming that \( \beta \) has property (**), we form \( \lambda = \lambda_j \), by omitting from \( F_k \) the intervals \( F^j_k \) of \( \mu \)-measure \( \geq \epsilon_j^\beta \). By Kolmogorov's estimate, \( \|S_T(\lambda_j)\|_1 \rightarrow 0 \), as \( j \rightarrow +\infty \) and \( T \rightarrow +\infty \), independently. Let now \( \int^* \) denote an integral over the domain \( |x - t| > \epsilon_j^a/2 \). Then
\[
\int^* |x - t|^{-1} \lambda_j(dt) = 0(e_j^{-\alpha}), \text{ if } \beta = 0,
\]
\[
\int^* |x - t|^{-1} \lambda_j(dt) = 0(e_j^{\beta - \alpha}) (\log \epsilon_j), \text{ if } 0 < \beta < \alpha.
\]
The first of these is obvious; the second is obtained by packing the subsets \( F^j_k \) as close to \( x \) as is consistent with the condition \( d(F_k, F_k) \geq \epsilon_j^\alpha \).

For each \( k \) such that \( \lambda_j(F^j_k) > \epsilon_j^\gamma \), we let \( \xi_k \) belong to \( F^j_k \) and consider the set defined by
\[
(S_k^j): \frac{1}{2} \lambda(F^j_k) \epsilon_j^\sigma < |x - \xi_k| < \lambda(F^j_k) \epsilon_j^\sigma,
\]
where \( \sigma = -\beta + 3\alpha/4 + 1/4 \), \( \tau = (1 + \gamma + \sigma)/2 \).

The number \( \lambda(F^j_k) \epsilon_j^\sigma \) lies between \( \epsilon_j^{\beta + \sigma} \) and \( \epsilon_j^{\gamma + \sigma} \); we note that \( \beta + \sigma > \alpha \), and \( \gamma + \sigma = 3/4 + \alpha/4 < 1 \). Moreover \( \epsilon_j^{-\tau} \epsilon_j = o(1) \), while \( \epsilon_j^{-\tau} \lambda(F^j_k) \epsilon_j^\sigma \rightarrow +\infty \).

For each \( k \) in question, the Lebesgue measure of \( S_k^j \) is asymptotically \( c\lambda(F^j_k) \epsilon_j^\sigma \), and the different sets are disjoint, because \( \lambda(F^j_k) \epsilon_j^\sigma = o(\epsilon_j^\sigma) \). We shall prove that \( |S_T(\lambda_j)| > c'\epsilon_j^{-\sigma} \) for a certain \( c' > 0 \), with \( T = \epsilon_j^{-\tau} \rightarrow +\infty \). This will prove that the total \( \mu \)-measure of the subsets \( F^j_k \), such that \( \epsilon_j^\gamma < \epsilon_j \leq \epsilon_j^\delta \), is \( o(1) \).

When \( x \in S_k^j \),
\[
|S_T(x) - \int_{F^j_k} D_T(x - t) \lambda(dt)| < \int_{F^j_k} |x - t|^{-1} \lambda(dt),
\]
and the error term on the right is \( o(\epsilon_j^{-\sigma}) \), because \( \sigma > \alpha - \beta \).
When $t \in F_k^l$, $t - \xi_k = o(x - \xi_k)$ because $\gamma + \sigma < 1$, and 
\[ \sin T(t - x) = \sin T(\xi_k - x) + o(1) \] because $\tau < 1$. This easily 
leads to the lower bound on $|S_T(x)|$.

Our construction is adapted from Kolmogorov's divergent Fourier 
series [31, Chapter VIII].

To complete our example, we must present a set $F$ that is also 
an $M_0$-set. This is known for various $M_0$-sets, but seems to occur 
explicitly in [1]: there exists a closed set $E \subseteq [0,1]$ and a sequence 
of integers $N_k \rightarrow +\infty$ such that

1. $|N_k x| < N_k^{-1} \pmod{1}$ for $x \in E$, $k \geq 1$,

2. The mapping $y = e^x$ transforms $E$ onto an $M_0$-set.

Then $y(E)$ is covered by intervals of length $\leq 2eN_k^{-2}$, whose 
distances are at least $(N_k^{-1} - 2N_k{-2})$.

In the remaining examples it is occasionally convenient to 
write $S_T(y)$ in place of $S_T(y, \mu)$, when $\mu = \mu_\theta$.

II. We present example (C) first, because (B) is based on an 
improvement in one of the inequalities used in (C). For each 
$n = 0, 1, 2, 3, \ldots$, $F_\theta$ is a union of $2^{n+1}$ sets $E_k$ of diameter 
$2\theta^{n+1}(1 - \theta)^{-1}$, and mutual distances at least 
$2\theta^{n+1}(1 - 2\theta) (1 - \theta)^{-1} \equiv c_1 \theta^{n+1} ; \mu(E_k) = 2^{-n-1}$.

The lower bound on the mutual distances gives a H"older condition 
on $\mu : \mu(B) \leq c_2 (\text{diam } B)^\alpha$, where $\alpha = - \log 2/\log \theta < 1$. 
If $\xi_k$ is the center of $E_k$, we have an identity 
$$ \int_{E_k} f(t) \mu(dt) = 2^{-n-1} \int f(\xi_k + \theta^{n+1} t) \mu(dt). $$

For each set $E_k$, we define the set $E_k^\sim$ by the inequality 
$d(x, E_k) < c_1 \theta^{n+1}/3$, so the sets $E_k^\sim$ have distances at least 
$2c_1 \theta^{n+1}/3$. If $x \in E_k^\sim$, then 
$$ |S_T(x, \mu) - \int_{E_k} D_T(x - t) \mu(dt)| < \int_{R - E_k} |x - t|^{-1} \mu(dt), $$
and in the last integral, $|x - t| \geq 2c_1 \theta^{n+1}/3$. Hence, by the 
H"older condition, the integral is $\leq c_3 (\theta^n)^{\alpha-1} = c_3 2^{-n} \theta^{-n}$. The 
principal term can be evaluated by the identity above, and simplified 
to the form $2^{-n} \theta^{-n-1} S_{T\theta^{n+1}}(\theta^{-n-1} x - \theta^{-n-1} \xi_k)$.
We observe that

$$\lim \int S_T(x, \mu) f(x) \, dx = \int f(x) \, \mu(dx),$$

for suitable test functions $f$; for example, this is true if $f$ and $f^\prime$ are integrable. Since $\mu$ is singular, we can find a test function $f$, such that $\|f\|_1 < 1$ and $|\int f(x) \, \mu(dx)| > 2c_3 + 2c_1^{-1}$. Hence

$$\max |D_T(\mu)| > 2c_3 + 2c_1^{-1} \text{ for large } T, \text{ say for } T > T_0.$$

Let $T > \theta^{-1} T_0$, and let $n \geq 0$ be chosen so that $T^* = \theta^{n+1} T$ satisfies the inequalities $T_0 < T^* < \theta^{-1} T_0$. Suppose that

$$|D_T(\theta^{-n-1} x - \theta^{-n-1} \xi_k)| > c_3 + c_1^{-1}.$$

Then $d(\theta^{-n-1} x - \theta^{-n-1} \xi_k, F_\theta) < c_1/3$, since $\pi > 3$, or

$$d(x, \xi_k + \theta^{n+1} F_\theta) < c_1 \theta^{n+1}/3, \text{ so } x \in E_\theta^\infty.$$ Hence

$$|D_T(x, \mu)| > c_3 \cdot 2^{-n-1} \theta^{-n-1} - c_3 2^{-n} \theta^{-n} = c_4 2^{-n} \theta^{-n}.$$

But it is easy to see that the set of $x$'s in question has measure at least $c_5 2^n \theta^n$, because $T_0 < T^* < \theta^{-1} T_0$, and the functions $D_T$ have derivatives bounded by $\theta^{-2} T_0^2$. Hence $\|D_T(\mu)\|_1^* > c_4 c_5$.

III. The example (B) requires a complicated construction, but relies in essence on small improvements on estimates already used. To estimate $S_T(\mu, x)$ we divide the range of integration into the subsets $\{|x-t| < T^{-1}\}$ and $\{|x-t| > T^{-1}\}$. The second yields an integral $O(T^{1-\alpha})$, by the Hölder condition, and the first yields $T \cdot O(T^{-\alpha}) = O(T^{1-\alpha})$ for the same reason (and the inequality $|D_T| < T$).

We give another estimate on $S_T(x, \mu)$ for large $T$, supposing that $\mu \in R$.

**Lemma 3.** — To each $\epsilon > 0$ there is a $T_0$ such that

$$|S_T(x, \mu)| < \epsilon d(x, F_\theta)^{-1}$$

whenever $T \geq T_0$ and $d = d(x, F_\theta) \geq \epsilon$.

**Proof.** — Let $\delta = d(x, F)$ and observe that

$$\delta S_T(x, \mu) = \pi^{-1} \int \sin T(x - t) \cdot \delta \cdot (x - t)^{-1} \mu(dt).$$

The function $g(t) = \delta \cdot (x - t)^{-1}$ is bounded by 1 on $F$, and
\[ |g(t_1) - g(t_2)| \leq \delta^{-1} |t_1 - t_2| \] for numbers \( t_1, t_2 \) in \( F_\theta \). Hence the conclusion follows from our assumption that \( \mu \in \mathbb{R} \) and the Tietze extension theorem.

The inequality of the Lemma can be written in a more useful way. When \( t \in F_\theta \), then \( |x - t| \leq d + 2 \leq d(1 + 2e^{-1}) \). Hence \( d(x,F_\theta)^{-1} \leq (1 + 2e^{-1}) \int |x - t|^{-1} \mu(dt) \). Suppose now that \( x \notin E_k \), so that \( d(\theta^{-n-1}x - \theta^{-n-1}x_k,F_0) \geq c_1 \theta^{n+1}/3 \). Using the identity for integrals over \( E_k \), we find the following estimate:

If \( x \notin E_k \) and \( T\theta^{n+1} > T_{00} \), then
\[
|\int_{E_k} D_T(x - t) \mu(dt)| < \epsilon \int_{E_k} |x - t|^{-1} \mu(dt).
\]

Consequently, when \( x \in E_k \) and \( T\theta^{n+1} \) is sufficiently large (depending on \( \epsilon > 0 \))
\[
|S_T(x,\mu) - 2^{-n-1} \theta^{-n-1} S_{T\theta^{n+1}}(\theta^{-n-1}x - \theta^{-n-1}x_k)| < \epsilon \theta^{n(\alpha-1)}.
\]

**Lemma 4.** — *To each \( \epsilon > 0 \) there is a \( \delta > 0 \) so that, when \( \theta^{-1} < \gamma < \delta T^{1-\alpha} \) then \( \gamma \{ |S_T(x,\mu)| > \gamma \} < \epsilon \).

**Proof.** — We choose \( n \geq 0 \) so that \( 1 < \theta^{n+1} Y^{1/1-\alpha} < \theta^{-1} \); this leads to the inequalities \( \theta^{n(\alpha-1)} > \gamma \), and \( T\theta^{n+1} > \delta^{-1} \). For fixed \( \ell \), we must estimate the Lebesgue measure of the set defined by
\[
|S_{T\theta^{n+1}}(\mu, \theta^{-n-1}x - \theta^{-n-1}x_k)| > \frac{1}{2} 2^{n+1} \theta^{n+1} \gamma.
\]

The right hand side exceeds \( \frac{1}{2} \gamma \); when \( T\theta^{n+1} \) is large, the measure of the set is at most \( \epsilon \theta^{n+1} \); the total for all \( \ell \) is at most \( \epsilon 2^{n+1} \theta^{n+1} < \epsilon Y^{-1} \). Hence \( \gamma \{ |S_T(x,\mu)| > \gamma \} < \epsilon \).

In view of the inequality \( |S_T(\mu,x)| = O(T^{1-\alpha}) \), the conclusion of the last lemma holds when \( Y > \delta^{-1} T^{1-\alpha} \), \( T > 1 \), for a certain \( \delta > 0 \).

In preparation for the next lemma, we recall the identity \((n = 1, 2, 3, \ldots)\)
\[
\int f(t) \mu(dt) \equiv 2^{-n} \sum_{k=1}^{2^n} \int f(\xi_k + \theta^n t) \mu(dt).
\]

We define \( \int f(t) \sigma_n(dt) \equiv 2^{-n} \sum_k \int f(\xi_k + \theta^{n+k} t) \mu(dt) \). Then
\( \sigma_n = g_n \cdot \mu \), where \( g_n \geq 0 \), \( g_n \) is continuous on \( F_\theta \) and takes the values 0 and \( 2^k (1 \leq k \leq 2^n) \). Using the formula for \( \sigma_n \) we get an identity

\[
S_T(x, \sigma_n) = 2^{-n} \theta^{-n} \sum_k \theta^{-k} S_{T \theta^n + k} (\theta^{-n-k} x - \theta^{-n-k} \xi_k).
\]

**Lemma 5.** To each \( \epsilon > 0 \), there is an \( N > 1 \) such that \( \limsup_{T \to +\infty} \| S_T(\sigma_n) \|^* < \epsilon \), if \( n \geq N \).

**Proof.** In calculating \( \limsup_{T \to +\infty} \| S_T(\sigma_n) \|^* \) we can omit \( x \)'s outside \( (-3,3) \), because \( \sigma_n \in \mathbb{R} \). In an obvious notation we write

\[
\sigma_n = \sum_k \sigma_{n,k},
\]

and observe that, for \( T > T_n, \epsilon \)

\[
|S_T(\sigma_n)| < \max_k |S_T(\sigma_{n,k})| + \epsilon/12.
\]

When \( Y > \epsilon/6 \) (the others are trivial, since we suppose that \( |x| < 6 \)),

\[
m\{ |S_T(\sigma_n)| > 2Y \} \leq \sum_k m\{ |S_T(\sigma_{n,k})| > Y \}
\]

\[
= \sum_k \theta^{n+k} m\{ |S_{T \theta^n + k}(x, \mu)| > 2^n \theta^{n+k} Y \}.
\]

Each summand is \( O(2^{-n} Y^{-1}) \) by Kolmogorov's inequality; if \( T \theta^{n+k} > 1 \), then the \( k \)-th term exceeds \( \epsilon 2^{-n} Y \) only if

\[
\delta (T \theta^{n+k})^{1-\alpha} < Y < \delta^{-1} (T \theta^{n+k})^{1-\alpha},
\]

by Lemma 4 and the remark after it, and this inequality occurs for at most \( 2(1 - \alpha)^{-1} \cdot \log \delta/\log \theta \) indices \( k = 1, \ldots, 2^n \). (We assume that \( Y > \theta^{-1} \), since \( S_T(\sigma_n) \to 0 \) almost everywhere as \( T \to +\infty \).) This proves our lemma.

A further property of \( \sigma_n \), obtained simply by increasing \( n \), is the inequality \( |\sigma_n(I) - \mu(I)| < \epsilon \) for all intervals \( I \).

The next lemma establishes a property of the functional \( \| \|_1^* \) to simplify the remaining calculations.

**Lemma 6.** Let \( a_i = \| f_i \|_1^* \), \( 1 \leq i \leq N \). Then

\[
\| \Sigma f_i \|_1^* \leq (\Sigma a_i^{1/2})^2.
\]

**Proof.** Let \( 0 \leq t_i \leq 1 \), and \( \Sigma t_i = 1 \). Then

\[
m\{ |\Sigma f_i| \geq Y \} \leq \Sigma m\{ |f_i| \geq t_i Y \} \leq \Sigma t_i^{-1} Y^{-1} a_i.
\]
The minimum of the sum is \( Y^{-1} \left( \sum a_i^{1/2} \right)^2 \). With a little more effort, we can obtain the bound \( c(1-p)^{-1} \left( \sum a_i^p \right)^{1/p} \), \( 0 < p < 1 \).

We are now in a position to construct the measure \( \lambda \). We shall find probability measures \( \lambda_k = f_k \mu \), with \( f_k \geq 0, \int f_k \, d\mu = 1 \), such that \( \|S_T(\lambda_k)\|_*^* < k^{-1} \) for \( T > T_k > T_{k-1} \ldots \) and \( |\hat{\lambda}_k(u)| < k^{-2} \) for \( u > T_k \). Lemma 5 provides \( \lambda_1 \); let us suppose that \( \lambda_k \) and \( T_k \) are known. We find \( \sigma_k \) so that \( |\sigma_k(I) - \lambda_k(I)| < k^{-1}(1+T_k)^{-2} \) and \( \|S_T(\sigma_k)\|_*^* < k^{-4}/25 \), and \( |\hat{\sigma}_k(u)| < k^{-1} \), for \( u > T_{k+1}^0 > T_k \). (The construction of \( f_{k+1} \mu \) from \( f_k \mu \) follows Lemma 5). We now set \( \lambda_{k+1} = (1 - k^{-1/2}) \lambda_k + k^{-1/2} \sigma_k \); by Lemma 6, we have for \( T > T_{k+1}^0 \)
\[
\|S_T(\lambda_{k+1})\|_*^* \leq (1 - k^{-1/2})^{1/2} k^{-1/2} + k^{-2}/5 .
\]
When \( k = 1 \), the last bound is \( 1/5 \), while \( (k + 1)^{-1} = \frac{1}{2} \). For \( k \geq 2 \), we need the inequality
\[
(1 - k^{-1/2})^{1/2} k^{-1/2} + k^{-2}/5 < (k + 1)^{-1/2} ,
\]
which can be verified with the aid of calculus. Clearly, we have \( |\hat{\lambda}_{k+1}(u)| < (k + 1)^{-2} \) for \( T > T_{k+1}^0 \); we take \( T_{k+1} = T_{k+1}^0 + T_{k+1}^0 \).

By the construction, and integration by parts, \( |\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leq k^{-3/2} (1 + T_k)^{-2} |u| \); consequently \( |\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leq k^{-3/2} \) unless \( |u| > 1 + T_k \). However, if \( |u| > T_{k+1} > T_k \), then \( |\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| < 2k^{-2} \). Since \( |\hat{\lambda}_k - \hat{\lambda}_{k+1}| < 2k^{-1/2} \), we have a limit \( \varphi(u) \), with
\[
|\varphi - \hat{\lambda}_k| = O(k^{-1/2}) .
\]
Hence \( \varphi = \hat{\lambda} \), with \( \lambda \) carried by \( F_\theta \) and \( \lambda \in \mathbb{R} \).

In verifying that \( \lim \|S_T(\lambda)\|_*^* = 0 \) we can calculate the weak norms over \((-3,3)\). Suppose that \( T_{k-1} \leq T \leq T_k \); then
\[
|S_T(\lambda_k) - S_T(\lambda)| = O(k^{-1/2}) .
\]
Since \( T \geq T_{k-1} \), \( \|S_T(\lambda_{k-1})\|_*^* < (k - 1)^{-1} \); and finally
\[
\|S_T(\lambda_k) - S_T(\lambda_{k-1})\|_*^* = O(k^{-1/2}) .
\]
Hence \( \|S_T(\lambda)\|_*^* = O(k^{-1/2}) \) over \((-3,3)\).
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Manuscrit reçu le 23 février 1981.

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