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THE CLASS OF CONVOLUTION OPERATORS
ON THE MARCINKIEWICZ SPACES

by Ka-Sing LAU (*)

1. Introduction.

Throughout the paper, the functions we consider will be complex valued, Borel measurable on $\mathbb{R}$. For $1 \leq p < \infty$, we will let

$$H^p \left( \mathbb{R}^n \right) = \left\{ f : \|f\|_{H^p} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} |f|^p \right)^{1/p} < \infty \right\}$$

and

$$\mathcal{V}^p = \left\{ g : \|g\|_{\mathcal{V}^p} = \lim_{\epsilon \to 0^+} \left( \frac{1}{2\epsilon} \int_{-\infty}^{\infty} |g(u + \epsilon) - g(u - \epsilon)|^p \, du \right)^{1/p} < \infty \right\}.$$ 

The space $H^p$ is called the Marcinkiewicz space. The space $\mathcal{V}^p$ was introduced by Hardy and Littlewood [3] in order to study the fractional derivatives and is called the integrated Lipschitz class. By identifying functions whose difference has zero norm, it was proved that both $H^p$ and $\mathcal{V}^p$ are Banach spaces [4], [8]. These spaces have also been studied in detail in [2], [3], [7], [10], [11], [12]. Let $W^p$ denote the class of functions $f$ in $H^p$ such that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f|^p$$

exists; then $W^p$ is a "non-linear" closed subspace of $H^p$. In [13], Wiener introduced the integrated Fourier transformation $g = W(f)$ of an $f$ in $W^2$ as

$$g(u) = \frac{1}{2\pi} \left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \right) f(x) \frac{e^{-ixu}}{-ix} \, dx$$

$$+ \frac{1}{2\pi} \int_{-1}^{1} f(x) \frac{e^{-ixu} - 1}{-ix} \, dx.$$  \hspace{1cm} (1.1)

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We call this transform the Wiener transformation. By using a deep Tauberian theorem, he showed that
\[ \| f \|_{\mathcal{W}_2} = \| W(f) \|_{\mathcal{W}_2}, \quad f \in \mathcal{W}_2. \]
Recently, this result has been extended by Lee and the author [8] to include the fact that the Wiener transformation \( W : \mathcal{W}_2 \to \mathcal{W}_2 \) is a surjective isomorphism. Moreover, the exact isomorphic constants have also been obtained. The theorem is an analog of the Plancherel theorem in the classical \( L^2 \) case. For \( 1 < p < 2, \frac{1}{p} + \frac{1}{p'} = 1 \), \( W \) also defines a bounded linear operator from \( \mathcal{M}^p \) into \( \mathcal{V}^{p'} \).

It is the purpose of this paper to study the convolution operators on the Marcinkiewicz space \( \mathcal{M}^p \), \( 1 \leq p < \infty \), and on the closed subspace \( \mathcal{M}_r^p \) of regular functions \( f \) (i.e.,
\[ \lim_{T \to \pm \infty} \frac{1}{T} \int_{-T}^{T} |f|^p = 0 \]
for \( a > 0 \)). Some results related to this subject can be found in [2], [14], [15].

In [2], Bertrandias showed that for each bounded regular Borel measure \( \mu \) on \( \mathbb{R} \), the convolution operator \( \Phi_\mu : \mathcal{M}^p \to \mathcal{V}^p \) given by \( \Phi_\mu(f) = \mu * f \) is well defined and \( \| \Phi_\mu \|_{\mathcal{M}^p} \leq \| \mu \| \).
In § 2, we show that if \( \mu \) satisfies \( \int |x| d|\mu| < \infty \), then the restriction map \( \Phi_\mu : \mathcal{M}_r^p \to \mathcal{M}_r^p \) satisfies
\[ \lim_{T \to \infty} \frac{1}{2T} \int_{\mathbb{R}} |(x_T \Phi_\mu - \Phi_\mu x_T) f|^p = 0, \]
where \( x_T \) is the characteristic function of \([ -T, T] \). This is used to prove that for any bounded regular Borel measure \( \mu \),
\[ \| \Phi_\mu \|_{\mathcal{M}_r^p} = \| \Phi_\mu \|_{L^p}, \]
where \( \| \Phi_\mu \|_{L^p} \) is the norm of the convolution operator \( \Phi_\mu \) on \( L^p (= L^p(\mathbb{R})) \) (Theorem 2.4).

Let \( \mathcal{S}_{\mathcal{M}_r^p} \) (\( \mathcal{S}_{L^p} \)) denote the norm closure of the family of convolution operators on \( \mathcal{M}_r^p \) (\( L^p \), respectively). It follows from the result mentioned above that \( \mathcal{S}_{\mathcal{M}_r^p} \) is isometrically isomorphic to \( \mathcal{S}_{L^p} \).
However, under the strong operator topologies, the structures of the two spaces are quite different. We prove that in $L^p$, the strong operator sequential convergence and the norm convergence coincide (Theorem 2.6).

In § 3, we consider the convolution operator under the Wiener transformation $W: L^p \rightarrow L^{p'}$, $1 < p < 2$. One of the difficulties in defining the multiplication operators on $L^p$ is that even for a very "nice" function $h$, the pointwise multiplication

$$(h \cdot g)(u) = h(u) \cdot g(u), \quad g \in L^p$$  \hspace{1cm} (1.2)$$
does not give a function in $L^p$. Let

$$D^{1/p} = \{ h : h(u + e) - h(u) = o(e^{1/p}) \text{ uniformly on } u \},$$
it is shown that if $g \in L^p \cap L^p$ and $h \in D^{1/p}$, then (1.2) defines a function in $L^p$. In [8, Theorem 3.3], it was proved that for each $g \in L^p$, there exists a $g' \in L^p \cap L^p$ such that $\|g - g'\|_{L^p} = 0$. Hence, for the above $h$, $h \cdot g$ can be defined to be the equivalence class in $L^p$ containing $h \cdot g'$ (defined by (1.2)) where $g' \in L^p \cap L^p$ and $\|g - g'\|_{L^p} = 0$. The main result of this section is that for $1 < p \leq 2$ and for any bounded regular Borel measure $\mu$ such that the Fourier-Stieltjes transformation $\hat{\mu}$ is in $D^{1/p}$, $\frac{1}{p} + \frac{1}{p'} = 1$, then $W$ yields

$$W(\mu * f) = \hat{\mu} \cdot W(f), \quad f \in L^p.$$  

In particular, if $\mu$ satisfies $\int_R |x| d|\mu| < \infty$, then $\hat{\mu} \in D^{1/p'}$ and the above equality holds.

In § 4, the results of § 3 are used to prove a Tauberian theorem on $M^2$. If $\mu$ is a bounded regular Borel measure on $R$ such that $\hat{\mu} \in D^{1/2}$ and $\hat{\mu}(u) \neq 0$ $\forall u \in R$, and if $f \in M^2$ satisfies

$$\|\mu * f\|_{M^2} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} |\mu * f|^2 \right)^{1/2} = 0,$$

then for any continuous measure $\nu \in M$ such that $\hat{\nu} \in D^{1/2}$,

$$\|\nu * f\|_{M^2} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} |\nu * f|^2 \right)^{1/2} = 0.$$  

This improves a result of Wiener [15, Theorem 29].
2. The Convolution Operators.

Let \( \mathcal{M}^p \), \( \mathcal{Y}^{-p} \) be defined as above. When there is no confusion, we will use the same notation \( f \in \mathcal{M}^p (\mathcal{Y}^{-p}) \) to denote the function \( f \) on \( \mathbb{R} \) as well as the equivalence class of functions in \( \mathcal{M}^p (\mathcal{Y}^{-p}) \), respectively whose difference from \( f \) has zero norm.

Let \( \Phi \) be a bounded linear operator from a Banach space \( X \) into \( X \) and let \( \| \Phi \|_X \) denote the norm of \( \Phi \) on \( X \).

**Proposition 2.1.** Let \( X \) be a closed subspace of \( \mathcal{M}^p \) such that \( L^p \subseteq X \) and let \( \Phi : X \rightarrow X \) be a linear map. Suppose \( \Phi \) satisfies the following conditions:

i) the restriction of \( \Phi \) on \( L^p \) defines a bounded linear operator \( \Phi : L^p \rightarrow L^p \),

ii) for each \( f \in X \), \( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |(X_T \Phi - X_T f) f|^p = 0 \).

Then \( \| \Phi \|_X \leq \| \Phi \|_{L^p} \).

**Proof.** Let \( f \in X \). Then
\[
\left( \frac{1}{2T} \int_{-T}^{T} |\Phi(f)|^p \right)^{1/p} \\
\leq \left( \frac{1}{2T} \int_{-T}^{T} |\Phi \cdot X_T f|^p \right)^{1/p} + \left( \frac{1}{2T} \int_{-T}^{T} |(X_T \Phi - X_T f) f|^p \right)^{1/p} \\
\leq \| \Phi \|_{L^p} \cdot \left( \frac{1}{2T} \int_{-T}^{T} |f|^p \right)^{1/p} + \left( \frac{1}{2T} \int_{-T}^{T} |(X_T \Phi - X_T f) f|^p \right)^{1/p}.
\]
Taking the limit supremum on \( T \) yields
\[
\| \Phi(f) \|_{L^p} \leq \| \Phi \|_{L^p} \cdot \| f \|_{L^p}
\]
and \( \| \Phi \|_X \leq \| \Phi \|_{L^p} \cdot \| f \|_{L^p} \).

Let \( \mathcal{M} \) be the class of bounded, regular Borel measures on \( \mathbb{R} \) and let \( \mathcal{M}_1 \) be the dense subspace of \( \mu \in \mathcal{M} \) such that
\[
\int_{\mathbb{R}} |x| d|\mu| < \infty.
\]
In [2, p. 19], Bertrandias showed that for each \( \mu \in \mathcal{M} \), the convolution operator \( \Phi_\mu : \mathcal{M}^p \rightarrow \mathcal{M}^p \) can be defined as the \( \mathcal{M}^p \)-limit
of the functions $\int_A^B f(x - y) d\mu(y)$ as $A, B \to \infty$, $f \in \mathcal{M}^p$. Since $\mathcal{M}^p \subset \mathcal{M}^1$ and

$$\frac{1}{2T} \int_{-T}^T \int_{-\infty}^\infty |f(x - y)| \, d|\mu|(y) \, dx$$

$$= \int_{-\infty}^\infty \frac{1}{2T} \int_{-T}^T |f(x - y)| \, dx \, d|\mu|(y) < \infty,$$

the integral $\int_{-\infty}^\infty f(x - y) d\mu(y)$ exists for almost all $x$. We can write the pointwise expression of $\Phi_\mu(f)$ as

$$\Phi_\mu(f)(x) = (\mu * f)(x) = \int_{-\infty}^\infty f(x - y) d\mu(y).$$

In the following, the convolution operators on the closed subspace $\mathcal{M}^p_r$ of regular functions $f$ (i.e. $\lim_{T \to \pm \infty} \frac{1}{T} \int_{-T}^T |f|^p = 0$ for $a > 0$) in $\mathcal{M}^p$ will be considered. Note that $f \in \mathcal{M}^p_r$ if and only if $\lim_{T \to \pm \infty} \frac{1}{T} \int_{-T}^{T+a} |f|^p = 0$. Also $\mathcal{W}^p \subset \mathcal{M}^p$. It is easy to show that if $\mu \in \mathcal{M}$, $f \in \mathcal{M}^p_r$, then $\mu * f \in \mathcal{M}^p_r$.

**Lemma 2.2.** Let $\mu \in \mathcal{M}_1$ and let $\Phi_\mu : \mathcal{M}^p_r \to \mathcal{M}^p_r$ be the convolution operator. Then $\Phi_\mu$ satisfies

$$\lim_{T \to \infty} \frac{1}{2T} \int_R |(\chi_T \Phi_\mu - \Phi_\mu \chi_T) f|^p = 0, \quad f \in \mathcal{M}^p_r.$$  

**Proof.** Let $f \in \mathcal{M}^p_r$ and let $\|\mu\| = 1$. For any $\varepsilon > 0$, there exists an $a > 0$ such that

$$\int_{R \setminus [-a,a]} |y| \, d|\mu| < \varepsilon$$

and a $T_0 > 1$ such that for $|T| > T_0$,

$$\frac{1}{2T} \int_{-T}^{T+a} |f|^p < \varepsilon$$

and for $T > T_0$,

$$\frac{1}{2T} \int_{-T}^{T} |f|^p \leq \|f\|^{p}_{\mathcal{M}^p} + \varepsilon.$$  

Now for $T > T_0$,  

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\[
\int_{-\infty}^{\infty} |(x_\alpha \Phi_\alpha - \Phi_\alpha x_\alpha) f|^p \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(x_\alpha(x) - x_\alpha(x - y)) f(x - y) d\mu(y)|^p dx \\
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(x_\alpha(x) - x_\alpha(x - y)) f(x - y)|^p d |\mu| (y) dx \\
= \int \int_E |f(x - y)|^p d |\mu| (y) dx
\]
where \( E = E_1 \cup E_2 \cup E_3 \cup E_4 \) with
\[
E_1 = \{ (x, y) : -T < x < T, \quad x + T < y \}, \\
E_2 = \{ (x, y) : -T < x < T, \quad y < x - T \}, \\
E_3 = \{ (x, y) : T < x, \quad x - T < y < x + T \},
\]
and
\[
E_4 = \{ (x, y) : x < -T, \quad x - T < y < x + T \}.
\]
On the region \( E_1 \), we have
\[
\int \int_{E_1} |f(x - y)|^p d |\mu| (y) dx \\
\leq \int_0^T \int_{-T}^{y - T} |f(x - y)|^p dx d |\mu| (y) \\
+ \int_T^{\infty} \int_{-T}^{T} |f(x - y)|^p dx d |\mu| (y) \\
\leq \left( \int_0^T d |\mu| \right) \left( \int_{-T - a}^{y - T} |f(z)|^p dz \right) \\
+ \int_a^{\infty} \int_{-(T + y)}^{T + y} |f(z)|^p dz d |\mu| (y) .
\]
This implies that
\[
\frac{1}{2T} \int \int_{E_1} |f(x - y)|^p d |\mu| (y) dx \\
\leq \varepsilon + (\| f \|_{\mathcal{A}^p}^p + \varepsilon) \int_a^{\infty} \frac{y + T}{T} d |\mu| (y) \\
\leq \varepsilon + 2 (\| f \|_{\mathcal{A}^p}^p + \varepsilon) \varepsilon . \tag{2.1}
\]
Similarly, we can show that the inequality (2.1) also holds for \( E_i \), \( i = 2,3,4 \). This completes the proof. \( \Box \)

It follows from Proposition 2.1 and Lemma 2.2 that
\[
\| \Phi \|_{\mathcal{A}^p_r} \leq \| \Phi \|_{L^p}.
\]
To obtain the reverse inequality, the following is required.
LEMMA 2.3. — Let $\mu \in M_1$ and let $f \in L^p$. For any $\varepsilon > 0$, there exists an $\widetilde{f} \in \mathcal{M}_r^p$ such that

1) $\| \widetilde{f} \|_{\mathcal{M}_r^p}^p \leq \| f \|_{L^p}^p + \varepsilon$,

2) $\| \mu * \widetilde{f} \|_{\mathcal{M}_r^p}^p \geq \| \mu * f \|_{L^p}^p$.

Proof. — Without loss of generality, we may assume that $\text{supp } f \subseteq [-A, A]$, $\text{supp } \mu \subseteq [-B, B]$ and $A, B > 1$. Let $C = A + B$, then $\text{supp}(\mu * f) \subseteq [-C, C]$.

Let $T_1 = C$ and let $f_1 = f$. Suppose that $T_{n-1}$, $f_{n-1}$ have been chosen, choose $T_n$ such that

$$T_n > T_{n-1} + 2nC, \quad \frac{T_n}{T_n + 2nC} \geq \left(1 - \frac{1}{n}\right)$$

and

$$\frac{1}{T_n - C} \int_0^{T_n} \left| \sum_{m=1}^{n-1} f_m \right|^p < \frac{\varepsilon}{2}.$$ 

Let

$$f_n = \frac{T_n}{n} \sum_{k=0}^{n-1} g_k,$$

where

$$g_k(x) = f(x - T_n - 2kC).$$

Since each $f_n$ is composed of $n$ disjoint copies of $f$ and all of the $f_n$'s are disjoint, it follows that the sequence $\{ \mu * f_n \}$ has the same property. Let

$$\widetilde{f} = 2^{1/p} \sum_{n=1}^{\infty} f_n.$$ 

To see that $\widetilde{f} \in \mathcal{M}_r^p$, observe that $\widetilde{f}$ is supported by

$$E = \bigcup_{n=1}^{\infty} [T_n - C, T_n + (2n - 1) C],$$

and that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} | \widetilde{f} |^p = \frac{1}{2T} \int_{T}^{T} | \widetilde{f} |^p.$$ 

If $n_0$ is such that

$$\frac{T_{n_0}}{T_{n_0} - C} \| f \|_{L^p}^p \leq \| f \|_{L^p}^p + \frac{\varepsilon}{2},$$

then for $n > n_0$ and for $T \in [T_n - C, T_n + (2n - 1) C]$,
\[ \frac{1}{2T} \int_{-T}^{T} |\tilde{f}|^p \leq \frac{1}{T} \int_{-T}^{T} \left| \sum_{m=1}^{n} f_m \right|^p \]
\[ \leq \frac{1}{T} \int_{-T}^{T} |f_n|^p + \frac{\varepsilon}{2} \]
\[ \leq \frac{Tn}{nT} \int_{-T}^{T} \sum_{k=0}^{n-1} |g_k|^p \, dx + \frac{\varepsilon}{2} \]
\[ \leq \frac{Tn}{Tn - C} \|f\|_{L^p}^p + \frac{\varepsilon}{2} \]
\[ \leq \|f\|_{L^p}^p + \varepsilon. \]

Moreover, for any \( T \) such that \( T_n - C \leq T \leq T_{n+1} - C \),
\[ \frac{1}{2T} \int_{-T}^{T+1} |\tilde{f}| \leq \frac{Tn}{nT} \|f\|_{L^p} \leq \frac{1}{n} \cdot \frac{Tn}{Tn - C} \|f\|_{L^p}. \]
Hence \( \tilde{f} \in \mathcal{M}^p \) and satisfies i). To prove ii), we let
\[ T = T_n + (2n - 1) C. \]
Then
\[ \frac{1}{2T} \int_{-T}^{T} |\mu \ast \tilde{f}|^p = \frac{1}{2T} \int_{-T}^{T} \left| \sum_{m=1}^{n} \mu \ast f_m \right|^p \]
\[ \geq \frac{1}{2T} \int_{-T}^{T} |\mu \ast f_n|^p \]
\[ \geq \frac{Tn}{Tn + (2n - 1) C} \|\mu \ast f\|_{L^p}. \]
This implies that
\[ \|\mu \ast \tilde{f}\|_{\mathcal{M}^p} \geq \|\mu \ast f\|_{L^p} \]
\[ \square \]

THEOREM 2.4. — Let \( 1 \leq p < \infty \) and let \( \mu \in \mathcal{M} \). Then the convolution operator \( \Phi_\mu : \mathcal{M}^p \rightarrow \mathcal{M}^p \) satisfies \( \|\Phi_\mu\|_{\mathcal{M}^p} = \|\Phi_\mu\|_{L^p} \).

\[ \square \]

Proof. — It follows from Proposition 2.1, Lemma 2.2 and Lemma 2.3 that \( \|\Phi_\mu\|_{\mathcal{M}^p} = \|\Phi_\mu\|_{L^p} \) for \( \mu \in \mathcal{M}_1 \). For \( \mu \in \mathcal{M} \), there exists a sequence \( \{\mu_n\} \) in \( \mathcal{M}_1 \) which converges to \( \mu \). Since
\[ \|\Phi_\mu - \Phi_{\mu_n}\|_{\mathcal{M}^p} \leq \|\Phi_\mu - \Phi_{\mu_n}\|_{L^p} \leq \|\mu - \mu_n\|, \]
it follows that
\[ \|\Phi_\mu\|_{\mathcal{M}^p} = \lim_{n \rightarrow \infty} \|\Phi_{\mu_n}\|_{\mathcal{M}^p} = \lim_{n \rightarrow \infty} \|\Phi_{\mu_n}\|_{L^p} = \|\Phi_\mu\|_{L^p}. \]
\[ \square \]
Let $\mathcal{J}_{r, p}$ denote the norm closure of the class of convolution operators on $\mathcal{M}_r^p(L^p)$, respectively. Theorem 2.4 implies that $\mathcal{J}_{r, p}$ and $\mathcal{J}_{L, p}$ are isometrically isomorphic. However, under the strong operator topologies, the two classes of operators are different (Theorem 2.6).

**Lemma 2.5.** — Let $\{\Phi_{\mu_n}\}$ be a sequence in $\mathcal{J}_{r, p}$. Suppose $\{\Phi_{\mu_n}\}$ converges to zero under the strong operator topology. Then $\{\Phi_{\mu_n}\}$ converges to zero under the norm topology.

**Proof.** — If the lemma were not true, then it follows from Theorem 2.4 and by passing to subsequence, we can assume that there exists a sequence $\{f_n\}$ in $L^p$ and an $a > 0$ such that

$$\|f_n\|_{L^p} = 1 \quad \text{and} \quad \|\mu_n \ast f_n\|_{L^p} > a \quad \forall n \in \mathbb{N}.$$  

We will construct an $\tilde{f} \in \mathcal{M}_r^p$ such that

$$\|\mu_n \ast \tilde{f}\|_{\mathcal{M}_r^p} > a \quad \forall n \in \mathbb{N}.$$  

This contradicts the hypothesis that $\{\Phi_{\mu_n}\}$ converges to zero under the strong operator topology.

Without loss of generality assume that for each $n$,

$$\text{supp } f_n \subseteq [-A_n, A_n], \quad \text{supp } \mu_n \subseteq [-B_n, B_n],$$  

and $\{A_n\}$, $\{B_n\}$ are increasing. Let $C_n = A_n + B_n$. In the following, we will define two sequences $\{T_n\}$ and $\{h_n\}$. Let $T_1 = C_1$, $h_1 = f_1$. Given $T_{n-1}$, $h_{n-1}$, choose $T_n$ such that

$$T_n > T_{n-1} + 2nC_n + C_n, \quad \frac{T_n}{T_n + (2n + 1)C_n} \geq 1 - \frac{1}{n}$$  

and

$$\frac{1}{T_n} \int_0^{T_n} \left| \sum_{m=1}^{n-1} h_m \right|^p < 1.$$  

Let

$$h_n(x) = \frac{T_n}{n} \sum_{k=1}^{n} f_k(x - T_n - 2(k - 1)C_n)$$  

and let

$$\tilde{f} = 2^{1/p} \sum_{n=1}^{\infty} h_n,$$  

then the same proof as in Lemma 2.3 shows that $\tilde{f} \in \mathcal{M}_r^p$ and

$$\|\mu_n \ast \tilde{f}\|_{\mathcal{M}_r^p} > a.$$  

$\square$
The following theorem follows immediately from Lemma 2.5.

**THEOREM 2.6.** — Let \( \mathcal{F}_p \) be the closure of the family of convolution operators on \( \mathcal{M}_p \). Then \( \mathcal{F}_p \) is a Banach algebra such that the strong operator sequential convergence and the norm convergence coincide.

Note that under the strong operator topology, \( \mathcal{F}_p \) is metrizable on bounded sets, hence Theorem 2.6 does not hold for \( \mathcal{F}_p \).

3. The Multipliers.

In this section, we will consider the convolution operator under the Wiener transformation. First, we will define the operators on \( \mathcal{V}_p \) of multiplying by scalar functions. We need the following proposition which was proved in [8].

**PROPOSITION 3.1.** — Let \( 1 < p < \infty \). Then for any \( g \in \mathcal{V}_p \), there exists a \( g' \in \mathcal{V}_p \cap L^p \) such that \( \| g - g' \|_{\mathcal{V}_p} = 0 \).

The proposition amounts to saying that by identifying functions whose difference has zero norm, each equivalence class has a representation in \( L^p \).

For each \( t \in \mathbb{R} \), we use \( \tau_t \) to denote the translation operator defined by

\[
(\tau_t g)(u) = g(t + u)
\]

where \( g \) is a function on \( \mathbb{R} \). For each \( g \in \mathcal{V}_p \), we can rewrite the definition of \( \| g \|_{\mathcal{V}_p} \) as

\[
\| g \|_{\mathcal{V}_p} = \lim_{\epsilon \to 0^+} (2\epsilon)^{-1/p} \| \tau_{\epsilon} g - \tau_{-\epsilon} g \|_{L^p} = \lim_{\epsilon \to 0^+} \epsilon^{-1/p} \| \tau_{\epsilon} g - g \|_{L^p}.
\]

Let \( \mathcal{D}^{1/p} \) be the class of bounded functions on \( \mathbb{R} \) such that

\[
h(u + \epsilon) - h(u) = o(\epsilon^{1/p})
\]

uniformly on \( u \). Let \( h \in \mathcal{D}^{1/p} \), let \( g \in \mathcal{V}_p \cap L^p \) and let \( h \cdot g \) be the pointwise multiplication of \( h \) and \( g \). Then

\[
e^{-1/p} \| \tau_{\epsilon}(h \cdot g) - h \cdot g \|_{L^p} \leq e^{-1/p} \| h \cdot (\tau_{\epsilon} g - g) \|_{L^p} + e^{-1/p} \| (\tau_{\epsilon} h - h) \cdot \tau_{\epsilon} g \|_{L^p}.
\]

(3.1)
Note that
\[
\lim_{e \to 0^+} e^{-1/p} \| (\tau_e h - h) \cdot \tau_e g \|_{L^p} = \| \lim_{e \to 0^+} e^{-1/p} (h - \tau_{-e} h) \cdot g \|_{L^p} \quad \text{(by the dominated convergence theorem)}
\]
\[
= 0. \quad (3.2)
\]

Hence, (3.1) and (3.2) imply
\[
\| h \cdot g \|_{\mathcal{Y}^p} \leq \| h \|_\infty \cdot \| g \|_{\mathcal{Y}^p}.
\]

It also follows from the above argument that if \( g \) and \( g' \) are in \( \mathcal{Y}^p \cap L^p \), then \( h \cdot g = h \cdot g' \) in \( \mathcal{Y}^p \). We define for \( h \in \mathcal{D}^{1/p} \) and for each \( g \in \mathcal{Y}^p \), the multiplication operator \( \Psi_h(g) \) to be the equivalence class in \( \mathcal{Y}^p \) containing \( h \cdot g' \) where \( g' \in \mathcal{Y}^p \cap L^p \) and \( \| g - g' \|_{\mathcal{Y}^p} = 0 \). We still use \( h \cdot g \) to denote \( \Psi_h(g) \).

Remark. – For an arbitrary \( g \in \mathcal{Y}^p \), the pointwise multiplication \( h \cdot g \) is not necessarily a function in \( \mathcal{Y}^p \). For example, let \( h(u) = e^{iu} \) and let \( g(u) = 1, \ u \in \mathbb{R} \), then the pointwise multiplication \( h \cdot g \) is not in \( \mathcal{Y}^p \).

Proposition 3.2. – Let \( 1 < p < \infty \) and let \( h \in \mathcal{D}^{1/p} \). Then the operator \( \Psi_h : \mathcal{Y}^p \to \mathcal{Y}^p \) defined above is a bounded linear operator with \( \| \Psi_h \|_{\mathcal{Y}^p} \leq \| h \|_\infty \). Moreover,
\[
\| \Psi_h(g) \|_{\mathcal{Y}^p} = \lim_{e \to 0^+} e^{-1/p} \| h \cdot (\tau_e g - g) \|_{L^p}.
\]

Proof. – We need only prove the last formula. The expressions (3.1) and (3.2) imply that
\[
\| \Psi_h(g) \|_{\mathcal{Y}^p} \leq \lim_{e \to 0^+} e^{-1/p} \| h \cdot (\tau_e g - g) \|_{L^p}.
\]

The reverse inequality is obtained by interchanging the first two terms of (3.1) and applying (3.2) again. \( \square \)

For each \( \mu \in M_1 \), it follows that
\[
\dot{\mu}'(u) = \lim_{e \to 0^+} \frac{\dot{\mu}(u + e) - \dot{\mu}(u)}{e} = \lim_{e \to 0^+} \frac{1}{e} \int_{-\infty}^{\infty} (e^{-i(u+e)x} - e^{-iux}) \, d\mu(x) \]
\[
= -i \int_{-\infty}^{\infty} e^{-iux} \cdot xd\mu(x).
\]

Hence, \( \dot{\mu}(u + e) - \dot{\mu}(u) = o(e^{1/p}) \) uniformly in \( u \), i.e. \( \dot{\mu} \in \mathcal{D}^{1/p} \).
COROLLARY 3.3. - Let $1 < p < \infty$ and let $\mu \in \mathbb{M}$ such that $\hat{\mu} \in \mathcal{D}^{1/p}$. Then the operator $\Psi_\mu : \gamma^p \rightarrow \gamma^p$ is a bounded linear operator with $\|\Psi_\mu\|_{y^p} \leq \|\hat{\mu}\|_{\infty}$. In particular, if $\mu \in M_1$, then $\mu$ satisfies the inequality.

Let $W$ be the Wiener transformation defined by (1.1).

THEOREM 3.4 [8]. - The Wiener transformation $W$ defines a bounded linear operator from $\mathcal{M}^p$ into $\gamma^p$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$.

In particular, if $p = 2$, then $W$ is an isomorphism from $\mathcal{M}^2$ onto $\gamma^2$ with

$$\|W\| = \left(\int_0^\infty h(x) \, dx\right)^{1/2}, \quad \|W^{-1}\| = \left(\max_{x > 0} \tilde{h}(x)\right)^{-1/2},$$

where

$$h(x) = \frac{2 \sin^2 x}{\pi x^2} \quad \text{and} \quad \tilde{h}(x) = \sup_{t > x} h(t), \quad x > 0.$$

LEMMA 3.5. - Let $1 < p < \infty$ and let $h \in \mathcal{D}^{1/p}$. Suppose $g \in \gamma^p$ and $g' \in \gamma^p \cap L^p$ are such that $\|g - g'\|_{\gamma^p} = 0$. Then

$$\lim_{\epsilon \to 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) - (\tau_\epsilon (h \cdot g') - \tau_{-\epsilon} (h \cdot g'))\|_{L^p} = 0$$

(where the involved multiplications are pointwise multiplication).

Proof. - Observe that

$$\lim_{\epsilon \to 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) - \tau_\epsilon(h \cdot g') - \tau_{-\epsilon} (h \cdot g')\|_{L^p} \leq \lim_{\epsilon \to 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon (g - g') - \tau_{-\epsilon} (g - g'))\|_{L^p} + \lim_{\epsilon \to 0^+} \epsilon^{-1/p} \| (\tau_{-\epsilon} h - h) \cdot \tau_{-\epsilon} g'\|_{L^p} + \lim_{\epsilon \to 0^+} \epsilon^{-1/p} \| (\tau_{-\epsilon} h - h) \cdot \tau_{-\epsilon} g'\|_{L^p}.$$

The first term is not greater than

$$\|h\|_\infty \lim_{\epsilon \to 0^+} \epsilon^{-1/p} \|\tau_\epsilon(g - g') - \tau_{-\epsilon} (g - g')\|_{L^p} \quad \text{which is equal to} \quad \|h\|_\infty \cdot \|g - g'\|_{\gamma^p} \quad \text{and by hypothesis, it equals}$$
zero. By an argument similar to (3.2), the second and the third term are also zero. This completes the proof of the lemma.

For an \( f \in L^p \), \( 1 < p \leq 2 \), we will use \( \hat{f} \) to denote the Fourier transformation of \( f \) in \( L^p' \). It is well known that for the above \( f \),
\[
\left( \int \left| \hat{f}(u) \right|^p \frac{du}{\sqrt{2\pi}} \right)^{1/p'} \leq \left( \int \left| f(x) \right|^p \frac{dx}{\sqrt{2\pi}} \right)^{1/p'}.
\]

**THEOREM 3.6.** Let \( 1 < p < 2, \frac{1}{p} + \frac{1}{p'} = 1 \). Then for any \( f \in M^p, \mu \in M \) such that \( \bar{\mu} \in \mathbb{Q}^{1/p'} \),
\[
W(\mu \ast f) = \bar{\mu} \cdot Wf \quad \text{in} \quad \nu^{p'}.
\]

**Proof.** First consider the case that \( \mu \) has bounded support, say, \( \text{supp} \mu \subseteq [-A, A] \). Without loss of generality assume that \( \| \mu \| = 1 \) and let
\[
W(f) = g \quad \text{and} \quad W(\mu \ast f) = g_1.
\]

In view of Lemma 3.5, it suffices to show that
\[
\lim_{\epsilon \to 0^+} \epsilon^{-1/p'} \left\| (\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) - \bar{\mu} \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) \right\|_{L^{p'}} = 0.
\]
Since \( (\tau_\epsilon g - \tau_{-\epsilon} g) \) is the Fourier transformation of
\[
h(x) = \sqrt{\frac{2}{\pi}} f(x) \frac{\sin \epsilon x}{x},
\]
it follows that \( (\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) \) is the Fourier transformation of
\[
h_1(x) = \sqrt{\frac{2}{\pi}} (\mu \ast f)(x) \frac{\sin \epsilon x}{x},
\]
and both \( h_1 \) and \( h \) are in \( L^p \) (cf. [8, Theorem 5.5]). Hence
\[
(2\epsilon)^{-1/p'} \left\| (\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) - \bar{\mu} \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) \right\|_{L^{p'}}
\]
\[
= (2\epsilon)^{-1/p'} \left\| (h_1 - h)^* \right\|_{L^{p'}}
\]
\[
= (2\epsilon)^{-1/p'} \left( \sqrt{2\pi} \int_{-\infty}^{\infty} |(h_1 - h)^*|^{p'} \frac{du}{\sqrt{2\pi}} \right)^{1/p'}
\]
\[
\leq (2\epsilon)^{-1/p'} (2\pi)^{1/2p'} \left( \int_{-\infty}^{\infty} |h_1 - h|^{p} \frac{du}{\sqrt{2\pi}} \right)^{1/p}
\]
\[
= \left( \frac{1}{\pi e^{p-1}} \int_{-\infty}^{\infty} \int_{-A}^{A} f(x-y) \left( \frac{\sin \epsilon x}{x} - \frac{\sin \epsilon (x-y)}{x-y} \right) d\mu(y) \right)^p dx \right)^{1/p}
\]
\begin{align*}
&\leq \left( \frac{1}{\pi e^{p-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-y)|^p \left| \frac{\sin \varepsilon x}{x} - \frac{\sin \varepsilon (x-y)}{x-y} \right| d|\mu|(y) \, dx \right)^{1/p} \\
&\leq \left( \frac{1}{\pi e^{p-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-y)|^p \left( \frac{8e |y|^p}{|x| + |y|} \right) d|\mu|(y) \, dx \right)^{1/p} \\
&\leq 8\pi^{-1/p} \cdot e^{1/p} \left( \int_{|y|<\lambda} \int_{-\infty}^{\infty} |f(x-y)|^p \frac{1}{|x|^p + 1} \, dx \, d|\mu|(y) \right) + \int_{1<|y|<\lambda} \left( \int_{-\infty}^{\infty} |f(x-y)|^p \frac{1}{|x|^p + 1} \, dx \right) |y|^p \, d|\mu|(y)
\end{align*}

The fact that $M^p \subseteq L^p \left( \mathbb{R}, \frac{dx}{|x|^p + 1} \right)$ [8, Proposition 2.1] implies that the last two terms of the above inequality are bounded. Hence

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-1/p'} \| (\tau_{\varepsilon} g_1 - \tau_{-\varepsilon} g_1) - \mu \cdot (\tau_{\varepsilon} g - \tau_{-\varepsilon} g) \|_{L^{p'}} = 0.$$ 

This completes the proof of the theorem for measures $\mu$ with bounded support. Now, for any $\mu \in M_1$, there exists a sequence of $\{\mu_n\}$ with bounded support such that $\|\mu_n - \mu\|_1 \to 0$ as $n \to \infty$. Corollary 3.3 implies

$$\|\Psi_{\mu_n} - \Psi_\mu\|_{y^{p'}} \leq \|\hat{\mu}_n - \hat{\mu}\|_{\infty} \leq \|\mu_n - \mu\|.$$ 

Hence

$$W(\mu * f) = \lim_{n \to \infty} W(\mu_n * f) = \lim_{n \to \infty} \hat{\mu}_n \cdot W(f) = \hat{\mu} \cdot W(f). \quad \Box$$ 

Let $\mu \in M$ and define the multiplication operator $\Psi_\mu : \mathcal{Y}^p \to \mathcal{Y}^p$ as the limit of $\Psi_{\mu_n}$, $\mu_n \in \mathcal{D}^{1/p}$.

**COROLLARY 3.7.** - Let $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. For each $\mu \in M$, let $\Phi_\mu$ be the convolution operator of $\mu$ on $M^p$ and let $\Psi_\mu$ be the multiplication operator on $\mathcal{Y}^p$. Then for any $f \in M^p$,

$$W(\Phi_\mu f) = \Psi_\mu (W(f)).$$ 

Let $\mathcal{Y}^2_r = W(M^2_r)$, then the following result follows from Theorem 2.4, Corollary 3.3, Theorem 3.4 and Theorem 3.6.

**COROLLARY 3.8.** - For each $\mu \in M$, we have

$$C^{-1} \| \Phi_\mu \|_{M^p_r} \leq \| \Psi_\mu \|_{M^2_r} \leq \| \Phi_\mu \|_{M^2_r} = \| \hat{\mu} \|_{\infty}$$

where $C = \| W \| \cdot \| W^{-1} \|$.
4. A Tauberian Theorem.

In [15, Theorem 29], Wiener proved a Tauberian theorem on $\mathcal{M}^2$. In this section, by making use of his idea and the results in the previous section, we can simplify his argument and extend the theorem.

**Lemma 4.1.** Let $\mu \in \mathcal{M}$ such that $\hat{\mu} \in \mathcal{D}^{1/2}$ and $\hat{\mu}(u) \neq 0$ for all $u \in \mathbb{R}$. If $f \in \mathcal{M}^2$ is such that $\|\mu \ast f\|_{\mathcal{M}^2} = 0$. Then $g = W(f)$ satisfies

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-C}^{C} |g(u + \epsilon) - g(u)|^2 \, du = 0 \quad \forall C > 0.$$

**Proof.** Since $\hat{\mu}$ is continuous and $\hat{\mu} \neq 0$, there exists a $Q > 0$ such that $|\hat{\mu}(u)| > Q$ for all $u \in [-C, C]$. Hence

$$\lim_{\epsilon \to 0^+} \frac{Q^2}{\epsilon} \int_{-C}^{C} |g(u + \epsilon) - g(u)|^2 \, du$$

$$\leq \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} |\hat{\mu}(u)|^2 |g(u + \epsilon) - g(u)|^2 \, du$$

$$= \|W(\mu \ast f)\|^2_{\mathcal{M}^2} \text{ (by Proposition 3.2 and Theorem 3.6)}$$

$$\leq \|W\|_{\mathcal{M}^2}^2 \cdot \|\mu \ast f\|_{\mathcal{M}^2}^2$$

$$= 0.$$ 

**Lemma 4.2.** Let $\nu$ be a continuous measure in $\mathcal{M}$ such that $\hat{\nu} \in \mathcal{D}^{1/2}$. Let $f \in \mathcal{M}^2$ and let $g = W(f)$. Then

$$\lim_{C \to \infty} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \int_{-C}^{C} + \int_{C}^{\infty} \right) |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0.$$

**Proof.** We will estimate the following limit:

$$\lim_{\eta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - e^{i\eta u} - 1 \right|^2 |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2.$$

Since $\nu$ is a continuous measure, $\lim_{|u| \to \infty} \hat{\nu}(u) = 0$. Also note that

$$\left| 1 - e^{i\eta u} - 1 \right|$$

is bounded, and for any $A > 0$,
\[
\lim_{\eta \to 0^+} \left| 1 - \frac{e^{i\eta u} - 1}{i u \eta} \right| = 0 \text{ uniformly for } u \in [-A, A].
\]

For \( \epsilon_0 > 0 \), there exists \( A_0 \) such that for \( A \geq A_0 \), \( \| \hat{\nu}(u) \| \leq \frac{\epsilon_0}{K_1} \) where \( K_1 (> 1) \) is the bound of \( \left| 1 - \frac{e^{i\eta u} - 1}{i u \eta} \right| \). There exists \( \eta_0 \) such that for \( 0 < \eta < \eta_0 \)

\[
\left| 1 - \frac{e^{i\eta u} - 1}{i u \eta} \right| < \frac{\epsilon_0}{K_2}, \quad u \in [-A_0, A_0],
\]

where \( K_2 (> 1) \) is a bound of \( \hat{\nu} \) in \([-A_0, A_0]\). Hence, for \( 0 < \eta < \eta_0 \),

\[
\left| 1 - \frac{e^{i\eta u} - 1}{i u \eta} \right| \cdot |\hat{\mu}(u)| < \epsilon_0, \quad u \in \mathbb{R},
\]

and

\[
\lim_{\eta \to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{i\eta u} - 1}{i u \eta} \right|^2 |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 \leq \epsilon_0 \| g \|_{y^2}.
\]

This implies

\[
\lim_{\eta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{i\eta u} - 1}{i u \eta} \right|^2 |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0.
\]

Since \( \left| 1 - \frac{e^{i\eta u} - 1}{i u \eta} \right| > \frac{1}{2} \) for any \( \eta > 4 \), we have

\[
\lim_{\eta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \int_{-\infty}^{-A_0} + \int_{A_0}^{\infty} \right) |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0. \quad \Box
\]

**Theorem 4.3.** Let \( \mu \in M \) such that \( \hat{\mu} \in \mathcal{D}^{1/2} \) and \( \hat{\mu}(u) \neq 0 \) for all \( u \) in \( \mathbb{R} \). Suppose \( f \in \mathcal{M}^2 \) satisfies

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\mu * f|^2 = 0.
\]

Then for any continuous measure \( \nu \in M \) such that \( \hat{\nu} \in \mathcal{D}^{1/2} \),

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\nu * f|^2 = 0.
\]

**Proof.** Lemma 4.1 implies that for any \( C > 0 \),

\[
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-C}^{C} |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0.
\]
Also by Lemma 4.2,
\[
\lim_{c \to \infty} \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( \int_{-\infty}^{-c} + \int_{c}^{\infty} \right) |\hat{\nu}(u)|^2 |g(u + \varepsilon) - g(u)|^2 = 0.
\]
This implies that \( \| \hat{\nu} \cdot g \|_{\mathcal{F}'} = 0 \). By Theorem 3.4 and Theorem 3.6, \( \| \nu * f \|_{\mathcal{F}'} = 0 \).

\[ \square \]

5. Some Remarks.

In Section 2, we proved that the convolution operator \( \Phi : \mathcal{M}^p \to \mathcal{M}^p \) satisfies \( \| \Phi_{\mu} \|_{\mathcal{M}^p} = \| \Phi_{\mu} \|_{L^p} \), we do not know whether or not \( \Phi_{\mu} : \mathcal{M}^p \to \mathcal{M}^p \) will satisfy the same equality.

An operator \( \Phi : L^p \to L^p \) is called a multiplier if \( \Phi \tau_t = \tau_t \Phi \) for \( t \in \mathbb{R} \). The relationship of multipliers and the equation \( \Phi(f) = h \cdot f \) for some bounded function \( h \) on \( \mathbb{R} \) is generally well known. Also, the class of multipliers on \( L^p \) equals the strong-operator closure of the class of convolution operators. However, nothing is known for the multipliers on \( \mathcal{M}^p \). It would be nice to have complete characterizations of the multiplier on \( \mathcal{M}^p \), especially on \( \mathcal{M}^2 \).

In Section 4, we can only prove the Tauberian theorem on \( \mathcal{M}^2 \) (Theorem 4.3). For \( 1 < p < 2 \), the Wiener transformation is well defined. All the proofs in Section 4 will go through except the last step in Theorem 4.3. It depends on the following statement which has to be justified:

For \( 1 < p < 2 \), the Wiener transformation \( W : \mathcal{M}^p \to \mathcal{F}^{p'} \) is one to one.

Note that the statement is true for the Fourier transformation from \( L^p \) to \( L^{p'} \), \( 1 < p < 2 \).

In our Tauberian Theorem, we have to assume that
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\mu * f|^2 = 0.
\]
We do not know whether the conclusion holds if we let $f \in \mathcal{W}^2$ and replace the zero by a positive number. Also, we do not know whether the condition on $\mu$ and $\nu$ in Theorem 4.3 can be relaxed.

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