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L^p-INEQUALITIES FOR THE LAPLACIAN AND UNIQUE CONTINUATION

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1. Introduction.

Unique continuation properties for solutions of partial differential equations or inequalities have been studied by various authors (see Hörmander [7], Chapter 8 for references). Let P, Q_1, \dots, Q_ν be partial differential operators in \mathbf{R}^n with constant coefficients, each of order less than or equal to m , and Ω an open connected subset of \mathbf{R}^n . We say that the differential inequality

$$|Pf(x)| \leq \sum_{j=1}^{\nu} |v_j(x)| |Q_j f(x)| \quad (1)$$

has (i) the unique continuation property in the class $H_{loc}^{m,p}(\Omega)$ if, whenever $f \in H_{loc}^{m,p}(\Omega)$ satisfies (1) (in the sense of distributions) and $f(x) = 0$ in some open, non-empty subset of Ω , one has $f \equiv 0$ on Ω , (ii) the weak unique continuation property if, whenever $f \in H^{m,p}(\Omega)$ satisfies (1) and $f(x) = 0$ in the complement of some compact subset of Ω , one has $f \equiv 0$. An important application of the weak unique continuation property concerns the proof of the non-existence of positive eigenvalues of self-adjoint Schrödinger operators, i.e. of partial differential operators of the form $-\Delta + v(x)$ in $L^2(\mathbf{R}^n)$, $n \geq 2$. We refer to [2,4] for details on this application.

Until very recently the coefficients v_j appearing in the differential inequalities under investigation were required to be locally in L^∞ . For second order operators this restriction has been relaxed in three recent papers by Berthier [2], Georgescu [4] and Schechter

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and Simon [8] to a condition of the type $v_j \in L_{loc}^w(\mathbb{R}^n)$ for suitable $w < \infty$. Berthier [2] uses analytic Fredholm theory in Hilbert space to obtain weak unique continuation for solutions of the Schrödinger equation with $v \in L_{loc}^w(\mathbb{R}^n)$ for $w > \max(n-2, n/2)$. Georgescu [4] proves generalizations of Hörmander inequalities between weighted Sobolev spaces; these imply unique continuation if the coefficients v_j of the first order derivatives are in $L_{loc}^{2n-1}(\mathbb{R}^n)$ and the coefficient v of the zero order term is in $L_{loc}^w(\mathbb{R}^n)$ with $w \geq \max(2, (2n-1)/3)$ (the second order term is $-\Delta$); the method is applicable to higher order operators. Schechter and Simon [8] use an inequality of the type

$$\| |x|^k f \|_p \leq c \| |x|^k \Delta f \|_q \quad (k = 0, \pm 1, \pm 2, \dots). \quad (2)$$

This is obtained by reduction to a corresponding one-dimensional inequality by expanding f in surface spherical harmonics, as was done in earlier publications where, however, only the case $p = q = 2$ was considered (e.g. Heinz [6]). The inequality (2) obtained in [8] implies unique continuation for Schrödinger operators if $v \in L_{loc}^w(\mathbb{R}^n)$ with $w > 1$ for $n = 1, 2$, $w > (2n-1)/3$ for $n = 3, 4, 5$ and $w \geq n-2$ for $n \geq 6$.

In the present paper we adopt the method of Schechter and Simon. Our principal result is a generalization of their basic inequality indicated above (Theorem 1.1 of [8], Theorem 1 and its Corollary in this paper). When applied to the problem of unique continuation for Schrödinger operators, our result improves those of [4] and [8] in 3 and 4 dimensions, in which we obtain the condition that is expected to be optimal; our condition for unique continuation is $v \in L_{loc}^w(\mathbb{R}^n)$ with $w > \max(n-2, n/2)$ ($w = n-2$ if $n \geq 5$).

The following lemma illustrates the relation between an inequality of the type (2) and unique continuation. Its proof will be indicated in Section 4. We denote by $B(R, x)$ the ball

$$B(R, x) = \{y \in \mathbb{R}^n \mid |y - x| < R\}.$$

LEMMA 1. — *Let P, Q_1, \dots, Q_p be partial differential operators with constant coefficients in \mathbb{R}^n , each of order less than or equal to m , and such that: if $G \subset \mathbb{R}^n$ is any open connected set, $f \in C^\infty(G)$, $Q_1 f = \dots = Q_p f = 0$ on G and f vanishes on an open, non-empty subset of G , then $f \equiv 0$. Suppose that there exist*

- i) a constant $c < \infty$, a number $R \in (0, \infty)$ and a subset Γ of \mathbf{R} having $+\infty$ as an accumulation point,
- ii) numbers $q, p_1, \dots, p_\nu \in [1, \infty]$ with $q \leq p_j$ for all j ,
- iii) a continuous, radial, strictly decreasing function $\varphi : B(R, 0) \setminus \{0\} \rightarrow \mathbf{R}$ such that, for all $f \in C_0^\infty(\mathbf{R}^n)$ having compact support in $B(R, 0) \setminus \{0\}$ and all $\kappa \in \Gamma$,

$$\sum_{j=1}^{\nu} \|e^{\kappa\varphi} Q_j f\|_{L^{p_j}(\mathbf{R}^n)} \leq c \|e^{\kappa\varphi} P f\|_{L^q(\mathbf{R}^n)}. \tag{3}$$

Let Ω be an open connected subset of \mathbf{R}^n and assume that $v_j \in L_{loc}^{w_j}(\Omega)$ ($j = 1, \dots, \nu$), where $1/w_j = 1/q - 1/p_j$. Then the differential inequality (1) has the unique continuation property in the class $H_{loc}^{m,q}(\Omega)$.

The organization of our paper is as follows. In Section 2 we deduce our basic inequality (Theorem 1) by reduction to a one-dimensional inequality. The latter will be proven in Section 3, and applications to unique continuation are given in Section 4. The following notations will be used : $\mathbf{R}_+ = (0, \infty)$ is the positive real half line, Δ the Laplacian in \mathbf{R}^n ($n \geq 2$) and $D = -id/dr$ (acting on functions of a real variable $r \in \mathbf{R}_+$). For $q \in [1, \infty]$, we denote by $q' = q/(q - 1)$ the conjugate exponent. $L^p(\Omega, \mathcal{B}; d\mu)$ denotes the L^p -space of functions from Ω to the Banach space \mathcal{B} . If $\mathcal{B} = \mathbf{C}$, we write $L^p(\Omega; d\mu)$, and if $d\mu$ is just Lebesgue measure, we write $L^p(\Omega, \mathcal{B})$. $H^{2,p}(\Omega)$ are the Sobolev spaces (in the terminology of Adams [1]), and $H_c^{2,p}(\Omega)$ is the subspace of $H^{2,p}(\Omega)$ of functions having compact support in Ω .

2. Some inequalities in L^p -spaces.

In this section we derive inequalities of the type (3) for the case where P is the Laplacian and Q_j the identity operator. As pointed out, the problem will be reduced to obtaining a similar inequality in one variable by expanding functions defined on \mathbf{R}^n ($n \geq 2$) in a series of surface spherical harmonics.

2.1. We first recall some facts about spherical coordinates in \mathbf{R}^n . Let S^{n-1} be the unit sphere in \mathbf{R}^n , σ_{n-1} its surface and Δ_S the

spherical Laplacian. We denote by the letter ω the points on S^{n-1} and by $d\omega$ the usual invariant measure on S^{n-1} induced by Lebesgue measure on \mathbf{R}^n ; the spaces $L^p(S^{n-1})$ are constructed with this measure. The restriction of $-\Delta_S$ to $C^\infty(S^{n-1})$ is essentially self-adjoint in $L^2(S^{n-1})$, and its closure $-\bar{\Delta}_S$ is a positive operator with purely discrete spectrum equal to $\{\ell(\ell + n - 2) \mid \ell = 0, 1, 2, \dots\}$. The dimension a_ℓ of the eigenprojection P_ℓ associated with the ℓ -th eigenvalue satisfies

$$c_n^{-1}(\ell + 1)^{n-2} \leq a_\ell \leq c_n(\ell + 1)^{n-2} \tag{4}$$

for some constant c_n . The elements of $P_\ell L^2(S^{n-1})$ coincide with the spherical harmonics of degree ℓ [9; p.138 ff.]. For each $\ell = 0, 1, 2, \dots$, we fix an orthonormal basis $\{Y_{\ell m}\}_{m=1}^{a_\ell}$ of the space $P_\ell L^2(S^{n-1})$.

Let $f : \mathbf{R}^n \rightarrow \mathbf{C}$. We denote by Uf the function defined on $\mathbf{R}_+ \times S^{n-1}$ by

$$(Uf)(r, \omega) = r^{1/2(n-1)} f(r\omega). \tag{5}$$

For sufficiently regular f one has

$$[U(-\Delta f)](r, \omega) = \left[-\frac{d^2}{dr^2} + r^{-2} \left(\frac{1}{4}(n-1)(n-3) - \Delta_S \right) \right] (Uf)(r, \omega). \tag{6}$$

For $f \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, we set

$$f_{\ell m}(r) = r^{1/2(n-1)} \int_{S^{(n-1)}} d\omega \overline{Y_{\ell m}(\omega)} f(r\omega), \quad r \in \mathbf{R}_+. \tag{7}$$

For fixed r and ℓ , we view the sequence

$$f_\ell(r) = \{f_{\ell 1}(r), f_{\ell 2}(r), \dots, f_{\ell a_\ell}(r), 0, 0, \dots\}$$

as a vector in the infinite-dimensional Hilbert space $\ell_+^2 \equiv \ell^2(\mathbf{Z}_+)$, and similarly for $Y_\ell(\omega) = \{Y_{\ell 1}(\omega), \dots, Y_{\ell a_\ell}(\omega), 0, 0, \dots\}$. The norm in ℓ_+^2 will be denoted by $\|\cdot\|$ and the scalar product between two vectors g_1 and g_2 in ℓ_+^2 by $g_1 \cdot g_2$. In this notation we then have

$$(Uf)(r, \omega) = \sum_{\ell=0}^\infty f_\ell(r) \cdot Y_\ell(\omega) \tag{8}$$

and

$$[U(-\Delta f)](r, \omega) = \sum_{\ell=0}^\infty [D^2 + \tilde{\ell}(\tilde{\ell} + 1)r^{-2}] f_\ell(r) \cdot Y_\ell(\omega), \tag{9}$$

where $\tilde{\ell} = \ell + \frac{1}{2}(n - 3)$ and the series are convergent at least in the $L^2(S^{n-1})$ sense for each $r \in \mathbf{R}_+$. The norm of $Y_\ell(\omega)$ in \mathcal{L}_+^2 is independent of ω and given by (see [9 ; Cor. IV.2.9])

$$\|Y_\ell(\omega)\| = a_\ell^{1/2} \sigma_{n-1}^{-1/2}. \tag{10}$$

2.2. Next we recall some inequalities proved by Schechter and Simon [8]. To each $g \in L^2(S^{n-1})$ we may associate as above a sequence $\{g_\ell\}_{\ell=0}^\infty$ of vectors in \mathcal{L}_+^2 such that $g_{\ell m} = 0$ for $m > a_\ell$ and

$$g_{\ell m} = \int_{S^{(n-1)}} d\omega \overline{Y_{\ell m}(\omega)} g(\omega). \quad (1 \leq m \leq a_\ell). \tag{11}$$

Clearly

$$\|g\|_{L^2(S^{n-1})}^2 = \sum_{\ell=0}^\infty \|g_\ell\|^2 = \sum_{\ell=0}^\infty \|a_\ell^{-1/2} g_\ell\|^2 a_\ell. \tag{12}$$

Also, (10) implies that

$$\sup_{\ell > 0} a_\ell^{-1/2} \|g_\ell\| \leq \sigma_{n-1}^{-1/2} \|g\|_{L^1(S^{n-1})}. \tag{13}$$

By using a vector-valued form of the Stein-Weiss interpolation theorem (e.g. [10 ; Ch. 1.18]) one obtains from (12) and (13) by interpolation that [8]

$$\left(\sum_{\ell=0}^\infty \|a_\ell^{-1/2} g_\ell\|^{q'} a_\ell \right)^{1/q'} \leq \sigma_{n-1}^{1/2-1/q} \|g\|_{L^q(S^{n-1})}, \tag{14}$$

for any $q \in [1, 2]$ and each $g \in L^q(S^{n-1})$, and that

$$\|h\|_{L^p(S^{n-1})} \leq \sigma_{n-1}^{1/p-1/2} \left(\sum_{\ell=0}^\infty \|a_\ell^{1/p'-1/2} h_\ell\|^{p'} \right)^{1/p'} \tag{15}$$

for any $p \in [2, \infty]$ and each $h \in L^p(S^{n-1})$.

2.3. We now show how an inequality of the type (3) in n dimensions can be obtained from a corresponding one-dimensional inequality. We set $S(a, b) = \{x \in \mathbf{R}^n \mid 0 \leq a < |x| < b \leq \infty\}$ and notice that

$$\|f\|_{L^p(S(a,b))} = \|r^{(n-1)/p} f\|_{L^p((a,b), L^p(S^{n-1}))}. \tag{16}$$

LEMMA 2. — Let $0 \leq a < b \leq \infty$, $1 \leq q \leq 2 \leq p < \infty$, $w = (1/q - 1/p)^{-1}$ and $\varphi, \psi : (a, b) \rightarrow \mathbf{R}$ continuous. Assume there is a sequence $\{\theta_\ell\}_{\ell=0}^\infty$ of non-negative numbers such that

$\Theta \equiv \left(\sum_{\ell=0}^{\infty} a_{\ell} \theta_{\ell}^w \right)^{1/w} < \infty$ and such that, for each $g : (a, b) \rightarrow \mathbf{C}^{a_{\ell}}$ of class C_0^{∞} and each ℓ :

$$\begin{aligned} & \| r^{(n-1)(1/p-1/2)} e^{\varphi} g \|_{L^p((a,b), \mathbf{C}^{a_{\ell}})} \\ & \leq \theta_{\ell} \| r^{(n-1)(1/q-1/2)} e^{\psi} [D^2 + \tilde{\ell}(\tilde{\ell} + 1) r^{-2}] g \|_{L^q((a,b), \mathbf{C}^{a_{\ell}})}, \end{aligned} \quad (17)$$

where $\tilde{\ell} = \ell + \frac{1}{2}(n-3)$. Then one has for each $f \in H_c^{2,q}(S(a,b))$:

$$\| e^{\varphi} f \|_{L^p(\mathbf{R}^n)} \leq \sigma_{n-1}^{-1/w} \Theta \| e^{\psi} \Delta f \|_{L^q(\mathbf{R}^n)}. \quad (18)$$

Proof. — We set $L_s = D^2 + s(s+1)r^{-2}$ and first assume that $f \in C_0^{\infty}(S(a,b))$. Then (18) is obtained by the following sequence of six inequalities, where we use successively: (1) the inequality (15), (2) Jessen's inequality ([3 ; VI.11.14] ; notice that $p' < p$), (3) the hypothesis (17), (4) the Hölder inequality (notice that $1/p' = 1/w + 1/q'$), (5) Jessen's inequality ($q' > q$) and (6) the inequality (14):

$$\begin{aligned} \| e^{\varphi} f \|_p &= \| r^{(n-1)(1/p-1/2)} e^{\varphi} \sum_{\ell=0}^{\infty} f_{\ell} \cdot Y_{\ell} \|_{L^p((a,b), L^p(S^{n-1}))} \\ &\leq \sigma_{n-1}^{1/p-1/2} \left\| \left(\sum_{\ell=0}^{\infty} \| a_{\ell}^{1/p'-1/2} r^{(n-1)(1/p-1/2)} e^{\varphi} f_{\ell} \|^{p'} \right)^{1/p'} \right\|_{L^p(a,b)} \\ &\leq \sigma_{n-1}^{1/p-1/2} \left(\sum_{\ell=0}^{\infty} \| a_{\ell}^{1/p'-1/2} r^{(n-1)(1/p-1/2)} e^{\varphi} f_{\ell} \|_{L^p((a,b), \mathbb{R}_+^2)}^{p'} \right)^{1/p'} \\ &\leq \sigma_{n-1}^{1/p-1/2} \left(\sum_{\ell=0}^{\infty} \| \theta_{\ell} a_{\ell}^{1/p'-1/2} r^{(n-1)(1/q-1/2)} e^{\psi} L_{\tilde{\ell}} f_{\ell} \|_{L^q((a,b), \mathbb{R}_+^2)}^{p'} \right)^{1/p'} \\ &\leq \sigma_{n-1}^{1/p-1/2} \left(\sum_{\ell=0}^{\infty} a_{\ell} \theta_{\ell}^w \right)^{1/w} \left(\sum_{\ell=0}^{\infty} \| a_{\ell}^{1/p'-1/2-1/w} r^{(n-1)(1/q-1/2)} e^{\psi} L_{\tilde{\ell}} f_{\ell} \|_{L^q((a,b), \mathbb{R}_+^2)}^{q'} \right)^{1/q'} \\ &\leq \sigma_{n-1}^{1/q-1/2-1/w} \Theta \left\| \left(\sum_{\ell=0}^{\infty} \| a_{\ell}^{1/q'-1/2} r^{(n-1)(1/q-1/2)} e^{\psi} L_{\tilde{\ell}} f_{\ell} \|^{q'} \right)^{1/q'} \right\|_{L^q(a,b)} \\ &\leq \sigma_{n-1}^{-1/w} \Theta \| r^{(n-1)(1/q-1/2)} e^{\psi} \sum_{\ell=0}^{\infty} L_{\tilde{\ell}} f_{\ell} \cdot Y_{\ell} \|_{L^q((a,b), L^q(S^{n-1}))} \\ &= \sigma_{n-1}^{-1/w} \Theta \| e^{\psi} \Delta f \|_q. \end{aligned}$$

The inequality (18) can now be extended from $C_0^{\infty}(S(a,b))$ to $H_c^{2,q}(S(a,b))$ by a density argument, which is given in a more general context in part (i) of the proof of Lemma 1 (Section 4). \square

2.4. The one-dimensional inequality (17) in Lemma 2 becomes particularly simple if one chooses φ of the form $\varphi(r) = \alpha \log r$, since then $\exp \varphi(r) = r^\alpha$. We therefore consider inequalities of the type

$$\|r^t f\|_{L^p(\mathbb{R}_+, \ell_+^2)} \leq c(s, t, \epsilon) \|r^{t+\epsilon} [D^2 + s(s+1)r^{-2}] f\|_{L^q(\mathbb{R}_+, \ell_+^2)},$$

where f is a ℓ_+^2 -valued function of class C_0^∞ . Our result on this is contained in the following proposition, the proof of which will be given in Section 3.

PROPOSITION 1. — Let $1 \leq q \leq p \leq \infty$, $1/w = 1/q - 1/p$ and $\epsilon = 2 - 1/w$. Let \mathcal{H} be a separable Hilbert space. Then for any $s, t \in \mathbb{R}$, $f: \mathbb{R}_+ \rightarrow \mathcal{H}$ of class $C_0^\infty(\mathbb{R}_+, \mathcal{H})$ we have

$$\|r^t f\|_{L^p(\mathbb{R}_+, \mathcal{H})} \leq (w')^{-1/w'} |2s + 1|^{-1/w} |t - s + 1/p|^{-1/w'} \cdot |t + s + 1 + 1/p|^{-1/w'} \|r^{t+\epsilon} [D^2 + s(s+1)r^{-2}] f\|_{L^q(\mathbb{R}_+, \mathcal{H})}. \quad (19)$$

For $s = -1/2$ one alternatively has

$$\|r^t f\|_{L^p(\mathbb{R}_+, \mathcal{H})} \leq 2^\epsilon e^{-1} (w')^{-1/w'} |t + 1/2 + 1/p|^{-\epsilon} \cdot \|r^{t+\epsilon} [D^2 + s(s+1)r^{-2}] f\|_{L^q(\mathbb{R}_+, \mathcal{H})}. \quad (20)$$

We now give the principal result of our paper.

THEOREM 1. — Let $1 \leq q \leq 2 \leq p < \infty$, $1/w = 1/q - 1/p$, $\mu = 2 - n/w$ and assume that $w > n/2$ (i.e. $\mu > 0$). Then one has for any $\tau \in \mathbb{R}$ and all $f \in H_c^{2,q}(\mathbb{R}^n \setminus \{0\})$:

$$\| |x|^\tau f \|_{L^p(\mathbb{R}^n)} \leq c(\tau) \| |x|^{\tau+\mu} \Delta f \|_{L^q(\mathbb{R}^n)}. \quad (21)$$

The constant $c(\tau)$ is finite provided that

$$(\tau - \ell + 2 - n/p') \cdot (\tau + \ell + n/p) \neq 0$$

for each $\ell = 0, 1, 2, \dots$, and it is given by

$$c(\tau) = \sigma_{n-1}^{-1/w} (w')^{-1/w'} \left[\sum_{\ell=0}^{\infty} \frac{a_\ell}{2\ell + n - 2} |(\tau - \ell + 2 - n/p')(\tau + \ell + n/p)|^{-w+1} \right]^{1/w}. \quad (22)$$

(For $n = 2$, the first term in the series (22) (i.e. $\ell = 0$) is infinite and must be replaced by $2^{2w-1} e^{-1} |\tau + 2/p|^{-2w+1}$. If $w = \infty$ (i.e. $p = q = 2$), one has instead of (22)

$$c(\tau) = \sup_{\ell > 0} |(\tau - \ell + 2 - n/2)(\tau + \ell + n/2)|^{-1}.$$

Proof. — This follows immediately from Lemma 2 and Proposition 1 by taking $\varphi(r) = \tau \log r$, $\psi(r) = (\tau + \mu) \log r$,

$$t = \tau + (n - 1)(1/p - 1/2), \quad \epsilon = 2 - 1/w, \quad s = \tilde{\chi} = \ell + 1/2(n - 3)$$

and noticing that $w/w' = w - 1$. The convergence of the series defining $c(\tau)$ follows from the estimate (4) for a_q and the condition $w > n/2$ which implies that $w - 1 > 1/2(n - 2)$. \square

COROLLARY. — Let $1 \leq q \leq 2 \leq p < \infty$, $1/w = 1/q - 1/p$, and assume $w > n/2$. Let $R < \infty$ and let $B(R, 0)$ be the ball $\{x \in \mathbf{R}^n \mid |x| < R\}$. Then one has for any $\tau \in \mathbf{R}$ and all $f \in H_c^{2,q}(B(R, 0) \setminus \{0\})$:

$$\| |x|^\tau f \|_{L^p(B(R,0))} \leq c(\tau) R^{2-n/w} \| |x|^\tau \Delta f \|_{L^q(B(R,0))}. \quad (23)$$

3. Proof of proposition 1.

In this section we prove Proposition 1. We begin with a preliminary result which is a slight extension of a lemma given in Hardy, Littlewood and Polya [5 ; No 319].

LEMMA 3. — Let $K : \mathbf{R}_+ \times \mathbf{R}_+ \longrightarrow \mathbf{C}$ be a homogeneous function of degree $-1/w'$, where $1 \leq w \leq \infty$ and $w' = w/(w - 1)$. Let \mathcal{H} be a Hilbert space and denote also by K the integral operator from $L^q(\mathbf{R}_+, \mathcal{H})$ to $L^p(\mathbf{R}_+, \mathcal{H})$ defined by

$$(Kf)(r) = \int_0^\infty K(r, u) f(u) du \quad (r \in \mathbf{R}_+). \quad (24)$$

If $1 \leq q \leq p \leq \infty$ and $q^{-1} - p^{-1} = w^{-1}$, then the norm of the operator K satisfies the inequality

$$\|K\|_{q \rightarrow p} \leq \left(\int_0^\infty r^{-1+w'/p} |K(r, 1)|^{w'} dr \right)^{1/w'}. \quad (25)$$

Proof. — If G is a locally compact abelian group, $d\gamma$ the Haar measure on G , then Young's inequality states that, if $1 \leq p, q, m \leq \infty$ and $p^{-1} = q^{-1} + m^{-1} - 1$,

$$\|k * g\|_{L^p(G, \mathcal{H}; d\gamma)} \leq \|k\|_{L^m(G; d\gamma)} \|g\|_{L^q(G, \mathcal{H}; d\gamma)}, \quad (26)$$

where

$$(k * g)(\gamma) = \int_G k(\gamma\gamma'^{-1})g(\gamma')d\gamma' \quad (\gamma, \gamma' \in G). \quad (27)$$

We apply this for the multiplicative group \mathbf{R}_+ , with Haar measure $r^{-1}dr$ ($dr =$ Lebesgue measure) and $k(r) = r^{1/p}K(r, 1)$. We obtain from (27) that

$$\begin{aligned} r^{1/p}(Kf)(r) &= r^{1/p} \int_0^\infty K(r, u)f(u)du \\ &= r^{1/p} \int_0^\infty u^{-1/w'}K\left(\frac{r}{u}, 1\right)f(u)du \\ &= r^{1/p} \int_0^\infty u^{1/w}\left(\frac{u}{r}\right)^{1/p}k(ru^{-1})f(u)\frac{du}{u} \\ &= [k * (u^{1/q}f)](r). \end{aligned} \quad (28)$$

Since $\|g\|_{L^p(\mathbf{R}_+, \mathscr{A}; dr)} = \|r^{1/p}g\|_{L^p(\mathbf{R}_+, \mathscr{A}; dr/r)}$, (28) and (26) imply that

$$\|Kf\|_{L^p(\mathbf{R}_+, \mathscr{A}; dr)} \leq \|k\|_{L^m(\mathbf{R}_+; dr/r)} \|f\|_{L^q(\mathbf{R}_+, \mathscr{A}; dr)},$$

with $m^{-1} = p^{-1} - q^{-1} + 1 = w'^{-1}$, i.e. $m = w'$. Inserting the definition of $k(r)$, we obtain (25). \square

Proof of Proposition 1. — Let $f \in C_0^\infty(\mathbf{R}_+, \mathscr{A})$. We define \hat{f} by $\hat{f}(r) = L_s f(r) = [D^2 + s(s + 1)r^{-2}]f(r)$. Integrating by parts, one finds that

$$-(2s + 1)f(r) = r^{s+1} \int_0^r u^{-s} \hat{f}(u) du + r^{-s} \int_r^\infty u^{s+1} \hat{f}(u) du. \quad (29)$$

Also, since $[D^2 + s(s + 1)r^{-2}]r^{-s} = [D^2 + s(s + 1)r^{-2}]r^{s+1} = 0$, one has

$$\int_0^\infty u^{-s} \hat{f}(u) du = \int_0^\infty u^{s+1} \hat{f}(u) du = 0. \quad (30)$$

We denote by χ_Δ the characteristic function of the set $\Delta \subset \mathbf{R}_+$ and introduce the following notations: $\kappa_+ = +1$, $\kappa_- = -1$, $\chi_+ = \chi_{[1, \infty)}$, $\chi_- = \chi_{(0, 1]}$ and

$$K_{\alpha\beta}(r, u) = \left(\frac{r}{u}\right)^{t+s+1} u^{1-\epsilon} \left[\kappa_\alpha \chi_\alpha\left(\frac{r}{u}\right) - \kappa_\beta \left(\frac{r}{u}\right)^{-2s-1} \chi_\beta\left(\frac{r}{u}\right) \right] \quad (31)$$

for $\alpha, \beta = +$ or $-$. In this notation, we find from (29) and (30) that $r^t f(r)$ may be expressed in either of the four following ways ($\alpha, \beta = +$ or $-$, $s \neq -1/2$).

$$r^t f(r) = -(2s + 1)^{-1} \int_0^\infty K_{\alpha\beta}(r, u) u^{t+\epsilon} \hat{f}(u) du. \quad (32)$$

Hence

$$\|r^t f\|_{L^p(\mathbb{R}_+, \mathcal{A})} \leq |2s + 1|^{-1} \|K_{\alpha\beta}\|_{q \rightarrow p} \|r^{t+\epsilon} \hat{f}\|_{L^q(\mathbb{R}_+, \mathcal{A})}. \quad (33)$$

In order to prove (19), it suffices to choose one of the four representations for $r^t f$ given in (32) (the choice will depend on the values of s, t and p) and to estimate the corresponding norm $\|K_{\alpha\beta}\|_{q \rightarrow p}$.

Each $K_{\alpha\beta}$ is homogeneous of degree $1 - \epsilon = -1 + 1/w = -1/w'$. One therefore gets from Lemma 3 that

$$\begin{aligned} & \|K_{\alpha\beta}\|_{q \rightarrow p} \\ & \leq \left(\int_0^\infty r^{w'(t+s+1+1/p)-1} |\kappa_\alpha \chi_\alpha(r) - \kappa_\beta r^{-2s-1} \chi_\beta(r)|^{w'} dr \right)^{1/w'}. \end{aligned} \quad (34)$$

A slightly weaker but more convenient inequality is obtained by using the fact that

$$|\kappa_\alpha \chi_\alpha(r) - \kappa_\beta r^{-2s-1} \chi_\beta(r)|^{w'} \leq |\chi_\alpha(r) - r^{-w'(2s+1)} \chi_\beta(r)| \quad (35)$$

(if $\alpha \neq \beta$, then $\chi_\alpha(r) \neq 0 \iff \chi_\beta(r) = 0$, so that (35) is evident; if $\alpha = \beta$, (35) follows from the inequality $|1 - \gamma|^\rho \leq |1 - \gamma^\rho|$ valid for $\gamma \geq 0$, $\rho \geq 1$). We then get

$$\begin{aligned} & \|K_{\alpha\beta}\|_{q \rightarrow p} \\ & \leq \left(\int_0^\infty |r^{w'(t+s+1+1/p)-1} \chi_\alpha(r) - r^{w'(t-s+1/p)-1} \chi_\beta(r)| dr \right)^{1/w'}. \end{aligned} \quad (36)$$

We now indicate how α and β must be chosen for given s, t and p in the order for the integral in (36) to be finite :

- i) if $t + 1/p < s$ and $t + 1/p < -s - 1$: $\alpha = \beta = +$,
- ii) if $t + 1/p < s$ and $t + 1/p > -s - 1$: $\alpha = -, \beta = +$,
- iii) if $t + 1/p > s$ and $t + 1/p < -s - 1$: $\alpha = +, \beta = -$,
- iv) if $t + 1/p > s$ and $t + 1/p > -s - 1$: $\alpha = \beta = -$.

The integral on the *r.h.s.* of (36) is easy to calculate. In all four cases (i) – (iv) one finds that it is equal to

$$(w')^{-1/w'} |2s + 1|^{1/w'} |t - s + 1/p|^{-1/w'} |t + s + 1 + 1/p|^{-1/w'} . \tag{37}$$

Inserting the estimate thus obtained for $\|K_{\alpha\beta}\|_{q \rightarrow p}$ into (33) and noticing that $-1 + 1/w' = -1/w$, one obtains (19).

The proof of (20) follows the same lines. Here one uses

$$-f(r) = r^{1/2} \log r \int_0^r u^{1/2} \hat{f}(u) du + r^{1/2} \int_r^\infty u^{1/2} \log u \hat{f}(u) du$$

and $(D^2 - r^{-2}/4) r^{1/2} = (D^2 - r^{-2}/4) r^{1/2} \log r = 0$. Since $s = -s - 1$, only the cases (i) and (iv), i.e. $\alpha = \beta$, are possible. The expression for $K_{\alpha\alpha}$ is now

$$K_{\alpha\alpha}(r, u) = \kappa_\alpha \left(\frac{r}{u}\right)^{t+1/2} u^{1-\epsilon} \log\left(\frac{r}{u}\right) \chi_\alpha\left(\frac{r}{u}\right).$$

By using the inequality $|\log z| \leq (e\delta)^{-1} z^{\pm\delta}$ for $z \lesssim 1$ respectively and any $\delta > 0$ and taking $\delta = 1/2 |t + 1/2 + 1/p|$ in the estimate of $\|K_{\alpha\alpha}\|_{q \rightarrow p}$, one arrives at (20). \square

Remark. – One may ask if the determination of the constants appearing in front of the norms on the r.h.s. of (19) and (21) is optimal. We have the following results about this: (a) if $1 \leq p = q < \infty$ (i.e. $w = \infty$ and $\epsilon = 2$), $s \neq -1/2$ and

$$(t - s + 1/p)(t + s + 1 + 1/p) \neq 0,$$

then the constant in (19) is optimal. This can be shown by using a result given in [9; § I.4.2]. (b) if $p = q = 2$, then the constant $c(\tau)$ in (21) is also optimal.

4. The unique continuation property.

We first give the proof of Lemma 1 and then a result about unique continuation for Schrödinger operators.

Proof of Lemma 1. – (i) We first show that the inequality (3) holds for each f in $H_c^{m,q}(B(R, 0) \setminus \{0\})$. By [1; Lemma 3.15], there is a $a \in (0, R)$ and a sequence $\{f_k\}$ in $C_0^\infty(S(a, R))$ converging to f in $H^{m,q}(R^n)$. Then, by (3),

$$\begin{aligned}
\sum_{j=1}^{\nu} \|Q_j(f_i - f_k)\|_{L^{p_j}(\mathbf{R}^n)} &\leq e^{-\kappa\varphi(R)} \sum_{j=1}^{\nu} \|e^{\kappa\varphi} Q_j(f_i - f_k)\|_{L^{p_j}(\mathbf{R}^n)} \\
&\leq e^{-\kappa\varphi(R)} e^{\kappa\varphi(a)} \|P(f_i - f_k)\|_{L^q(\mathbf{R}^n)} \\
&\leq e^{\kappa\varphi(a) - \kappa\varphi(R)} \|f_i - f_k\|_{H^{m,q}(\mathbf{R}^n)}.
\end{aligned}$$

Hence, for each j , $\{Q_j f_k\}_k$ is a Cauchy sequence in $L^{p_j}(\mathbf{R}^n)$. Its limit is $Q_j f$ (since $f_k \rightarrow f$ also in $\mathfrak{S}'(\mathbf{R}^n)$, hence $Q_j f_k \rightarrow Q_j f$ in $\mathfrak{S}'(\mathbf{R}^n)$). If one now writes the inequality (3) for f_k and lets k tend to infinity, one obtains (3) for the limit function f , since $e^{\kappa\varphi}$ is bounded on $S(a, b)$.

(ii) Assume that $f \in H_{\text{loc}}^{m,q}(\Omega)$ vanishes in an open neighbourhood U of some point $x_0 \in \Omega$. Denote by \bar{B}_a the ball $B_a = B(a, x_0)$. Choose ρ such that $0 < \rho < R$, $\bar{B}_\rho \subset \Omega$ and $c \|v_j\|_{L^{p_j}(B_\rho)} < 1$, where c is the constant appearing in (3). Let

$\delta \in (0, 1/2\rho)$ be such that $B_{2\delta} \subseteq U$. We claim that the hypotheses of the lemma imply that $f = 0$ on $B_{\rho-\delta}$. By connecting an arbitrary point $x \in \Omega$ with x_0 by a smooth curve in Ω , one can then deduce by a simple argument that $f(x) = 0$ at each $x \in \Omega$.

To verify our claim, let $\eta \in C_0^\infty(\Omega \cap B_R)$ be such that $\eta(x) = 1$ for $x \in B_\rho$, and set $g = \eta f$. We have $g \in H_c^{m,q}(B_R \setminus \{x_0\})$. Define φ_0 by $\varphi_0(x) = \varphi(x - x_0)$. By a change of variables, one deduces from the hypothesis (3) and (i) above that

$$\sum_{j=1}^{\nu} \|e^{\kappa\varphi_0} Q_j h\|_{L^{p_j}(\mathbf{R}^n)} \leq c \|e^{\kappa\varphi_0} P h\|_{L^q(\mathbf{R}^n)} \quad (38)$$

for all $h \in H_c^{m,q}(B_R \setminus \{x_0\})$, in particular for $h = g$.

From (38), (1) and the Hölder inequality we now obtain that

$$\begin{aligned}
\sum_{j=1}^{\nu} \|e^{\kappa\varphi_0} Q_j f\|_{L^{p_j}(B_\rho)} &\leq \sum_{j=1}^{\nu} \|e^{\kappa\varphi_0} Q_j g\|_{L^{p_j}(\Omega)} \\
&\leq c \|e^{\kappa\varphi_0} P g\|_{L^q(\Omega)} \\
&\leq c \|e^{\kappa\varphi_0} P f\|_{L^q(B_\rho)} + c \|e^{\kappa\varphi_0} P g\|_{L^q(\Omega \setminus B_\rho)}
\end{aligned}$$

$$\begin{aligned} &\leq c \sum_{j=1}^{\nu} \|v_j e^{\kappa\varphi_0} Q_j f\|_{L^q(B_\rho)} \\ &\qquad\qquad\qquad + c \|e^{\kappa\varphi_0} P g\|_{L^q(\Omega \setminus B_\rho)} \\ &\leq c \sum_{j=1}^{\nu} \|v_j\|_{L^{w_j}(B_\rho)} \|e^{\kappa\varphi_0} Q_j f\|_{L^{p_j}(B_\rho)} \\ &\qquad\qquad\qquad + c \|e^{\kappa\varphi_0} P g\|_{L^q(\Omega \setminus B_\rho)}. \end{aligned} \tag{39}$$

Let $\alpha_j = 1 - c \|v_j\|_{L^{w_j}(B_\rho)}$. Since φ is strictly decreasing, we obtain from (39) that

$$\sum_{j=1}^{\nu} \alpha_j \left\| \left(\frac{\exp \varphi_0}{\exp \varphi(\rho)} \right)^\kappa Q_j f \right\|_{L^{p_j}(B_\rho)} \leq c \|P g\|_{L^q(\Omega \setminus B_\rho)} < \infty.$$

Since $\alpha_j > 0$ and $[\exp \varphi(x)/\exp \varphi(\rho)]^\kappa \rightarrow +\infty$ for each $x \in B_\rho$ as $\kappa \rightarrow \infty$ in Γ , we must have $Q_j f = 0$ on B_ρ for each $j = 1, \dots, \nu$.

Now choose $\varphi \in C_0^\infty(B(1, 0))$ such that $\int \varphi(x) dx = 1$ and put $\varphi_\epsilon(x) = \epsilon^{-n} \varphi(\epsilon^{-1} x)$. For $0 < \epsilon < \delta$, consider the distribution f_ϵ on $B_{\rho-\delta}$ given by $f_\epsilon = \varphi_\epsilon * f$. Clearly $f_\epsilon \in C^\infty(B_{\rho-\delta})$, $f_\epsilon \rightarrow f$ in $\mathcal{D}'(B_{\rho-\delta})$ as $\epsilon \rightarrow 0$ and $f_{\epsilon|B_\delta} = 0$. Also $Q_j f_\epsilon = \varphi_\epsilon * Q_j f = 0$ on $B_{\rho-\delta}$ for each $j = 1, \dots, \nu$. It follows that $f_\epsilon = 0$ on $B_{\rho-\delta}$ by one of the hypotheses of the lemma, whence $f = 0$ on $B_{\rho-\delta}$. \square

THEOREM 2. — *Let Ω be an open connected subset of \mathbf{R}^n and $v \in L_{loc}^w(\Omega)$ with $w > n/2$ if $n = 2, 3, 4$ and $w \geq n - 2$ if $n \geq 5$. Then the differential inequality $|\Delta f(x)| \leq |v(x)| |f(x)|$ has the unique continuation property in $H_{loc}^{2,q}(\Omega)$, where $q = 1$ if $w \leq 2$ and $q = 2w/(w + 2)$ if $w \geq 2$.*

Proof. — We use Lemma 1 with $\varphi(r) = -\log r$, $q = 1$ if $w \leq 2$, $q = 2w/(w + 2)$ if $w \geq 2$ and $p = (1/q - 1/w)^{-1}$. We take κ of the form $\kappa = \kappa_m = n/p + 1/2 + m$, $m = 1, 2, 3, \dots$. The inequality (3) can be verified by using (23), with $\tau = -\kappa_m$. (23) requires that $w > n/2$. Furthermore, the constant c in (3) must

be independent of κ . Thus w must be such that $c(-\kappa_m) \leq c_0 < \infty$ for all m , where $c(\kappa)$ is given by (22). A necessary condition for this to hold is that $w \geq n - 2$, since terms with ℓ close to $\kappa_m - n/p$ in (22) are of the order $O(m^{(n-2-w)/w})$ as $m \rightarrow \infty$.

That the conditions $w \geq n - 2$ and $w > n/2$ are also sufficient may be seen by comparing the series in (22) to an integral. Indeed, using the inequality (4), one finds that

$$\begin{aligned} & \sum_{\ell=2}^{\infty} \frac{a_{\ell}}{2\ell + n - 2} |(-\kappa_m - \ell + 2 - n/p) (-\kappa_m + \ell + n/p)|^{-w+1} \\ & \leq k \int_{\Delta_1 \cup \Delta_2} u^{n-3} |(m + u + n - 1)(m - u)|^{-w+1} du \\ & = k m^{n-2w} \int_{\Delta'_1 \cup \Delta'_2} y^{n-3} |(1 + y + (n - 1)/m)(1 - y)|^{-w+1} dy, \end{aligned} \tag{40}$$

where k is a constant which is independent of m ,

$$\begin{aligned} \Delta_1 &= [1/2, m - 1/2], \quad \Delta_2 = [m + 1/2, \infty), \\ \Delta'_1 &= [1/(2m), 1 - 1/(2m)] \quad \text{and} \quad \Delta'_2 = [1 + 1/(2m), \infty). \end{aligned}$$

For $w \neq 2$, the term on the *r.h.s.* of (40) is bounded by

$$k m^{n-2w} c_{n,w} (1 + m^{w-2} + \delta_{n_2} \log m),$$

which is $O(1)$ as $m \rightarrow \infty$ provided that $w > n/2$ and $n - w - 2 \leq 0$. The terms with $\ell = 0$ and $\ell = 1$ in the series (22) are $O(1)$ or $o(1)$ for each $w \geq 1$. □

Remark. — In the case $n = 3$, Theorem 2 says that the inequality $|\Delta f| \leq |v| |f|$ has the unique continuation property in the class $H_{loc}^{2,1}(\Omega)$ if $v \in L_{loc}^w(\Omega)$ for some $w > 3/2$. It is important that we succeeded to prove this in the class $H_{loc}^{2,1}$ and not only in $H_{loc}^{2,2}$ for example. In fact, suppose v is in $L_{loc}^w(\mathbb{R}^3)$ with $w > 3/2$ and satisfies suitable conditions at infinity. Then one can define the self-adjoint operator $-\Delta + v$ in $L^2(\mathbb{R}^3)$ as a sum of quadratic forms. If $f \in L^2(\mathbb{R}^3)$ is an eigenvector of this self-adjoint operator, then one will have $f \in H^{1,2}(\mathbb{R}^3)$, and nothing more in general ($H^{1,2}$ is identical with the form domain of $-\Delta + v$). By Sobolev inequalities, $H^{1,2}(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, so that $f \in L^6(\mathbb{R}^3)$. Then, by the Hölder inequality, $vf \in L_{loc}^q(\mathbb{R}^3)$ for some $q > 6/5$, and $q \rightarrow 6/5$ when $w \rightarrow 3/2$. It follows that $\Delta f \in L_{loc}^q(\mathbb{R}^3)$

(because $(-\Delta + v)f = \lambda f$, $\lambda \in \mathbf{R}$, implies that $|\Delta f| = |(v - \lambda)f|$). Hence $f \in H_{\text{loc}}^{2,q}(\mathbf{R}^3)$ for some $q > 6/5$, and, if $w \rightarrow 3/2$, then $q \rightarrow 6/5$. This shows that one cannot suppose more than $f \in H_{\text{loc}}^{2,6/5}(\mathbf{R}^3)$. In conclusion, if one wants to apply a unique continuation property to the problem of non-existence of positive eigenvalues of $-\Delta + v$ in $n = 3$ dimensions, one must have this property at least in the class $H_{\text{loc}}^{2,6/5}(\mathbf{R}^3)$.

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