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<http://www.numdam.org/item?id=AIF_1981__31_3_147_0>
SPHERICAL SUMMATION:
A PROBLEM OF E. M. STEIN

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In this paper we present a proof of a conjecture formulated by E.M. Stein [1], page 5, about the spherical summation operators. We obtain a stronger version of the Carleson-Sjölin theorem [2] and, as a corollary, we obtain a.e. convergence for lacunary Bochner-Riesz means.

With \( \lambda > 0 \) let \( T^\lambda_R \) denote the Fourier multiplier operator given by

\[
(T^\lambda_R f)(\xi) = (1 - |\xi|^2/R^2)^\lambda \hat{f}(\xi) \quad \text{for} \quad f \in \mathcal{S}(\mathbb{R}^2),
\]

and let \( \{R_j\} \) be any sequence of positive numbers.

**Theorem 1.** — Given \( \lambda > 0 \) and \( \frac{4}{3 + 2\lambda} < p < \frac{4}{1 - 2\lambda} \) there exists some positive constant \( C_{\lambda,p} \) such that

\[
\left\| \sum_j |T^\lambda_{R_j} f_j|^2 \right\|^{1/2}_p \leq C_{\lambda,p} \left\| \sum_j |f_j|^2 \right\|^{1/2}_p .
\]

Let \( T_* f = \sup_j |T^\lambda_{2^j} f| \). The methods developed to prove Theorem 1 yield, as an easy consequence, the following result.

**Theorem 2.** — For \( \lambda > 0 \) and \( \frac{4}{3 + 2\lambda} < p < \frac{4}{1 - 2\lambda} \) there exists some constant \( C'_{\lambda,p} \) such that

\[
\| T_* f \|_p \leq C'_{\lambda,p} \| f \|_p .
\]
As a result we have, for \( f \in L^p(\mathbb{R}^2) \)
\[
f(x) = \lim_{j \to \infty} T^{2j}_f(x) \quad \text{for a.e. } x \in \mathbb{R}^2.
\]

As part of the machinery in the proofs of Theorems 1 and 2 we shall make use of the two following results, whose proofs can be found in [3] and [4].

Given a real number \( N > 1 \) consider the family \( B \) of all rectangles with eccentricity \( N \) and arbitrary direction, and let \( M \) be the associated maximal operator
\[
Mf(x) = \sup_{x \in \mathbb{R} \in B} \frac{1}{|R|} \int_R |f(x)| \, dx.
\]

**Theorem 3.** — There exist constants \( C, \alpha \) independent of \( N \) such that
\[
\|Mf\|_2 \leq C |\log N|^{\alpha} \|f\|_2.
\]

Consider a disjoint covering of \( \mathbb{R}^n \) by a lattice of congruent parallelepipeds \( \{Q_{\nu}\}_{\nu \in \mathbb{Z}^n} \) and the associated multiplier operators
\[
(P_{\nu} f)^\wedge = \chi_{Q_{\nu}} \hat{f}.
\]

**Theorem 4.** — For each \( s > 1 \) there exists a constant \( C_s \) such that, for every non negative, locally integrable function \( \omega \) and every \( f \in \mathcal{S}(\mathbb{R}^n) \) we have
\[
\int_{\mathbb{R}^n} \sum_{\nu} |P_{\nu} f(x)|^2 \omega(x) \, dx \leq C_s \int_{\mathbb{R}^n} |f(x)|^2 A_s \omega(x) \, dx
\]
where \( A_s g = [M(g^s)]^{1/s} \) and \( M \) denotes the strong maximal function in \( \mathbb{R}^n \).

**Proof of Theorem 1.** — Suppose that \( \phi : \mathbb{R} \to \mathbb{R} \) is a smooth function supported in \([-1, +1]\), and consider the family of multipliers \( S^\delta_f \) defined by
\[
(S^\delta_f)^\wedge (\xi) = \phi(\delta^{-1}(R_f^{-1} |\xi| - 1)) \hat{f}(\xi)
\]
and also, for a fixed \( \delta > 0 \), consider the family
\[
(T^n_f)^\wedge (\xi) = \psi_n(\arg(\xi)) (S^\delta_f)^\wedge (\xi)
\]
where the \( \psi_n \) are a smooth partition of the unity on the circle,
\[ 1 = \sum_{n=1}^{N} \psi_n; \]

\( \psi_n \) is supported on \( \left| \frac{N}{2\pi} \theta - n \right| \leq 1 \) and \( N = [\delta^{-1/2}] \), so that the support of \((T_j^n f)^\sim\) is much like a rectangle with dimensions \( R_j \delta \times R_j \delta^{1/2} \).

There are three main steps in our proof.

a) The same argument of ref. [3] allows us to reduce theorem 1 to prove the following inequality

\[ \left\| \left\| \sum_{j} \left| S_j f_j \right|^2 \right|^{1/2} \right\|_4 \leq C \left( \log \delta \right)^\beta \left\| \sum_{j} \left| f_j \right|^2 \right|^{1/2} \right\|_4. \]  \hspace{1cm} (1)

b) With adequate decompositions of the multipliers and geometric arguments, we prove

\[ \left\| \left\| \sum_{j} \left| S_j^\delta f_j \right|^2 \right|^{1/2} \right\|_4 \leq C' \left( \log \delta \right) \left\| \sum_{j,n} \left| T_j^n f_j \right|^2 \right|^{1/2} \right\|_4. \]  \hspace{1cm} (2)

c) An estimate of the kernels of \( T_j^n \), together with theorems 3 and 4 yields,

\[ \left\| \left\| \sum_{j,n} \left| T_j^n f_j \right|^2 \right|^{1/2} \right\|_4 \leq C'' \left( \log \delta \right)^\alpha \left\| \sum_{j} \left| f_j \right|^2 \right|^{1/2} \right\|_4. \]  \hspace{1cm} (3)

We refer to [3] for a) and begin with part b).

Fixed \( \delta > 0 \), we select just one dyadic interval \( 2^k < R \leq 2^{k+1} \) out of each \( \log_2 \delta \) correlative intervals, and we allow in the left hand side of (2) only those indices \( j \) for which \( R_j \) lays in a selected interval. Also we only take one \( T_j^n \) for each 4 correlative indices \( n \), and only those supported in the angular sector \( |\sin \theta| \leq 1/2 \). All these operations will contribute with the factor \( 24 \left( \log_2 \delta \right) \) to the inequality (2).

The left hand side of (2) is less than the 4th rooth of twice

\[ \sum_{R_j < R_k} \int \left| \left( \sum_n T_j^n f_j \right) \left( \sum_m T_k^m f_k \right) \right|^2 \]  \hspace{1cm} (4)

and now we only have two kinds of pairs \((j,k)\) : either \( R_j < R_k \leq 2R_j \) or \( R_j \leq \delta R_k \). Let's denote \( \Sigma^I \) and \( \Sigma^II \) the two corresponding halves of (4). We have
\[ \Sigma^1 = \int \left| \sum_{n,m} \left( T^n_j f_j \right)^* (T_k^m f_k)^* \right|^2 \leq 4 \Sigma^1 \int \left| \sum_{n \leq m} \left( T^n_j f_j \right)^* (T_k^m f_k)^* \right|^2. \]

Now an easy geometric argument shows that, for fixed \( j, k \), the supports of \( \left( T^n_j f_j \right)^* (T_k^m f_k)^* \) are disjoint for different pairs \( n \leq m \), so that we have

\[ \Sigma^1 \leq 4 \int \Sigma^1 \sum_{n \leq m} |(T^n_j f_j)^* (T_k^m f_k)^*|^2 \leq 4 A \quad (5) \]

with

\[ A = \left\| \sum_{j} \left| T^n_j f_j \right|^2 \right\|^{1/2} \left\| T^n_j f_j \right\|^{4}. \]

For the pairs \( (j, k) \) in \( \Sigma^{11} \) we have

\[ \Phi = \text{supp} \left| (T^n_j f_j)^* (T_k^m f_k)^* \right| \cap \text{supp} \left| (T^n_j f_j)^* (T_k^m f_k)^* \right| \]

if \( m_1 \neq m_2 \), because \( R_j \leq \delta R_k \), so that

\[ \Sigma^{11} = \int \sum_{n} \left| \left( \sum_{j} T^n_j f_j \right) T_k^m f_k \right|^2 \leq \left( \int \left( \sum_{j} \left| T^n_j f_j \right|^2 \right)^2 \right)^{1/2} \left( \int \left( \sum_{j} \left| T_k^m f_k \right|^2 \right)^2 \right)^{1/2} \leq \sqrt{2} |\Sigma^1| + |\Sigma^{11}|^{1/2} A^{1/2}. \quad (6) \]

From (5) and (6) we obtain (2).

Now we come into part c). First we observe that for each fixed \( j \) it is possible to choose two grids of parallelepipeds as the one in theorem 3 and such that each of the multipliers \( T^n_j \) is supported within one of the parallelepipeds, let's call it \( Q^n_j \). If \( (P_j^n f)^* = \chi_{Q^n_j} \hat{f} \) is the corresponding multiplier operator, we have

\[ T^n_j f_j = T^n_j P^n_j f_j. \]

Furthermore, an integration by parts arguments shows that each of the kernels of the \( T^n_j \) is majorized by a sum

\[ C \sum_{\nu=0}^{\infty} 2^{-\nu} \frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} \]

where the \( R^n_{\nu,j} \) are rectangles with dimensions \( 2^\nu \delta^{-1} \times 2^\nu \delta^{-1/2} \) and \( C \) is independent of \( n, j \) or \( \delta > 0 \). Therefore in order to
estimate A we only have to estimate uniformly in \( \nu \) the \( L^4 \)-norm of
\[
\left| \sum_{j,n} \frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} * (P^n_j f_j) \right|^2 \overset{1/2}.
\]

Or, what amounts to the same, the \( L^2 \)-norm of its square. If \( \omega > 0 \) is in \( L^2(\mathbb{R}^2) \) we have
\[
\sum_{j,n} \int \frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} * (P^n_j f_j) (x) \, d\omega(x) \, dx 
\leq \sum_{j,n} \int |P^n_j f_j(\nu)|^2 \left[ \frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} * \omega \right] (\nu) \, d\nu 
\leq \sum_{j,n} \int |P^n_j f_j(\nu)|^2 M \omega(\nu) \, d\nu 
\leq 2 C_s \sum_j \int |f_j(\nu)|^2 A_4 (M \omega)(\nu) \, d\nu 
\leq C_s \left\| \sum_j |f_j|^2 \right\|_{L^4}^2 \left\| M \omega \right\|_2 
\leq C |\log \delta|^\alpha \left\| \sum_j |f_j|^2 \right\|_{L^4}^{1/2} \left\| \omega \right\|_2,
\]
by successive applications of theorems 4 and 3. This estimate proves (3).

**Proof of Theorem 2.** — With the same notations of the preceding proof, let now \( R_j = 2^j \). We have
\[
T^\ast_k f(x) \leq \sup_j |\overline{T^\ast_k f(x)}| + \sup_j |(T^\ast_k - \overline{T^\ast_k}) f(x)| 
\leq \left| \sum_j |\overline{T^\ast_k f(x)}|^2 \right|^{1/2} + Cf^\ast (x)
\]
where \( T^\ast_j - \overline{T^\ast_j} \) stands for a \( C^\infty \) central core of the multiplier \( T^\ast_j \) and \( f^\ast \) is the Hardy-Littlewood maximal function.

By the same arguments of part a) in the preceding proof we may reduce ourselves to prove
\[
\left\| \sum_j |S_j |f|^2 \right\|_{L^4}^{1/2} \leq C |\log \delta|^\alpha \|f\|_4
\]
for some constants \( C, \alpha \), independent of \( \delta > 0 \).
We define the operators $U_j$ by

$$U_j f(x, y) = x^{j-1} \hat{f}(x, y),$$

and apply the methods in parts b) and c) above to obtain the inequality

$$\left\| \sum_j S_n^5 f \right\|_4^{1/2} \leq C |\log \delta|^{1/2} \left\| \sum_j |U_j f|^2 \right\|_4^{1/2},$$

which yields (7) by the classical Littlewood-Paley theory.

BIBLIOGRAPHY


Manuscrit reçu le 1er décembre 1980.