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SPHERICAL SUMMATION : A PROBLEM OF E. M. STEIN

by A. CÓRDOBA and B. LÓPEZ-MELERO

In this paper we present a proof of a conjecture formulated by E.M. Stein [1], page 5, about the spherical summation operators. We obtain a stronger version of the Carleson-Sjölin theorem [2] and, as a corollary, we obtain a.e. convergence for lacunary Bochner-Riesz means.

With $\lambda > 0$ let T_R^λ denote the Fourier multiplier operator given by

$(T_R^\lambda f)\hat{}(\xi) = (1 - |\xi|^2/R^2)_+^\lambda \hat{f}(\xi)$ for $f \in \mathcal{S}(\mathbf{R}^2)$, and let $\{R_j\}$ be any sequence of positive numbers.

THEOREM 1. – *Given* $\lambda > 0$ *and* $\dfrac{4}{3+2\lambda} < p < \dfrac{4}{1-2\lambda}$ *there exists some positive constant* $C_{\lambda,p}$ *such that*

$$\Big\| \Big| \sum_j |T_{R_j}^\lambda f_j|^2 \Big|^{1/2} \Big\|_p \leqslant C_{\lambda,p} \Big\| \Big| \sum_j |f_j|^2 \Big|^{1/2} \Big\|_p .$$

Let $T_* f = \sup_j |T_{2^j}^\lambda f|$. The methods developed to prove Theorem 1 yield, as an easy consequence, the following result.

THEOREM 2. – *For* $\lambda > 0$ *and* $\dfrac{4}{3+2\lambda} < p < \dfrac{4}{1-2\lambda}$ *there exists some constant* $C'_{\lambda,p}$ *such that*

$$\| T_*^\lambda f \|_p \leqslant C'_{\lambda,p} \| f \|_p .$$

As a result we have, for $f \in L^p(\mathbf{R}^2)$

$$f(x) = \lim_j T^{\lambda}_{2^j} f(x) \qquad \text{for a.e.} \quad x \in \mathbf{R}^2 .$$

As part of the machinery in the proofs of Theorems 1 and 2 we shall make use of the two following results, whose proofs can be found in [3] and [4].

Given a real number $N > 1$ consider the family B of all rectangles with eccentricity N and arbitrary direction, and let M be the associated maximal operator

$$Mf(x) = \sup_{x \in R \in B} \frac{1}{|R|} \int_R |f(x)|\, dx .$$

THEOREM 3. – *There exist constants* C, α *independent of* N *such that*

$$\| Mf \|_2 \leqslant C\, |\log N|^{\alpha}\, \| f \|_2 .$$

Consider a disjoint covering of $\mathbf{R}^n$ by a lattice of congruent parallelepipeds $\{Q_\nu\}_{\nu \in Z^n}$ and the associated multiplier operators

$$(P_\nu f)\hat{} = \chi_{Q_\nu} \hat{f} .$$

THEOREM 4. – *For each* $s > 1$ *there exists a constant* C_s *such that, for every non negative, locally integrable function* ω *and every* $f \in \mathcal{S}(\mathbf{R}^n)$ *we have*

$$\int_{\mathbf{R}^n} \sum_\nu |P_\nu f(x)|^2\, \omega(x)\, dx \leqslant C_s \int_{\mathbf{R}^n} |f(x)|^2\, A_s\, \omega(x)\, dx$$

where $A_s g = [M(g^s)]^{1/s}$ *and* M *denotes the strong maximal function in* $\mathbf{R}^n$.

Proof of Theorem 1. – Suppose that $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ is a smooth function supported in $[-1, +1]$, and consider the family of multipliers S_j^δ defined by

$$(S_j^\delta f)\hat{}\,(\xi) = \phi(\delta^{-1}(R_j^{-1}\, |\xi| - 1))\, \hat{f}(\xi)$$

and also, for a fixed $\delta > 0$, consider the family

$$(T_j^n f)\hat{}\,(\xi) = \psi_n(\arg(\xi))\, (S_j^\delta f)\hat{}\,(\xi)$$

where the ψ_n are a smooth partition of the unity on the circle,

$$1 = \sum_{n=1}^{N} \psi_n ;$$

ψ_n is supported on $\left|\frac{N}{2\pi}\theta - n\right| \leqslant 1$ and $N = [\delta^{-1/2}]$, so that the support of $(T_j^n f)^\wedge$ is much like a rectangle with dimensions $R_j\delta \times R_j\delta^{1/2}$.

There are three main steps in our proof.

a) The same argument of ref. [3] allows us to reduce theorem 1 to prove the following inequality

$$\left\| \left| \sum_j |S_j^\delta f_j|^2 \right|^{1/2} \right\|_4 \leqslant C\,|\log\delta|^\beta \left\| \left| \sum_j |f_j|^2 \right|^{1/2} \right\|_4 . \tag{1}$$

b) With adequate decompositions of the multipliers and geometric arguments, we prove

$$\left\| \left| \sum_j |S_j^\delta f_j|^2 \right|^{1/2} \right\|_4 \leqslant C'\,|\log\delta| \left\| \left| \sum_{j,n} |T_j^n f_j|^2 \right|^{1/2} \right\|_4 . \tag{2}$$

c) An estimate of the kernels of T_j^n, together with theorems 3 and 4 yields,

$$\left\| \left| \sum_{j,n} |T_j^n f_j|^2 \right|^{1/2} \right\|_4 \leqslant C''\,|\log\delta|^\alpha \left\| \left| \sum_j |f_j|^2 \right|^{1/2} \right\|_4 . \tag{3}$$

We refer to [3] for a) and begin with part b).

Fixed $\delta > 0$, we select just one dyadic interval $2^k < R \leqslant 2^{k+1}$ out of each $|\log_2\delta|$ correlative intervals, and we allow in the left hand side of (2) only those indices j for which R_j lays in a selected interval. Also we only take one T_j^n for each 4 correlative indices n, and only those supported in the angular sector $|\sin\theta| \leqslant 1/2$. All these operations will contribute with the factor 24 $|\log_2\delta|$ to the inequality (2).

The left hand side of (2) is less than the 4th rooth of twice

$$\sum_{R_j \leqslant R_k} \int \left| \left(\sum_n T_j^n f_j\right)\left(\sum_m T_k^m f_k\right) \right|^2 \tag{4}$$

and now we only have two kinds of pairs (j, k) : either $R_j \leqslant R_k \leqslant 2R_j$ or $R_j \leqslant \delta R_k$. Let's denote Σ^{I} and Σ^{II} the two corresponding halves of (4). We have

$$\Sigma^{\mathrm{I}} = \Sigma^{\mathrm{I}} \int \Big| \sum_{n,m} (T_j^n f_j)\hat{} * (T_k^m f_k)\hat{} \Big|^2$$
$$\leqslant 4\, \Sigma^{\mathrm{I}} \int \Big| \sum_{n \leqslant m} (T_j^n f_j)\hat{} * (T_k^m f_k)\hat{} \Big|^2 .$$

Now an easy geometric argument shows that, for fixed j, k, the supports of $(T_j^n f_j)\hat{} * (T_k^m f_k)\hat{}$ are disjoint for different pairs $n \leqslant m$, so that we have

$$\Sigma^{\mathrm{I}} \leqslant 4 \int \Sigma^{\mathrm{I}} \sum_{n \leqslant m} |(T_j^n f_j)\hat{} * (T_k^m f_k)\hat{}|^2 \leqslant 4\,\mathrm{A} \tag{5}$$

with
$$\mathrm{A} = \Big\| \Big| \sum_{j,n} |T_j^n f_j|^2 \Big|^{1/2} \Big\|_4^4 .$$

For the pairs (j, k) in Σ^{II} we have

$$\phi = \operatorname{supp} |(T_j^{n_1} f_j)\hat{} * (T_k^{m_1} f_k)\hat{}| \cap \operatorname{supp} |(T_j^{n_2} f_j)\hat{} * (T_k^{m_2} f_k)\hat{}|$$

if $m_1 \neq m_2$, because $R_j \leqslant \delta R_k$, so that

$$\Sigma^{\mathrm{II}} = \Sigma^{\mathrm{II}} \int \sum_m \Big| \Big(\sum_n T_j^n f_j\Big) T_k^m f_k \Big|^2$$
$$\leqslant \Big| \int \Big(\sum_j \Big| \sum_n T_j^n f_j \Big|^2\Big)^2 \Big|^{1/2} \Big| \int \Big(\sum_{k,m} |T_k^m f_k|^2\Big)^2 \Big|^{1/2}$$
$$\leqslant \sqrt{2}\; |\Sigma^{\mathrm{I}} + \Sigma^{\mathrm{II}}|^{1/2}\, \mathrm{A}^{1/2} . \tag{6}$$

From (5) and (6) we obtain (2).

Now we come into part c).

First we observe that for each fixed j it is possible to choose two grids of parallelepipeds as the one in theorem 3 and such that each of the multipliers T_j^n is supported within one of the parallelepipeds, let's call it Q_j^n. If $(P_j^n f)\hat{} = \chi_{Q_j^n} \hat{f}$ is the corresponding multiplier operator, we have

$$T_j^n f_j = T_j^n P_j^n f_j .$$

Furthermore, an integration by parts arguments shows that each of the kernels of the T_j^n is majorized by a sum

$$C \sum_{\nu=0}^{\infty} 2^{-\nu} \frac{1}{|R_{\nu,j}^n|} \chi_{R_{\nu,j}^n}$$

where the $R_{\nu,j}^n$ are rectangles with dimensions $2^\nu \delta^{-1} \times 2^\nu \delta^{-1/2}$ and C is independent of n, j or $\delta > 0$. Therefore in order to estimate A we only have to estimate uniformly in ν the L^4-norm of

$$\left| \sum_{j,n} \left| \frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} * (P^n_j f_j) \right|^2 \right|^{1/2}.$$

Or, what amounts to the same, the L^2-norm of its square. If $\omega \geqslant 0$ is in $L^2(\mathbf{R}^2)$ we have

$$\begin{aligned}
&\sum_{j,n} \int \left| \frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} * (P^n_j f_j)(x) \right|^2 \omega(x)\, dx \\
&\quad \leqslant \sum_{j,n} \int |P^n_j f_j(y)|^2 \left[\frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} * \omega \right](y)\, dy \\
&\quad \leqslant \sum_{j,n} \int |P^n_j f_j(y)|^2\, M\omega(y)\, dy \\
&\quad \leqslant 2\, C_s \sum_j \int |f_j(y)|^2\, A_s(M\omega)(y)\, dy \\
&\quad \leqslant C'_s \left\| \left| \sum_j |f_j|^2 \right|^{1/2} \right\|_4^2 \|M\omega\|_2 \\
&\quad \leqslant C\, |\log \delta|^\alpha\, C'_s \left\| \left| \sum_j |f_j|^2 \right|^{1/2} \right\|_4^2 \|\omega\|_2 ,
\end{aligned}$$

by successive applications of theorems 4 and 3. This estimate proves (3).

Proof of Theorem 2. – With the same notations of the preceding proof, let now $R_j = 2^j$. We have

$$\begin{aligned}
T^\lambda_* f(x) &\leqslant \sup_j |\overline{T^\lambda_j} f(x)| + \sup_j |(T^\lambda_j - \overline{T^\lambda_j}) f(x)| \\
&\leqslant \left| \sum_j |\overline{T^\lambda_j} f(x)|^2 \right|^{1/2} + C f^*(x)
\end{aligned}$$

where $T^\lambda_j - \overline{T^\lambda_j}$ stands for a C^∞ central core of the multiplier T^λ_j and f^* is the Hardy-Littlewood maximal function.

By the same arguments of part a) in the preceding proof we may reduce ourselves to prove

$$\left\| \left| \sum_j |S^\delta_j f|^2 \right|^{1/2} \right\|_4 \leqslant C\, |\log \delta|^\alpha\, \|f\|_4 \tag{7}$$

for some constants C, α, independent of $\delta > 0$.

We define the operators U_j by

$$\widehat{U_j f}(x, y) = \chi_{\{2^{j-1} \leq x \leq 2^j\}} \hat{f}(x, y),$$

and apply the methods in parts b) and c) above to obtain the inequality

$$\left\| \left| \sum_j |S_j^\delta f|^2 \right|^{1/2} \right\|_4 \leq C \, |\log \delta|^\alpha \left\| \left| \sum_j |U_j f|^2 \right|^{1/2} \right\|_4$$

which yields (7) by the classical Littlewood-Paley theory.

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