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THE VALUE-DISTRIBUTION OF LACUNARY SERIES AND A CONJECTURE OF PALEY

by Takafumi MURAI

1. Introduction.

The purpose of this paper is to establish the following

THEOREM 1. – *For any real number* $q > 1$, *there exist two positive numbers* ϵ *and* ρ, *depending only on* q, *with the following property: For every convergent (Hadamard) lacunary power series*

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}, \quad n_{k+1}/n_k \geqslant q \tag{1}$$

in the open unit disk $\mathbf{D} = \{z \,;\, |z| < 1\}$ *satisfying*

$$|c_k| < \epsilon \sum_{j=k+1}^{\infty} |c_j| \quad (k \geqslant 1) \tag{2}$$

and every complex number α *satisfying*

$$|\alpha| < \rho \sum_{k=1}^{\infty} |c_k|, \tag{3}$$

$f(z)$ *takes* α *infinitely often in* $\mathbf{D}$, *where* $\sum_{k=1}^{\infty} |c_k|$ *need not be convergent.*

As immediate consequence, we have the following two corollaries.

COROLLARY 2. – *An unbounded lacunary power series in* $\mathbf{D}$ (*) *takes every complex value infinitely often.*

(*) A lacunary series $\sum_{k=1}^{\infty} c_k z^{n_k}$, $n_{k+1}/n_k \geqslant q > 1$ in $\mathbf{D}$ is unbounded if and only if $\sum_{k=1}^{\infty} |c_k| = +\infty$.

COROLLARY 3. – *Let* $f(z)$ *be as in Theorem 1. If* $\sum_{k=1}^{\infty} |c_k| < +\infty$, *then* $f(e^{it})$, $0 \leqslant t < 2\pi$, *is a Peano curve, that is,* $\{f(e^{it}); 0 \leqslant t < 2\pi\}$ *contains an open set.*

The problem whether Corollary 2 is valid or not was raised by R.E.A.C. Paley in [10]. G. Weiss and M. Weiss showed that a lacunary power series $f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}$, $n_{k+1}/n_k \geqslant q > 1$ takes every complex value infinitely often in $\mathbf{D}$, if $f(z)$ is unbounded and $q \geqslant q_0$ (= about 100) ([13]). W.H.J. Fuchs showed that the assertion holds if $\limsup_{k\to\infty} |c_k| > 0$ ([4], [5]). I.L. Chang showed that the assertion holds, with $\mathbf{D}$ replaced by a sector $\{z \in \mathbf{D}; \alpha < \arg z < \beta\}$, if $\sum_{k=1}^{\infty} |c_k|^{2+\eta} = +\infty$ for some $\eta > 0$ ([3]). (See Remark 21 in this paper.) Other approaches to this problem are given in [1] and [2]. The first part of this paper gives a detailed proof of the result announced in [9].

A function $f(z)$ is said to possess the Peano curve property, if it has the property stated in Corollary 3. The Peano curve property was first discussed by R. Salem and A. Zygmund in [11]. Corollary 3 is not new. (See [7].) Our theorem is a solution to the above problem and useful to discuss the Peano curve property of lacunary power series.

2. Preliminaries.

We denote by $D(\omega, r)$ the open disk with center ω and radius r.

LEMMA 4 ([4]). – *Let* ℓ *be a positive integer and* $g(\zeta)$ *an analytic function in* $D(\omega, r)$ *such that* $|g^{(\ell)}(\omega)| \geqslant y_1$ *and* $|g^{(\ell)}(\zeta)| \leqslant y_2$ $(\zeta \in D(\omega, r))$. *Then*

$$g(D(\omega, r)) \supset D(g(\omega), \overline{\eta}(\ell)\, r^{\ell} y_1^{\ell+1} y_2^{-\ell}),$$

where $\overline{\eta}(\ell)$ *is a constant depending only on* ℓ.

LEMMA 5 ([12]). – *If a lacunary power series*

$$h(z) = \sum_{k=1}^{\infty} a_k z^{m_k}, \; m_{k+1}/m_k \geqslant q > 1$$

satisfies the conditions $\lim_{k\to\infty} a_k = 0$ *and* $\sum_{k=1}^{\infty} |a_k| = +\infty$, *then, for every complex number* α, *there exists a point* t_α *in* $[0, 2\pi)$ *such that* $\lim_{r\uparrow 1} h(re^{it}\alpha) = \lim_{k\to\infty} \sum_{j=1}^{k} a_j e^{im_j t}\alpha = \alpha$.

Let $\mathcal{L}$ be a straight line not passing through the origin. We say that a point ζ is situated to the right of $\mathcal{L}$ if it is contained in the closed half-plane limited by $\mathcal{L}$ which does not contain the origin. We denote by $\mathcal{L}(\zeta, r)$ the straight line of distance (from the origin) r, which is perpendicular to the ray $\{\zeta x \,;\, x \geq 0\}$.

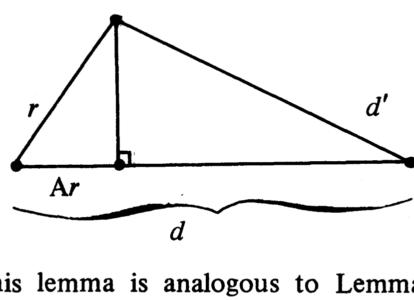
LEMMA 6 (Lemma 4 in [12]). – *There exist two constants* $0 < A = A_q \leq 1$ *and* $B = B_q \geq 1$ *depending only on* $q > 1$ *with the following property: For every lacunary polynomial*

$$Q(t) = \sum_{k=1}^{n} a_k e^{im_k t}, \quad m_{k+1}/m_k \geq q,$$

every straight line $\mathcal{L}$ *of distance (from the origin)* $A \sum_{k=1}^{n} |a_k|$ *and every interval* I *in* $[0, 2\pi)$ *of length* B/m_1, *there exists a point* ξ *in* I *such that* $Q(\xi)$ *is situated to the right of* $\mathcal{L}$.

LEMMA 7. – *Let* d, d', r, A *be as in (Fig.). If* $0 < A \leq 1$ *and* $d \geq \{(A^2 + 1)/A\}r$, *then* $d' \leq d - (A/2)r$.

(Fig.)



Proof. – This lemma is analogous to Lemma 6 in [12]. Since $d'^2 \leq (d - Ar)^2 + r^2$, we have

$$d^2 - d'^2 \geq 2Ard - (A^2 + 1)r^2 = Ar\{2d - (A^2 + 1)/A \cdot r\} \geq 0,$$

and hence

$$d - d' \geq Ar\{2d - (A^2 + 1)/A \cdot r\}/2d = Ar\{1 - (A^2 + 1)/A \cdot (r/2d)\} \geq (A/2)r,$$

and the lemma follows.

LEMMA 8. – *Let* $P(\zeta) = \sum_{k=1}^{n} a_k \exp(m_k \zeta)$ *be an analytic function satisfying* $m_{k+1}/m_k \geq q > 1$. *Then, for every complex number* ω, *there exists a non-negative integer* $\ell = \ell(\omega; P)$ *with* $\ell \leq \sigma n \log n$ $(\sigma = \sigma_q = 10(1 + 2/\log q))$ *such that*

$$|P^{(\ell)}(\omega)| \geq 1/2 \cdot m_k^{\ell}\, |a_k| \exp(m_k \mathrm{Re}\omega) \quad (1 \leq k \leq n). \tag{4}$$

Proof. – In the case where $n = 1$, (4) evidently holds with $\ell = 0$. Suppose $n \geq 2$ and set

$$\begin{cases} P_\ell = \sum_{k=1}^{n} m_k^{\ell} a_k \exp(m_k \omega) \\ \alpha_{\ell,k} = m_k^{\ell}\, |a_k| \exp(m_k \mathrm{Re}\omega) \quad (1 \leq k \leq n) \\ \alpha_{\ell,n+1} = 0 \\ \nabla_\ell = \max_{1 \leq k \leq n+1} \alpha_{\ell,k} \quad (\ell \geq 0). \end{cases} \tag{5}$$

Let λ be the first integer such that $q^k \geq 5n$ $(k \geq 1)$. Then $2\lambda n \leq \sigma n \log n$. Hence it is sufficient to show that, for some $\mu\,(0 \leq \mu \leq n)$, $(\S)_\mu$: $|P_{2\lambda\mu}| \geq 1/2 \cdot \nabla_{2\lambda\mu}$.

Put $j_0 = 1$. Then the following two cases are possible:

$$(*)_0 \quad \alpha_{0,j_0} > 5n\alpha_{0,k} \quad (j_0 < k \leq n+1)$$

$$(*)_0' \quad \alpha_{0,j_0} \leq 5n\alpha_{0,k} \quad \text{for some } j_0 < k \leq n+1.$$

If $(*)_0$, then $(\S)_0$ evidently holds. If $(*)_0'$, then a set $\{k > j_0 ; \alpha_{\lambda,k} \geq \alpha_{\lambda,j_0}\}$ is not empty, according to $q^\lambda \geq 5n$. Let j_1 be the first integer in this set. Then the following two cases are possible:

$$(*)_1 \quad \alpha_{2\lambda,j_1} > 5n\alpha_{2\lambda,k} \quad (j_1 < k \leq n+1)$$

$$(*)_1' \quad \alpha_{2\lambda,j_1} \leq 5n\alpha_{2\lambda,k} \quad \text{for some } j_1 < k \leq n+1.$$

If $(*)_1$, then $(\S)_1$ holds, since

$$\begin{aligned} |P_{2\lambda}| &\geq \alpha_{2\lambda,j_1} - \sum_{k>j_1} \alpha_{2\lambda,k} - \sum_{k<j_1} \alpha_{\lambda,k} m_k^{\lambda} \\ &\geq 4/5 \cdot \alpha_{2\lambda,j_1} - \alpha_{\lambda,j_1} \sum_{k<j_1} m_k^{\lambda} = 4/5 \cdot \alpha_{2\lambda,j_1} - \alpha_{2\lambda,j_1} \sum_{k<j_1} (m_k/m_{j_1})^{\lambda} \\ &\geq 4/5 \cdot \alpha_{2\lambda,j_1} - (q^\lambda - 1)^{-1} \alpha_{2\lambda,j_1} \geq 1/2 \cdot \alpha_{2\lambda,j_1} = 1/2 \cdot \nabla_{2\lambda}. \end{aligned}$$

If $(*)'_1$, then a set $\{k > j_1 ; \alpha_{3\lambda,k} \geqslant \alpha_{3\lambda,j_1}\}$ is not empty. Let j_2 be the first integer in this set. Then the following two cases are possible :

$$(*)_2 \quad \alpha_{4\lambda,j_2} > 5n\alpha_{4\lambda,k} \quad (j_2 < k \leqslant n+1)$$

$$(*)'_2 \quad \alpha_{4\lambda,j_2} \leqslant 5n\alpha_{4\lambda,k} \quad \text{for some } j_2 < k \leqslant n+1 .$$

If $(*)_2$, then $(\S)_2$ holds. If $(*)'_2$, then we define j_3 and consider corresponding two cases $(*)_3$, $(*)'_3$ by the same manner as above. If $(*)_3$, then $(\S)_3$ holds. We repeat this discussion.

Since $j_0 < j_1 < \cdots \leqslant n$, there exists $0 \leqslant \nu \leqslant n$ such that $(*)'_\nu$ does not occur. This signifies that $(\S)_\mu$ holds for some $0 \leqslant \mu \leqslant n$.

LEMMA 9. – *Let $(m_k)_{k=1}^\infty$ be a sequence of positive integers satisfying $m_{k+1}/m_k \geqslant q > 1$ and $(b_k)_{k=1}^\infty$ a sequence of non-negative numbers satisfying*

$$b_k < 1/2 \sum_{j=k+1}^{\infty} b_j \quad (k \geqslant 1), \ \lim_{k\to\infty} b_k = 0, \tag{6}$$

where $\sum_{k=1}^{\infty} b_k$ need not be convergent. For every positive integer Γ, we put

$$\begin{cases} u_k = m_k^\Gamma b_k, \quad \mathrm{U}_k = \max\{u_j ; j < k\}, \ \mathrm{U}_1 = 0 \\ v_k = m_k^{-\Gamma} b_k, \ \mathrm{V}_k = \sum_{j>k} v_j = \sum_{j>k} m_j^{-\Gamma} b_j \quad (k \geqslant 1). \end{cases} \tag{7}$$

We denote by $\mathcal{R} = \{k_\nu\}_{\nu=1}^\infty$ $(k_{\nu+1} > k_\nu)$ the totality of all integers k for which $u_k \geqslant \mathrm{U}_k$ and $v_k \geqslant \mathrm{V}_k$.

If Γ satisfies

$$1 - q^{-\Gamma} - (q^\Gamma - 1)^{-1} \geqslant 3/4, \tag{8}$$

then

$$\sum_{\mu=\nu}^{\infty} b_{k_\mu} \geqslant 1/2 \sum_{k=k_\nu}^{\infty} b_k \quad (\nu \geqslant 1), \tag{9}$$

where (9) signifies $\sum_{\mu=1}^{\infty} b_{k_\mu} = +\infty$, if $\sum_{k=1}^{\infty} b_k = +\infty$.

Proof. – We first show $\limsup_{k\to\infty} u_k = +\infty$. If $\sum_{k=1}^{\infty} b_k = +\infty$, then $\left(\sum_{k=1}^{\infty} m_k^{-\Gamma}\right) \sup_k u_k \geqslant \sum_{k=1}^{\infty} b_k = +\infty$, and hence $\limsup_{k\to\infty} u_k = +\infty$.

If $\sum_{k=1}^{\infty} b_k < +\infty$, then, for every $k \geqslant 1$,

$$b_k \leqslant 1/2 \sum_{j=k+1}^{\infty} b_j \leqslant 1/2 \sum_{j=k}^{\infty} b_j = 1/2 \sum_{j=k}^{\infty} m_j^{-\Gamma} m_j^{\Gamma} b_j$$

$$\leqslant 1/2 \sum_{j=k}^{\infty} m_j^{-\Gamma} \sup_{j\geqslant k} u_j = 1/2 \sum_{j=k}^{\infty} (m_k/m_j)^{\Gamma} m_k^{-\Gamma} \sup_{j\geqslant k} u_j$$

$$\leqslant \{2(1-q^{-\Gamma})\}^{-1} m_k^{-\Gamma} \sup_{j\geqslant k} u_j ,$$

and hence $u_k \leqslant \{2(1-q^{-\Gamma})\}^{-1} \sup_{j\geqslant k} u_j \leqslant 2/3 \cdot \sup_{j\geqslant k} u_j$, which gives $\limsup_{k\to\infty} u_k = +\infty$.

Let $\{K_\nu\}_{\nu=1}^{\infty}$ $(K_{\nu+1} > K_\nu)$ be the totality of all integers k for which $u_k \geqslant U_k$. Then $\mathcal{R} \subset \{K_\nu\}_{\nu=1}^{\infty}$. For every k satisfying $K_n \leqslant k < K_{n+1}$, we have $u_k \leqslant u_{K_n}$, and hence

$$b_k \leqslant (m_{K_n}/m_k)^{\Gamma} b_{K_n} \leqslant q^{\Gamma(K_n - k)} b_{K_n} .$$

Therefore

$$\sum_{k=K_\nu}^{K_\mu} b_k = \sum_{n=\nu}^{\mu-1} \sum_{K_n<k<K_{n+1}} b_k + b_{K_\mu} \leqslant \sum_{n=\nu}^{\mu-1} b_{K_n} \sum_{K_n\leqslant k<K_{n+1}} q^{\Gamma(K_n-k)} \tag{10}$$

$$+ b_{K_\mu} \leqslant (1-q^{-\Gamma})^{-1} \sum_{n=\nu}^{\mu} b_{K_n} \quad (\nu \leqslant \mu) .$$

Let $\mathcal{R}'$ denote the totality of all integers k for which $v_k < V_k$. If $\mathcal{R}'$ is empty, then $\{K_\nu\}_{\nu=1}^{\infty} = \mathcal{R}$ and (9) follows from (10).

Suppose $\mathcal{R}' \neq \Phi$. We have, for every $k \in \mathcal{R}'$,

$$b_k \leqslant \sum_{j>k} (m_k/m_j)^{\Gamma} b_j \leqslant \sum_{j>k} q^{\Gamma(k-j)} b_j ,$$

and hence

$$\sum_{K_\nu\leqslant k\leqslant K_\mu, k\in\mathcal{R}'} b_k \leqslant \sum_{k=K_\nu}^{K_\mu} \sum_{j>k} q^{\Gamma(k-j)} b_j$$

$$\leqslant (q^{\Gamma}-1)^{-1} \left\{ \sum_{k=K_\nu}^{K_\mu} b_k + \sum_{k>K_\mu} q^{\Gamma(K_\mu+1-k)} b_k \right\} \tag{11}$$

$$= (q^{\Gamma}-1)^{-1} \sum_{k=K_\nu}^{K_\mu} b_k + o(1) \quad (\nu \leqslant \mu) .$$

Remove $\mathcal{R}'$ from $\{K_\nu\}_{\nu=1}^{\infty}$. Then the resulting set equals $\mathcal{R}$. By (10) and (11), we have

$$\sum_{\nu \leqslant n \leqslant \mu, \mathrm{K}_n \notin \mathcal{R}'} b_{\mathrm{K}_n} \geqslant \sum_{n=\nu}^{\mu} b_{\mathrm{K}_n} - \sum_{\mathrm{K}_\nu \leqslant k \leqslant \mathrm{K}_\mu, k \in \mathcal{R}'} b_k$$

$$\geqslant \{1 - q^{-\Gamma} - (q^{\Gamma} - 1)^{-1}\} \sum_{k=\mathrm{K}_\nu}^{\mathrm{K}_\mu} b_k + o(1)$$

$$\geqslant 1/2 \sum_{k=\mathrm{K}_\nu}^{\mathrm{K}_\mu} b_k + o(1) \quad (\nu \leqslant \mu).$$

Letting μ tend to infinity, we have (9).

3. The case where $\sum_{k=1}^{\infty} |c_k| = +\infty$.

Let $f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}$, $n_{k+1}/n_k \geqslant q > 1$ be a lacunary power series in **D** satisfying $\sum_{k=1}^{\infty} |c_k| = +\infty$. By the Fuchs result in [4], we may assume the condition $\lim_{k\to\infty} c_k = 0$. We consider an analytic function

$$\mathrm{F}(\zeta) = f(e^{\zeta}) = \sum_{k=1}^{\infty} c_k \exp(n_k \zeta)\,(^*) \tag{12}$$

in a domain $\mathrm{U} = \{\zeta\,;\, \mathrm{Re}\zeta < 0\}$ and shall show that it takes every complex value infinitely often in $\mathrm{U}^* = \mathrm{U} \cap \{\zeta\,;\, 0 \leqslant \mathrm{Im}\zeta < 2\pi\}$. We use two fixed integers γ, N, depending only on q, which are defined as follows.

DEFINITION 10. – Let $\gamma = \gamma_q$ *be an integer satisfying (8)* $(\Gamma = \gamma)$ *and* $\mathrm{N} = \mathrm{N}_q$ an integer satisfying

$$q^{-\mathrm{N}+1}(q-1)^{-1} \leqslant 1/8e \tag{13}$$

$$\mathrm{H}(x, \mathrm{N}\,;\, \gamma, \sigma) = \exp\{(2\gamma + 1 + 4\sigma\mathrm{N}^2)\log x - x\} \leqslant 1/8e \tag{14}$$

for all $x \geqslant q^{\mathrm{N}}$, *where* $\sigma = \sigma_q$ *is the constant in Lemma 8.*

Now we define $u_k, \mathrm{U}_k, v_k, \mathrm{V}_k, \mathcal{R} = \{k_\nu\}_{\nu=1}^{\infty}$ by $m_k = n_k$, $b_k = |c_k|$, $\Gamma = \gamma$ in Lemma 9. Then we have the following

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(*) The author expresses the thanks to Prof. W.H.J. Fuchs, who suggested to use this transform.

LEMMA 11. – *For every complex number* ω *with* $\mathrm{Re}\omega = -1/n_k$, *there exists an integer* $L = L(\omega; F)$ *with* $\gamma + 1 \leqslant L \leqslant \gamma + 1 + 4\sigma N^2$ *such that*

$$|F^{(L)}(\omega)| \geqslant 1/2e \cdot \{n_k^L |c_k| - 1/4 \cdot n_k^{L-\gamma} U_k - 1/4 \cdot n_k^{L+\gamma} V_k\} \qquad (15)$$

$$|F^{(L)}(\zeta)| \leqslant C\{n_k^L |c_k| + n_k^{L-\gamma} U_k + n_k^{L+\gamma} V_k\}$$
$$(\zeta \in D(\omega, (1 - q^{-1})/n_k)), \qquad (16)$$

where $C = 1/(q-1) + w!\, q^w \quad (w = 2\gamma + 1 + 4\sigma N^2)$.

Proof. – To define $L(\omega; F)$, we consider an analytic function $P_k(\zeta) = \Sigma_j^0\ n_j^{\gamma+1} c_j \exp(n_j \zeta)$, where Σ_j^0 denotes the summation over all j satisfying $q^{-N} < n_j/n_k < q^N$. Then the number of terms of $P_k(\zeta)$ is at most $2N$. By Lemma 8, there exists a non-negative integer $\ell = \ell(\omega; P_k)$ with $\ell \leqslant \sigma(2N)\log(2N) \leqslant 4\sigma N^2$ such that

$$|P_k^{(\ell)}(\omega)| \geqslant 1/2 \cdot n_k^\ell \{n_k^{\gamma+1} |c_k|\} \exp(n_k \mathrm{Re}\omega) = 1/2e \cdot n_k^{\gamma+1+\ell} |c_k|. \ (17)$$

Then we put $L = \gamma + 1 + \ell(\omega; P_k)$. Evidently

$$\gamma + 1 \leqslant L \leqslant \gamma + 1 + 4\sigma N^2 .$$

(15): Put $\phi_k(\zeta) = \Sigma_j' c_j \exp(n_j \zeta)$ and $\Phi_k(\zeta) = \Sigma_j'' c_j \exp(n_j \zeta)$, where Σ_j' denotes the summation over all j satisfying $n_j/n_k \leqslant q^{-N}$ (; if such j's do not exist, $\phi_k(\zeta) \equiv 0$,) and Σ_j'' the summation over all j satisfying $n_j/n_k \geqslant q^N$. Then

$$F^{(L)}(\zeta) = \phi_k^{(L)}(\zeta) + P_k^{(\ell)}(\zeta) + \Phi_k^{(L)}(\zeta).$$

We have

$$|\phi_k^{(L)}(\omega)| \leqslant \Sigma_j' n_j^L |c_j| \leqslant \Sigma_j' n_j^{L-\gamma} U_k = \Sigma_j' (n_j/n_k)^{L-\gamma} n_k^{L-\gamma} U_k \qquad (18)$$
$$\leqslant \Sigma_j' (n_j/n_k)\, n_k^{L-\gamma} U_k \leqslant q^{-N+1}(q-1)^{-1} n_k^{L-\gamma} U_k \leqslant 1/8e \cdot n_k^{L-\gamma} U_k ,$$

according to (13). We have

$$|\Phi_k^{(L)}(\omega)| \leqslant \Sigma_j'' n_j^L |c_j| \exp(n_j \mathrm{Re}\omega)$$
$$= n_k^L\, \Sigma_j'' \{(n_k/n_j)^\gamma |c_j|\} \{(n_j/n_k)^{L+\gamma} \exp(-n_j/n_k)\} \qquad (19)$$
$$\leqslant n_k^L \{\Sigma_j''(n_k/n_j)^\gamma |c_j|\} \sup\{H(n_j/n_k, N; \gamma, \sigma);\ n_j/n_k \geqslant q^N\}$$
$$\leqslant 1/8e \cdot n_k^{L+\gamma} V_k ,$$

according to (14). Thus we have, from (17), (18) and (19),

$$|F^{(L)}(\omega)| \geqslant |P_k^{(\ell)}(\omega)| - |\phi_k^{(L)}(\omega)| - |\Phi_k^{(L)}(\omega)|$$
$$\geqslant 1/2e \cdot \{n_k^L |c_k| - 1/4 \cdot n_k^{L-\gamma} U_k - 1/4 \cdot n_k^{L+\gamma} V_k\}.$$

(16): Put $\psi_k(\zeta) = \sum_{j<k} c_j \exp(n_j \zeta)$ and $\Psi_k(\zeta) = \sum_{j>k} c_j \exp(n_j \zeta)$, where $\psi_k(\zeta) \equiv 0$ if $k = 1$. Then $F(\zeta) = \psi_k(\zeta) + c_k \exp(n_k \zeta) + \Psi_k(\zeta)$. Let $\zeta \in D(\omega, (1 - q^{-1})/n_k)$. We have evidently

$$|\{c_k \exp(n_k \zeta)\}^{(L)}| \leqslant n_k^L |c_k| \leqslant C n_k^L |c_k| .$$

By the same manner as in (18), we have $|\psi_k^{(L)}(\zeta)| \leqslant \sum_{j<k} (n_j/n_k)\, n_k^{L-\gamma} U_k$. The right-hand side is dominated by $(q - 1)^{-1} n_k^{L-\gamma} U_k \leqslant C n_k^{L-\gamma} U_k$. We have

$$\begin{aligned} |\Psi_k^{(L)}(\zeta)| &\leqslant \sum_{j>k} n_j^L |c_j| \exp(-n_j/qn_k) \\ &= n_k^L \sum_{j>k} \{(n_k/n_j)^\gamma |c_j|\} \{(n_j/n_k)^{L+\gamma} \exp(-n_j/qn_k)\} \\ &\leqslant (L + \gamma)!\ q^{L+\gamma} n_k^L \sum_{j>k} (n_k/n_j)^\gamma |c_j| \leqslant C n_k^{L+\gamma} V_k . \end{aligned}$$

These estimates give (16).

LEMMA 12. – *For every complex number* ω *with* $\operatorname{Re}\omega = -1/n_{k_\nu}$, *we have* $F(D(\omega, (1 - q^{-1})/n_{k_\nu})) \supset D(F(\omega), \eta |c_{k_\nu}|)$, *where* $\eta = \eta_q$ *is a constant depending only on* q.

Proof. – Let $L = L(\omega; F)$ be the integer in Lemma 11. Since $u_{k_\nu} \geqslant U_{k_\nu}$ and $v_{k_\nu} \geqslant V_{k_\nu}$, we have $|F^{(L)}(\omega)| \geqslant 1/4e \cdot n_{k_\nu}^L |c_{k_\nu}|$ and $|F^{(L)}(\zeta)| \leqslant 3C n_{k_\nu}^L |c_{k_\nu}|$ $(\zeta \in D(\omega, (1 - q^{-1})/n_{k_\nu}))$. Hence Lemma 4 shows that $F(D(\omega, (1 - q^{-1})/n_{k_\nu}))$ contains the open disk with center $F(\omega)$ and radius

$$\begin{aligned} &\overline{\eta}(L) \{(1 - q^{-1})/n_{k_\nu}\}^L \{1/4e \cdot n_{k_\nu}^L |c_{k_\nu}|\}^{L+1} \{3C n_{k_\nu}^L |c_{k_\nu}|\}^{-L} \\ &\qquad = \overline{\eta}(L) \{(1 - q^{-1})/12eC\}^L (4e)^{-1} |c_{k_\nu}| (= \eta'(L) |c_{k_\nu}|, \text{say}). \end{aligned}$$

Putting $\eta = \min\{\eta'(\overline{\ell})\ ;\ \gamma + 1 \leqslant \overline{\ell} \leqslant \gamma + 1 + 4\sigma N^2\}$, we have the required inclusion.

Now we show that, for a given complex number a, $F(\zeta)$ takes a infinitely often in U^*. For the sake of simplicity, we assume $a = 0$; in fact, the following discussion will be independent of the given number a. Let us remember the notation $\mathfrak{K} = \{k_\nu\}_{\nu=1}^{\infty}$. By Lemma 9, we have $\sum_{\nu=1}^{\infty} |c_{k_\nu}| = +\infty$. Put

$r_\nu = -1/n_{k_\nu}$, $O_\nu = \{\zeta\,;\, \mathrm{Re}\zeta < r_\nu\}$, $O_\nu^* = O_\nu \cap \{\zeta\,;\, 0 \leqslant \mathrm{Im}\zeta < 2\pi\}$ $(\nu \geqslant 1)$. For a given $\nu' \geqslant 1$, we assume that $F(\zeta)$ does not take 0 in $U^* - O_{\nu'}^*$. (Since $F(\zeta) = F(\zeta + 2\pi i)$ $(\zeta \in U)$, this equals $F(\zeta) \neq 0$ in $U - O_{\nu'}$.) If this assumption leads to a contradiction, it yields that $F(\zeta)$ takes 0 in $U^* - O_\nu$.

To show a contradiction, we put

$\delta = \min\{|F(\zeta)|\,;\, \mathrm{Re}\zeta = r_{\nu'}\} = \min\{|f(z)|\,;\, |z| = e^{r_{\nu'}}\}$, $R_\nu = \overline{O}_\nu - O_{\nu'}$,

$\delta_\nu = \min\{|F(\zeta)|\,;\, \zeta \in R_\nu\} = \min\{|f(z)|\,;\, e^{r_{\nu'}} \leqslant |z| \leqslant e^{r_\nu}\}$ $(\nu > \nu')$. Then δ, δ_ν are positive, according to our hypothesis. By Lemma 5, $\lim_{\nu \to \infty} \delta_\nu = 0$, and hence there exists $\nu'' > \nu'$ such that $\delta_\nu < \delta$ $(\nu \geqslant \nu'')$. Choose a sequence $(\omega_\nu)_{\nu=\nu''}^{\infty}$ $(\omega_\nu \in R_\nu)$ such that $\delta_\nu = |F(\omega_\nu)|$. Let $\nu \geqslant \nu''$. By the minimum modulus principle, $\mathrm{Re}\omega_\nu = r_\nu$. By Lemma 12,

$$F(D(\omega_\nu, (1 - q^{-1})/n_{k_\nu})) \supset D(F(\omega_\nu), \eta\, |c_{k_\nu}|).$$

Note that $r_\nu + (1 - q^{-1})/n_{k_\nu} = -1/qn_{k_\nu} \leqslant r_{\nu+1}$. Since $F(\zeta)$ does not take 0 in $R_{\nu+1}$, we have

$$\delta_{\nu+1} \leqslant \min\{|F(\zeta)|\,;\, \zeta \in D(\omega_\nu, (1 - q^{-1})/n_{k_\nu})\} \leqslant \delta_\nu - \eta\, |c_{k_\nu}|,$$

that is, $\delta_\nu - \delta_{\nu+1} \geqslant \eta\, |c_{k_\nu}|$. Therefore

$$\delta_{\nu''} = \sum_{\nu=\nu''}^{\infty} (\delta_\nu - \delta_{\nu+1}) \geqslant \eta \sum_{\nu=\nu''}^{\infty} |c_{k_\nu}| = +\infty,$$

which is a contradiction. Hence $F(\zeta)$ takes 0 in $U^* - O_{\nu'}^*$. Since $\nu' \geqslant 1$ is arbitrary, the proof is completed.

4. The cas where $\sum_{k=1}^{\infty} |c_k| < +\infty$.

We need the following

LEMMA 13. – *There exist three constants* $\overline{\epsilon} = \overline{\epsilon}_q$ $(0 < \overline{\epsilon} \leqslant 1/2)$, $\rho = \rho_q$, $W = W_q$ *depending only on* $q > 1$ *with the following property: For every lacunary power series* $S(t) = \sum_{k=1}^{\infty} a_k e^{im_k t}$, $m_{k+1}/m_k \geqslant q$ *satisfying*

$$|a_k| \leqslant \overline{\epsilon} \sum_{j=k+1}^{\infty} |a_j| < +\infty \quad (k \geqslant 1) \tag{20}$$

and every complex number a *satisfying*

$$|a| \leqslant \rho \sum_{k=1}^{\infty} |a_k|, \tag{21}$$

there exist a sufficiently large integer E *and a corresponding point* θ_E *in* $[0, 2\pi)$ *such that*

$$|a - S_E(\theta_E)| \leqslant W|a_E|, \quad \textit{where} \quad S_E(t) = \sum_{k=1}^{E} a_k e^{im_k t} \tag{22}$$

$$|a_k| \leqslant W|a_E| \quad (k \geqslant E) \tag{23}$$

$$G_{E-1} = \sum_{k=1}^{E-1} |a_k| q^{k-(E-1)} \leqslant W|a_E|. \tag{24}$$

We postpone the proof of this lemma to the next section. In this section, we show that Theorem 1 follows from this lemma.

Let $f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}$, $n_{k+1}/n_k \geqslant q > 1$ be a lacunary power series in $\mathbf{D}$. For a while, we assume the condition (20), replacing a_k by c_k, where $\bar{\epsilon}$ is not a required constant in (2) and ϵ will be determined later.

As in the preceding section, we deal with $F(\zeta) = f(e^\zeta)$ and use two fixed integers $\tilde{\gamma}, \tilde{N}$, depending only on q, which are defined as follows.

DEFINITION 14. – *Let* $\tilde{\gamma} = \tilde{\gamma}_q$ *be an integer satisfying (8)* $(\Gamma = \tilde{\gamma})$ *and* $W(q^{\tilde{\gamma}} - 1)^{-1} \leqslant 1$. *Let* $\tilde{N} = \tilde{N}_q$ *be an integer satisfying (13) and (14), with* γ *replaced by* $\tilde{\gamma}$.

Now we define $u_k, U_k, v_k, V_k, \mathcal{R} = \{k_\nu\}_{\nu=1}^{\infty}$ by $m_k = n_k$, $b_k = |c_k|$, $\Gamma = \tilde{\gamma}$ in Lemma 9. Then Lemma 11 holds, with γ, N replaced by $\tilde{\gamma}, \tilde{N}$. Hence Lemma 4 gives

LEMMA 15. – *For every complex number* ω *with* $\mathrm{Re}\,\omega = -1/n_{k_\nu}$, *we have* $F(D(\omega, (1 - q^{-1})/n_{k_\nu})) \supset D(F(\omega), \tilde{\eta}|c_{k_\nu}|)$, *where* $\tilde{\eta} = \tilde{\eta}_q$ *is a constant depending only on* q.

Let a be a complex number satisfying $|a| \leqslant \rho \sum_{k=1}^{\infty} |c_k|$. We define r_ν, O_ν, O_ν^* $(\nu \geqslant 1)$ as above. For a given $\nu' \geqslant 1$, we assume that $F(\zeta)$ does not take a in $U^* - O_{\nu'}^*$ (, that is, $F(\zeta) \neq a$ in

$U - O_{\nu'})$. Under this assumption, we shall show an inequality, which will contradict (2) for a sufficiently small ϵ.

To show such an inequality, we shall apply Lemma 13 to $S(t) = f(e^{it})$. Put $\delta = \min\{|F(\zeta) - a|;\ \mathrm{Re}\zeta = r_{\nu'}\}$, $R_\nu = \overline{O}_\nu - O_{\nu'}$, $\delta_\nu = \min\{|F(\zeta) - a|;\ \zeta \in R_\nu\}$ $(\nu > \nu')$. Then δ, δ_ν are positive. Note that

$$\lim_{k\to\infty} \max_t |F(-1/n_k + it) - S_k(t)| = \lim_{k\to\infty} \left\{ \sum_{j=1}^{k} (n_j/n_k)\,|c_j| + \sum_{j=k+1}^{\infty} |c_j| \right\} = 0.$$

By Lemma 13, $\lim_{\nu\to\infty} \delta_\nu = 0$. For every integer E in Lemma 13, we have $E \in \mathfrak{K}$, since

$$U_E = \max\{n_k^{\tilde\gamma}|c_k|;\ 1 \leqslant k \leqslant E-1\} \leqslant \sum_{k=1}^{E-1} n_k^{\tilde\gamma}|c_k|$$
$$= n_E^{\tilde\gamma} \sum_{k=1}^{E-1} (n_k/n_E)^{\tilde\gamma}|c_k| \leqslant n_E^{\tilde\gamma} \sum_{k=1}^{E-1} |c_k|\, q^{\tilde\gamma(k-E)}$$
$$\leqslant q^{-\tilde\gamma} n_E^{\tilde\gamma} G_{E-1} \leqslant W q^{-\tilde\gamma} n_E^{\tilde\gamma}|c_E| = W q^{-\tilde\gamma} u_E \leqslant u_E$$

and

$$V_E = \sum_{k=E+1}^{\infty} n_k^{-\tilde\gamma}|c_k| \leqslant W|c_E| \sum_{k=E+1}^{\infty} n_k^{-\tilde\gamma}$$
$$= W v_E \sum_{k=E+1}^{\infty} (n_E/n_k)^{\tilde\gamma} \leqslant W(q^{\tilde\gamma} - 1)^{-1} v_E \leqslant v_E.$$

Hence there exists ν'' such that $E = k_{\nu''}$ is an integer in Lemma 13 and $\delta_\nu < \delta$ $(\nu \geqslant \nu'')$. By the same discussion as in the preceding section, we have $\delta_{\nu''} \geqslant \tilde\eta \sum_{\nu=\nu''}^{\infty} |c_{k_\nu}|$. By Lemma 9, we have

$$\sum_{\nu=\nu''}^{\infty} |c_{k_\nu}| \geqslant 1/2 \sum_{k=E}^{\infty} |c_k| \geqslant 1/2 \sum_{k=E+1}^{\infty} |c_k|.$$

Let θ_E denote the corresponding point with $E = k_{\nu''}$ in Lemma 13. Then we have, with $W' = 2W + 1 + W/(q-1)$,

$$\delta_{\nu''} \leqslant |a - F(-1/n_E + i\theta_E)|$$
$$\leqslant |a - S_E(\theta_E)| + \sum_{k=1}^{E} |c_k|(1 - e^{-n_k/n_E}) + \sum_{k=E+1}^{\infty} |c_k|\, e^{-n_k/n_E}$$
$$\leqslant W|c_E| + \sum_{k=1}^{E} (n_k/n_E)|c_k| + W|c_E| \sum_{k=E+1}^{\infty} e^{-n_k/n_E}$$
$$\leqslant W|c_E| + \{G_{E-1} + |c_E|\} + W/(q-1)\cdot|c_E|$$
$$\leqslant \{2W + 1 + W/(q-1)\}|c_E| = W'|c_E|.$$

Hence we have

$$W' | c_E | \geqslant \tilde{\eta}/2 \sum_{k=E+1}^{\infty} | c_k | . \tag{25}$$

Now we put $\epsilon = \min \{\bar{\epsilon}, \tilde{\eta}/3W'\}$ and, in addition to the above assumption, we suppose that $f(z)$ satisfies (2). Then (25) shows a contradiction, and hence $F(\zeta)$ takes α in $U^* - O^*_{\nu'}$. Since $\nu' \geqslant 1$ is arbitrary, $F(\zeta)$ takes α infinitely often in U^*. Since α is arbitrary as long as $|\alpha| \leqslant \rho \sum_{k=1}^{\infty} |c_k|$, the proof is completed.

5. Proof of Lemma 13.

It remains only to prove Lemma 13. For the proof of this lemma, we use fixed constants $A, B, K, Z, \bar{\epsilon}$, depending only on q, which are defined as follows.

DEFINITION 16. – *Let* $A = A_q$ *and* $B = B_q$ *be the constants in Lemma 6. Let* $K = K_q$, $Z = Z_q$ *be two positive integers and* $\bar{\epsilon} = \bar{\epsilon}_q$ *a positive number such that*

$$2\bar{\epsilon}KZ(A^2 + 1)/A \leqslant \min \{A/8, A^2/16B\} \tag{26}$$

$$Y_q = 3A/16 + (A/8 + 2\bar{\epsilon}KZ) + Bq^{-K+1}(q-1)^{-1}(A^2/16B + 1) \tag{27}$$
$$- \{A/2 \cdot (1 - \bar{\epsilon}K - 2/Z) - (\bar{\epsilon}K + 2/Z)\} \{1 - (A/8 + 2\bar{\epsilon}KZ)\} < 0$$

$$A/2 - 2/Z - Bq^{-K+1}(q-1)^{-1}(1 + 2/Z) > 0 . \tag{28}$$

Such a 3-tuple $(K, Z, \bar{\epsilon})$ exists, since we can choose K, Z such that (27) and (28) are valid, with $\bar{\epsilon}$ replaced by 0, and after the choice of K, Z, we can choose $\bar{\epsilon}$ in such a way that the required inequalities are valid.

Now let $S(t) = \sum_{k=1}^{\infty} a_k e^{im_k t}$, $m_{k+1}/m_k \geqslant q > 1$ be a lacunary power series satisfying (20), where $\bar{\epsilon}$ is the constant given above. For the sake of simplicity, we write, for a power series $R(t) = \sum_{n=0}^{\infty} \hat{R}(n) e^{int}$, $\|R\| = \sum_{n=0}^{\infty} |\hat{R}(n)|$. We shall divide $S(t)$ into polynomials $\bar{\Delta}_1(t), \bar{\Delta}_2(t), \cdots; \Delta_1(t), \Delta_2(t), \cdots$, where the number of terms of each $\bar{\Delta}_m(t)$ is K and that of each $\Delta_m(t)$ is less than or equal to $K(2Z-2)$. Let $\widetilde{\Delta}_{\ell}(t) = \sum_{K(\ell-1)<k\leqslant K\ell} a_k e^{im_k t}$ $(\ell \geqslant 1)$. Choose a sequence $(\ell_m)_{m=1}^{\infty}$ of positive integers such that $\|\widetilde{\Delta}_{\ell_m}\| = \min\{\|\widetilde{\Delta}_{\ell}\| \,;\, Z(m-1) < \ell \leqslant Zm\}$. We put

$$\overline{\Delta}_m(t) = \widetilde{\Delta}_{\ell_m}(t),\ \Delta_m(t) = \sum_{\ell_{m-1}<\ell<\ell_m} \widetilde{\Delta}_{\ell}(t) \quad (m \geqslant 1,\ \ell_0 = 0),$$

where $\Delta_1(t) \equiv 0$, if $\ell_1 = 1$. Note that

$$\|\overline{\Delta}_m\| \leqslant 1/Z \cdot (\|\Delta_m\| + \|\Delta_{m+1}\|) \quad (m \geqslant 1). \tag{29}$$

We put

$$\begin{cases} \nu_m = (\text{the largest exponent occurring in } \Delta_m(t)) \\ T_m(t) = \sum_{k=1}^{\nu_m} a_k e^{im_k t}, \quad T_0(t) \equiv 0 \\ g_m = \sum_{k=1}^{\nu_m} |a_k|\, q^{k-\nu_m}, \quad g_0 = 0 \quad (m \geqslant 1), \end{cases} \tag{30}$$

where $\nu_1 = g_1 = 0$, $T_1(t) \equiv 0$, if $\Delta_1(t) \equiv 0$. Now we break up the proof of Lemma 13 into several steps.

LEMMA 17. – *For any* $m \geqslant 1$,

$$\|\overline{\Delta}_m\| + \|\Delta_{m+1}\| \leqslant 2\overline{\epsilon} KZ\, \|S - T_m\| \tag{31}$$

$$\sum_{r=m}^{\infty} \|\overline{\Delta}_r\| \leqslant (\overline{\epsilon}K + 2/Z)\, \|S - T_m\| \tag{32}$$

$$\sum_{r=m+1}^{\infty} \|\Delta_r\| \geqslant (1 - \overline{\epsilon}K - 2/Z)\, \|S - T_m\| \tag{33}$$

$$\sum_{r=m}^{\infty} g_r \leqslant q(q-1)^{-1} \{g_m + \|S - T_m\|\}. \tag{34}$$

Proof. – (31): By (20), we have, for every $k \geqslant \nu_m$, $|a_k| \leqslant \overline{\epsilon}\, \|S - T_m\|$. Since the number of terms of $\overline{\Delta}_m + \Delta_{m+1}$ is less than $2KZ$, we have (31).

(32): By (29), we have

$$\sum_{r=m+1}^{\infty} \|\overline{\Delta}_r\| \leqslant 1/Z \sum_{r=m+1}^{\infty} (\|\Delta_r\| + \|\Delta_{r+1}\|) \leqslant 2/Z \cdot \|S - T_m\|.$$

Since the number of terms of $\overline{\Delta}_m$ is K, we have

$$\|\overline{\Delta}_m\| \leqslant \overline{\epsilon}\mathrm{K}\,\|\mathrm{S}-\mathrm{T}_m\|,$$

according to (20). From these inequalities (32) follows.

(33): Since $\|\mathrm{S}-\mathrm{T}_m\| = \sum_{r=m+1}^{\infty} \|\Delta_r\| + \sum_{r=m}^{\infty} \|\overline{\Delta}_r\|$, (33) follows from (32).

(34): Suppose $\nu_m \neq 0$. Then we have

$$\begin{aligned}\sum_{r=m}^{\infty} g_r &= \sum_{r=m}^{\infty}\sum_{k=1}^{\nu_r} |a_k|\, q^{k-\nu_r} \leqslant \sum_{n=\nu_m}^{\infty}\sum_{k=1}^{n} |a_k|\, q^{k-n}\\ &= \sum_{k=1}^{\nu_m} |a_k| \sum_{n=\nu_m}^{\infty} q^{k-n} + \sum_{k=\nu_m+1}^{\infty} |a_k| \sum_{n=k}^{\infty} q^{k-n}\\ &= q(q-1)^{-1}\left\{\sum_{k=1}^{\nu_m} |a_k|\, q^{k-\nu_m} + \sum_{k=\nu_m+1}^{\infty} |a_k|\right\}\\ &= q(q-1)^{-1}\{g_m + \|\mathrm{S}-\mathrm{T}_m\|\}.\end{aligned}$$

Suppose $\nu_m = 0$. Then $m=1$ and $\nu_1 = 0$. Since $g_1 = 0$, $\mathrm{T}_1(t) \equiv 0$, we have

$$\begin{aligned}\sum_{r=1}^{\infty} g_r = \sum_{r=2}^{\infty} g_r &\leqslant q(q-1)^{-1}\{g_2 + \|\mathrm{S}-\mathrm{T}_2\|\}\\ &\leqslant q(q-1)^{-1}\{g_1 + \|\mathrm{S}-\mathrm{T}_1\|\}.\end{aligned}$$

LEMMA 18. – *Suppose that there exist a non-negative integer* J *and a corresponding point* s_{J} *in* $[0, 2\pi)$ *such that*

$$|\alpha - \mathrm{T}_{\mathrm{J}}(s_{\mathrm{J}})| \leqslant \mathrm{A}/8 \cdot \|\mathrm{S}-\mathrm{T}_{\mathrm{J}}\|$$

and $g_{\mathrm{J}} \leqslant \mathrm{A}^2/16\mathrm{B}\cdot\|\mathrm{S}-\mathrm{T}_{\mathrm{J}}\|$. *Then there exist a pair* (j', J'), $\mathrm{J} < j' < \mathrm{J}'$ *of integers and corresponding points* $s_{j'}, \cdots, s_{\mathrm{J}'}$ *in* $[0, 2\pi)$ *verifying the following conditions: with* $\lambda_m = |\alpha - \mathrm{T}_m(s_m)|$ $(j' \leqslant m \leqslant \mathrm{J}')$,

$$\lambda_m \geqslant (\mathrm{A}^2+1)/\mathrm{A}\cdot\|\Delta_{m+1}\| \quad (j' \leqslant m < \mathrm{J}') \tag{35}$$

$$\lambda_m \leqslant \lambda_{m-1} - \mathrm{A}/2\cdot\|\Delta_m\| + \|\overline{\Delta}_{m-1}\| + \mathrm{B}q^{-\mathrm{K}} g_{m-1} \quad (j' < m < \mathrm{J}') \tag{36}$$

$$\begin{aligned}(\mathrm{A}^2+1)/\mathrm{A}\cdot\|\Delta_{\mathrm{J}'+1}\| > \lambda_{\mathrm{J}'} = \lambda_{\mathrm{J}'-1} &- \mathrm{A}/2\cdot\|\Delta_{\mathrm{J}'}\|\\ &+ \|\overline{\Delta}_{\mathrm{J}'-1}\| + \mathrm{B}q^{-\mathrm{K}} g_{\mathrm{J}'-1}\end{aligned} \tag{37}$$

$$g_{\mathrm{J}'} \leqslant (\mathrm{A}^2+1)/\mathrm{A}\cdot\|\Delta_{\mathrm{J}'+1}\|. \tag{38}$$

Proof. – (Definition of j') : Let j' be the first integer satisfying $\|T_m - T_J\| \geqslant A/8 \cdot \|S - T_J\|$ $(m \geqslant J)$. We show the following inequalities

$$X_m = A\|T_m - T_J\| - Bg_J \geqslant (A^2 + 1)/A \cdot \|\Delta_{m+1}\| \quad (m \geqslant j') \tag{39}$$

and

$$\begin{aligned}\overline{Y} = \{3A/16 \cdot \|S - T_J\| + \|T_{j'} - T_J\|\} \\ - \{A/2 \cdot (1 - \overline{\epsilon}K - 2/Z) - (\overline{\epsilon}K + 2/Z)\} \|S - T_{j'}\| \\ + Bq^{-K+1}(q-1)^{-1} \{g_{j'} + \|S - T_{j'}\|\} < 0.\end{aligned} \tag{40}$$

Let $m \geqslant j'$. Since $\|T_m - T_J\| \geqslant \|T_{j'} - T_J\| \geqslant A/8 \cdot \|S - T_J\|$ and $Bg_J \leqslant A^2/16 \cdot \|S - T_J\|$, we have, from (26) and (31),

$$\begin{aligned}X_m \geqslant A^2/16 \cdot \|S - T_J\| \geqslant A^2/16 \cdot \|S - T_m\| \geqslant A^2/32\overline{\epsilon}KZ \cdot \|\Delta_{m+1}\| \\ \geqslant (A^2 + 1)/A \cdot \|\Delta_{m+1}\|,\end{aligned}$$

and hence (39).

Since

$$\begin{aligned}\|T_{j'} - T_J\| &= \|T_{j'-1} - T_J\| + \|\overline{\Delta}_{j'-1}\| + \|\Delta_{j'}\| \\ &\leqslant A/8 \cdot \|S - T_J\| + 2\overline{\epsilon}KZ \|S - T_{j'-1}\| \leqslant (A/8 + 2\overline{\epsilon}KZ) \|S - T_J\|,\end{aligned}$$

$$\|S - T_{j'}\| = \|S - T_J\| - \|T_{j'} - T_J\| \geqslant \{1 - (A/8 + 2\overline{\epsilon}KZ)\} \|S - T_J\|$$

and

$$g_{j'} + \|S - T_{j'}\| \leqslant g_J + \|S - T_J\| \leqslant (A^2/16B + 1) \|S - T_J\|,$$

we have $\overline{Y}/\|S - T_J\|$

$$\begin{aligned}&\leqslant 3A/16 + (A/8 + 2\overline{\epsilon}KZ) + Bq^{-K+1}(q-1)^{-1}(A^2/16B + 1) \\ &\quad - \{A/2 \cdot (1 - \overline{\epsilon}K - 2/Z) - (\overline{\epsilon}K + 2/Z)\} \{1 - (A/8 + 2\overline{\epsilon}KZ)\} \\ &= Y_q < 0.\end{aligned}$$

(Definition of $s_{j'}$): Applying Lemma 6 to $Q(t) = T_{j'}(t) - T_J(t)$, $\mathscr{L} = \mathscr{L}(-\alpha + T_J(s_J), A\|Q\|)$ and $I = (s_J - B/\mu, s_J + B/\mu)$ (μ: the smallest exponent in $Q(t)$), we choose $s_{j'}$ in I so that

$$|\{\alpha - T_J(s_J)\} - \{T_{j'}(s_{j'}) - T_J(s_{j'})\}| \geqslant A\|T_{j'} - T_J\|.$$

Since

$$\begin{aligned}\lambda_{j'} &= |\alpha - T_{j'}(s_{j'})| \\ &= |\{\alpha - T_J(s_J)\} - \{T_{j'}(s_{j'}) - T_J(s_{j'})\} - \{T_J(s_{j'}) - T_J(s_J)\}| \\ &\geqslant A\|T_{j'} - T_J\| - |s_{j'} - s_J| \left\| \frac{d}{dt} T_J(\cdot) \right\| \\ &\geqslant A\|T_{j'} - T_J\| - B/\mu \sum_{k=1}^{\nu_J} m_k |a_k| \geqslant A\|T_{j'} - T_J\| - Bg_J \\ &= X_{j'} \geqslant (A^2 + 1)/A \cdot \|\Delta_{j'+1}\|,\end{aligned}$$

(35) holds for $m = j'$. Let us remark

$$\begin{aligned}\lambda_{j'} &= |\{\alpha - T_J(s_J)\} - \{T_{j'}(s_{j'}) - T_J(s_{j'})\} - \{T_J(s_{j'}) - T_J(s_J)\}| \\ &\leqslant |\alpha - T_J(s_J)| + \|T_{j'} - T_J\| + |s_{j'} - s_J| \left\| \frac{d}{dt} T_J(\cdot) \right\| \\ &\leqslant A/8 \cdot \|S - T_J\| + \|T_{j'} - T_J\| + Bg_J \\ &\leqslant 3A/16 \cdot \|S - T_J\| + \|T_{j'} - T_J\| .\end{aligned} \tag{41}$$

(Definition of J'): Applying first Lemma 6 to $Q(t) = \Delta_{j'+1}(t)$, $\mathcal{L} = \mathcal{L}(\alpha - T_{j'}(s_{j'}), A\|Q\|)$ and $I = (s_{j'} - B/\overline{\mu}, s_{j'} + B/\overline{\mu})$ ($\overline{\mu}$: the smallest exponent in $Q(t)$) and using next Lemma 7, we choose θ' in I so that $|\{\alpha - T_{j'}(s_{j'})\} - \Delta_{j'+1}(\theta')| \leqslant \lambda_{j'} - A/2 \cdot \|\Delta_{j'+1}\|$. Then $|\alpha - T_{j'+1}(\theta')|$

$$\begin{aligned}&= |\{\alpha - T_{j'}(s_{j'}) - \Delta_{j'+1}(\theta')\} - \overline{\Delta}_{j'}(\theta') - \{T_{j'}(\theta') - T_{j'}(s_{j'})\}| \\ &\leqslant \lambda_{j'} - A/2 \cdot \|\Delta_{j'+1}\| + \|\overline{\Delta}_{j'}\| + |\theta' - s_{j'}| \left\| \frac{d}{dt} T_{j'}(\cdot) \right\| \\ &\leqslant \lambda_{j'} - A/2 \cdot \|\Delta_{j'+1}\| + \|\overline{\Delta}_{j'}\| + Bq^{-K} g_{j'} \quad (= \widetilde{T}, \text{ say}).\end{aligned}$$

We distinguish the following two cases:

(a) $\max_\theta |\alpha - T_{j'+1}(\theta)| < \widetilde{T}$,

(b) $\max_\theta |\alpha - T_{j'+1}(\theta)| \geqslant \widetilde{T}$.

If (a), we choose $s_{j'+1}$, with the aid of Lemma 6, so that $|\alpha - T_{j'+1}(s_{j'+1})| \geqslant A\|T_{j'+1}\|$. Then we have, from (39),

$$\begin{aligned}|\alpha - T_{j'+1}(s_{j'+1})| \geqslant A\|T_{j'+1}\| &\geqslant A\|T_{j'+1} - T_J\| \\ &\geqslant X_{j'+1} \geqslant (A^2 + 1)/A \cdot \|\Delta_{j'+2}\|,\end{aligned}$$

and hence (35) and (36) hold for $m = j' + 1$. If (b), we choose $s_{j'+1}$, by the continuity of $|\alpha - T_{j'+1}(\cdot)|$, so that $|\alpha - T_{j'+1}(s_{j'+1})| = \widetilde{T}$. Then (36) holds for $m = j' + 1$.

If $\lambda_{j'+1} < (A^2 + 1)/A \cdot \|\Delta_{j'+2}\|$, then we put $J' = j' + 1$. If $\lambda_{j'+1} \geqslant (A^2 + 1)/A \cdot \|\Delta_{j'+2}\|$, we find $s_{j'+2}$ in the same manner as we found $s_{j'+1}$. We continue this process until we reach an integer J' satisfying (37). Such an integer exists; otherwise, we have, from Lemma 17, (36), (40) and (41),

$$0 \leqslant \liminf_{r\to\infty} \lambda_r \leqslant \liminf_{r\to\infty} (\lambda_{r-1} - A/2\cdot \|\Delta_r\| + \|\overline{\Delta}_{r-1}\| + Bq^{-K}g_{r-1})$$
$$\leqslant \liminf_{r\to\infty} \{\lambda_{r-2} - A/2\cdot(\|\Delta_{r-1}\| + \|\Delta_r\|)$$
$$+ (\|\overline{\Delta}_{r-2}\| + \|\overline{\Delta}_{r-1}\|) + Bq^{-K}(g_{r-2} + g_{r-1})\}$$
$$\leqslant \cdots \leqslant \lambda_{j'} - A/2 \sum_{r=j'+1}^{\infty} \|\Delta_r\| + \sum_{r=j'}^{\infty} \|\overline{\Delta}_r\| + Bq^{-K}\sum_{r=j'}^{\infty} g_r$$
$$\leqslant \{3A/16\cdot \|S - T_J\| + \|T_{j'} - T_J\|\} - A/2\cdot(1 - \overline{\epsilon}K - 1/Z)\|S - T_{j'}\|\}$$
$$+ (\overline{\epsilon}K + 2/Z)\|S - T_{j'}\| + Bq^{-K+1}(q-1)^{-1}\{g_{j'} + \|S - T_{j'}\|\}$$
$$= \overline{Y} < 0,$$

which is a contradiction.

(Proof of (38)): Since $\lambda_{J'_{-1}} \geqslant (A^2+1)/A\cdot\|\Delta_{J'}\|$ $(0 < A \leqslant 1)$, we have $\lambda_{J'_{-1}} - A/2\cdot\|\Delta_{J'}\| \geqslant \|\Delta_{J'}\|$. Hence we have, from (37),

$$g_{J'} = \sum_{k=1}^{\nu_{J'}} |a_k|\, q^{k-\nu_{J'}} \leqslant \|\Delta_{J'}\| + \|\overline{\Delta}_{J'_{-1}}\| + q^{-K}g_{J'_{-1}}$$
$$\leqslant \{\lambda_{J'_{-1}} - A/2\cdot\|\Delta_{J'}\|\} + \|\overline{\Delta}_{J'_{-1}}\| + Bq^{-K}g_{J'_{-1}} = \lambda_{J'}$$
$$\leqslant (A^2+1)/A\cdot\|\Delta_{J'_{+1}}\|.$$

LEMMA 19. – *Let* $|\alpha| \leqslant A/8\cdot\|S\|$. *Then there exist a sufficiently large integer* J'' *and a corresponding point* $s_{J''}$ *in* $[0, 2\pi)$ *such that* $|\alpha - T_{J''}(s_{J''})| \leqslant (A^2+1)/A\cdot\|\Delta_{J''_{+1}}\|$ *and*

$$g_{J''} \leqslant (A^2+1)/A\cdot\|\Delta_{J''_{+1}}\|.$$

Proof. – Since $|\alpha| \leqslant A/8\cdot\|S\|$ and $g_0 = 0$, the integer $J = 0$ satisfies the conditions in Lemma 18 (, where $s_J = 0$). Hence there exist J' and a corresponding point $s_{J'}$ such that $\lambda_{J'} \leqslant (A^2+1)/A\cdot\|\Delta_{J'_{+1}}\|$ and $g_{J'} \leqslant (A^2+1)/A\cdot\|\Delta_{J'_{+1}}\|$. By (26) and (31), we have

$$(A^2+1)/A\cdot\|\Delta_{J'_{+1}}\| \leqslant 2\overline{\epsilon}KZ(A^2+1)/A\cdot\|S - T_{J'}\| \leqslant \begin{cases} A/8\cdot\|S - T_{J'}\| \\ A^2/16B\cdot\|S - T_{J'}\|, \end{cases}$$

and hence $\lambda_{J'} \leqslant A/8\cdot\|S - T_{J'}\|$ and $g_{J'} \leqslant A^2/16B\cdot\|S - T_{J'}\|$. This implies that $(J', s_{J'})$ also satisfies the conditions in Lemma 18. Repeating this discussion, we obtain a required $(J'', s_{J''})$.

LEMMA 20. – *There exists a constant* $\overline{W} = \overline{W}(q, A, B, K, Z)$ *with the following property: For every complex number* α $(|\alpha| \leqslant A/8 \cdot \|S\|)$ *and the associated integer* J'' *with* α *in Lemma 19, there exists a point* t_F *in* $[0, 2\pi)$ *such that* $|\alpha - T_F(t_F)| \leqslant \overline{W} \|\Delta_F\|$ *and* $g_F \leqslant \overline{W} \|\Delta_F\|$, *where* F *is the first integer satisfying* $\|\Delta_m\| = \max_{r \geqslant J''} \|\Delta_r\| \ (m \geqslant J'')$.

Proof. – Set $\overline{W}' = (A^2 + 1)/A + (1 + 2/Z) q(q-1)^{-1}$ and $\overline{W} = \max\{\overline{W}', (A^2+1)/A + 1 + 3/Z + Bq^{-K+1}(q-1)^{-1}(\overline{W}' + 1/Z)\}$.

We shall show that $\overline{W}$ is a required constant. If $F = J''$, we put $t_F = s_{J''}$, where $s_{J''}$ is a point corresponding to J''. Then the required inequalities evidently hold. Suppose $F \neq J''$. We have, for every $J'' \leqslant m \leqslant F$,

$$\begin{aligned} g_m &= \sum_{k=1}^{\nu_m} |a_k| q^{k-\nu_m} \leqslant g_{J''} + \sum_{r=J''+1}^{m} \|\Delta_r\| q^{r-m} + \sum_{r=J''}^{m-1} \|\overline{\Delta}_r\| q^{r-m} \\ &\leqslant g_{J''} + q(q-1)^{-1} \|\Delta_F\| + 1/Z \sum_{r=J''}^{m-1} (\|\Delta_r\| + \|\Delta_{r+1}\|) q^{r-m} \qquad (42) \\ &\leqslant g_{J''} + (1 + 2/Z) q(q-1)^{-1} \|\Delta_F\| \leqslant (A^2+1)/A \cdot \|\Delta_{J''+1}\| \\ &\quad + (1 + 2/Z) q(q-1)^{-1} \|\Delta_F\| \leqslant \overline{W}' \|\Delta_F\| . \end{aligned}$$

In particular, $g_F \leqslant \overline{W}' \|\Delta_F\| \leqslant \overline{W} \|\Delta_F\|$.

For the choice of t_F, we define inductively points $\{t_m\}_{m=J''+1}^{F}$ in $[0, 2\pi)$ such that, with $\overline{\lambda}_m = |\alpha - T_m(t_m)|$ $(J'' + 1 \leqslant m \leqslant F)$,

$$\left\{ \begin{aligned} &\overline{\lambda}_m \leqslant \{(A^2+1)/A + 1\} \|\Delta_m\| + \|\overline{\Delta}_{m-1}\| \quad \text{or} \\ &\overline{\lambda}_m \leqslant \overline{\lambda}_{m-1} - A/2 \cdot \|\Delta_m\| + \|\overline{\Delta}_{m-1}\| + Bq^{-K} g_{m-1} . \end{aligned} \right. \qquad (43)$$

Set $t_{J''+1} = s_{J''}$. Then we have

$$\begin{aligned} \overline{\lambda}_{J''+1} = |\alpha - T_{J''+1}(s_{J''})| &\leqslant |\alpha - T_{J''}(s_{J''})| + \|\overline{\Delta}_{J''}\| + \|\Delta_{J''+1}\| \\ &\leqslant \{(A^2+1)/A + 1\} \|\Delta_{J''+1}\| + \|\overline{\Delta}_{J''}\| . \end{aligned}$$

Suppose that $t_{J''+1}, \cdots, t_{m-1}$ have been defined. If

$$\overline{\lambda}_{m-1} < (A^2+1)/A \cdot \|\Delta_m\| ,$$

we put $t_m = t_{m-1}$. Then

$$\overline{\lambda}_m \leqslant \overline{\lambda}_{m-1} + \|\overline{\Delta}_{m-1}\| + \|\Delta_m\| \leqslant \{(A^2+1)/A + 1\} \|\Delta_m\| + \|\overline{\Delta}_{m-1}\| .$$

If $\overline{\lambda}_{m-1} \geqslant (A^2+1)/A \cdot \|\Delta_m\|$, then, using Lemma 6 and 7, we choose a point t_m in $(t_{m-1} - B/\widetilde{\mu}, t_{m-1} + B/\widetilde{\mu})$ ($\widetilde{\mu}$: the smallest exponent in Δ_m) so that

$$|\{\alpha - T_{m-1}(t_{m-1})\} - \Delta_m(t_m)| \leqslant \overline{\lambda}_{m-1} - A/2 \cdot \|\Delta_m\| .$$

Then $\overline{\lambda}_m \leqslant \overline{\lambda}_{m-1} - A/2 \cdot \|\Delta_m\| + \|\overline{\Delta}_{m-1}\| + Bq^{-K} g_{m-1}$. Thus $\{t_m\}_{m=J''+1}^{F}$ are defined.

Next we show that t_F is a required point. Let j'' be the last integer satisfying $\overline{\lambda}_m \leqslant \{(A^2+1)/A + 1\} \|\Delta_m\| + \|\overline{\Delta}_{m-1}\|$ $(J''+1 \leqslant m \leqslant F)$. If $j'' = F$, then

$$\overline{\lambda}_F \leqslant \{(A^2+1)/A + 1\} \|\Delta_F\| + \|\overline{\Delta}_{F-1}\| \leqslant \{(A^2+1)/A + 1\} \|\Delta_F\| + 1/Z \cdot (\|\Delta_{F-1}\| + \|\Delta_F\|) \leqslant \{(A^2+1)/A + 1 + 2/Z\} \|\Delta_F\| \leqslant \overline{W} \|\Delta_F\| .$$

Hence the required inequality holds. Suppose $j'' \neq F$. Put $d = \sum_{m=j''+1}^{F} \|\Delta_m\|$, $\overline{d} = \sum_{m=j''}^{F-1} \|\overline{\Delta}_m\|$ and $\overline{g} = \sum_{m=j''}^{F-1} g_m$. Then

$$\overline{d} \leqslant 1/Z \sum_{m=j''}^{F-1} (\|\Delta_m\| + \|\Delta_{m+1}\|) \leqslant 2/Z \cdot d + 1/Z \cdot \|\Delta_{j''}\| \leqslant 2/Z \cdot d + 1/Z \cdot \|\Delta_F\|$$

and

$$\overline{g} \leqslant q(q-1)^{-1} \{g_{j''} + \|T_{F-1} - T_{j''}\|\} \leqslant q(q-1)^{-1} \{\overline{W}' \|\Delta_F\| + d + \overline{d}\} \leqslant q(q-1)^{-1}(1 + 2/Z) d + q(q-1)^{-1}(\overline{W}' + 1/Z) \|\Delta_F\| .$$

By these inequalities and (28), we have

$$\begin{aligned}
\overline{\lambda}_F &\leqslant \overline{\lambda}_{F-1} - A/2 \cdot \|\Delta_F\| + \|\overline{\Delta}_{F-1}\| + Bq^{-K} g_{F-1} \\
&\leqslant \cdots \leqslant \overline{\lambda}_{j''} - A/2 \cdot d + \overline{d} + Bq^{-K} \overline{g} \\
&\leqslant \overline{\lambda}_{j''} - A/2 \cdot d + (2/Z \cdot d + 1/Z \cdot \|\Delta_F\|) \\
&\qquad + Bq^{-K+1}(q-1)^{-1} \{(1 + 2/Z) d + (\overline{W}' + 1/Z) \|\Delta_F\|\} \\
&\leqslant \overline{\lambda}_{j''} + \{1/Z + Bq^{-K+1}(q-1)^{-1}(\overline{W}' + 1/Z)\} \|\Delta_F\| \\
&\leqslant \{(A^2+1)/A + 1\} \|\Delta_{j''}\| + \|\overline{\Delta}_{j''-1}\| \\
&\qquad + \{1/Z + Bq^{-K+1}(q-1)^{-1}(\overline{W}' + 1/Z)\} \|\Delta_F\| \\
&\leqslant \{(A^2+1)/A + 1\} \|\Delta_{j''}\| + 1/Z \cdot (\|\Delta_{j''-1}\| + \|\Delta_{j''}\|) \\
&\qquad + \{1/Z + Bq^{-K+1}(q-1)^{-1}(\overline{W}' + 1/Z)\} \|\Delta_F\| \\
&\leqslant \{(A^2+1)/A + 1 + 3/Z + Bq^{-K+1}(q-1)^{-1}(\overline{W}' + 1/Z)\} \|\Delta_F\| \\
&\leqslant \overline{W} \|\Delta_F\| .
\end{aligned}$$

This completes the proof of this lemma.

Now we give three constants $\overline{\epsilon}, \rho, W$ in Lemma 13. Let $\overline{\epsilon}$ be the constant given in Definition 16. Put $\rho = A/8$ and $W = \max\{2KZ(\overline{W}+1), q^{2KZ}\overline{W}2KZ\}$.

Let $|\alpha| \leqslant \rho \sum_{k=1}^{\infty} |a_k| = A/8 \cdot \|S\|$ and (F, t_F) be as in Lemma 20. We choose an integer E such that a_E is one of coefficients having the largest modulus in Δ_F. Then $\|\Delta_F\| \leqslant 2KZ\,|a_E|$. Put $\theta_E = t_F$. Then (E, θ_E) is a required pair, since

$$\begin{cases} |\alpha - S_E(\theta_E)| \leqslant |\alpha - T_F(t_F)| + \|T_F - S_E\| \leqslant (\overline{W}+1)\,\|\Delta_F\| \\ \qquad\qquad\qquad \leqslant 2KZ(\overline{W}+1)\,|a_E| \leqslant W\,|a_E| \\ \sup\limits_{k>E} |a_k| \leqslant \sup\limits_{m>F} \|\Delta_m\| = \|\Delta_F\| \leqslant 2KZ\,|a_E| \leqslant W\,|a_E| \\ G_{E-1} = \sum\limits_{k=1}^{E-1} |a_k|\, q^{k-(E-1)} \leqslant q^{\nu_F - E+1} g_F \leqslant q^{2KZ}\,\overline{W}\,\|\Delta_F\| \\ \qquad\qquad\qquad \leqslant q^{2KZ}\,\overline{W}2KZ\,|a_E| \leqslant W\,|a_E|. \end{cases}$$

This completes the proof of Lemma 13.

Remark 21. – We also know that an unbounded lacunary power series $f(z)$ takes every complex value infinitely often in every sector $\{z \in D\,;\, \alpha < \arg z < \beta\}$. In fact, let us note that a set $\{t \in [0, 2\pi)\,;\, \lim\limits_{r\to 1} |f(re^{it})| = +\infty\}$ is dense in $[0, 2\pi)$, if $\lim\limits_{k\to\infty} c_k = 0$ ([12]). Hence we may assume

$$\lim_{r\to 1} |f(re^{i\alpha})| = \lim_{r\to 1} |f(re^{i\beta})| = +\infty.$$

The proof is now along the same line as the proof of Theorem 1.

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