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JEAN ESTERLÉ

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A COMPLEX-VARIABLE PROOF OF THE WIENER TAUBERIAN THEOREM

by Jean ESTERLE (*)

1. Introduction.

Denote by $\hat{f}$ the Fourier transform of an element f of $L^1(R)$ and put, for every subset S of $L^1(R)$

$$Z(S) = \{t \in R \mid \hat{f}(t) = 0 \quad \text{for every} \quad f \in S\}.$$

The Wiener tauberian theorem states that if a closed ideal I of $L^1(R)$ satisfies $Z(I) = \varnothing$ then $I = L^1(R)$. In fact, as every character of $L^1(R)$ is a function of the form $f \to \hat{f}(x)$, where x is some element of R, the fact that $Z(I) = \varnothing$ is equivalent to the fact that I is not contained in the kernel of any character of $L^1(R)$ which means that the quotient algebra $L^1(R)/I$ is radical.

We thus see that the Wiener tauberian theorem may be stated as follows.

Let I be a closed ideal of $L^1(R)$. If the quotient algebra $L^1(R)/I$ is radical, then $I = L^1(R)$.

The classical proof of the Wiener tauberian theorem depends on the study of the Fourier transform. We will give here a very different proof, based upon the Ahlfors-Heins theorem for bounded analytic functions on the right-hand half-plane. Using this theorem we show that if $(a^t)_{\mathrm{Re}\, t > 0}$ is an analytic semigroup in a radical Banach algebra $\mathscr{R}$ satisfying $\sup_{\mathrm{Re}\, t \geqslant 1} e^{-\|t\|^\alpha}.\|a^t\| < +\infty$ for some $\alpha < 1$ then $a^t = 0$ for $\mathrm{Re}\, t > 0$.

Using an analytic extension of the fundamental semigroup of the heat equation for the real line, it is easy to see that $L^1(R)$ possesses an analytic semi group $(a^t)_{\mathrm{Re}\, t > 0}$ such that $\lim_{\substack{t > 0 \\ t \to 0}} \|f * a^t - f\| = 0$ for every $f \in L^1(R)$ and such that $\|a^t\| \leqslant \sqrt{|t|}$ if $\mathrm{Re}\, t \geqslant 1$.

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It follows then from the above result that if $L^1(R)/I$ is radical for some closed ideal I of $L^1(R)$ then $L^1(R)/I = \{0\}$ and $I = L^1(R)$, which proves the Wiener tauberian theorem for $L^1(R)$. A similar result holds as well known for $L^1(G)$ if G is any locally compact abelian group.

If G is not abelian, several extensions are possible. Eymard defined in [2] the Fourier algebra $A(G)$ for arbitrary G. The characters of $A(G)$ are the maps $f \to f(x)$, where $x \in G$, and if we put, for $S \subseteq A(G)$, $Z(S) = \{x \in G | f(x) = 0 \text{ for every } f \in A(G)\}$, it is true that $Z(I) \neq \emptyset$ for every closed ideal I of $A(G)$ distinct of $A(G)$.

In another direction H. Leptin distinguishes in [4] the strong and weak Wiener property for non commutative locally compact groups. A locally compact group G is said to have the strong Wiener property if every two-sided proper closed ideal I of $L^1(G)$ is annihilated by some $*$-irreductible representation of $L^1(G)$ on a Hilbert space, and G is said to have the weak Wiener property if $I = L^1(G)$ for every closed two sided ideal I of $L^1(G)$ such that $L^1(G)/I$ is radical. These two properties are equivalent if $L^1(G)$ is symmetric [3], and G has the strong Wiener property if G is compact, or if G is a connected nilpotent Lie group [4]. The argument used here shows that G has the weak Wiener property if $L^1(G)$ possesses an analytic semi group $(a^t)_{\mathrm{Re}\, t > 0}$ such that $\sup_{\mathrm{Re}\, t \geqslant 1} e^{-\|t\|^\alpha} \|a^t\| < +\infty$ for some $\alpha < 1$, $\lim_{\substack{t \to 0 \\ t > 0}} \|f * a^t - f\| = 0$ for every $f \in L^1(G)$.

I do not know any conditions on G which ensure the existence of such a semi group, but it follows from a general result of Sinclair [5] about Banach algebras with two-sided approximate identities that if $L^1(G)$ is separable then $L^1(G)$ possesses an analytic semigroup $(a^t)_{\mathrm{Re}\, t > 0}$ such that $\|a^t\| \leqslant 1$ for $t > 0$ and such that $\lim_{\substack{t \to 0 \\ t > 0}} \|f * a^t - f\| = 0$ for every $f \in L^1(G)$.

2. Analytic semigroups in radical Banach algebras.

We will show in this section that if an analytic semigroup $(a^t)_{\mathrm{Re}\, t > 0}$ in a radical Banach algebra $\mathscr{R}$ satisfies $\sup_{\mathrm{Re}\, t \geqslant 1} e^{-\|t\|^\alpha} \|a^t\| < +\infty$ for some $\alpha < 1$ then the semigroup $(a^t)_{\mathrm{Re}\, t}$ is the zero semigroup.

This fact is an easy consequence of the following classical result of the theory of analytic functions.

THEOREM 2.1. — (*Ahlfors Heins theorem*). *Let* g *be a continuous bounded function over the closed right-hand half plane* $\Pi = \{t \in \mathrm{C} | \mathrm{Re}\, t \geqslant 0\}$ *which is analytic for* $\mathrm{Re}\, t > 0$. *If* g *is not identically* 0, *there exists* $c \in \mathrm{R}$ *such that*

$$\limsup_{r \to \infty} \frac{\log |g(re^{i\theta})|}{r} = c \cos \theta$$

for every $\theta \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$.

Proof. — See [1], chapter 7 where this result is proved under a weaker but more complicated hypothesis on g. In fact a stronger conclusion holds. There exists a set S of outer capacity zero such that $\lim_{r \to \infty} \frac{\log |g(re^{i\theta})|}{r} = c \cos \theta$ for $|\theta| < \frac{\pi}{2}$, $\theta \notin \mathrm{S}$.

The proof is based upon a rather complicated study of Blaschke products over the half-plane.

COROLLARY 2.2. — *Let* g *be a continuous function over the closed half plane* $\Pi = \{t \in \mathbf{C}/\mathrm{Re}\, t \geqslant 0\}$ *which is analytic for* $\mathrm{Re}\, t > 0$ *and satisfies*

$$\sup_{\mathrm{Re}\, t \geqslant 0} e^{-\|t\|^{\alpha}} |g(t)| < +\infty$$

for some $\alpha < 1$. *If* $\limsup_{t \to \infty} \frac{\log |g(t)|}{t} = -\infty$, *then* g *is identically zero.*

Proof. — Put, for $\mathrm{Re}\, t \geqslant 0 : h(t) = e^{-t^{\beta}}.g(t)$ where $\alpha < \beta < 1$. We have

$$\mathrm{Re}(t^{\beta}) = |t|^{\beta} \cos [\beta \operatorname{Arg} t] \geqslant |t|^{\beta} \cos \frac{\beta\pi}{2}.$$

So there exists $\delta \in \mathrm{R}$ such that $|t|^{\alpha} - \mathrm{Re}\,(t^{\beta}) \leqslant \delta$ for $\mathrm{Re}\, t \geqslant 0$, and we have, for $\mathrm{Re}\, t > 0$ and some $\mathrm{K} > 0$

$$|h(t)| \leqslant \mathrm{K} e^{|t|^{\alpha} - \mathrm{Re}\,(t^{\beta})} \leqslant \mathrm{K} e^{\delta}.$$

We thus see that h is bounded. Also $\lim_{t \to \infty} \frac{\log e^{-t^{\beta}}}{t} = 0$, so

$$\limsup_{t \to \infty} \frac{\log |h(t)|}{t} = \limsup_{t \to \infty} \frac{\log |g(t)|}{t} = -\infty$$

and the Alhfors Heins theorem implies that h is identically zero. So g is identically zero, and the corollary is proved.

We have the following easy lemma.

LEMMA 2.3. – *Let $\mathcal{R}$ be a radical Banach algebra, and let $(a^t)_{t>0}$ be a continuous semigroup in $\mathcal{R}$ over the positive reals. Then* $\limsup_{t\to\infty} \dfrac{\log \|a^t\|}{t} = -\infty$.

Proof. – For $t > 0$ denote by $[t]$ the unique integer n satisfying $n \leqslant t < n+1$. There exists $M > 0$ such that $\|a^t\| \leqslant M$ for every $t \in [1, 2]$.

We have, for $t \geqslant 2$

$$\|a^t\| \leqslant \|a^{[t]-1}\| \, \|a^{t-[t]+1}\| \leqslant M\|a^{[t]-1}\|.$$

As $\mathcal{R}$ is radical, $\lim_{t\to\infty} \|a^{[t]-1}\|^{\frac{1}{[t]-1}} = 0$, so

$$\limsup_{t\to\infty} \frac{\log \|a^{[t]-1}\|}{[t]-1} = -\infty, \ \limsup \frac{\log \|a^{[t]-1}\|}{t} = -\infty$$

and

$$\limsup_{t\to\infty} \frac{\log \|a^t\|}{t} \leqslant \limsup_{t\to\infty} \frac{\log M}{t} + \limsup_{t\to\infty} \frac{\log \|a^{[t]-1}\|}{t} = -\infty.$$

This proves the lemma.

We now obtain the following theorem.

THEOREM 2.4. – *Let $\mathcal{R}$ be a radical Banach algebra and let $(a^t)_{\mathrm{Re}\, t>0}$ be an analytic semigroup in $\mathcal{R}$ over the right-hand open half plane.*

If $\sup_{\mathrm{Re}\, t\geqslant 1} e^{-|t|^\alpha}\|a^t\| < +\infty$ for some $\alpha < 1$, the semigroup $(a^t)_{\mathrm{Re}\, t>0}$ is the zero semi group.

Proof. – Let ℓ be a continuous linear form over $\mathcal{R}$. Put, for $\mathrm{Re}\, t > -1$

$$g(t) = \ell(a^{t+1}).$$

Choose some element β of $]\alpha, 1[$. We have

$$\sup_{\mathrm{Re}\, t\geqslant 0} e^{-|t|^\beta}|g(t)| \leqslant \|\ell\| \sup_{\mathrm{Re}\, t\geqslant 0} e^{-|t|^\beta + |t+1|^\alpha} \sup_{\mathrm{Re}\, t\geqslant 1} e^{-|t|^\alpha}\|a^t\| < +\infty.$$

As g is analytic for $\operatorname{Re} t > -1$, g is continuous for $\operatorname{Re} t \geqslant 0$ and analytic for $\operatorname{Re} t > 0$. As $\mathscr{R}$ is radical, we have $\limsup_{t\to\infty} \frac{\log \|a^t\|}{t} = -\infty$, so

$$\limsup_{t\to\infty} \frac{\log |g(t)|}{t} \leqslant \limsup_{t\to\infty} \frac{\log \|\ell\| + \log \|a\|}{t} + \limsup_{t\to\infty} \frac{\log \|a^t\|}{t} = -\infty .$$

It follows then from corollary 2.2 that $g(t) = 0$ for $\operatorname{Re} t \geqslant 0$, so in fact $g(t) = 0$ for $\operatorname{Re} t > -1$, and $\ell(a^t) = 0$ for every continuous linear form over $\mathscr{R}$ and every complex number t such that $\operatorname{Re} t > 0$. This shows that $(a^t)_{\operatorname{Re} t>0}$ is the zero semi group and the theorem is proved.

3. The Wiener tauberian theorem and the heat equation.

Put, for $\operatorname{Re} t > 0, x \in \mathrm{R}$:

$$a^t(x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{t}}.$$

Clearly $a^t \in \mathrm{L}^1(\mathrm{R})$, and we have

$$\|a^t\| = \frac{1}{\sqrt{\pi|t|}} \int_{-\infty}^{\infty} e^{-\frac{\operatorname{Re} t x^2}{|t|^2}} dx = \sqrt{\frac{|t|}{\operatorname{Re} t}}.$$

So $\|a^t\| = 1$ if $t > 0$, and $\|a^t\| \leqslant \sqrt{|t|}$ if $\operatorname{Re} t \geqslant 1$. A routine well-known computation shows that $a^{t+t'} = a^t * a^{t'}$ for $t > 0$, $t' > 0$ and that $\lim_{\substack{t\to 0\\ t>0}} \|f * a^t - f\| = 0$ for every $f \in \mathrm{L}^1(\mathrm{R})$. Also the map $t \to a^t$ is an analytic map from the right-hand open half plane into $\mathrm{L}^1(\mathrm{R})$. Fix $t_0 > 0$. The analytic function $t \to a^{t_0+t} - a^{t_0} * a^t$ vanishes over the positive reals, so $a^{t_0+t} = a^{t_0} * a^t$ for $t_0 > 0$, $\operatorname{Re} t > 0$. Now fix t_1 such that $\operatorname{Re} t_1 > 0$. The analytic function $t \to a^{t+t_1} - a^t * a^{t_1}$ vanishes for $t > 0$, so $a^{t+t_1} = a^t * a^{t^1}$ for $\operatorname{Re} t > 0$, $\operatorname{Re} t_1 > 0$.

The Wiener tauberian theorem follows now easily from theorem 2.4.

THEOREM 3.1. — *Let* I *be a closed ideal of* $\mathrm{L}^1(\mathrm{R})$. *If the quotient algebra* $\mathrm{L}^1(\mathrm{R})/\mathrm{I}$ *is radical, then* $\mathrm{I} = \mathrm{L}^1(\mathrm{R})$.

Proof. — Let $(a^t)_{\operatorname{Re} t>0}$ be the analytic semi group in $\mathrm{L}^1(\mathrm{R})$ introduced above. For $\operatorname{Re} t > 0$ denote by $\bar{a}^t$ the image of a^t in $\mathrm{L}^1(\mathrm{R})/\mathrm{I}$.

The semi group $(\bar{a}^t)_{\mathrm{Re}\,t>0}$ is an analytic semi group in $L^1(R)/I$, and we have $\|\bar{a}^t\| \leqslant \|a^t\| \leqslant \sqrt{|t|}$ for $\mathrm{Re}\,t \geqslant 1$.

It follows from theorem 2.4 that $(\bar{a}^t)_{\mathrm{Re}\,t>0}$ is the zero semi group. In particular $a^t \in I$ if $t > 0$. Now let f be any element of $L^1(R)$. Then $f * a^t \in I$ for every $t > 0$ and as I is closed we have $f = \lim_{\substack{t\to 0\\ t>0}} f * a^t \in I$. So $I = L^1(R)$, and the theorem is proved.

The semi group $(a^t) > 0$ is well known and often called the « fundamental semi group of the heat equation ». If we denote by $u(x,t)$ the temperature of the point x of an homogeneous line at the instant t, we have

$$u(x,t) = (a^t * f)(x)$$

where

$$f(x) = u(x,0).$$

We thus see that the Wiener tauberian theorem follows from the fact that the fundamental semi group of the heat equation possesses an analytic extension to the right hand open half plane which grows slowly at infinity for $\mathrm{Re}\,t \geqslant 1$.

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Jean Esterle,

Laboratoire associé au C.N.R.S.
U.E.R. de Mathématiques et Informatique
Université de Bordeaux I
351, Cours de la Libération
33405 Talence (France).