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<http://www.numdam.org/item?id=AIF_1980__30_1_159_0>

CLASSIFICATION

OF CONNECTED UNIMODULAR LIE GROUPS

WITH DISCRETE SERIES

by NGUYEN HUU ANH

Let G be a separable locally compact group with center Z. An irreducible representation[1] of G is said to be a member of the discrete series if it has some non zero matrix coefficient which is square integrable modulo Z. In this article we introduce the notion of *H-groups* and prove that the discrete series exist for these groups. Moreover, apart from some technical requirements, it will be proved that a connected *unimodular* Lie group G with center Z such that the center of $G/\mathrm{Rad}\,G$ is finite has discrete series if and only if G may be written as $G = HS$, $H \cap S = Z^0$, where H is a H-group with center Z^0 and S is a connected reductive Lie group with discrete series such that $\mathrm{Cent}(S)/Z$ is compact.

In particular, if G is the semidirect product of a simply connected solvable Lie group and a connected semisimple Lie group with finite center, then the subgroup S may be chosen so that it is the direct product of a vector group T such that $T/T \cap Z$ is compact and a connected semisimple Lie group with discrete series. Thus in view of the results of Harish-Chandra on the discrete series of semisimple Lie groups (cf. [6]), the problem of classifying connected unimodular Lie groups with discrete series is completely solved. Finally it should be noted that the results in this article contain our previous result on algebraic groups (cf. [2]) as a special case, for H will be proved to be nilpotent if G is locally algebraic (see Lemma 2.2).

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(1) *Warning:* "representation" means "continuous unitary representation" in a Hilbert Space.

Mackey's machinery on group extensions plays again a very important role. The strictly ergodic case may be avoided thanks to Theorem 1.1 which is interesting in its own right. The notion of H-groups is introduced in section 2. Theorem 2.9 is essential for carrying out the theory of Pfaffian·polynomial exactly as in [15]. Theorem 2.12 contains the most important results in the theory of representations of H-groups. In section 3 we treat the most general case. Theorems 3.4 and 3.6 are the main results of the article. Finally in section 4. we give another characterization of connected unimodular Lie groups with discrete series. In particular Theorem 4.4 gives the solution of a conjecture stated in [15].

In the following we continue to use the notations of [1] and [2]. In particular $\mathfrak{h}$, $\mathfrak{z}$,... are the Lie algebras of the Lie groups $H, Z,...$; $(\mathfrak{h}/\mathfrak{z})^*$ is identified with the annihilator of $\mathfrak{z}$ in $\mathfrak{h}^*$, and similarly $(H/Z)^\smallfrown$ is identified with the annihilator of Z in $\hat{H}$. The neutral component of a Lie group G is denoted by G^0 , and the simply connected groups are always supposed to be connected.

1. Some preliminary results.

Recall that if G is a separable locally compact group, Z a closed normal subgroup of G, χ a character of Z, then a representation π of G is said to be a χ-representation if $\pi|Z \simeq$ mult χ. If moreover π has a non zero matrix coefficient which is square integrable modulo Z with respect to the right Haar measure of G/Z, then we say that π is square integrable mod Z. In this case we have Schur's orthogonality relation and can define the formal degree $d(\pi)$ of π (cf. [1], [9], [13]).

THEOREM 1.1. — *Let* G *be a separable locally compact unimodular group. Let* $Z \subset K$ *be closed normal abelian subgroups of* G. *Let* χ *be a character of* Z, *and* π *be a square integrable* mod Z *irreducible* χ-*representation of* G. *Then the quasi orbit of* G/K *in* $\hat{K}$ *defined by* $\pi|K$ *is transitive.*

Proof. — Since $\hat{K}$ is smooth and of type I, we may assume that the system of imprimitivity for π is the canonical one associated

to a measure μ belonging to the quasi orbit defined by $\pi | K$ (cf. [8], Theorems 7.6 and 5.6). In particular the projection valued measure P belonging to $\pi | K$ is just the multiplication by characteristic functions on $\hat{K}$. Moreover there is a measurable cocycle c on $G \times \hat{K}$ taking values in the unitary group of a Hilbert space $\mathcal{H}$ such that for all g in G and all functions f in $L^2(\hat{K}, d\mu) \otimes \mathcal{H}$ we have:

$$\pi(g)f(\lambda) = c(g, \lambda) \, \rho(g, \lambda)^{1/2} \, f(\lambda g) \quad \mu\text{-a.e. } \lambda \qquad (1.1)$$

where ρ is a Borel function such that for each g, $\rho(g, .)$ is the Radon-Nikodym derivative of the translate μ^g of μ with respect to μ.

Now the action of K on $\hat{K}$ is trivial. Therefore it follows from (1.1) that for all k in K and for all f we have:

$$\pi(k) f(\lambda) = c(k, \lambda) f(\lambda) \quad \text{a.e. } \lambda$$

this together with the fact that P is precisely the projection-valued measure associated to the direct integral decomposition of $\pi | K$ into the characters λ show that $c(k, \lambda) = \lambda(k) \, \mathrm{Id}_{\mathcal{H}}$ for all k in K, and for almost all λ in $\hat{K}$.

On the other hand by the same argument as in [1] we see that μ — viewed as a measure on the G-invariant subset $\lambda_0(K/Z)\hat{\ }$ — is absolutely continuous with respect to the translate of the Haar measure $d\lambda$ of $(K/Z)\hat{\ }$ by λ_0, where λ_0 is some fixed extension of χ to K. Furthermore $d\lambda$ is relatively invariant in the sense of [14]. Hence we may replace μ by the restriction to its support of the translate of $d\lambda$ by λ_0. This being done, we have

$$(\pi(kg) f_1, f_2) = \int_{\hat{K}} \rho(g, \lambda)^{1/2} \, (c(k, \lambda) \, c(g, \lambda) \, f_1(\lambda g), f_2(\lambda)) \, d\mu(\lambda)$$

$$= \lambda_0(k) \int_{(K/Z)\hat{\ }} \lambda(k) \, F_g(\lambda) \, d\lambda \qquad (1.2)$$

where $F_g(\lambda) = \rho(g, \lambda_0 \lambda)^{1/2} \, (c(g, \lambda_0 \lambda) \, f_1(\lambda_0 \lambda g), f_2(\lambda_0 \lambda))$ if $\lambda_0 \lambda \in \operatorname{supp} \mu$ and is 0 otherwise.

Note that F_g is integrable with respect to $d\lambda$. Therefore, if we denote by $\hat{F}_g(k)$ its Fourier transform, then

$$\int_{G/Z} |(\pi(g) f_1, f_2)|^2 \, dg = \int_{G/K} \int_{K/Z} |(\pi(kg) f_1, f_2)|^2 \, dk \, d\tilde{g}$$

$$= \int_{G/K} \int_{K/Z} |\hat{F}_g(k)|^2 \, dk \, d\tilde{g} \qquad (1.3)$$

where dg and $d\widetilde{g}$ are Haar measures of G/Z and G/K respectively.

Now it follows from (1.3) that $\hat{F}_g(\cdot)$ is square integrable for almost all g in G. Therefore, if the Haar measure of K/Z is properly normalized, then we have by the Plancherel formula on K/Z:

$$\int_{G/Z} |(\pi(g)f_1, f_2)|^2 \, dg = \int_{G/K} \int_{(K/Z)^\wedge} |F_g(\lambda)|^2 \, d\lambda \, d\widetilde{g} \qquad (1.4)$$

$$= \int_{G/K} \int_{\hat{K}} \rho(g, \lambda) |(c(g, \lambda) f_1(\lambda g), f_2(\lambda))|^2 \, d\mu(\lambda) \, d\widetilde{g}.$$

On the other hand by Schur's orthogonality relation:

$$\int_{G/Z} |(\pi(g)f_1, f_2)|^2 \, dg = d(\pi)^{-1} \|f_1\|^2 \|f_2\|^2. \qquad (1.5)$$

Let us apply (1.4) and (1.5) for $f_1 = \varphi \otimes u$, $f_2 = \psi \otimes v$, where φ and ψ belong to $L^2(\hat{K}, d\mu)$, and u, v are in $\mathcal{H}$ such that $\|u\| = \|v\| = 1$. We have:

$$\int_{\hat{K}} |\psi(\lambda)|^2 \, d\mu(\lambda) \int_{G/K} \rho(g, \lambda) |\varphi(\lambda g)|^2 |(c(g, \lambda)u, v)|^2 \, d\widetilde{g}$$

$$= d(\pi)^{-1} \|\varphi\|^2 \|\psi\|^2.$$

In particular, we have, for μ almost all λ in $\hat{K}$:

$$\int_{G/K} \rho(g, \lambda) |\varphi(\lambda g)|^2 |(c(g, \lambda)u, v)|^2 \, d\widetilde{g} = d(\pi)^{-1} \|\varphi\|^2.$$

Substituting successively for φ in the above equality the characteristic functions of a countable separating family of Borel subsets of $\hat{K}$, we see that for all u, v belonging to a fixed orthonormal basis $\mathcal{B}$ of $\mathcal{H}$ and for almost all λ in $\hat{K}$:

$$\int_{G/K} \rho(g, \lambda) \varphi(\lambda g) |(c(g, \lambda)u, v)|^2 \, d\widetilde{g} = d(\pi)^{-1} \int_{\hat{K}} \varphi(\nu) \, d\mu(\nu) \quad (1.6)$$

where φ is an arbitrary non negative measurable function on $\hat{K}$.

Assume that μ is not concentrated in the orbit λG, where λ satisfies (1.6). Then the characteristic function of λG is μ-negligible, and hence we have according to (1.6):

$$\rho(g, \lambda) |(c(g, \lambda)u, v)|^2 = 0$$

for all u, v in $\mathcal{B}$ and for almost all g in G. This is a contradiction.   QED

Remark. – Apparently the only assumption needed on π is that all of its matrix coefficients are square integrable mod Z and satisfy Schur's orthogonality relation. However, as proved in [3], this condition is equivalent to the fact that π is a square integrable mod Z *irreducible* representation.

LEMMA 1.2. – *Let ρ be a representation of a connected Lie group G in a finite-dimensional real vector space V. Let λ be a V-valued cocycle of G with respect to ρ. Assume that there is $c_0 \in \text{Cent}(G)$ such that $\rho(c_0) = \alpha(c_0) . \text{Id}_V$, $\alpha(c_0) \neq 1$. Then λ is a coboundary. In fact we have:*

$$\lambda(g) = (\rho(g) - \text{Id}_V)\frac{\lambda(c_0)}{\alpha(c_0) - 1} .$$

Proof. – We have:

$$\lambda(g) + \rho(g)\lambda(c_0) = \lambda(gc_0) = \lambda(c_0 g) = \lambda(c_0) + \rho(c_0)\lambda(g).$$

Therefore

$$\lambda(g) = (\rho(g) - \text{Id}_V)\frac{\lambda(c_0)}{\alpha(c_0) - 1}. \qquad\qquad \text{QED}$$

PROPOSITION 1.3. – *Let G, ρ, V, λ be as above. Assume also that G is reductive and ρ is irreducible. Then either (i) ρ is trivial and $\lambda(G) = V$, $\dim V = 1$, or (ii) λ is a coboundary.*

Proof. – If $\lambda = 0$, then it is a coboundary and (i) does not hold. Thus suppose that $\lambda \neq 0$. First assume that ρ is trivial so that $\dim V = 1$. Then $\lambda(G)$ is plainly a linear subspace of V because G is connected and hence $\lambda(G) = V$. Note that λ is not a coboundary in this case.

Now assume that ρ is non trivial. Put $N = \ker \rho$. Then λ is a homomorphism from N into the abelian group V. Therefore if $\lambda(N) \neq 0$, we may find an element n_0 in $N \cap \text{Cent}(G)$ such that $\lambda(n_0) \neq 0$. We have

$$\lambda(g) + \rho(g)\lambda(n_0) = \lambda(gn_0) = \lambda(n_0 g) = \lambda(n_0) + \lambda(g)$$

i.e. $\qquad\qquad \rho(g)\lambda(n_0) = \lambda(n_0) \qquad \forall g \in G.$

This however contradicts the fact that ρ is a non trivial irreducible representation of G. Therefore $\lambda(N) = 0$, and λ may be viewed as a cocycle with respect to the representation of G/N induced from ρ, i.e. we may assume that ρ is faithful. Recall that ρ is irreducible so that $\rho(G)$ is the neutral component of a real algebraic subgroup of $GL(V)$. By Lemma 1.2 it is sufficient to consider the case when this algebraic group contains no R-split torus in its center, i.e. when the center of $\rho(G)$ is compact. Since every finite dimensional representation of such a group is completely reducible, λ is a coboundary (cf. [12], Théorème 2, exposé 4).

QED

2. Representation theory of H-groups.

DEFINITION 2.1. — *Let H be a connected Lie group, and Z a central subgroup of H. Then H is said to be a H-group if there exists a linear form ℓ in $\mathfrak{h}^*$ such that the orbit of ℓ in $\mathfrak{h}^*$ via the coadjoint representation of H is the hyperplane $\ell + (\mathfrak{h}/\mathfrak{z})^*$.*

Remark. — It follows immediately from the above definition that the symplectic form associated to $\ell : B_\ell(\xi, \eta) = \ell([\xi, \eta])$ is non degenerate on $\mathfrak{h}/\mathfrak{z}$, i.e. $\mathfrak{z}$ is the Lie algebra of the centralizer G_ℓ of ℓ in G. Since the orbit $\ell + (\mathfrak{h}/\mathfrak{z})^*$ is simply connected we must have $Z = G_\ell = \text{Cent}(H)$. In particular Z is connected. Moreover H is simply connected if and only if Z is simply connected.

LEMMA 2.2. — *Let H be a H-group with center Z. Then H is solvable. If we assume further that H is locally algebraic, then H is nilpotent.*

Proof. — Let $\mathfrak{s}$ be a maximal semisimple subalgebra of $\mathfrak{h}$ so that $\mathfrak{h}$ contains an $\mathfrak{s}$-invariant subspace $\overline{\mathfrak{h}}$ complementing $\mathfrak{z}$. Let $\ell \in \mathfrak{h}^*$ be such that the H-orbit of ℓ in $\mathfrak{h}^*$ is $\ell + (\mathfrak{h}/\mathfrak{z})^*$. Then there exists ℓ_1 in this orbit such that $\ell_1 = 0$ on $\overline{\mathfrak{h}}$. Since the symplectic form B_{ℓ_1} is also non degenerate on $\mathfrak{h}/\mathfrak{z}$ and $\mathfrak{s}$ is orthogonal to $\overline{\mathfrak{h}}$ with respect to B_{ℓ_1} we see that $\mathfrak{s} = 0$, i.e. $\mathfrak{h}$ is solvable. In case H is locally algebraic we may take $\mathfrak{s}$ to be

the reductive subalgebra in a fixed Levi's decomposition of $\mathfrak{h}$. Then the above argument shows that $\mathfrak{s} = 0$, i.e. $\mathfrak{h}$ is nilpotent.   QED

LEMMA 2.3. — *Let G be a Lie group, Z a central subgroup of G . Assume that G has a normal subgroup H which is a H-group with center Z . Then there exists a closed subgroup S such that*

$$G = HS, \quad H \cap S = Z . \qquad (2.1)$$

Moreover S may be chosen so that $S^0 = S \cap G^0$.

Proof. — Let $\ell \in \mathfrak{h}^*$ be such that the H-orbit of ℓ in $\mathfrak{h}^*$ is precisely $\ell + (\mathfrak{h}/\mathfrak{z})^*$. Put $S = \{x \in G | \ell \circ \mathrm{Ad}_{\mathfrak{h}}(x) = \ell\}$. Then   S clearly satisfies (2.1). Note that $S \cap G^0$ is the centralizer of ℓ in G^0 and hence $S \cap G^0$ is connected since the G^0-orbit $\ell + (\mathfrak{h}/\mathfrak{z})^*$ is simply connected. Hence $S \cap G^0 = S^0$.   QED

LEMMA 2.4. — *Let G, H, S, Z, ℓ be as above. Assume also that G is connected and that there is an extension ℓ_1 of ℓ to $\mathfrak{g}$ such that the S-orbit of $\ell_1|\mathfrak{s}$ is $\ell_1|\mathfrak{s} + (\mathfrak{s}/\mathfrak{z})^*$. Then the G-orbit of ℓ_1 in $\mathfrak{g}^*$ is $\ell_1 + (\mathfrak{g}/\mathfrak{z})^*$. In particular G is a H-group with center Z .*

Proof. — Let $\ell_2 \in \mathfrak{g}^*$ be such that $\ell_2|\mathfrak{z} = \ell_1|\mathfrak{z}$. We have to prove that ℓ_2 belongs to the same G-orbit as ℓ_1 . By the assumption, there exist x_1 in H and x_2 in S such that

$$(\ell_1 \circ \mathrm{Ad}\, x_1)|\mathfrak{h} = \ell \circ \mathrm{Ad}\, x_1 = \ell_2|\mathfrak{h}$$

and $(\ell_2 \circ \mathrm{Ad}\, x_1^{-1})|\mathfrak{s} = (\ell_1 \circ \mathrm{Ad}\, x_2)|\mathfrak{s}$.

Then we have $\ell_2 = \ell_1 \circ \mathrm{Ad}(x_2 x_1)$.   QED

Example 2.5. — In view of the results in [10] and [15] we see that the class of simply connected H-groups contains that of unimodular exponential groups with discrete series and hence that of simply connected nilpotent Lie groups with discrete series. On the other hand using Lemma 2.4 we may construct non-type I H-groups. In particular these groups are not of exponential type. Let indeed $\mathfrak{h}$ be the five dimensional Heisenberg algebra with basis ζ_1, ξ_i, η_i $(i = 1, 2)$ such that the only non zero brackets satisfied by the

members of this basis are:

$$[\xi_i, \eta_i] = \zeta_1, \quad i = 1, 2.$$

Let $\hat{s}$ be the three dimensional Heisenberg algebra with basis ζ_2, ξ_3, η_3 such that $[\xi_3, \eta_3] = \zeta_2$. Let $\widetilde{g}$ be the semi direct product of $\mathfrak{h}$ and $\hat{s}$ such that ζ_2 and ξ_3 commute with the elements of $\mathfrak{h}$ and:

$$[\eta_3, \xi_1] = \eta_1, \quad [\eta_3, \eta_1] = -\xi_1, \quad [\eta_3, \xi_2] = \theta\eta_2, \quad [\eta_3, \eta_2] = -\theta\xi_2,$$
$$[\eta_3, \zeta_1] = 0$$

where θ is a fixed irrational number. Let g be the quotient algebra of $\widetilde{g}$ by the ideal $\mathbf{R}(\zeta_1 - \zeta_2)$. Let G be the simply connected Lie group with Lie algebra g. Let H, S be the analytic subgroups of G corresponding to $\mathfrak{h}$, $\hat{s}$ respectively, where $\mathfrak{h}$, $\hat{s}$ are identified with Lie subalgebras of g so that ζ_1 and ζ_2 are identified with the same element ζ in the center $\mathfrak{z}$ of g. We have $G = HS$, $H \cap S = Z$, where Z is the analytic subgroup of G corresponding to $\mathfrak{z}$. Thus G is a H-group by Lemma 2.4. However G is not of type I since it is the central extension of the group G/Z which is isomorphic to the direct product of $\mathbf{R}$ and the five dimensional Mautner's group.

LEMMA 2.6. — *Let $\mathfrak{h}$ be a Lie algebra and $\mathfrak{z}$ an ideal of $\mathfrak{h}$. Assume that there exists ℓ in $\mathfrak{h}^*$ such that the symplectic form B_ϱ is non degenerate on $\mathfrak{h}/\mathfrak{z}$. Let τ be an automorphism of $\mathfrak{h}$ leaving $\mathfrak{z}$ invariant such that τ splits over $\mathbf{R}$, and that $\mathfrak{h}$ has a τ-invariant subspace $\overline{\mathfrak{h}}$ complementing $\mathfrak{z}$ such that $\ell(\overline{\mathfrak{h}}) = 0$. Finally assume that $\tau|\mathfrak{z}$ has at most two eigenvalues: α and $-\alpha$. Then $|\det \tau| = |\alpha|^q$, where $q = \dfrac{1}{2} \dim \mathfrak{h}/\mathfrak{z} + \dim \mathfrak{z}$.*

Proof. — Let $\{\xi_i\}$ be a basis of $\overline{\mathfrak{h}}$ consisting of eigenvectors of τ corresponding to the eigenvalues $\{\alpha_i\}$. We have:

$$\tau[\xi_i, \xi_j] = [\tau\xi_i, \tau\xi_j] = \alpha_i \alpha_j [\xi_i, \xi_j].$$

Hence $|\alpha_i| |\alpha_j| = |\alpha|$ if $[\xi_i, \xi_j] \notin \overline{\mathfrak{h}}$.

It follows that $|\alpha_i| |\alpha_j| = |\alpha|$ if $B_\varrho(\xi_i, \xi_j) \neq 0$ i.e. $|\tau| B_\varrho = |\alpha| B_\varrho$, where $|\tau|$ is the linear transformation of $\overline{\mathfrak{h}}$ such that $|\tau|\xi_i = |\alpha_i|\xi_i$, $\forall i$, and the action of $|\tau|$ on B_ϱ as a symplectic form on $\overline{\mathfrak{h}}$ is

induced from that of $|\tau|$ on $\overline{\mathfrak{h}}$. Thus

$$\det |\tau| = \mathrm{Pf}(|\tau| B_\varrho)(\mathrm{Pf}(B_\varrho) = |\alpha|^{\frac{1}{2}\dim \mathfrak{h}/\mathfrak{z}}$$

where Pf is the Pfaffian of B_ϱ with respect to a fixed volume on $\overline{\mathfrak{h}}$. Finally

$$|\det \tau| = \det |\tau| \, |\det(\tau|_\mathfrak{z})| = |\alpha|^q .$$ QED

LEMMA 2.7. – Let $\mathfrak{h}$, $\mathfrak{z}$, τ be as above. Assume that there exists $\alpha \neq \pm 1$ such that $\mathrm{spec}(\tau|_\mathfrak{z}) \subset \{1, -1, \alpha, -\alpha\}$ and $\mathrm{spec}(\tau|_\mathfrak{z}) \cap \{\alpha, -\alpha\} \neq \emptyset$. Then $|\det \tau| \neq 1$.

Proof. – Let $|\tau|$ be as above. It is sufficient to prove that $\det |\tau|$ is not equal to any negative integral power of $|\alpha|$. Let $|\tau|^t = e^{t \log |\tau|}$ be the one parameter subgroup of $\mathrm{GL}(\overline{\mathfrak{h}})$ generated by $\log |\tau|$; then as in the proof of Lema 2.6 we have

$$|\alpha_i|^t \, |\alpha_j|^t = |\alpha|^{t k_{ij}} \quad \text{if} \quad B_\varrho(\xi_i, \xi_j) \neq 0$$

where $k_{ij} = 0$ or 1.

Thus $\det |\tau|^t = \mathrm{Pf}(|\tau|^t B_\varrho)/\mathrm{Pf}(B_\varrho)$ is a polynomial in $|\alpha|^t$. Therefore by the unique factorization in the polynomial ring $\mathbf{R}[X]$, $\det |\tau|^t$ is not equal to any negative integral power of $|\alpha|^t$ and hence $\det |\tau|$ is not equal to any negative integral power of $|\alpha|$.
                  QED

PROPOSITION 2.8. – *Let* H *be a connected Lie group. Let* $\mathfrak{z}$ *be an ideal in* $\mathfrak{h}$. *Assume that there exists* ℓ *in* $\mathfrak{h}^*$ *such that the* H-*orbit of* ℓ *in* $\mathfrak{h}^*$ *is* $\ell + (\mathfrak{h}/\mathfrak{z})^*$. *Let* τ *be an automorphism of* $\mathfrak{h}$ *leaving* $\mathfrak{z}$ *invariant.*

i) *If* $\mathrm{spec}(\tau|_\mathfrak{z})^c \subset \{\lambda \in \mathbf{C} : |\lambda| = 1\}$, *where* $(\tau|_\mathfrak{z})^c$ *is the complexification of* $(\tau|_\mathfrak{z})$, *then* $|\det \tau| = 1$.

ii) *If* $\mathrm{spec}(\tau|_\mathfrak{z})^c \subset \{\lambda \in \mathbf{C} : |\lambda| = 1\} \cup \{\alpha, -\alpha\}$ *and* $\mathrm{spec}(\tau|_\mathfrak{z})^c \cap \{\alpha, -\alpha\} \neq \emptyset$, *where* $\alpha \in \mathbf{R} \setminus \{1, -1\}$, *then* $|\det \tau| \neq 1$.

Proof. – Let T be the neutral component of the smallest real algebraic subgroup of $\mathrm{Aut}(\mathfrak{h})$ containing τ. Then T is the direct product of a unipotent real algebraic group U and the neutral

component S of a real algebraic toral subgroup of $\text{Aut}(\mathfrak{h})$ (cf. [5]).
Moreover S is the almost direct product of a connected **R**-split
subgroup S_1 and a compact connected subgroup S_2 (cf. [11]).
Since the elements of U and S_2 clearly have determinant 1, and
all of their eigenvalues corresponding to eigen vectors in $\mathfrak{z}^c$ have
absolute values 1, we may assume that τ belongs to S_1. In other
words we may assume that τ is split over **R**. It follows then that
$\mathfrak{h}$ has a τ-invariant subspace $\overline{\mathfrak{h}}$ complementing $\mathfrak{z}$. Since the H-
orbit of ℓ is $\ell + (\mathfrak{h}/\mathfrak{z})^*$ we may assume that $\ell(\overline{\mathfrak{h}}) = 0$. Now
the statements i) and ii) follow immediately from Lemmas 2.6 and
2.7, for the symplectic form B_ℓ is non degenerate on $\overline{\mathfrak{h}}$.

<div align="right">QED</div>

Now let us consider another characterization of H-groups
which is more convenient for later applications since it only involves
the Lie algebras in question.

THEOREM 2.9. — *Let H be a connected solvable Lie group,
and Z a closed connected central subgroup of H. For every ℓ
in $\mathfrak{h}^*$, the following statements are equivalent:*

i) *The H-orbit of ℓ in $\mathfrak{h}^*$ is $\ell + (\mathfrak{h}/\mathfrak{z})^*$*

ii) *H is unimodular and the associated symplectic form B_ℓ
is non degenerate on $\mathfrak{h}/\mathfrak{z}$.*

Proof. — We proceed by induction on $\dim H$. First note that
if i) or ii) hold then Z is the neutral component of the center of H.
Hence we will assume that it is so in the following. Moreover by
factoring out $\exp(\ker \ell|_\mathfrak{z})$ if necessary, we may assume that
$\dim Z = 1$ and ℓ is non zero on $\mathfrak{z}$. Let N be the nilradical of H.

α) First assume that N is isomorphic to a Heisenberg group
with center Z. Since $\ell|_\mathfrak{z} \neq 0$, the N-orbit of $\ell|\mathfrak{n}$ is $\ell|\mathfrak{n} + (\mathfrak{n}/\mathfrak{z})^*$
(cf. [15]). Hence by Lemma 2.3, $H = NS$, $N \cap S = Z$, where
$S = \{x \in H : (\ell \circ \text{Ad } x)|\mathfrak{n} = \ell|\mathfrak{n}\}$. Moreover by Proposition 2.8:
$|\det \text{Ad}_\mathfrak{n}(x)| = 1$, $\forall x \in S$. On the other hand since N is unimo-
dular and $S/Z \simeq H/N$ is abelian we see that H is already unimodular
in this case. Thus we have proved i) $\Longrightarrow$ ii). Now assume that B_ℓ
is non degenerate on $\mathfrak{h}/\mathfrak{z}$. Then it is obvious that $B_{\ell|\mathfrak{s}}$ is non
degenerate on $\mathfrak{s}/\mathfrak{z}$, i.e. $\mathfrak{s}$ is a Heisenberg algebra with center $\mathfrak{z}$.
Therefore by Lemma 2.4 the H-orbit of ℓ in $\mathfrak{h}^*$ is $\ell + (\mathfrak{h}/\mathfrak{z})^*$.

β) Assume that the center of N is still equal to Z but N is not isomorphic to any Heisenberg group. Then as in the proof of Proposition 4.2 of [1] there is a closed connected normal abelian subgroup K of H such that K/Z is contained in the center of N/Z and the N-orbit of $\ell \mid \mathfrak{k}$ is $\ell \mid \mathfrak{k} + (\mathfrak{k}/\mathfrak{z})^*$. Put

$$H_0 = \{x \in H : (\ell \circ \mathrm{Ad}\, x) \mid \mathfrak{k} = \ell \mid \mathfrak{k}\}.$$

Then H_0 is connected. Moreover since $\mathfrak{h}_0$ is the annihilator of $\mathfrak{k}$ with respect to B_ℓ, $\mathfrak{h}/\mathfrak{h}_0$ may be identified with the dual $\mathfrak{k}/\mathfrak{z}$ via B_ℓ if the latter is assumed to be non degenerate on $\mathfrak{h}/\mathfrak{z}$. For all x in H_0, ξ in $\mathfrak{k}$ and η in $\mathfrak{h}$, we have

$$B_\ell(\mathrm{Ad}(x)\,\xi,\,\mathrm{Ad}(x)\eta) = \ell \circ \mathrm{Ad}(x)\,([\xi,\,\eta])$$

$$= \ell([\xi,\,\eta])$$

$$= B_\ell(\xi,\,\eta).$$

In particular the representation of H_0 in $\mathfrak{h}/\mathfrak{h}_0$ induced by the adjoint representation is contragredient to that induced in $\mathfrak{k}/\mathfrak{z}$. This being set we have

$$\det \mathrm{Ad}\, x = \det \mathrm{Ad}_{\mathfrak{h}/\mathfrak{z}}(x)$$
$$= \det \mathrm{Ad}_{\mathfrak{h}/\mathfrak{h}_0}(x)\ \det \mathrm{Ad}_{\mathfrak{h}_0/\mathfrak{k}}(x)\ \det \mathrm{Ad}_{\mathfrak{k}/\mathfrak{z}}(x).$$

Hence

$$\det \mathrm{Ad}\, x = \det \mathrm{Ad}_{\mathfrak{h}_0/\mathfrak{k}}(x) \qquad \forall x \in H_0. \tag{2.3}$$

Thus let us assume i). Then it follows that the H_0-orbit of $\ell \mid \mathfrak{h}_0$ in $\mathfrak{h}_0^*$ is $\ell \mid \mathfrak{h}_0 + (\mathfrak{h}_0/\mathfrak{k})^*$. Hence H_0/K is unimodular by the induction hypothesis (strictly speaking we should factor out $\exp \ker(\ell \mid \mathfrak{k})$ before applying the induction hypothesis, and then go back to H_0 via the canonical homomorphism $H_0 \longrightarrow H_0/\exp \ker(\ell \mid \mathfrak{k})$. However this kind of argument is trivial and will be omitted in the following). Now it follows from (2.3) that: $|\det \mathrm{Ad}\, x| = 1 \quad \forall x \in H_0$.

Moreover it is clear that $H = NH_0$ and $\det \mathrm{Ad}\, x = 1$, $\forall x \in N$. Therefore H is unimodular.

Conversely assume ii). Then it follows from (2.3) that H_0/K is unimodular. Moreover $B_{\ell \mid \mathfrak{h}_0}$ is plainly non degenerate on $\mathfrak{h}_0/\mathfrak{k}$. Therefore the H_0-orbit of $\ell \mid \mathfrak{h}_0$ in $\mathfrak{h}_0^*$ is $\ell \mid \mathfrak{h}_0 + (\mathfrak{h}_0/\mathfrak{k})^*$ according to the induction hypothesis. Let $\ell_1 \in \mathfrak{h}^*$ be such that $\ell_1 \mid \mathfrak{z} = \ell \mid \mathfrak{z}$.

We have to prove that ℓ_1 belongs to the same H-orbit as ℓ. Since the H-orbit of $\ell | \mathfrak{f}$ in $\mathfrak{f}^*$ is $\ell | \mathfrak{f} + (\mathfrak{f}/\mathfrak{z})^*$ we may assume that $\ell_1 | \mathfrak{f} = \ell | \mathfrak{f}$ without loss of generality. Then there exists x_1 in H_0 such that $\ell_1 | \mathfrak{h}_0 = (\ell \circ \mathrm{Ad}\, x_1) | \mathfrak{h}_0$. Finally since $\mathfrak{f}/\mathfrak{z}$ is contained in the center of $\mathfrak{n}/\mathfrak{z}$ and since $\mathfrak{h}/\mathfrak{h}_0 \simeq \mathfrak{n}/\mathfrak{n} \cap \mathfrak{h}_0$ may be identified with the dual of $\mathfrak{f}/\mathfrak{z}$ via B_ℓ we see that

$$\{\ell \circ \mathrm{Ad}\, x / x \in K\} = \ell + (\mathfrak{h}/\mathfrak{h}_0)^* \,.$$

Hence there exists x_2 in K such that $\ell_1 \circ \mathrm{Ad}\,(x_1^{-1}) = \ell \circ \mathrm{Ad}(x_2)$, i.e. $\ell_1 = \ell \circ \mathrm{Ad}(x_2 x_1)$.

γ) Finally assume that Z is properly contained in the center of N. Then $(\mathrm{Cent}\, N)/Z$ must be a vector group. Let us choose K to be minimal among the connected normal subgroups of H lying in the center of N and properly containing Z so that the adjoint representation induces an irreducible representation ρ of H in $\mathfrak{f}/\mathfrak{z}$. Note that K is closed in N since N/Z is simply connected. Now $\lambda(x) = (\ell \circ \mathrm{Ad}\, x) | \mathfrak{f} - \ell | \mathfrak{f}$ is a cocycle of H with respect to the contragredient representation $\check{\rho}$ of ρ. Since both $\check{\rho}$ and λ are trivial on N, λ may be viewed as a cocycle of the abelian group H/N, and hence by Proposition 1.3 λ must be of the form $\lambda(x) = \check{\rho}(x)\ell_0 - \ell_0$ for some $\ell_0 \in (\mathfrak{f}/\mathfrak{z})^*$ unless $\check{\rho}$ is the identity representation. In the latter case $\dim(\mathfrak{f}/\mathfrak{z}) = 1$ and $\lambda(H) = (\mathfrak{f}/\mathfrak{z})^*$, i.e. $\ell \circ \mathrm{Ad}_\mathfrak{f}(H) = \ell | \mathfrak{f} + (\mathfrak{f}/\mathfrak{z})^*$. In any case put

$$H_0 = \{x \in H : (\ell \circ \mathrm{Ad}\, x) | \mathfrak{f} = \ell | \mathfrak{f}\}$$

and
$$\overline{K} = \exp \ker(\ell | \mathfrak{f})\,.$$

Then H_0 is a closed normal subgroup of H and $\overline{Z} = K/\overline{K}$ is a closed connected central subgroup of $\widetilde{H}_0 = H_0^0/\overline{K}$.

Let us assume i). Then the H-orbit of $\ell | \mathfrak{f}$ in $\mathfrak{f}^*$ is $\ell | \mathfrak{f} + (\mathfrak{f}/\mathfrak{z})^*$. In this case $\check{\rho}$ and hence ρ must be the identity representation of H. Note that $H_0^0 = H_0$. Now it is easy to see that the H_0-orbit of $\ell | \mathfrak{h}_0$ is $\ell | \mathfrak{h}_0 + (\mathfrak{h}_0/\mathfrak{f})^*$. Moreover each $\mathrm{Ad}_\mathfrak{f}(x)$, $x \in H$, is unipotent because ρ is trivial. Therefore we have, according to Proposition 2.8

$$|\det \mathrm{Ad}_{\mathfrak{h}_0}(x)| = 1\,, \quad \forall x \in H$$

i.e. H is unimodular since H/H_0 is abelian.

Conversely assume that ii) holds. Then the H-orbit of $\ell|\mathfrak{h}$ in $\mathfrak{k}^*$ is clearly open in $\ell|\mathfrak{k} + (\mathfrak{k}/\mathfrak{z})^*$. Thus in case $\check{\rho}$ is not the identity representation we have $\check{\rho}(H) = \mathbf{R}^+$ or $\check{\rho}(H) = \mathbf{R}^+ \times \mathbf{T}$. Hence there exists $x_0 \in H$ such that $(\mathrm{Ad}\, x_0)|\mathfrak{k}/\mathfrak{z} = \alpha\, \mathrm{Id}$, $\alpha \neq \pm 1$, $\alpha \in \mathbf{R}$. On the other hand $\check{\rho}(H_0)$ and hence $\rho(H_0)$ reduces to the identity. Therefore

$$\det \mathrm{Ad}_{\mathfrak{k}/\mathfrak{z}}(x) = 1, \quad \forall x \in H_0$$

i.e.

$$\det \mathrm{Ad}_{\mathfrak{k}}(x) = 1, \quad \forall x \in H_0 .$$

This together with the fact that H_0^0 is a closed normal subgroup of the unimodular group H implies that H_0^0/K is unimodular, i.e. $\widetilde{H}_0 = H_0^0/\overline{K}$ is unimodular. Since the form $B_{\ell|\mathfrak{h}_0}$ is plainly non degenerate on $\mathfrak{h}_0/\mathfrak{k}$, it follows from the induction hypothesis that the H_0^0-orbit of $\ell|\mathfrak{h}_0$ is $\ell|\mathfrak{h}_0 + (\mathfrak{h}_0/\mathfrak{k})^*$. Recall that $\mathrm{Ad}_{\mathfrak{k}}(x_0)$ has exactly two eigenvalues: 1 and α. Therefore Proposition 2.8 implies: $|\det \mathrm{Ad}_{\mathfrak{h}_0}(x_0)| \neq 1$, i.e. $|\det \mathrm{Ad}(x_0)| \neq 1$ since $\det \mathrm{Ad}_{\mathfrak{h}/\mathfrak{h}_0}(x_0) = 1$. This however contradicts the fact that H is unimodular. Thus $\check{\rho}$ and hence ρ is trivial and the H-orbit of $\ell|\mathfrak{k}$ is $\ell|\mathfrak{k} + (\mathfrak{k}/\mathfrak{z})^*$. Note that $H_0^0 = H_0$ in this case. Now by the same argument as above we see that the H_0-orbit of $\ell|\mathfrak{h}_0$ is $\ell|\mathfrak{h}_0 + (\mathfrak{h}_0/\mathfrak{k})^*$ and the K-orbit of ℓ in $\mathfrak{k}^*$ is $\ell + (\mathfrak{h}/\mathfrak{h}_0)^*$. These two facts together imply that the H-orbit of ℓ in $\mathfrak{h}^*$ is $\ell + (\mathfrak{h}/\mathfrak{z})^*$ as in the last part of case β).   QED

Remark. — The notations being as above, assume that $\dim Z = 1$. It follows immediately from the proof of Theorem 2.9 that if the nilradical N of H is not isomorphic to a Heisenberg group with center Z, then there is a closed normal abelian subgroup K of H lying in N such that K/Z is contained in the center of N/Z and the H-orbit of $\ell|\mathfrak{k}$ is $\ell|\mathfrak{k} + (\mathfrak{k}/\mathfrak{z})^*$. Moreover there is a linear subspace V of $\mathfrak{h}$ dual to $\mathfrak{k}/\mathfrak{z}$ with respect to B_ℓ such that $(x, \xi) \longmapsto x \exp \xi$ is a diffeomorphism of $H_0 \times V$ onto H. Assume indeed $\mathrm{Cent}(N) = Z$. Then $H = N H_0$, and the N-orbit of $\ell|\mathfrak{k}$ is $\ell|\mathfrak{k} + (\mathfrak{k}/\mathfrak{z})^*$. In this case V may be chosen to be a subspace of $\mathfrak{n}$ complementing $\mathfrak{n} \cap \mathfrak{h}_0$. On the other hand if $\mathrm{Cent}(N) \neq Z$ then $\dim \mathfrak{k}/\mathfrak{z} = 1$ and $\mathrm{Ad}(H)$ reduces to the identity on $\mathfrak{k}/\mathfrak{z}$ and $\mathfrak{z}$ respectively. In this case $\mathfrak{h}_0$ is an ideal of codimension 1 and V can be any one-dimensional subspace complementing $\mathfrak{h}_0$.

COROLLARY 2.10. – *Let* G *be a Lie group,* H *a closed normal subgroup of* G *which is a* H-*group with center* Z, *and* S *a Lie subgroup of* G *such that* G = HS, H ∩ S = Z. *Assume that for each* x *in* S, *the eigenvalues of* Ad x *corresponding to eigenvectors in* $\mathfrak{z}^c$ *have absolute values* 1. *Then* G *is unimodular if and only if* S *is unimodular.*

Proof. – It follows from Theorem 2.9 and Proposition 2.8 that:

$$|\det \mathrm{Ad}_{\mathfrak{h}}(x)| = 1 \quad \forall x \in G.$$

Moreover the last assumption implies that G (resp. S) is unimodular iff G/Z (resp. S/Z) is unimodular. Finally G/Z is plainly the semi direct product of H/Z and S/Z. Hence the Corollary follows from the standard results in the integration theory of Lie groups.   QED

Example 2.11. – To see the necessity of the assumption that H is unimodular in Theorem 2.9, let us take H = $\mathbf{R}^+ \times \mathbf{R}^2$ with multiplication:

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 x_2, x_2^{-1} y_1 + y_2,$$
$$(x_2^{-1} - 1)y_1 + z_1 + z_2).$$

Then H is a three-dimensional solvable Lie group with center Z = {(1, 0, z) : z ∈ **R**}. Note that H is not unimodular. In fact the modular function of H is $\Delta(x, y, z) = x$. Now let {ξ, η, ζ} be the canonical basis of $\mathfrak{h}$ = $\mathbf{R}^3$ so that {ζ} is a basis of $\mathfrak{z}$. Let $\ell \in \mathfrak{h}^*$ be such that $\ell(\xi) = \ell(\eta) = 0$ and $\ell(\zeta) = 1$. Then we have $\ell([\xi, \eta]) = \ell(\eta + \zeta) = 1$, i.e. B_ℓ is non degenerate on $\mathfrak{h}/\mathfrak{z}$. However H is not a H-group as indicated by Theorem 2.9. Of course one may prove this directly by observing that the projection on $\mathfrak{f}^*$ of the H-orbit of ℓ in $\mathfrak{h}^*$ consists of those linear forms ℓ_1 on $\mathfrak{f}$ such that $\ell_1(\zeta) = 1$ and $\ell_1(\eta) \geqslant -1$, where $\mathfrak{f} = \mathbf{R}\eta + \mathbf{R}\zeta$.

We turn now to the representation theory of H-groups. It turns out that it is very much similar to that of nilpotent Lie groups with discrete series in [15]. First note that thanks to Theorem 2.9 we may define the Pfaffian polynomial on $\mathfrak{z}^*$ exactly as in [15]. Let indeed P(ℓ) denotes the Pfaffian of the two-form B_ℓ associated to $\ell \in \mathfrak{h}^*$ with respect to some fixed volume element $d\alpha$

on $\mathfrak{h}/\mathfrak{z}$. More explicitly we have $\Lambda^n B_\varrho = P(\ell)\, d\alpha$, where $n = \dfrac{1}{2}$ dim $\mathfrak{h}/\mathfrak{z}$ and $\Lambda^n B_\varrho$ is the n-th exterior power of B_ϱ . Now we may prove as in [15] that $P(\ell)$ depends only on the restriction of ℓ on $\mathfrak{z}$ and hence $P(\ell)$ may be viewed as a polynomial function on $\mathfrak{z}^*$. The only facts needed are that H is unimodular and that the two statements i) and ii) in Theorem 2.9 are equivalent. This being set let $\mathscr{R} = \mathscr{R}_H \subset \mathfrak{z}^*$ be the complement of the nullset of $P(\ell)$ in $\mathfrak{z}^*$. We have :

THEOREM 2.12. – *Let* H *be a simply connected* H-*group with center* Z. *For every* ℓ *in* $\mathscr{R}$, *put* $\chi_\varrho = \exp \sqrt{-1}\, \ell$. *Then* H *has a unique (up to a unitary equivalence) irreducible* χ_ϱ-*representation* T^ϱ. *Moreover every* χ_ϱ-*representation of* H *is equivalent to a multiple of* T^ϱ. *In particular* $\mathrm{ind}_{Z\uparrow H}\, \chi_\varrho \simeq \mathrm{mult}\, T^\varrho$ *and hence* T^ϱ *is square integrable* mod Z .

Proof. – First note that by Proposition 1.2 of [1] (see also [10]), if $\mathrm{ind}_{Z\uparrow H}\, \chi_\varrho \simeq \mathrm{mult}\, T^\varrho$ then T^ϱ is square integrable mod Z . Hence it remains to prove the existence of an irreducible χ_ϱ-representation of H for each $\ell \in \mathscr{R}$ such that every χ_ϱ-representation π is equivalent to a multiple of T^ϱ . Furthermore by using the central decomposition of π we may assume that π is a factor. Let us proceed by induction on dim H . As usual we suppose dim Z = 1 and ℓ is non zero on $\mathfrak{z}$.

α) First assume that the nilradical N of H is isomorphic to a Heisenberg group with center Z . Then it follows from the proof of Theorem 2.9 case α) that H = NS, $N \cap S = Z$ where S is a Heisenberg group with center Z . Consider the semi direct product $G = N \circledS S$. Let $\nu : G \longrightarrow H$ be the canonical homomorphism so that $\nu(n, s) = ns$. Let T_1^ϱ and T_2^ϱ be the unique members of the discrete series of N and S respectively such that $T_1^\varrho | Z \simeq \mathrm{mult}\, \chi_\varrho$ and $T_2^\varrho | Z \simeq \mathrm{mult}\, \chi_\varrho$ (cf. [1]). Note that S is defined to be the set of those elements of H leaving fixed an arbitrary but fixed extension ℓ_1 of ℓ to $\mathfrak{n}$, or equivalently the set of those elements leaving invariant a subspace $\overline{\mathfrak{n}}$ of $\mathfrak{n}$ complementing $\mathfrak{z}$. Thus the function J_ϱ in [1] which is defined and continuous on $\mathrm{Sp}(\overline{\mathfrak{n}}, B_{\varrho_1})$ has continuous square roots on S since S is simply connected. It follows then from Lemma 3.1 of [1] that T_1^ϱ may be extended to

an irreducible representation of G denoted again by T_1^ϱ. Moreover

$$\pi \circ \nu(n, s) = T_1^\varrho(n, s) \otimes \sigma(s) \quad \forall(n, s) \in G$$

where σ is some factor representation of S such that $\sigma|Z \simeq \text{mult } \chi_\varrho$. But then $\sigma \simeq \text{mult } T_2^\varrho$ and hence

$$\pi \circ \nu \simeq \text{mult } T^\varrho \circ \nu \tag{2.4}$$

where T^ϱ is the irreducible χ_ϱ-representation of H defined by the irreducible representation $T_1^\varrho(n, s) \otimes T_2^\varrho(s)$ of G by factoring out ker ν. The factorization is possible since:

$$T_1^\varrho(z, z^{-1}) \otimes T_2^\varrho(z^{-1}) = T_1^\varrho(z) \otimes T_2^\varrho(z^{-1}) = \chi_\varrho(z) \chi_\varrho(z^{-1}) \text{ Id} = \text{Id}$$

i.e. $T_1^\varrho(n, s) \otimes T_2^\varrho(s)$ is trivial on ker ν. Now π is equivalent to a multiple of T^ϱ as indicated by (2.4).

β) Let us consider now the case in which N is not isomorphic to a Heisenberg group with center Z. By the remark of Theorem 2.9 there is a closed connected normal abelian subgroup K of H lying in N such that the H-orbit of an arbitrary extension ℓ_1 of ℓ to $\mathfrak{k}$ is $\ell_1 + (\mathfrak{k}/\mathfrak{z})^*$, i.e. the H-orbit of $\lambda_1 = \exp \sqrt{-1} \ell_1$ in $\hat{K}$ is $\lambda_1(K/Z)\hat{}$. This implies in particular that the quasi orbit defined by $\pi|K$ is concentrated in this orbit. Therefore according to Theorem 8.1 of [8], π is induced from a factor representation σ of the subgroup

$$H_0 = \{x \in H : \lambda_1(xkx^{-1}) = \lambda_1(k) \quad \forall k \in K\}$$
$$= \{x \in H : \ell_1 \circ \text{Ad}_\mathfrak{k}(x) = \ell_1\}.$$

Note that $H_0/\exp \ker \ell_1$ is a H-group as indicated by the proof of Theorem 2.9. Therefore it follows from the induction hypothesis that σ is equivalent to a multiple of the unique irreducible representation $\overline{T}^{\varrho_1}$ of H_0 such that $\overline{T}^{\varrho_1}|K \cong \text{mult } \lambda_1$. Finally this representation together with the orbit $\lambda_1(K/Z)\hat{}$ determine the unique irreducible representation T^ϱ of H such that $T^\varrho|Z \simeq \text{mult } \chi_\varrho$. More explicity $T^\varrho \simeq \text{ind}_{H_0 \uparrow H} \overline{T}^{\varrho_1}$. Hence

$$\pi \simeq \text{ind}_{H_0 \uparrow H} (\text{mult } \overline{T}^{\varrho_1}) \simeq \text{mult } T^\varrho. \qquad \text{QED}$$

PROPOSITION 2.13. − *Let* H, Z, ℓ *be as above. Put* $\dim H/Z = 2p$. *Assume that the Haar measure* dh *of* H/Z *is defined by a positive invariant* $2p$-*form* ω *such that its value at the identity in* H/Z, $(\omega)_1$

is the volume element on $\mathfrak{h}/\mathfrak{z}$ *used in the definition of* $P(\ell)$. *Then the formal degree of the irreducible representation* T^ℓ *is given by* $d(T^\ell) = |P(\ell)|$.

Proof. — The notations being as in Theorem 2.12, let us assume first that case α) in the proof of that Theorem holds so that

$$T^\ell \circ \nu(n, s) = T_1^\ell(n, s) \otimes T_2^\ell(s).$$

By Theorem 4 of [15] the Haar measures dn and ds of N/Z and S/Z respectively may be normalized so that

$$d(T_i^\ell) = |P_i(\ell)| \qquad i = 1, 2$$

where $P_1(\ell)$ and $P_2(\ell)$ are the Pfaffian polynomial functions on $\mathfrak{z}^*$ defined by the structures of the Heisenberg groups N and S respectively. Note that H/Z is the semi direct product of N/Z and S/Z. This allows us to normalize dh so that $dh = dn\,ds$. This being done we have $d(T^\ell) = d(T_1^\ell)\,d(T_2^\ell)$. Let α, β denote the invariant forms on N/Z, S/Z defining the Haar measures dn, ds respectively. Put dim N/Z $= 2q$, dim S/Z $= 2r$. We have $p = q + r$. Let ℓ_1 be the extension of ℓ to $\mathfrak{n}$ so that $\ell_1(\bar{\mathfrak{n}}) = 0$ as in the proof of Theorem 2.12 case α). Let ℓ_2 be an arbitrary extension of ℓ to $\mathfrak{s}$, and ℓ_3 the unique extension of ℓ to $\mathfrak{h}$ so that it coincides with ℓ_1 and ℓ_2 on $\mathfrak{n}$ and $\mathfrak{s}$ respectively. Since $\mathfrak{n}$ is clearly orthogonal to $\mathfrak{s}$ with respect to B_{ℓ_3} we have

$$\Lambda^p B_{\ell_3} = (\Lambda^q B_{\ell_1}) \wedge (\Lambda^r B_{\ell_2})$$

i.e.

$$P(\ell)(\omega)_1 = P_1(\ell)\,P_2(\ell)\,(\alpha)_1 \wedge (\beta)_1$$

i.e.

$$P(\ell) = P_1(\ell)\,P_2(\ell).$$

Hence

$$d(T^\ell) = |P(\ell)|.$$

Now assume that case β) in the proof of Theorem 2.12 holds so that $T^\ell = \text{ind}_{H_0 \uparrow H} \overline{T}^{\ell_1}$. Hence it follows from Theorem 1.3 of [1] that:

$$d(T^\ell) = d(\overline{T}^{\ell_1}). \tag{2.5}$$

Recall that the Haar measure dh_0 of H_0/K is so normalized that

$$dh = dh_0\,d\bar{h}\,dk \tag{2.6}$$

where dk is a fixed Haar measure of K/Z and $d\bar{h}$ is the quasi

invariant measure on $H_0 \backslash H$ which is transported from the trans-
late $d(\lambda_1, \lambda)$ of the Haar measure $d\lambda$ of $(K/Z)^\frown$. Here we have
normalized the Haar measure $d\lambda$ in such a way that the Plancherel
formula on (K/Z), $(K/Z)^\frown$ holds for the pair of measures dk, $d\lambda$.
Let ω_0, θ be the invariant forms defining dh_0, dk respectively.
Let V be the subspace appeared in the remark of Theorem 2.9.
Recall that V is dual to $\mathfrak{k}/\mathfrak{z}$ via $B_{\overline{\ell}}$ where $\overline{\ell}$ is a fixed extension
of ℓ_1 to $\mathfrak{h}$. Let us normalize the Lebesgue measure $d\xi$ of V
in such a way that the Plancherel formula on $\mathfrak{k}/\mathfrak{z}$, V holds for
the pair of measures $(\theta)_1$, $d\xi$. This being set, we have according
to (2.6):

$$(\omega)_1 = (\omega_0)_1 \wedge d\xi \wedge (\theta)_1 . \qquad (2.7)$$

On the other hand since $\mathfrak{h}_0$ is orthogonal to $\mathfrak{k}$ with respect to
$B_{\overline{\ell}}$ we have

$$\Lambda^p B_{\overline{\ell}} = \pm (\Lambda^q B_{\overline{\ell} | \mathfrak{h}_0}) \wedge d\xi \wedge (\theta)_1 \qquad (2.8)$$

where $\dim H/Z = 2p$, $\dim H_0/K = 2q$. Now it follows from (2.7)
and (2.8) that

$$P(\ell) = \pm P(\ell_1). \qquad (2.9)$$

Finally we have by the induction hypothesis that $d(\overline{T}^{\ell_1}) = |P(\ell_1)|$.
This together with (2.5) and (2.9) show that $d(T^\ell) = |P(\ell)|$.

                                                                QED

   Remark. — The proof of Proposition 2.13 is somewhat more
complicated than that of Theorem 4 of [15] since we have to treat
separately case α). On the other hand it is shorter than the latter
because we do not have to use the Plancherel formula on K/Z which
has already been used in the proof of Theorem 1.3 of [1].

   PROPOSITION 2.14. — *The map* $\phi : \ell \longmapsto T^\ell$ *is a bijection of*
$\mathcal{L}$ *onto the set of (equivalent classes of) square integrable* mod Z
irreducible representations of H. *Moreover this map is a homeo-*
morphism from the natural topology of $\mathcal{L}$ *to the Fell topology*
on representations.

   Proof. — The proof is essentially the same as that of Theorem 2
of [15]. The only thing remaining to be checked is the fact that ϕ
is surjective. Thus let π be a square integrable mod Z irreducible

χ_{ϱ}-representation of H. Then by Theorem 3.4 which will be proved in the next section independently of Proposition 2.14, H may be written as $H = H_1 S$, $H_1 \cap S = Z$, where H_1 is a H-group with center Z, S is a reductive subgroup such that $\text{Cent}(S)/Z$ is compact. Moreover there exists an extension ϱ_1 of ϱ to $\mathfrak{h}_1$ such that the H_1-orbit of ϱ_1 in $\mathfrak{h}_1^*$ is $\varrho_1 + (\mathfrak{h}_1/\mathfrak{z})^*$. Since H is solvable, S is in fact abelian. Thus S/Z is a simply connected compact abelian Lie group and hence S/Z reduces to the identity, i.e. $H = H_1$. Hence $\varrho \in \mathfrak{R}$.

<div align="right">QED</div>

Remark. — In Theorem 2.12, Corollary 2.13 and Proposition 2.14 we have supposed that H is simply connected. In the case H is not simply connected, then $\hat{Z}$ is identified via the map $\varrho \longmapsto \exp \sqrt{-1}\,\varrho$ with an additive subgroup of $\mathfrak{z}^*$ of the form $V \times L$, where V is a vector subgroup of $\mathfrak{z}^*$ and L is a discrete finitely generated free abelian subgroup of $\mathfrak{z}^*$ such that $\dim \mathfrak{z}^* = \dim V + \text{rank } L$. It is easy to see that the polynomial function $P(\varrho)$ is not identically zero on $V \times L$. Hence the intersection of $\mathfrak{R} = \mathfrak{R}_H$ with $V \times L$ is a non empty open set. Therefore the above results still hold for a non simply connected H-group H, where the set $\mathfrak{R}$ is to be replaced by the intersection of itself with $V \times L$.

3. The general case.

In this section we will consider the case of a general connected unimodular Lie group. First of all, for technical reasons, we have to consider non connected groups:

PROPOSITION 3.1. — *Let G be a unimodular Lie group with radical R. Assume that G is contained as a closed normal subgroup in a connected Lie group F such that: (i) F/G^0 is abelian, (ii) G contains a closed connected normal nilpotent subgroup N of F such that F/N is a reductive Lie group, (iii) $F/\text{Rad } F$ has finite center.*

Let Z be a closed central subgroup of F lying in G such that $Z \cap N = Z^0$ and N/Z^0 is simply connected. Finally let χ be a character of Z. Put $\varrho = - \sqrt{-1}\,d\chi$.

Under these conditions, if G *has a square integrable* mod Z *irreducible* χ-*representation* π, *then there exist a closed connected normal subgroup* H *of* F *such that* $N \subset H \subset R$, *and a linear form* ℓ_1 *on* $\mathfrak{h}$ *extending* ℓ *such that the* H-*orbit of* ℓ_1 *in* $\mathfrak{h}^*$ *is* $\ell_1 + (\mathfrak{h}/\mathfrak{z})^*$. *In particular* H *is a* H-*group with center* Z^0.

Proof. — We will proceed by induction on $\dim G$. First of all by restricting F a little we may assume that F/G^0 is generated by the one-parameter subgroups passing by the elements of G/G^0. This being done F is clearly unimodular. On the other hand, by replacing Z by some subgroup of finite index if necessary we may assume that Z is the direct product of Z^0 and a discrete finitely generated free abelian group $D \subset \operatorname{Rad} F$. As usual we suppose that $\dim Z \leqslant 1$, and $\ell \neq 0$ if $\mathfrak{z} \neq 0$.

α) Assume that $\operatorname{Cent}(N) = Z^0$. If N is isomorphic to some Heisenberg group we may take $H = N$. Otherwise let K_1 be a closed connected normal subgroup of F such that K_1/Z^0 is the center of N/Z^0. In virtue of Proposition 2.3 of [1] the center K of K_1 contains Z^0 strictly. Of course K is a closed connected normal abelian subgroup of F. Note that $K_1 D$ and hence KD are closed subgroups of G since they are inverse images via the canonical projection $G \longrightarrow G/D$ of closed subgroups of $ND/D \simeq N$ which are defined in a similar manner. Note also that the exponential map is a diffeomorphism of $\mathfrak{k}/\mathfrak{z}$ onto K/Z^0 since the latter is a simply connected abelian group. Now we may prove exactly as in Proposition 4.2 of [1] that the N-orbit of an arbitrary extension $\overline{\ell}$ of ℓ to $\mathfrak{k}$ is of the form $\overline{\ell} + (\mathfrak{k}/\mathfrak{z})^*$, i.e. the N-orbit and hence the G-orbit of $\overline{\chi}$ in $(KD)\hat{\ }$ is $\overline{\chi}(KD/Z^0 D)\hat{\ }$ where $\overline{\chi}$ is the unique character of KD extending χ such that $d\overline{\chi} = \sqrt{-1}\,\overline{\ell}$. Thus π is induced from a square integrable mod KD irreducible $\overline{\chi}$-representation σ of the subgroup $G_0 = G \cap F_0$ where

$$F_0 = \{x \in F : \overline{\chi}(xkx^{-1}) = \overline{\chi}(k), \quad \forall k \in KD\}$$
$$= \{x \in F : \overline{\ell} \circ \operatorname{Ad}_{\mathfrak{k}}(x) = \overline{\ell}\}.$$

As before the subgroups F_0, $N_0 = N \cap F_0$ are connected, and we have $\operatorname{Rad} F_0 = (\operatorname{Rad} F) \cap F_0$, $G_0^0 = G^0 \cap F_0$ since the orbit $\overline{\ell} + (\mathfrak{k}/\mathfrak{z})^*$ is simply connected. In particular F_0/G_0^0 is isomorphic to a subgroup of F/G^0 and is thus abelian. On the

other hand one may choose $\bar{\ell}$ so that it vanishes on the subspace of $\mathfrak{f}$ complementing $\mathfrak{z}$ and left invariant by a fixed maximal connected semisimple subgroup S of F. This implies that F_0 contains S, and hence F_0/N_0 is reductive since $F_0/N_0 \simeq F_0 N/N \subset F/N$. Moreover since $F/\operatorname{Rad} F$ is semisimple, $F_0/\operatorname{Rad} F_0$ is actually isomorphic to $F/\operatorname{Rad} F$. In particular $F_0/\operatorname{Rad} F_0$ also has finite center. Finally $KD/\exp \ker \bar{\ell} = (K/\exp \ker \bar{\ell}) D$ clearly satisfies the requirements of the subgroup Z. Hence it follows from the induction hypothesis that there exist a closed connected normal subgroup H_1 of F_0 such that $N_0 \subset H_1 \subset \operatorname{Rad} G_0$, and a linear form $\bar{\ell}_1$ on $\mathfrak{h}_1$ extending $\bar{\ell}$ such that the H_1-orbit of $\bar{\ell}_1$ in $\mathfrak{h}_1^*$ is $\bar{\ell}_1 + (\mathfrak{h}_1/\mathfrak{f})^*$. Now as in the proof of Theorem 2.12 case β) there is an extension ℓ_1 of $\bar{\ell}_1$ to $\mathfrak{h} = \mathfrak{n} + \mathfrak{h}_1$ such that the H-orbit of ℓ_1 in $\mathfrak{h}^*$ is $\ell_1 + (\mathfrak{h}/\mathfrak{z})^*$ where $H = NH_1$.

β) Now assume that the center of N contains Z^0 strictly. Let us choose a closed connected normal subgroup K of F lying in the center of N such that the adjoint representation of F induces an irreducible representation ρ of F in $\mathfrak{f}/\mathfrak{z}$. Note as above that KD is closed and the exponential map is a diffeomorphism of $\mathfrak{f}/\mathfrak{z}$ onto K/Z^0. Thus Theorem 1.1 together with Theorem 1.3 of [1] imply that the quasi orbit defined by $\pi|KD$ is concentrated in an orbit which is open in $\overline{\chi}(KD/Z^0 D)\hat{\ }$ where $\overline{\chi}$ is some extension of χ to KD, i.e. the G-orbit of $\bar{\ell} = -\sqrt{-1}\,d\overline{\chi}$ in $\mathfrak{f}^*$ is an open subset of $\bar{\ell} + (\mathfrak{f}/\mathfrak{z})^*$. Moreover π is induced from a square integrable mod KD irreducible $\overline{\chi}$-representation σ of the subgroup $G_0 = G \cap F_0$ where

$$F_0 = \{x \in F: \overline{\chi}(xkx^{-1}) = \overline{\chi}(k), \quad \forall k \in KD\}$$
$$= \{x \in F: \bar{\ell} \circ \operatorname{Ad}_\mathfrak{f}(x) = \bar{\ell}\}.$$

Suppose that ρ is non trivial. Since ρ may be viewed as a representation of the reductive group F/N, it follows from Lemma 1.3 that there is ℓ_0 in $(\mathfrak{f}/\mathfrak{z})^*$ such that $\bar{\ell} \circ \operatorname{Ad}_\mathfrak{f}(x) - \bar{\ell} = \check{\rho}(x)\ell_0 - \ell_0$, $\forall x \in F$, where $\check{\rho}$ is the representation of F in $(\mathfrak{f}/\mathfrak{z})^*$ contragredient to ρ. Let $\widetilde{F}$ be the smallest real algebraic subgroup of $GL((\mathfrak{f}/\mathfrak{z})^*)$ containing $\check{\rho}(F)$ and $\widetilde{F}_0 = \{\widetilde{x} \in \widetilde{F}: \widetilde{x}\ell_0 = \ell_0\}$. Then $\widetilde{F}_0$ and hence $\check{\rho}(F_0)$ has finitely many connected components. Moreover the projection $\overline{F}_0$ of F_0 in $F/\operatorname{Rad} F$ has finitely many connected components and $\overline{F}_0^0/\operatorname{Rad} \overline{F}_0$ has finite center. Therefore

$F_1 = (\text{Rad F}) \, F_0$ has finitely many connected components and $F_1^0/\text{Rad} \, F_1^0$ has finite center. Now $\overline{f}_0$ may be viewed as a sub-algebra of a fixed maximal semisimple subalgebra $\hat{s}$ of $\overline{f}$ which may be chosen to be in $\mathfrak{g}$ so that $d\check{p}$ induces an isomorphism of some nilpotent ideal $\mathfrak{n}_1$ of $\overline{f}_0$ such that $\overline{f}_0/\mathfrak{n}_1$ is reductive onto the Lie algebra $\mathfrak{u}_1$ of the unipotent radical U_1 of $\widetilde{F}_0$. In fact $\mathfrak{n}_1$ may be chosen to be in $\mathfrak{g}_0 \cap \hat{s}$ since $\mathfrak{u}_1$ is clearly contained in the maximal semisimple subalgebra $d\check{p}(\hat{s})$ of $d\check{p}(\overline{f})$. Therefore if N_1 denotes the analytic subgroup of G_0^0 corresponding to $\mathfrak{n}_1$ then $(\text{Rad F})N_1$ is a closed normal subgroup of F_1^0. Moreover $\ker \check{p} \cap N_1 = 0$ since the covering $N_1 \longrightarrow U_1$ is trivial, for U_1 is simply connected. In fact, we have $(\text{Rad F}) \cap N_1 = 0$ since $\ker \check{p} \cap N_1$ is easily seen to be of finite index in $(\text{Rad F}) \cap N_1$. In particular we have $ND \cap N_1 = 0$ and hence $NN_1 \cap D = 0$. Now NN_1/N is a direct factor of the connected Lie group $(\text{Rad F}) \, N_1/N$ and hence a topological direct factor. Thus NN_1/N is closed in $(\text{Rad F}) \, N_1/N$, i.e. NN_1 is a closed normal subgroup of G_0^0. Note that $[\overline{f}_1, \overline{f}_1] \subset \mathfrak{n} + [\overline{f}_0, \overline{f}_0] \subset \mathfrak{g} \cap \overline{f}_0 = \mathfrak{g}_0$. Hence F_1^0/G_0^0 is abelian. Moreover

$$\overline{f}_1/(\mathfrak{n} + \mathfrak{n}_1) = (\text{Rad} \, \overline{f}) + \overline{f}_0/(\mathfrak{n} + \mathfrak{n}_1) = (\text{Rad} \, \overline{f}) + \overline{f}_0/(\mathfrak{n} + \mathfrak{n}_1).$$

Therefore $\overline{f}_1/(\mathfrak{n} + \mathfrak{n}_1)$ is the direct product of the abelian central ideal $(\text{Rad} \, \overline{f}) + \mathfrak{n}_1/(\mathfrak{n} + \mathfrak{n}_1) \simeq (\text{Rad} \, \overline{f})/\mathfrak{n}$ and the reductive sub-algebra $(\overline{f}_0 + \mathfrak{n})/(\mathfrak{n} + \mathfrak{n}_1) \simeq \overline{f}_0/\mathfrak{n}_1$. Hence $\overline{f}_1/(\mathfrak{n} + \mathfrak{n}_1)$ is reductive. Note finally that the restriction of σ to $G_0 \cap F_1^0$ splits into at most finitely many irreducible components. Thus we are in a situation to apply the induction hypothesis and get a closed connected normal subgroup H_1 of $G_0 \cap F_1^0$ containing NN_1, and a linear form $\overline{\ell}_1$ of $\mathfrak{h}_1$ extending $\overline{\ell}$ such that the H_1-orbit of $\overline{\ell}_1$ in $\mathfrak{h}_1^*$ is $\overline{\ell}_1 + (\mathfrak{h}_1/\mathfrak{f})^*$. In particular $\overline{\ell}_1$ may be chosen to vanish on $(\mathfrak{h}_1 \cap \overline{r}) + (\mathfrak{h}_1 \cap \hat{s})$ where $\overline{r}$ is a $\hat{s}$-invariant subspace of r com-plementing $\mathfrak{f}$. But then $\overline{\ell}_1 \circ \text{Ad}_{\mathfrak{h}_1}(\xi) = 0$ for all ξ in $\mathfrak{n}_1$, i.e. $\mathfrak{n}_1 \subset \mathfrak{f}$. Thus $\mathfrak{n}_1 = 0$ and hence $\mathfrak{u}_1 = d\check{p}(\mathfrak{n}_1) = 0$, i.e. $\widetilde{F}_0$ is reductive. This implies that $\widetilde{F}/\widetilde{F}_0$ is a real *affine* algebraic variety. Therefore, according to Lemma 1 of [2], the center of $\widetilde{F}$ has a non trivial connected **R**-split subgroup. In particular there is x in Rad F such that $\rho(x) = c \, \text{Id}$ with $c \in \mathbf{R} \setminus \{1, -1\}$. Thus it follows from Proposition 2.8 that $|\det \text{Ad}_{\mathfrak{h}_1}(x)| \neq 1$, i.e. $|\det \text{Ad}_{\text{rad} \, \mathfrak{f}}(x)| \neq 1$ (note that Rad $\overline{F}/H_1$ is abelian). This however contradicts the fact

that F is unimodular. Thus ρ is trivial. Then it is easily seen that the R-orbit of $\bar{\ell}$ in $\mathfrak{f}^*$ is $\bar{\ell} + (\mathfrak{f}/_\mathfrak{z})^*$, and that there exists a closed one-parameter subgroup T of R such that R is the semi direct product of $R \cap F_0$ and T. Put $H = H_1 T$, and let ℓ_1 be an arbitrary extension of $\bar{\ell}_1$ to $\mathfrak{h}$. Then as in the proof of Theorem 2.9 case γ), we see that the H-orbit of ℓ_1 in $\mathfrak{h}^*$ is $\ell_1 + (\mathfrak{h}/_\mathfrak{z})^*$.

QED

COROLLARY 3.2. – *If* N *does not reduce to the identity then neither does* Z^0.

Now let us apply the above Proposition for a connected non semisimple Lie group G. It is clear that the nilradical N and the center Z of G satisfy the conditions of Proposition 3.1 if we take F = G. Hence there exist a closed connected normal subgroup H such that $N \subset H \subset R$, and a linear form ℓ_1 on $\mathfrak{h}$ extending ℓ such that the H-orbit of ℓ_1 in $\mathfrak{h}^*$ is $\ell_1 + (\mathfrak{h}/_\mathfrak{z})^*$. Put $S = \{x \in G : \ell_1 \circ \mathrm{Ad}_\mathfrak{h}(x) = \ell_1\}$. Then $G = HS$, $H \cap S = Z^0$ according to Lemma 2.2. Note that by Corollary 3.2, Z^0 does not reduce to the identity since G is not semisimple. As usual we assume that $\dim Z = 1$ and ℓ is non zero on $\mathfrak{z}$. Let $\bar{\mathfrak{h}} = \ker \ell_1$. Then S may be characterized as $S = \{x \in G : \mathrm{Ad}(x)\,\bar{\mathfrak{h}} \subset \bar{\mathfrak{h}}\}$.

Thus B_{ℓ_1} is a non degenerate symplectic form on $\bar{\mathfrak{h}}$ and $\mathrm{Ad}_{\bar{\mathfrak{h}}}(S)$ may be viewed as a subgroup of $Sp(\bar{\mathfrak{h}}, B_{\ell_1})$ so that the function J_ℓ in [1] is defined and continuous on S. Let $\nu : \widetilde{S} \longrightarrow S$ be the at most two sheeted covering of S on which J_ℓ has continuous square roots. Let us denote again by ν the covering $\nu : H \circledS \widetilde{S} \longrightarrow H \circledS S$. Note finally that π lifts to a square integrable mod $Z^0 \circledS Z$ irreducible representation of $H \circledS S$ and hence to a square integrable mod $Z^0 \circledS \nu^{-1}(Z)$ irreducible representation $\widetilde{\pi}$ of $H \circledS \widetilde{S}$. We have:

LEMMA 3.3. – *The member* T^ℓ *of the discrete series of* H *extends to an irreducible representation of* $H \circledS \widetilde{S}$ *denoted again by* T^ℓ. *Moreover* $\widetilde{\pi}$ *may be written in the form*

$$\widetilde{\pi}(h, \widetilde{s}) = T^\ell(h, \widetilde{s}) \otimes \sigma(\widetilde{s}) \quad \forall (h, \widetilde{s}) \in H \circledS \widetilde{S} \qquad (3.1)$$

where σ *is a square integrable* mod $\nu^{-1}(Z)$ *irreducible* $\widetilde{\chi}$-*representation of* $\widetilde{S}$, $\widetilde{\chi} = \chi \circ \nu |\nu^{-1}(Z)$. *In particular the discrete series of* $\widetilde{S}$ *exist.*

Proof. — The proof is just a copy of Theorem 4.5 of [1]. The only extra case to be considered is when N is isomorphic to a Heisenberg group with center Z^0. In this case, according to the proof of Theorem 2.9, H may be written as $H = NH_1$, $N \cap H_1 = Z^0$, and the representation $\widetilde{T}^{\varrho}$ obtained from T^{ϱ} by composing the latter with the canonical homomorphism $N \circledS H_1 \longrightarrow NH_1 = H$ is determined by a relation similar to (3.1) in which T^{ϱ} is replaced by the member T_1^{ϱ} of the discrete series of N, and σ by the member T_2^{ϱ} of the discrete series of H_1 which is also a Heisenberg group with center Z^0. Let J_{ϱ}' and J_{ϱ}'' be defined by the structures of the Heisenberg groups N and H_1 respectively in the same manner as J_{ϱ}. Then $J_{\varrho} = J_{\varrho}' J_{\varrho}''$ on S. Let $\check{\nu} : \check{S} \longrightarrow \widetilde{S}$ be the at most two sheeted covering of $\widetilde{S}$ such that J_{ϱ}' and hence J_{ϱ}'' have continuous square roots on $\check{S}$. Then T_1^{ϱ} and T_2^{ϱ} extend to irreducible representations of $(N \circledS H_1) \circledS \check{S} \simeq N \circledS (H_1 \circledS \check{S})$ and $H_1 \circledS \check{S}$ respectively. Let us denote these representations again by T_1^{ϱ} and T_2^{ϱ} respectively. Put

$$T_3^{\varrho}(n, h_1, \check{s}) = T_1^{\varrho}(n, h_1, \check{s}) \otimes T_2^{\varrho}(h_1, \check{s})$$
$$\forall (n, h_1, \check{s}) \in (N \circledS H_1) \circledS \check{S} .$$

If $\check{\nu}(\check{s}) = 1$ we have

$$T_3^{\varrho}(1, 1, \check{s}) = T_1^{\varrho}(1, 1, \check{s}) \otimes T_2^{\varrho}(1, \check{s})$$
$$= (J_{\varrho}' \circ \nu \circ \check{\nu})^{1/2}(\check{s}) \; (J_{\varrho}'' \circ \nu \circ \check{\nu})^{1/2}(\check{s}) . \mathrm{Id}$$
$$= (J_{\varrho} \circ \nu)^{1/2} \circ \check{\nu}(\check{s}) . \mathrm{Id} = \mathrm{Id}$$

and

$$T_3^{\varrho}(z, z^{-1}, 1) = T_1^{\varrho}(z, z^{-1}, 1) \otimes T_2^{\varrho}(z^{-1}, 1) = \widetilde{T}^{\varrho}(z, z^{-1}) = \mathrm{Id} ,$$
$$\forall z \in Z^0 .$$

Therefore T_3^{ϱ} defines, by passing to quotient, an irreducible representation of $H \circledS \widetilde{S}$ extending T^{ϱ}. Let us denote this representation again by T^{ϱ}. On the other hand $\widetilde{\pi}$ lifts to an irreducible representation π_1 of $(N \circledS H_1) \circledS \check{S}$. Hence by Lemma 3.1 of [1], π_1 may be written as

$$\pi_1(n, h_1, \check{s}) = T_1^{\varrho}(n, h_1, \check{s}) \otimes \pi_2(h_1, \check{s}) ,$$
$$\forall (n, h_1, \check{s}) \in (N \circledS H_1) \circledS \check{S} \qquad (3.2)$$

where π_2 is an irreducible representation of $H_1 \circledS \check{S}$ such that $\pi_2 | Z^0 \circledS 1 \simeq \mathrm{mult} \; \chi \otimes 1$. Thus π_2 may be written as

$$\pi_2(h_1, \check{s}) = T_2^\varrho(h_1, \check{s}) \otimes \check{\sigma}(\check{s}), \quad \forall (h_1, \check{s}) \in H_1 \textcircled{s} \check{S} \quad (3.3)$$

where $\check{\sigma}$ is an irreducible representation of $\check{S}$ such that $\check{\sigma} \mid (\nu \circ \check{\nu})^{-1}(Z) \simeq \text{mult } \check{\chi}, \; \check{\chi} = \chi \circ \nu \circ \check{\nu} \mid (\nu \circ \check{\nu})^{-1}(Z)$.

From (3.2) and (3.3) we get

$$\pi_1(n, h_1, \check{s}) = T_1^\varrho(n, h_1, \check{s}) \otimes T_2^\varrho(h_1, \check{s}) \otimes \check{\sigma}(\check{s})$$

$$= T_3^\varrho(n, h_1, \check{s}) \otimes \check{\sigma}(\check{s}) \quad \forall (n, h_1, \check{s}) \in (N \textcircled{s} H_1) \textcircled{s} \check{S}.$$

Since both π_1 and T_3^ϱ are trivial on $\ker \check{\nu}$ so is $\check{\sigma}$, and hence $\check{\sigma}$ defines by passing to quotient an irreducible representation σ of $\widetilde{S}$. Moreover

$$\widetilde{\pi}(h, \widetilde{s}) = T^\varrho(h, \widetilde{s}) \otimes \sigma(\widetilde{s}) \quad \forall (h, \widetilde{s}) \in H \textcircled{s} \widetilde{S}.$$

From this and the fact that $\widetilde{\pi}$ is square integrable mod $Z^0 \textcircled{s} \nu^{-1}(Z)$ it follows that σ is square integrable mod $\nu^{-1}(Z)$.

<div align="right">QED</div>

Let the notations be as above so that $G = HS$, $H \cap S = Z^0$. By Lemma 3.3 there is an at most two sheeted covering $\nu : \widetilde{S} \longrightarrow S$ such that $\widetilde{S}$ has a square integrable mod $\nu^{-1}(Z)$ irreducible representation. In particular $\text{Cent}(\widetilde{S})/\nu^{-1}(Z) \cap \text{Cent}(\widetilde{S})$ is compact, and hence so is $\text{Cent}(S)/Z$. Now it is easily seen that $\nu^{-1}(Z) \cap \text{Cent}(\widetilde{S})$ has finite index in $\nu^{-1}(Z)$. Therefore the discrete series of $\widetilde{S}$ exist. Suppose that S is not reductive. Then $\text{Rad } S$ and thus $\text{Rad } \widetilde{S}$ are two-step nilpotent Lie groups. Therefore it follows from Proposition 3.1 that $\text{Rad } \widetilde{S}$ is a Heisenberg group with center $\widetilde{Z}_1$, and hence $\text{Rad } S$ is a Heisenberg group with center $Z_1 = \nu(\widetilde{Z}_1)$. Let S_1 be a maximal connected semisimple subgroup of S, and V a S_1-invariant subspace of $\text{Rad } \mathfrak{s}$ complementing $\mathfrak{z}_1$. Then $\mathfrak{h}_1 = \mathfrak{z} + V$ is plainly an ideal of $\mathfrak{s}$ which is a Heisenberg algebra with center $\mathfrak{z}$. Let H_1 be the analytic subgroup of S corresponding to $\mathfrak{h}_1$. Then it is obvious that $\text{Rad } S/Z^0$ is the direct product of H_1/Z^0 and Z_1/Z^0. Therefore HH_1 is a closed normal subgroup of G. Moreover we have $\ell \in \mathscr{Q}_{HH_1}$ by Lemma 2.4. Hence by replacing H by HH_1 and S by Z_1S_1 if necessary we may assume that S is reductive. We have thus proved:

THEOREM 3.4. — *Let G be a connected unimodular Lie group with center Z. Assume that $G/\text{Rad } G$ has finite center. Let χ be*

a character of Z. *Put* $\ell = -\sqrt{-1}\,d\chi \in \mathfrak{z}^*$. *If* G *has a square integrable* mod Z *irreducible* χ-*representation, then* G *may be written as* $G = HS$, $H \cap S = Z^0$ *where* H *is a closed normal subgroup of* G *which is a H-group with center* Z^0 *such that* $\ell \in \mathscr{L}_H$, *and* S *is a connected reductive subgroup such that* $\mathrm{Cent}(S)/Z$ *is compact and that* S *has an at most two sheeted covering with discrete series.*

COROLLARY 3.5. — *Let* G *be the semi direct product of a simply connected solvable unimodular Lie group* R *and a connected semisimple Lie group* S *with finite center. Let* Z *be the center of* G *so that* $Z \cap R$ *is the direct product of* Z^0 *and a discrete finitely generated free abelian group* D. *If the discrete series of* G *exist then:*

i) S *has an at most two sheeted covering with discrete series in the ordinary sense. In particular* S *has a compact Cartan subgroup.*

ii) R *is the semi direct product of a H-group* H *with center* Z^0 *and a vector group* T *containing* D *such that* T/D *is compact.*

More precisely, let π *be a square integrable* mod Z *irreducible* χ-*representation of* G *where* $\chi \in \hat{Z}$. *Then* $\ell = -\sqrt{-1}\,d\chi \in \mathscr{L}_H$.

Proof. — By Theorem 3.4 G may be written as $G = HS_1$, $H \cap S_1 = Z^0$, where H is a H-group with prescribed properties and S_1 is a reductive subgroup having at most two sheeted covering with discrete series such that $\mathrm{Cent}(S_1)/Z$ is compact. Moreover by replacing S_1 by some conjugate we may assume that it contains S. Let T be the connected subgroup of $\mathrm{Cent}(S_1)$ generated by the one-parameter subgroups passing by the elements of D. Then S_1 is clearly the direct product of Z^0, T and S. Moreover T/D is compact since $\dim T = rk\,D$. Finally S has an at most two sheeted covering with discrete series since S_1 does. Note that S also has a finite covering which is acceptable in the sense of [7]. Hence S has a compact Cartan subgroup (cf. [6]).   QED

For the converse of Theorem 3.4 and Corollary 3.5 we have:

THEOREM 3.6. — *Let* G *be the semi direct product of a H-group* H *with center* Z *and a connected Lie group* S. *Assume*

that $\mathfrak{h}$ has a S-invariant subspace $\overline{\mathfrak{h}}$ complementing $\mathfrak{z}$ so that the center of G may be written as the direct product of Z and a subgroup D lying in the center of S. Then G has an at most two sheeted covering $\nu : \widetilde{G} = H \circledS \widetilde{S} \longrightarrow G$ such that every member $T^{\mathfrak{l}}$ of the discrete series of H can be extended to an irreducible representation of $\widetilde{G}$ denoted again by $T^{\mathfrak{l}}$. Moreover let $\chi \in (Z\nu^{-1}(D))^{\widehat{\ }}$ be such that $d\chi|_{\mathfrak{z}} = \sqrt{-1}\,\mathfrak{l}$. Then every χ-representation of $\widetilde{G}$ may be written as

$$\pi(h,\widetilde{s}) = T^{\mathfrak{l}}(h,\widetilde{s}) \otimes \sigma(\widetilde{s}) \quad \forall (h,\widetilde{s}) \in \widetilde{G} \qquad (3.4)$$

where σ is a χ_1-representation of $\widetilde{S}$, $\chi_1 = \chi|\nu^{-1}(D)$.

Finally π is square integrable $\mod Z\nu^{-1}(D)$ if and only if σ is square integrable $\mod \nu^{-1}(D)$.

Proof. — First note that it follows immediately from the definition of H-groups that ZS is precisely the set of those elements of G leaving $\overline{\mathfrak{h}}$ invariant. Therefore the center of G is equal to ZD as claimed. Now as in the proof of Lemma 3.3, the existence of the subspace $\overline{\mathfrak{h}}$ allows one to define the continuous function $J_{\mathfrak{l}}$ on S. Then the covering ν is chosen so that $J_{\mathfrak{l}} \circ \nu$ has continuous square roots on $\widetilde{S}$. The rest of the proof is carried out exactly as in Lemma 3.3 and Theorem 4.5 of [1].

QED

Remark. — 1) If the discrete series of G exist then $\widetilde{S}$ has square integrable $\mod \nu^{-1}(D)$ irreducible representations. Therefore by Theorem 3.4 S may be written as $S = H_1 S_1$, where H_1 is a H-group with center D^0 and S_1 is reductive. Now $HH_1 = (HD^0)(ZH_1)$ is a H-group with center ZD^0 by Lemma 2.4. Moreover ZS_1 is reductive, and hence G satisfies the statements of Theorem 3.4 as expected.

2) Suppose that S is reductive and $\text{Cent}(S)/D$ is compact. Then $\text{Ad}_{\mathfrak{h}}(S)$ is a connected reductive Lie group with compact center and hence the subspace $\overline{\mathfrak{h}}$ always exists. Furthermore if D is discrete then Theorem 3.6 may be viewed as a converse of Corollary 3.5. The members of the discrete series of $\widetilde{G}$ are described explicitly by (3.4).

3) Assume again that S is reductive and $\text{Cent}(S)/D$ is compact. Assume now that there is a (topological) isomorphism φ from Z

onto a closed subgroup of D, and that S has a square integrable mod D irreducible χ_1-representation σ, where $\chi_1 \in \hat{D}$ is such that $\ell = -\sqrt{-1}\, d(\chi_1 \circ \varphi) \in \mathcal{R}_H$. Let T^ℓ be the irreducible representation of $H \circledS S$ extending the member of the discrete series of H corresponding to ℓ. Let $\tilde{\sigma}$ be the pull back of σ to $\tilde{S}$. Then (3.4) gives a square integrable mod $Z \circledS \nu^{-1}(D)$ irreducible representation of $H \circledS \tilde{S}$ which is trivial on

$$\{(z_1, z_2) \in Z \circledS \tilde{Z} : \varphi(z_1) = \nu(z_2)\}$$

where $\tilde{Z} = (\nu^{-1}\varphi(Z))^0$. Hence by passing to quotient we obtain the following converse of Theorem 3.3:

COROLLARY 3.7. – *Let G be a connected Lie group with center Z. Suppose that $G = HS$, $H \cap S = Z^0$ where H is a H-group with center Z^0 and S is a connected reductive subgroup containing Z such that $\mathrm{Cent}(S)/Z$ is compact. Finally assume that S has a square integrable mod Z irreducible χ-representation where $\chi \in \hat{Z}$ is such that $-\sqrt{-1}\, d\chi \in \mathcal{R}_H$. Then G has an at most two sheeted covering with discrete series.*

Now combining Corollary 3.5 and the remark 2) above of Theorem 3.6 we have the following "algebraic" characterization of unimodular Lie groups with discrete series:

THEOREM 3.8. – *Let G be a connected unimodular Lie group with center Z. Assume that the radical of G is simply connected and that G has a maximal connected semisimple subgroup with finite center. Then G has a finite covering with discrete series if and only if it may be written as the semi direct product of a H-group with center Z^0 and a connected reductive subgroup which is itself the direct product of a vector group T such that $T/T \cap Z$ is compact and a connected semisimple subgroup with compact Cartan subgroup.*

Remark. – 1) The connected Lie groups with discrete series in which the conclusion of Theorem 3.8 does not hold are necessarily non unimodular. The "$ax + b$" group, and more generally the group given in the example of [2] are such.

2) In view of [16], every almost connected locally compact unimodular group with discrete series is an extension of a connected

unimodular Lie group with discrete series by a compact group. However there is no satisfactory characterization of such groups at the present moment.

4. On a conjecture of Wolf and Moore.

In this section we give another characterization of connected unimodular Lie groups with discrete series which, in the case of simply connected solvable Lie groups, solves the conjecture stated in [15].

THEOREM 4.1. — *Let* G *be the semidirect product of a simply connected unimodular solvable Lie group* R *and a connected semi-simple Lie group with finite center. Let* Z *be the center of* G, *and* χ *a character of* Z. *If* G *has a square integrable* mod Z *irreducible* χ-*representation then there exists a linear form* ℓ *on* $\mathfrak{g}$ *extending* $-\sqrt{-1}\,d\chi$ *such that*

(4.1) G_ℓ/Z *is compact, where* G_ℓ *is the centralizer of* ℓ *in* G.

Conversely, let ℓ *be a linear form on* $\mathfrak{g}$ *satisfying* (4.1). *Then* G *has a finite covering* $\widetilde{G}$ *with discrete series. In fact members of the discrete series of* $\widetilde{G}$ *may be chosen so that their restrictions to* Z^0 *are equivalent to multiples of the character* $\exp(\sqrt{-1}\,\ell|_{\mathfrak{z}})$ *of* Z^0.

Proof. — First assume that G has a square integrable mod Z irreducible χ-representation. Then, according to Corollary 3.5, G may be written in the form $G = H \circledS (T \times S)$ where H is a H-group with center Z^0 such that $-\sqrt{-1}\,d\chi \in \mathscr{D}_H$, T is a vector group such that $T/T \cap Z$ is compact and S is a connected semi-simple Lie group with compact Cartan subgroup. Note that T and S have been chosen so that $\ell_1 \circ \mathrm{Ad}_{\mathfrak{h}}(x) = \ell_1$, $\forall x \in T \times S$, where ℓ_1 is a fixed extension of $-\sqrt{-1}\,d\chi$ to $\mathfrak{h}$. Let ξ be a regular element of $\mathfrak{s}$ such that its centralizer C in S is a compact Cartan subgroup. Let ℓ be any extension of ℓ_1 to $\mathfrak{g}$ such that $\ell|_{\mathfrak{s}}$ is defined by ξ via the Killing form of $\mathfrak{s}$. Then it is clear that $G_\ell = Z^0 TC$ and hence G_ℓ/Z is compact.

Conversely suppose that there is some ℓ in $\mathfrak{g}^*$ satisfying (4.1). Then by an argument similar to that in the proofs of Proposition 3.1 and Theorem 3.3 which is omitted here, one finds that $G = HS$, $H \cap S = Z^0$, where H is a H-group such that the H-orbit of $\ell_1 = \ell \mid \mathfrak{h}$ in $\mathfrak{h}^*$ is $\ell_1 + (\mathfrak{h}/\mathfrak{z})^*$ and S is a connected reductive subgroup such that $\ell_1 \circ \mathrm{Ad}_\mathfrak{h}(x) = \ell_1$, $\forall x \in S$. In particular $S_{\ell_2} = G_\ell$, where $\ell_2 = \ell \mid \mathfrak{s}$ and S_{ℓ_2} is the centralizer of ℓ_2 in S. It follows that $\mathrm{Cent}(S)/Z$ is compact. Hence S may be written as $S = Z^0 \times T \times S'$ where T is a vector group such that $T/T \cap Z$ is compact and S' is a connected semisimple Lie group with finite center. Let $\ell_3 = \ell_2 \mid \mathfrak{s}'$; then we have $S_{\ell_2} = Z^0 \times T \times S'_{\ell_3}$, where S'_{ℓ_3} is the centralizer of ℓ_3 in S'. In particular S'_{ℓ_3} is compact. Let $\xi \in \mathfrak{s}'$ be such that ℓ_3 is defined by ξ via the Killing form of $\mathfrak{s}'$. If ξ is not semisimple then S'_{ℓ_3} contains unipotent elements so that it cannot be compact. Hence ξ must be semisimple. In particular S'_{ℓ_3} contains a Cartan subgroup of S' which is of course compact. Therefore G has a finite covering $\widetilde{G}$ with discrete series according to Theorem 3.8. In fact members of the discrete series of $\widetilde{G}$ with prescribed properties may be chosen as indicated by (3.4).

QED

In the case when G is a simply connected solvable Lie group $\widetilde{G} = G$, and we have the following more precise statement.

COROLLARY 4.2. — *Let G be a simply connected unimodular solvable Lie group with center Z. Let $\chi \in \hat{Z}$. Then G has a square integrable mod Z irreducible χ-representation if and only if there is an extension ℓ of $-\sqrt{-1}\, d\chi$ to $\mathfrak{g}$ such that*

$$(4.2) \quad G_\ell/Z \text{ is compact.}$$

Moreover if it is the case then every extension of $-\sqrt{-1}\, d\chi$ to $\mathfrak{g}$ will satisfy (4.2).

Proof. — It remains to prove the last assertion. Thus assume that $-\sqrt{-1}\, d\chi$ has an extension ℓ satisfying (4.2). Then as in the proof of Theorem 4.1, G may be written as $G = H \circledS T$, where H is a H-group with center Z^0 such that $\ell \mid \mathfrak{z} \in \mathscr{R}_H$ and T is a vector group such that $T/T \cap Z$ is compact. Let ℓ_1 be any extension of $-\sqrt{-1}\, d\chi$ to $\mathfrak{g}$. Then ℓ_1 belongs to the same

orbit as $\ell_2 \in \mathfrak{g}^*$ such that $\ell_2(\overline{\mathfrak{h}}) = 0$, where $\overline{\mathfrak{h}}$ is a fixed T-invariant subspace of $\mathfrak{h}$ complementing $\mathfrak{z}$. Note that such a subspace exists since $Ad_{\mathfrak{h}}(T)$ is compact. Now it is plain that $G_{\ell_2} = Z^0 T$ and hence G_{ℓ_2}/Z is compact. Therefore G_{ℓ_1}/Z is also compact since G_{ℓ_1} is conjugate to G_{ℓ_2}.

<div align="right">QED</div>

Let $G = H \circledS T$ be as above. Then, according to Theorem 3.5, every irreducible χ-representation of G may be written in the form

$$\pi(h, t) = \lambda(t) \, T^\ell(h, t) \quad \forall h \in H, \quad \forall t \in T \qquad (4.3)$$

where $\ell = - \sqrt{-1} \, d\chi$ and T^ℓ is a fixed extension of the member of the discrete series of H corresponding to ℓ and λ is a character of T. Moreover these representations are all square integrable mod Z and they constitute all possible extensions of that member of the discrete series of H to G. Thus we have:

COROLLARY 4.3. – *Let $G = H \circledS T$ be as above. Then for each $\ell \in \mathfrak{D}_H$, the irreducible representations of G whose restrictions to Z^0 are equivalent to multiples of $\exp \sqrt{-1} \ell$ belong to the discrete series of G. Moreover they are precisely the extensions of the member T^ℓ of the discrete series of H to G.*

Thanks to Corollary 4.3, we can now state the following Theorem which gives the solution to the conjecture of Wolf and Moore stated in [15]:

THEOREM 4.4. – *Let G be a simply connected unimodular solvable Lie group with center Z. Then the members of the discrete series of G if they exist are precisely the Auslander-Kostant's "induced" representations associated to the integral orbits $\mathcal{O}_\ell$ such that G_ℓ/Z is compact.*

Proof. – It follows from Corollary 4.2 that the existence of the discrete series of G is equivalent to the existence of ℓ in $\mathfrak{g}^*$ such that G_ℓ/Z is compact. Moreover if it is the case then G_ℓ is connected and hence the corresponding orbit is integral. Therefore the Theorem is trivial if the above equivalent conditions do not hold. Now assume that the discrete series of G exist so that G may be written as $G = H \circledS T$ as in Corollary 4.3. Let $\overline{\mathfrak{h}}$ be a T-invariant

subspace of $\mathfrak{h}$ complementing $\mathfrak{z}$. Then every $\ell \in \mathfrak{g}^*$ such that $\ell|_\mathfrak{z} \in \mathscr{R}_H$ is conjugate to an $\ell_1 \in \mathfrak{g}^*$ such that $\ell_1(\overline{\mathfrak{h}}) = 0$. On the other hand let $\ell_1, \ell_2 \in \mathfrak{g}^*$ be such that $\ell_1|_\mathfrak{z} = \ell_2|_\mathfrak{z}$ and that $\ell_1(\overline{\mathfrak{h}}) = \ell_2(\overline{\mathfrak{h}}) = 0$. Assume that $\ell_2 = \ell_1 \circ \mathrm{Ad}\, x$ for some $x \in G$. Then $\mathrm{Ad}(x)\,\overline{\mathfrak{h}} \subset \overline{\mathfrak{h}}$ and hence $x \in Z^0 T$. Moreover $\ell_1 = \ell_2$ on $\mathfrak{t}$ since T is abelian. Therefore $\ell_1 = \ell_2$. Thus we have proved that the set $\mathscr{S} = \{\ell \in \mathfrak{g}^* : \ell|_\mathfrak{z} \in \mathscr{R}_H \text{ and } \ell(\overline{\mathfrak{h}}) = 0\}$ is a cross section of the set of integral orbits $\mathcal{O}_\varrho$ such that G_ϱ/Z is compact.

For each $\ell_1 \in \mathscr{S}$, let η_{ϱ_1} be the character of $G_{\varrho_1} = Z^0 T$ such that $d\eta_{\varrho_1} = \sqrt{-1}\,\ell_1|_{\mathfrak{g}_{\varrho_1}}$. Let $\mathfrak{p}$ be a positive strongly admissible polarization at ℓ_1, and $\pi_1 = \mathrm{ind}(\eta_{\varrho_1}, \mathfrak{p})$ be the "induced" representation constructed by Auslander and Kostant in [4]. Note that π_1 is irreducible since ℓ_1 is integral. Let λ be a character of T and let us extend it to a character of G by letting $\lambda(H) = \{1\}$. Put $\ell = -\sqrt{-1}\,d\lambda$ and $\ell_2 = \ell_1 + \ell$. Then it is clear that $\ell_2 \in \mathscr{S}$. In particular $G_{\varrho_2} = Z^0 T = G_{\varrho_1}$, and hence $\mathfrak{p}$ is also a polarization at ℓ_2. Let $\pi_2 = \mathrm{ind}(\eta_{\varrho_2}, \mathfrak{p})$ be the corresponding irreducible representation of G. Let χ_{ϱ_1} and χ_{ϱ_2} be the characters of the group D introduced in [4] so that their restrictions to $Z^0 T$ are η_{ϱ_1} and η_{ϱ_2} respectively. Then π_1 and π_2 are subrepresentations of the ordinary induced representations $\mathrm{ind}_{D\uparrow G}\chi_{\varrho_1}$ and $\mathrm{ind}_{D\uparrow G}\chi_{\varrho_2}$ respectively. Now it is clear that $\varphi \longmapsto \lambda\varphi$ sets up a unitary equivalence between the representations $\lambda\,\mathrm{ind}_{D\uparrow G}\chi_{\varrho_1}$ and $\mathrm{ind}_{D\uparrow G}\chi_{\varrho_2}$. Moreover it follows immediately from the definition of π_1 and π_2 that this map carries the subspace of the representation space of $\lambda\,\mathrm{ind}_{D\uparrow G}\chi_{\varrho_1}$ corresponding to π_1 onto that corresponding to π_2. Hence we have

$$\pi_2 \simeq \lambda\,\pi_1. \tag{4.4}$$

Finally since $Z^0 \subset G_{\varrho_1} \subset D$ we see that the restriction of $\mathrm{ind}_{D\uparrow G}\chi_{\varrho_1}$ to Z^0 and hence the restriction of π_1 to Z^0 is equivalent to a multiple of $\exp\sqrt{-1}\,(\ell_1|_\mathfrak{z})$. In particular π_1 is a member of the discrete series of G. This together with (4.4) and (4.3) show that the family of all representations π_1 associated to $\ell_1 \in \mathscr{S}$ is precisely the discrete series of G.

QED

Remark. — In view of (4.3) and the description of the discrete series of H in section 2, our result is more precise than Theorem 3.5

of [10], at least for the unimodular case. Moreover in this case Theorem 4.4 contains that Theorem strictly since our groups are not necessary of type I (cf. example 2.5). Note also that in [17] J.Y. Charbonnel obtained a similar result even for non unimodular solvable groups. There is however a supplementary assumption, namely the regular representation is of type I.

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