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ON SCHWARTZ'S THEOREM
FOR THE MOTION GROUP

by Yitzhak Weit

1. Introduction.

Schwartz’s Theorem in the theory of mean periodic functions on the real line states that every closed, translation-invariant subspace of the space of continuous functions on $\mathbb{R}$ is spanned by the polynomial-exponential functions it contains [5]. In particular, every translation-invariant subspace contains an exponential function.

In [2] the two-sided analogue of this result was generalized to $\text{SL}_2(\mathbb{R})$. However, since [3] it is known that Schwartz’s Theorem fails to hold for $\mathbb{R}^n$, $n > 1$.

Our main goal is to show that the two-sided analogue of Schwartz’s Theorem holds for the motion group $\text{M}(2)$. That is, every closed, two-sided invariant subspace of $\text{C}(\text{M}(2))$ contains an irreducible invariant subspace and every such subspace is spanned by a class of functions which replace the polynomial-exponentials on $\mathbb{R}$.

It seems remarkable that the analogue of Schwartz’s Theorem holds for the three dimensional Lie groups $\text{SL}_2(\mathbb{R})$ and $\text{M}(2)$ while it fails to hold for $\mathbb{R}^2$.

In section 3 we verify the two-sided Schwartz’s Theorem for the motion group. In section 4 we consider the problem of one-sided spectral analysis. Finally, in section 5, we study some invariant subspaces of continuous functions on $\mathbb{R}^2$. It turns out that the one-sided Schwartz’s Theorem for the motion group is intimately connected with a problem of Pompeiu type [1, 4, 7].
2. Preliminaries and Notation.

Let $\mathbb{M}(2)$ denote the Euclidean motion group consisting of
the matrices $\begin{pmatrix} e^{ia} & z \\ 0 & 1 \end{pmatrix}$, $a \in \mathbb{R}$, $z \in \mathbb{C}$.

Let $C(\mathbb{M}(2))$ denote the space of all continuous functions on
$\mathbb{M}(2)$ with the usual topology of uniform convergence on compact
sets. Let $\mathcal{D}(\mathbb{R}^n)$ be the space of infinitely differentiable functions
on $\mathbb{R}^n$ endowed with the topology of uniform convergence of functions
and their derivatives on compacta. Let $\mathcal{D}'(\mathbb{R}^n)$ be the
dual of $\mathcal{D}(\mathbb{R}^n)$, the space of Schwartz distributions on $\mathbb{R}^n$ having
compact support. The pairing between $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ is denoted
by $T(f)$ for $f \in \mathcal{D}(\mathbb{R}^n)$ and $T \in \mathcal{D}'(\mathbb{R}^n)$, and for such $f$ and $T$ we
denote by $T * f$ the convolution of $T$ and $f$. For $T \in \mathcal{D}'(\mathbb{R}^n)$,
the Fourier transform of $T$ is defined by $\hat{T}(z) = T(e^{iz \cdot x})$ where
$z \in \mathbb{C}^n$, $x \in \mathbb{R}^n$ and $z \cdot x = z_1 x_1 + \ldots + z_n x_n$. By Paley-Wiener-
Schwartz Theorem, the space $\mathcal{D}'(\mathbb{R}^n)$ of Fourier transforms of
elements of $\mathcal{D}'(\mathbb{R}^n)$ is identified with the space of entire functions
of $n$ complex variables of exponential type which have polynomial
growth on the real subspace $\mathbb{R}^n$. The topology of $\mathcal{D}'(\mathbb{R}^n)$ is so
defined as to make the Fourier transform a topological isomorphism.

Let $\Pi$ denote the group of all rotations of $\mathbb{R}^2$. We denote
by $\mathcal{D}'_{(r)}(\mathbb{R}^2)$ the space of all $T \in \mathcal{D}'(\mathbb{R}^2)$ which satisfy
$T \circ \tau = T$
for every $\tau \in \Pi$. Let $\mathcal{D}'_{(r)}(\mathbb{R}^2)$ denote the space of Fourier
transforms of elements of $\mathcal{D}'_{(r)}(\mathbb{R}^2)$. We notice that each $f \in \mathcal{D}'_{(r)}(\mathbb{R}^2)$
is a function of $z_1^2 + z_2^2$ and that for any even function $g \in \mathcal{D}'(\mathbb{R})$
the function $\tilde{g}$ where $\tilde{g}(z_1, z_2) = g(\sqrt{z_1^2 + z_2^2})$ belongs to $\mathcal{D}'_{(r)}(\mathbb{R}^2)$. Let $\mathcal{D}_0(\mathbb{R}^2)$
denote the space of elements of $\mathcal{D}(\mathbb{R}^2)$ having compact
support and $\mathcal{D}'_0(\mathbb{R}^2)$ the space of radial functions in $\mathcal{D}_0(\mathbb{R}^2)$.

Let $C(\mathbb{R}^n)$ denote the space of continuous functions on $\mathbb{R}^n$
with the topology of uniform convergence on compacta and $C^{(r)}(\mathbb{R}^2)$
the radial functions in $C(\mathbb{R}^2)$. The dual of $C(\mathbb{R}^n)$ is the space
$M_0(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ of all complex-valued Radon measures having
compact support. Let $M_0^{(r)}(\mathbb{R}^2) = M_0(\mathbb{R}^2) \cap \mathcal{D}'_{(r)}(\mathbb{R}^2)$.

Finally, for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and $z = x + iy \in \mathbb{C}$ let
$(\lambda, z) = \lambda_1 x + \lambda_2 y$. 
3. Two-sided spectral synthesis.

The two-sided analogue of Schwartz's Theorem in spectral analysis for the motion group is stated in the following:

**Theorem 1.** — Every closed, two-sided invariant subspace of $C(M(\mathbb{R}^2))$ contains either a character of $M(2)$ or a function $g(\theta, \phi) = e^{i(\lambda_1, \lambda_2)} \phi$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and $\lambda_1^2 + \lambda_2^2 \neq 0$. The two-sided invariant subspace generated by $e^{i(\lambda, z)}$ where $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1^2 + \lambda_2^2 \neq 0$, is irreducible (minimal).

**Proof.** — For $f \in C(M(2))$, $f \neq 0$, let $V_f$ denote the closed subspace generated by the two-sided translates of $f$.

The subspace $V_f$ contains all the functions $g$ where

$$g(\theta, \phi) = f(\theta + \beta, u e^{i\theta} + e^{i\theta} \phi + w)$$

for every $\theta, \beta \in \mathbb{R}$ and $u, w \in \mathbb{C}$. Let $u = \theta = w = 0$ in (1).

Then, for a suitable $m \in \mathbb{Z}$ the function

$$\int_0^{2\pi} f(\theta + \beta, \phi) e^{-im\beta} d\beta = e^{i\alpha_m} \int_0^{2\pi} f(\theta, \phi) e^{-im\beta} d\beta$$

is non-zero and belongs to $V_f$. Let $N$ denote the translation-invariant and rotation-invariant subspace of $C(\mathbb{R}^2)$ generated by $g_1$.

By (1) the functions $e^{i\alpha_m}g_1(\theta + \beta, \phi + w)$ belongs to $V_f$ for every $\theta \in \mathbb{R}$ and $w \in \mathbb{C}$. That is, $V_f$ contains all functions $e^{i\alpha_m}\widetilde{g}(z)$ where $\widetilde{g} \in N$. In [1] it was proved that every closed, translation-invariant and rotation-invariant subspace of $C(\mathbb{R}^2)$ is spanned by the polynomial-exponential functions it contains. In particular, the subspace $N$ contains therefore an exponential function $e^{i(\lambda, z)}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and the function $h(\theta, \phi) = e^{i\alpha_m} e^{i(\lambda, z)}$ belongs to $V_f$. If $\lambda_1^2 + \lambda_2^2 = 0$ then the subspace $N$ contains the constant functions and $V_f$ contains therefore the character $e^{i\alpha_m}$. Suppose that $\lambda_1^2 + \lambda_2^2 \neq 0$.

Let $h_1 \in \mathcal{S}_0(\mathbb{R}^2)$ of the form $h_1(w) = h_2(r) e^{-i\theta m}$ where $w = re^{i\theta}$, and $h_2 \in \mathcal{S}_0^r(\mathbb{R}^2)$ such that $h_1(\lambda_1, \lambda_2) \neq 0$. 


Then the function:

\[ f(\lambda, z) = e^{i\lambda^1 z + i\lambda^2 w} - e^{i\lambda^1 z} = h, \quad \lambda^1, \lambda^2 \in \mathbb{R}^2 \]  

(here \( dw \) denotes Lebesgue measure on \( \mathbb{R}^2 \)) is non-zero and belongs to \( V^f \). It follows, by (1) and the analyticity of the elements of \( G^f(\mathbb{R}^2) \) that \( V^f \) contains all functions \( e^{i(\mu, z)} \) where \( \mu = (\mu^1, \mu^2) \in \mathbb{C}^2 \) such that \( \mu^1 + \mu^2 = \lambda^1 + \lambda^2 \). To prove the second part of the theorem, let \( g(z) = e^{i(\lambda, z)} \) where \( \lambda = (\lambda^1, \lambda^2) \in \mathbb{C}^2 \), \( \lambda^1 + \lambda^2 \neq 0 \). Firstly, we will show that \( V^g \) contains no character of \( M(2) \).

Suppose that \( e^{ima} \in V^g \) for some \( m \in \mathbb{Z} \). Let \( \mu \in C(M(2)) \), \( \mu(\lambda, z) = e^{-ima} \mu_1(z) \) where \( \mu_1 \in G^f(\mathbb{R}^2) \) such that \( \hat{\mu}_1(0, 0) \neq 0 \) and \( \hat{\mu}_1(0, 0) \neq 0 \). We have

\[ \int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z) dz = 0 \]

for every \( \theta \in \mathbb{R} \). Consequently, we deduce

\[ \int_{M(2)} e^{i(\lambda, e^{i\theta} z + we^{i\alpha})} e^{-ima} \mu_1(z) d\alpha dz \\
= \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z) dz \right] e^{i[(\lambda, we^{i\alpha}) - ma]} d\alpha = 0 \]

for every \( \theta \in \mathbb{R} \) and \( w \in \mathbb{C} \). Namely, \( \mu \) annihilates the subspace \( V^g \). On the other hand, we have

\[ \int_{M(2)} e^{ima} \mu(\lambda, z) d\alpha dz \neq 0 \]

which proves the irreducibility of \( V^g \). This completes the proof.

Schwartz's Theorem in spectral synthesis is described in the following:
THEOREM 2. — Every closed, two-sided invariant subspace of $C(M(2))$ is spanned by the functions as

$$g(e^{i\alpha}, z) = e^{ima}Q(Rez, Imz)e^{i(\lambda,z)}$$

that it contains. ($\lambda \in \mathbb{C}^2$ and $Q$ is polynomial).

Proof. — For $f \in C(M(2)), f \neq 0$ let $V$ denote the closed subspace generated by the two-sided translates of $f$. Obviously, $f$ is contained in the closed subspace generated by the functions:

$$e^{ima}P_m(z) = \int_0^{2\pi} f(e^{i(\alpha+\beta)}, z) e^{-ima}d\beta = e^{ima} \int_0^{2\pi} f(e^{i\beta}, z) e^{-ima}d\beta$$

where $m \in \mathbb{Z}$.

By [1], each function $e^{ima}P_m(z)$ is contained in the closed subspace spanned by the functions $e^{ima}Q(Rez, Imz)e^{i(\lambda,z)}$ where $Q(Rez, Imz)e^{i(\lambda,z)}$ is contained in the rotation-invariant and translation-invariant subspace of $C(R^2)$ generated by $P_m(z)$, and hence in the two-sided invariant subspace generated by $P_m(z)$ which completes the proof of the theorem.

4. One-sided spectral analysis.

One-sided spectral analysis of bounded functions on $M(2)$ was studied in [6].

Notation. — Let $\Gamma_w, w \in \mathbb{C}$, denote the closed subspace of $C(R^2)$ spanned by the functions $e^{i(\lambda_1 x + \lambda_2 y)}$ (of $(x, y) \in R^2$) where $\lambda_1^2 + \lambda_2^2 = w^2$. For the characterization of right-invariant subspaces of $C(M(2))$ we state the following:

THEOREM 3. — Every closed, right-invariant subspace of $C(M(2))$ contains a function as

$$g(e^{i\alpha}, z) = e^{ima}g_1(z), m \in \mathbb{Z}, g_1 \neq 0.$$ 

Moreover, if $g_1 \notin \Gamma_0$, then the closed right-invariant subspace generated by $g$ contains a function as $h(e^{i\alpha}, z) = g_2(z)$.

For $g_2 \in \Gamma_w$ and $g_1 \in \Gamma_0$ the closed right-invariant subspaces generated by $g_2$ and by $e^{ima}g_1(z)$ are irreducible.
Proof. – Let \( f \in V, f \neq 0 \), where \( V \) is a closed right-invariant subspace of \( C(M(2)) \). Then \( V \) contains all functions \( f^* \) such that \( f^*(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)}, z - e^{i\alpha}w) \) where \( \beta \in \mathbb{R} \) and \( w \in \mathbb{C} \). Hence, for a suitable \( m \in \mathbb{Z} \) the function
\[
\int_0^{2\pi} f(e^{i(\alpha+\beta)}, z) e^{-im\beta} d\beta = e^{ima} \int_0^{2\pi} f(e^{i\beta}, z) e^{-im\beta} d\beta = e^{ima} g_1(z)
\]
is non-zero and belongs to \( V \). Suppose that \( g_1 \notin \Gamma_0 \). Then if \( g_1 \) is a polynomial (in \( \text{Re} z \) and \( \text{Im} z \)) which is harmonic on \( \mathbb{R}^2 \) there exists a function \( h \in \mathcal{S}_0(\mathbb{R}^2), h(w) = \mu(r) e^{im\theta}, \mu \in \mathcal{S}_0^{\tau}(\mathbb{R}^2), w = re^{i\theta}, \) such that \( g_1 \neq h \).

Hence the function
\[
e^{ima} \int_{\mathbb{R}^2} g_1(z - e^{i\alpha}w) h(w) dw = \int_{\mathbb{R}^2} g_1(z - w) h(w) dw = g_2(z) \tag{3}
\]
is non-zero and belongs to \( V \).

Otherwise, the closed rotation-invariant and translation-invariant subspace generated by \( g_1 \) contains a function \( e^{i(\lambda, z)} \) where \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_1^2 + \lambda_2^2 \neq 0 \) \[1\]. Let \( h_1 \in \mathcal{S}_0(\mathbb{R}^2), h_1(w) = \mu_1(r) e^{im\theta} \) where
\[
\mu_1 \in \mathcal{S}_0^{\tau}(\mathbb{R}^2), w = re^{i\theta}, \text{ such that } h_1(\lambda_1, \lambda_2) \neq 0.
\]
There exists \( \beta \in \mathbb{R} \) such that \( h_{1,\beta} \neq g_1 \), where
\[
h_{1,\beta}(w) = h_1(e^{i\beta}w) = e^{im\beta} h_1(w).
\]
Hence, \( h_1 \neq g_1 \) and proceeding as in (3) we complete the proof of the first part of the theorem.

Let \( V_1 \) be the closed right-invariant subspace generated by \( g_2(z) \) where \( g_2 \in \Gamma_{w_0} \) for some \( w_0 \in \mathbb{C}, w_0 \neq 0 \). We may show, as in the proof of Theorem 1, that \( V_1 \) contains no functions as \( e^{ima} g_1(z) \) where \( g_1 \in \Gamma_0 \). Suppose now that \( g_3 \in V_1 \) where \( g_3 \in \Gamma_{w_1}, w_1 \in \mathbb{C} \). To derive the irreducibility of \( V_1 \) we will show that \( g_3 = C g_2 \) for some \( C \in \mathbb{C} \). Let \( \{ \Phi_n \} \) be a sequence in \( \mathcal{S}_0(\mathbb{R}^2) \) such that
\[
\int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n(w) dw \Rightarrow_{C(M(2))} g_3(z).
\]

Then we have
\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n(w) dw \right] d\alpha \Rightarrow_{C(M(2))} g_3(z).
\]
and
\[ \int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n^*(|w|) dw \overset{C(M(2))}{\longrightarrow} g_3(z) \] (5)

where
\[ \Phi_n^*(|w|) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{-i\alpha}w) d\alpha, \Phi_n^* \in \mathcal{B}_0^{(r)}, n = 1, 2, \ldots \] (6)

But for every \( n \) we have
\[ \int_{\mathbb{R}^2} g_2(z - w) \Phi_n^*|w| dw = \Phi_n^*(w_0) g_2(z). \]

Consequently, \( g_3 = Cg_2 \), as required. Similarly, we verify the irreducibility of the closed right-invariant subspace generated by \( g_1(z)e^{ima} \) where \( g_1 \in \Gamma_0 \).

**Remark 1.** — We don’t know whether Theorem 3 characterizes all the irreducible right invariant subspaces as it is not known whether the exponentials are the only functions of \( C(\mathbb{R}^n), n > 1 \) which generate irreducible translation-invariant subspaces. Whether every translation-invariant subspace of \( C(\mathbb{R}^n), n > 1 \) contains an irreducible subspace seems to be an open question.

**Remark 2.** — In view of Theorem 3 the right-sided analogue of Schwartz’s Theorem in spectral analysis of continuous functions may be formulated as the following question; does every closed, right-invariant subspace of \( C(M(2)) \) contain either a function as \( e^{ima}g_1(z) \) where \( g_1 \in \Gamma_0, g_1 \neq 0 \ m \in \mathbb{Z} \), or \( g_2(z) \) where \( g_2 \in \Gamma_w, g_2 \neq 0 \), for some \( w \in \mathbb{C} \)?

**Notation.** — Let \( \mu_R, R \geq 0 \), denote the normalized Lebesgue measure of the circle \( \{z : |z| = R\} \). For \( f \in C(\mathbb{R}^2) \) let \( N_f^{(r)} \) denote the closed subspace spanned by \( \{f \ast \mu_R : R \geq 0\} \) and \( \tau(f) \) the closed translation-invariant subspace generated by \( f \).

We deduce an equivalent form of the right-sided analogue of Schwartz’s Theorem (as formulated in Remark 2).

It is described in

**Theorem 4.** — The following statements are equivalent:

(i) The right-sided analogue of Schwartz’s Theorem holds for \( M(2) \).
(ii) Let \( f \in C(\mathbb{R}^2) \), \( f \neq 0 \). Then: (a) If \( \tau(f) \cap \Gamma_0 = \{0\} \) then there exists \( w \in C \) such that \( N_f^{(r)} \cap \Gamma_w \neq \{0\} \). (b) If \( \tau(f) \cap \Gamma_0 \neq \{0\} \) then, either \( N_f^{(r)} \cap \Gamma_w \neq \{0\} \) for some \( w \in C \), or, there exist \( m \in \mathbb{Z} \), \( g \in \Gamma_0 \), \( g \neq 0 \) and a sequence \( \psi_n \in \mathcal{S}_0^{(r)}(\mathbb{R}^2) \) such that

\[
 f * \phi_n \xrightarrow{C(\mathbb{R}^2)} g
\]

where \( \phi_n(r, \theta) = \psi_n(r) e^{-im\theta} \), \( n = 1, 2, \ldots \). (Here \( (r, \theta) \) are the polar coordinates in \( \mathbb{R}^2 \)).

Proof. – Suppose that the right-sided analogue of Schwartz’s Theorem holds for \( M(2) \). Let \( f \in C(M(2)) \) where \( f(e^{i\alpha}, z) = f(z) \). Suppose that \( \tau(f) \cap \Gamma_0 = \{0\} \). The closed right-invariant subspace \( W_f \) generated by \( f \) contains no function as \( e^{im\alpha}g(z) \neq 0 \) where \( g \in \Gamma_0 \) and \( m \in \mathbb{Z} \). Since, otherwise

\[
 \int_{\mathbb{R}^2} f(z - e^{ia}w) \mu_n(w) \, dw \xrightarrow{C(M(2))} e^{im\alpha}g(z)
\]

implies for \( \alpha = 0 \) that: \( f * \mu_n \xrightarrow{C(\mathbb{R}^2)} g \), a contradiction. Hence, \( W_f \) contains a function \( g_1(z) \) where \( g_1 \in \Gamma_w \), \( g_1 \neq 0 \). In other words, there exist \( \Phi_n \in \mathcal{S}_0(\mathbb{R}^2) \), \( n = 1, 2, \ldots \), such that

\[
 \int_{\mathbb{R}^2} f(z - e^{ia}w) \Phi_n(w) \, dw \xrightarrow{C(M(2))} g_1(z).
\]

Hence, by (5) we have:

\[
 \int_{\mathbb{R}^2} f(z - w) \Phi_n^*(|w|) \, dw \xrightarrow{C(M(2))} g_1(z)
\]

where \( \Phi_n^* \) are defined in (6). That is, \( g_1 \in N_f^{(r)} \) which yields (ii) (a).

Suppose now that \( \tau(f) \cap \Gamma_0 \neq \{0\} \). If \( W_f \cap \Gamma_v \neq \{0\} \) for some \( v \in C \) then, as proved above, \( N_f^{(r)} \cap \Gamma_v \neq \{0\} \) (here, the functions of \( \Gamma_v \) are looked upon as function on \( M(2) \)). Otherwise, the subspace \( W_f \) must contain a function as \( e^{im\alpha}g_2(z) \) where \( g_2 \in \Gamma_0 \), \( g_2 \neq 0 \), and \( m \in \mathbb{Z} \). Namely, there exists \( \phi_n \in \mathcal{S}_0(\mathbb{R}^2) \) such that

\[
 \int_{\mathbb{R}^2} f(z - e^{ia}w) \phi_n(w) \, dw \xrightarrow{C(M(2))} e^{im\alpha}g_2(z).
\]

Hence we have

\[
 \frac{1}{2\pi} \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} f(z - \xi) \phi_n(e^{-i\alpha} \xi) \, d\xi \right] e^{-im\alpha} \, d\alpha \longrightarrow g_2(z)
\]
which yields
\[ \frac{1}{2\pi} \int_{\mathbb{R}^2} f(z - \xi) \tilde{\phi}_n(\xi) \, d\xi \rightarrow g_2(z) \]

where \( \tilde{\phi}_n(\xi) = \tilde{\psi}_n(r) e^{-im\theta}, \tilde{\psi}_n(r) = \int_0^{2\pi} \phi(e^{-in\eta}r) e^{-im\eta} \, d\eta, \xi = re^{i\theta} \), and we have shown that (i) implies (ii).

Suppose now that (ii) holds. By Theorem 3 we have to show that for every \( f \in C(M(2)), f(e^{i\alpha}, z) = f(z), f \neq 0 \), the subspace \( W_f \) contains either a function \( g(z), g \neq 0, g \in \Gamma_w \), or, a function \( g(e^{i\alpha}, z) = e^{im\alpha} g_1(z) \) where \( g_1 \in \Gamma_0 \), \( g_1 \neq 0 \) and \( m = \mathbb{Z} \).

Let \( f \in C(\mathbb{R}^2), f \neq 0 \) and suppose that \( \mathcal{N}_f \cap \Gamma_w \neq \{0\} \) for some \( w \in \mathcal{C} \). Then, by definition, there exist \( \psi_n \in \mathcal{S}_0^{(r)}(\mathbb{R}^2) \) \( n = 1, 2, \ldots, \) and \( g \in \Gamma_w \) such that
\[ \int_{\mathbb{R}^2} f(z) \psi_n(\xi) \, d\xi = g(z). \]

But we have
\[ \int_{\mathbb{R}^2} f(z - e^{i\alpha} \xi) \psi_n(\xi) \, d\xi = \int_{\mathbb{R}^2} f(z - \xi) \psi_n(\xi) \, d\xi \quad \text{for} \quad n = 1, 2, \ldots, \]

which implies (i).

Finally, suppose that \( \tau(f) \cap \Gamma_0 \neq \{0\} \) and that \( \mathcal{N}_f \cap \Gamma_w = \{0\} \) for every \( w \in \mathcal{C} \). By (ii) (b) we have
\[ \int_{\mathbb{R}^2} f(z - e^{i\alpha} w) \phi_n(w) \, dw = \int_{\mathbb{R}^2} f(z - \xi) \phi_n(e^{-i\alpha} \xi) \, d\xi = e^{im\alpha} \int_{\mathbb{R}^2} f(z - \xi) \phi_n(\xi) \, d\xi \]

for \( n = 1, 2, \ldots, \) which yields, by (7)
\[ \int_{\mathbb{R}^2} f(z - e^{i\alpha} \xi) \psi_n(\xi) \, d\xi \xrightarrow{C(M(2))} e^{im\alpha} g(z). \]

This completes the proof.

5. Invariant subspaces of \( C(\mathbb{R}^2) \).

For \( f \in C(\mathbb{R}^2) \) we say that \( w \in \text{Sp}^{T.R.}(f), w \in \mathcal{C} \) if the translation-invariant and rotation-invariant subspace generated by \( f \) contains an exponential in \( \Gamma_w \). Actually, the fact announced in [1] that unless \( f = 0 \) we have \( \text{Sp}^{T.R.}(f) \neq \emptyset \) implies the main
results of [1] concerning the Pompeiu problem [4, 7]. By Theorem 4, the one-sided Schwartz’s Theorem for the motion group is intimately connected to the following problem:

For $f \in C(\mathbb{R}^2)$ we say that $w \in Sp^{(r)}(f)$, $w \in C$, if $N_f^{(r)} \cap \Gamma_w \neq \{0\}$, and that $0 \in Sp^{(r)}(f)$ if $N_f^{(r)} \cap \Gamma_0 \neq \{0\}$, where $\Gamma_0$ denotes the space of harmonic functions on $\mathbb{R}^2$. Suppose that $f \neq 0$. Does this imply that $Sp^{(r)}(f) \neq \emptyset$?

**Remark 3.** — We notice that for $f \in C(\mathbb{R}^2)$ we have $Sp^{(r)}(f) \subseteq Sp^{T. R.}(f)$.

**Remark 4.** — This question is connected to the following problem of Pompeiu type:

Determine for which family $P \subset M_0(\mathbb{R}^2)$, the only continuous function $f$ on $\mathbb{R}^2$ such that $T(f * \mu_R) = 0$ for all $T \in P$ and $R \geq 0$, is the zero function.

Let $J_n$ denote the $n$th Bessel function of the first kind. By definition, we deduce

$$J_n(r)e^{in\theta} = \frac{1}{2\pi r^n} \int_0^{2\pi} e^{ir\cos(\phi - \theta)} e^{in\phi} d\phi.$$  

Hence we have $J_n(wr)e^{in\theta} \in \Gamma_w$, $Sp^{(r)}(J_n(wr)e^{in\theta}) = \{w\}$ for $w \in C$, $w \neq 0$ and $N_f^{(r)}$ is one-dimensional where $I_n(r, \theta) = J_n(wr)e^{in\theta}$.

A partial answer to the above question is provided by the following result:

**Theorem 5.** — Let $f \in C(\mathbb{R}^2)$, $f \neq 0$ where

$$f(r, \theta) = \sum_{m=0}^{N} g_m(r)e^{im\theta}, \quad g_m \in C^{(r)}(\mathbb{R}^2) \quad (m = 0, 1, \ldots, N).$$

Then $Sp^{(r)}(f) \neq \emptyset$. If $0 \notin Sp^{(r)}(f)$ there exist $\lambda, a_m \in C$ $(m = 0, 1, \ldots, N)$, $\lambda \neq 0$, where $\sum_{m=0}^{N} |a_m| > 0$ such that

$$\sum_{m=0}^{N} a_m J_m(\lambda r)e^{im\theta} \text{ belongs to } N_f^{(r)}.$$  

Moreover, we have

$$Sp^{(r)}(f) = \bigcup_{m=0}^{N} Sp^{(r)}(g_m(r)e^{im\theta}).$$
The proof will be accomplished in several lemmas.

**LEMMA 6.** — Every proper closed ideal in $\mathfrak{g}'(\mathbb{R}^2)$ has a common zero in $\mathbb{C}^2$.

**Proof.** — Let $J$ be a proper closed ideal in $\mathfrak{g}'(\mathbb{R}^2)$ and suppose that the functions in $J$ have no common zeroes. Every $f \in J$ is a function of $z_1^2 + z_2^2$. That is, there exists an even entire function $Q_f$ of one complex variable such that

$$f(z_1, z_2) = Q_f(\sqrt{z_1^2 + z_2^2}) \text{ and } Q_f \in \mathfrak{g}'(\mathbb{R}).$$

Let $J^*$ be the ideal in $\mathfrak{g}'(\mathbb{R})$ generated by $\{Q_f : f \in J\}$.

Obviously, the functions in $J^*$ have no common zeroes. Thus, applying Schwartz's Theorem [5] we deduce that $J^* = \mathfrak{g}'(\mathbb{R})$. That is, there exists a sequence $\{P_n\}$ in $J^*$ converging to $1$ in $\mathfrak{g}'(\mathbb{R})$. Each $P_n$ must be of the form $\sum_{j=1}^{k} T_j(w) S_j(w)$ where each $T_j \in \mathfrak{g}'(\mathbb{R})$ and $S_j \in J$. But then the function

$$\sum_{j=1}^{k} T_j(w) S_j(w) + \sum_{j=1}^{k} T_j(-w) S_j(-w) = \sum_{j=1}^{k} (T_j(w) + T_j(-w)) S_j(w)$$

belongs to $J$ since each $T_j(w) + T_j(-w)$ belongs to $\mathfrak{g}'(\mathbb{R}^2)$. Hence, $Q_n(w) = \frac{1}{2} (P_n(w) + P_n(-w))$ belongs to $J$ and $Q_n \to 1$ in $\mathfrak{g}'(\mathbb{R}^2)$, a contradiction.

**LEMMA 7.** — Let $f \in C(\mathbb{R}^2)$ where $f(r, \theta) = g(r)e^{im\theta}$, $g \in C^{(r)}(\mathbb{R}^2)$, $g \neq 0$, $m \in \mathbb{Z}$. Then $Sp^{(r)}(f) \neq \emptyset$. If $0 \not\in Sp^{(r)}(f)$ there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $H \in \mathbb{N}^{(r)}$ where

$$H(r, \theta) = J_m(\lambda r) e^{im\theta}.$$

**Proof.** — We may assume that $f \in \mathcal{S}(\mathbb{R}^2)$. Let $M_f^{(r)}$ denote the closed subspace of $\mathcal{S}(\mathbb{R}^2)$ spanned by $\{f \ast \mu_R : R \geq 0\}$. For $m \in \mathbb{Z}$ let $\mathcal{S}_m(\mathbb{R}^2)$ denote the closed subspace of functions $s \in \mathcal{S}(\mathbb{R}^2)$ such that $s(r, \theta) = h(r)e^{im\theta}$. We have $M_f^{(r)} \subseteq \mathcal{S}_m(\mathbb{R}^2)$.

Let $\mathcal{S}_m'(\mathbb{R}^2) \subseteq \mathcal{S}'(\mathbb{R}^2)$ denote the dual of $\mathcal{S}_m(\mathbb{R}^2)$.

Let $M_f^{(r)\perp} = \{T \in \mathcal{S}_m'(\mathbb{R}^2) : T(f) = 0, f \in M_f^{(r)}\}$. 

Every element of $\hat{\delta}_m(R^2)$ is of the form $p(r)e^{im\theta}$ (as a function on $R^2$). Let $P = \{ p : \hat{T}(r, \theta) = p(r)e^{im\theta}, T \in M_f^{(r)} \}$.

We notice that all functions of $P$ are even or odd depending on $m$.

Let $k$ be the larger integer such that 0 is a zero of order $k$ for each $p \in P$. It follows that $\frac{p(w)}{w^k}$, $p \in P$, is an even entire function of $w$ and by complexification of $\frac{p(r)}{r^k}$

$$p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}$$

is an entire function on $C^2$. The space $J^* = \left\{ p^* : p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}, p \in P \right\}$ is therefore a closed ideal in $\hat{\delta}_m(R^2)$. If $0 \notin Sp^{(r)}(f)$, $J^*$ is a proper ideal.

Hence, by Lemma 6, there exists

$$\lambda^* = (\lambda_1, \lambda_2) \in C^2, \lambda_1^2 + \lambda_2^2 = \lambda^2 \neq 0$$

which is a common zero of $J^*$. Consequently, for each $T \in M_f^{(r)}$ we have $\hat{T}(w) = 0$ where $w = (w_1, w_2) \in C^2$, $w_1^2 + w_2^2 = \lambda^2$.

It follows that $T(Q) = 0$ for $T \in M_f^{(r)}$ where

$$Q(x, y) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{i\lambda_1(x \cos \phi + y \sin \phi)} e^{im\phi} d\phi.$$ But we have

$$Q(r, \theta) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{i\lambda r \cos (\phi - \theta)} e^{im\phi} d\phi = J_m(wr)e^{im\theta}.$$ Consequently, $Q \in M_f^{(r)} \cap \Gamma^\lambda$ which completes the proof.

**Notation.** — Let $C(R^2, C^N)$ denote the space of all continuous functions on $R^2$ which take values in $C^N$, with the usual topology. Let $M_0(R^2, C^N)$ be the dual of $C(R^2, C^N)$, the space of vector-valued measures having compact support. For $f \in C(R^2, C^N)$, (resp. $\mu \in M_0(R^2, C^N)$) let $(f)_n$ (resp. $(\mu)_n$) denote the nth coordinate of $f$ (resp. $\mu$). For $m = (m_1, m_2, \ldots, m_N) \in Z^N$ let $B_{(m)}$ denote
the closed subspace of $C(\mathbb{R}^2, \mathbb{C}^N)$ which consists of all functions $f$ where

$$(f)_n(r, \theta) = h_n(r) e^{im_n\theta} \quad n = 1, 2, \ldots, N.$$  

Let $B_{(m)}'$ be the dual of $B_{(m)}$, the space of all $\eta \in M_0(\mathbb{R}^2, \mathbb{C}^N)$ such that $(\eta)_n = \mu_n e^{-im_n\theta}$ where $\mu_n \in M_0^{(r)}(\mathbb{R}^2)$, $n = 1, 2, \ldots, N$. We will use the following equality:

$$(J_k(wr')e^{ik\theta'}) \ast (\mu(r')e^{im\theta'}) (r, \theta) = \phi(w)J_{k+m}(wr)e^{i(k+m)\theta} \quad (8)$$

where $\mu \in M_0^{(r)}(\mathbb{R}^2)$, $w \in \mathbb{C}$, and $\mu(r')e^{im\theta'}(r, \theta) = \phi(r)e^{im\theta}$. Finally, we notice that $M_0^{(r)}(\mathbb{R}^2)$ acts on $B_{(m)}$ by convolution. Namely, $f \in B_{(m)}$ and $\mu \in M_0^{(r)}(\mathbb{R}^2)$ imply that $f * \mu \in B_{(m)}$.

**Lemma 8.** - Every closed non-trivial subspace of $B_{(m)}$, invariant under $M_0^{(r)}(\mathbb{R}^2)$ contains an invariant one-dimensional subspace. Moreover, if $f \in B_{(m)}$ such that $\lambda \in Sp^{(r)}((f)_1)$, $\lambda \neq 0$, for some $n$, $1 \leq n \leq N$, then the closed subspace spanned by $\{f \ast \mu^R : R \geq 0\}$ contains a function $g \neq 0$, such that

$$(g)_n(r, \theta) = a_nJ_m(\lambda r)e^{imn\theta} \quad n = 1, 2, \ldots, N.$$  

**Proof.** - By induction on $N$ where the case $N = 1$ is provided by Lemma 7. Let $f \in B_{(m)}$ and suppose that $0 \neq \lambda \in Sp^{(r)}((f)_1)$. Let $V_f$ denote the closed subspace of $B_{(m)}$ spanned by $\{f \ast \mu^R : R \geq 0\}$ and $V'_f = \{\eta \in B_{(m)}' : \eta(g) = 0, g \in V_f\}$. We notice that for $\eta \in V'_f$ we have:

$$\sum_{n=1}^{N} (g_n(r)e^{imn\theta}) \ast (\mu_n e^{-imn\theta}) = 0 \quad (9)$$

where $(\eta)_n = \mu_n e^{-imn\theta}$ and $(f)_n = g_n(r)e^{imn\theta}$, $n = 1, 2, \ldots, N$.

Thus we may assume that there exists $\eta \in V'_f$ such that

$$(J_m(\lambda r)e^{imn\theta}) \ast (\mu N e^{-imn\theta}) \neq 0.$$  

Otherwise, the subspace $V_f$ contains a function $g^*$ such that $(g^*)_n = 0$ for $n = 1, 2, \ldots, N - 1$, and $(g^*)_N = J_m(\lambda r)e^{imn\theta}$ which completes the proof. To this end, let $h \in B_{(m')}$ where $(h)_n = (f)_n$ for $n = 1, 2, \ldots, N - 1$, $m' = (m_1, m_2, \ldots, m_{N-1})$ and $B_{(m')} \subset C(\mathbb{R}^2, \mathbb{C}^{N-1})$. By the induction hypothesis the subspace $V_h$ contains a function $h^* \neq 0$ such that
\((h^n)_n = b_n J_m(n) e^{imn \theta}\) for \(n = 1, 2, \ldots, N - 1\).

That is, there exists a sequence \(\{\phi_k\}, \phi_k \in M^r_0(\mathbb{R}^2)\), such that
\[
(g_n(r') e^{imn \theta} * \phi_k) (r, \theta) \xrightarrow{C(\mathbb{R}^2)} b_n J_m(n) e^{imn \theta}
\]

for \(n = 1, 2, \ldots, N - 1\), where \(\sum_{n=1}^{N-1} |b_n| > 0\). Let \(\psi_k \in M^r_0(\mathbb{R}^2)\) where
\[
\psi_k = \phi_k * \mu_N e^{-imn \theta} * \mu_N e^{imn \theta} \quad k = 1, 2, \ldots .
\]

Then by (8), (10) and (11) we obtain:
\[
g_n(r) e^{imn \theta} * \psi_k \xrightarrow{C(\mathbb{R}^2)} b_n J_m(n) e^{imn \theta} * \mu_N e^{-imn \theta} * \mu_N e^{imn \theta}
\]

for \(n = 1, 2, \ldots, N - 1\) where \(C_1 \in \mathbb{C}, C_1 \neq 0\).

For \(n = N\) we have by (9) and (8):
\[
g_N(r) e^{imN \theta} * \psi_k = g_N(r) e^{imN \theta} * \mu_N e^{-imN \theta} * \phi_k * \mu_N e^{imN \theta}
\]

Hence we obtain
\[
g_n(r) e^{imn \theta} * \psi_k \xrightarrow{C(\mathbb{R}^2)} \left[ \sum_{n=1}^{N-1} g_n(r) e^{imn \theta} * \mu_n e^{-imn \theta} \right] * \phi_k * \mu_N e^{imn \theta}.
\]

Similarly, we may prove that if \(0 \in Sp^r((f)_n)\) for some \(n, 1 \leq n \leq N\), then \(V_f\) contains a function \(g \neq 0\) such that:
\[(g)_n(r, \theta) = a_n r^m e^{imn \theta} \quad n = 1, 2, \ldots, N.\]

**Proof of Theorem 5.** Let \(h \in B(m), B(m) \subset C(\mathbb{R}^2, \mathbb{C}^{N+1})\) where \(m = (0,1, \ldots, N)\) and \((h)_n(r, \theta) = g_{n-1}(r) e^{i(n-1) \theta}\), \(n = 1, 2, \ldots, N + 1\), and suppose that \(\lambda \in Sp^r((h)_k), \lambda \neq 0\), for some \(k_0, 1 \leq k_0 \leq N + 1\). Then by Lemma 8, there exists a sequence \(\{\phi_k\}, \phi_k \in M^r_0(\mathbb{R}^2) k = 1, 2, \ldots,\) such that
\[
(g_{n-1}(r') e^{i(n-1) \theta} * \phi_k)(r, \theta) \xrightarrow{C(\mathbb{R}^2)} a_{n-1} J_{n-1}(\lambda r) e^{i(n-1) \theta}
\]
for \( n = 1, 2, \ldots, N + 1 \) where \( \sum_{n=0}^{N} |a_n| > 0 \). Hence, we have

\[
\left[ \left( \sum_{n=0}^{N} g_n(r') e^{in\theta} \right) \phi_k \right](r, \theta) \xrightarrow{\mathcal{C}(\mathbb{R}^2) \rightarrow k \to \infty} \sum_{n=0}^{N+1} a_n J_n(\lambda r) e^{in\theta}.
\]

If \( 0 \in \text{Sp}^{(r)}((h)_{k_0}) \) then, similarly, \( N_f^{(r)} \) contains \( g \in \tilde{\Gamma}_0 \), \( g \neq 0 \), where \( g(r, \theta) = \sum_{n=0}^{N} b_n r^n e^{i n \theta} \). Finally, we may easily prove that \( \text{Sp}^{(r)}(f) \subseteq \bigcup_{m=0}^{N} \text{Sp}^{(r)}(g_m e^{i m \theta}) \) and the result follows.

**COROLLARY 6.** — Let \( f \in \mathcal{C}(\mathbb{R}^2) \), \( f \neq 0 \) where

\[
f(r, \theta) = \sum_{m=0}^{N} g_m(r) e^{i m \theta}, \quad g_m \in \mathcal{C}^{(r)}(\mathbb{R}^2) \quad (m = 0, 1, \ldots, N).
\]

Then the translation-invariant closed subspace \( \tau(f) \) generated by \( f \) contains an exponential function.

**Proof.** — If \( 0 \in N_f^{(r)} \) then \( \tau(f) \) contains a polynomial and hence \( 1 \in \tau(f) \). Otherwise, by Theorem 5, \( g \in \tau(f) \), \( g \neq 0 \) where:

\[
g(r, \theta) = \sum_{m=0}^{N} a_m J_m(\lambda r) e^{i m \theta}
\]

for some \( \lambda, \ a_m \in \mathbb{C}, \ \lambda \neq 0, (m = 0, 1, \ldots, N) \).

The subspace \( \tau(f) \) contains therefore all the functions \( h \) where

\[
h(x, y) = (g * \mu)(x, y)
\]

\[
= \lim_{C \to 0} \int_{\mathbb{R}^2} \left[ \int_0^{2\pi} e^{i \lambda (x - \alpha) \cos \phi + (y - \beta) \sin \phi} e^{i m \phi} d\mu(\alpha, \beta) \right] d\mu(\alpha, \beta)
\]

\[
= \lim_{C \to 0} \int_{\mathbb{R}^2} \left[ \int_0^{2\pi} e^{i \lambda (x \cos \phi + y \sin \phi)} e^{i m \phi} d\phi \right] d\mu(\alpha, \beta)
\]

for every \( \mu \in M_0(\mathbb{R}^2) \) where \( C \in \mathbb{C}, \ C \neq 0 \).

Thus \( \tau(f) \) contains all the functions \( u \) where

\[
u(x, y) = \sum_{m=0}^{N} a_m \int_0^{2\pi} \sin(\phi) e^{i \lambda (x \cos \phi + y \sin \phi)} e^{i m \phi} d\phi
\]

for every \( s \in \mathcal{C}[0, 2\pi] \), \( s(0) = s(2\pi) \). For a sequence \( \{s_n\} \) converg-
ing to the Dirac mass $\delta_{\phi_0}$ concentrated in $\phi_0$ where $\sum_{m=0}^{N} a_m e^{im\phi_0} \neq 0$, we obtain, by passing to the limit, that $v \in \tau(f)$ where

$$v(x, y) = \left( \sum_{m=0}^{N} a_m e^{im\phi_0} \right) e^{i(x \lambda \cos \phi_0 + y \lambda \sin \phi_0)}$$

which completes the proof.

**Remark 5.** — To this end we may introduce the following proof to the fact that every translation-invariant and rotation-invariant closed subspace of $C(R^2)$ contains an exponential function [1]. Let $R_f$ denote the closed translation-invariant and rotation-invariant subspace generated by $f \neq 0$. Then, for a suitable $m \in \mathbb{Z}$ the function $g$ where

$$g(r, \theta) = \int_0^{2\pi} f(r, \theta + \beta) e^{-im\beta} d\beta = e^{im\theta} \int_0^{2\pi} f(r, \beta) e^{-im\beta} d\beta = e^{im\theta} f_1(r)$$

is non-zero and belongs to $R_f$. Let $\mu \in M_0^r(R^2)$ where $\mu(f) \neq 0$. Hence the function $g_1 = g \ast (\mu e^{-im\theta})$ is non-zero and belongs to $R_f \cap C(R^2)$.

By Lemma 6, or by Lemma 7 for $m = 0$, there exists $\lambda \in C$ such that $J_0(\lambda r) \in R_f$. Arguing as in the proof of Corollary 6, we deduce that $R_f$ contains the exponentials $e^{i(x \lambda \cos \phi + y \lambda \sin \phi)}$ for every $\phi \in R$ and the result follows.

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