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HARMONIC INTERPOLATING SEQUENCES, L^p AND BMO

by John B. GARNETT

Let (z_ν) be a sequence in the upper half plane. If $1 < p \leqslant \infty$ and if $y_\nu^{1/p} f(z_\nu) = a_\nu$, $\nu = 1, 2, \ldots$ has solution $f(z)$ in the class of Poisson integrals of L^p functions for any sequence $(a_\nu) \in l^p$, then we show that (z_ν) is an interpolating sequence for H^∞. If $f(z_\nu) = a_\nu$, $\nu = 1, 2, \ldots$ has solution in the class of Poisson integrals of BMO functions whenever $(a_\nu) \in l^\infty$, then (z_ν) is again an interpolating sequence for H^∞. A somewhat more general theorem is also proved and a counterexample for the case $p \leqslant 1$ is described.

1. Let $z_\nu = x_\nu + iy_\nu$, $y_\nu > 0$ be a sequence in the upper half plane U, and let

$$P_\nu(t) = \frac{1}{\pi} \frac{y_\nu}{(t - x_\nu)^2 + y_\nu^2}$$

be the Poisson kernel for z_ν. When

$$\int f(t)(1 + t^2)^{-1}\, dt < \infty$$

we write $f(z_\nu) = \int f(t) P_\nu(t)\, dt$ and when $1 \leqslant p \leqslant \infty$ we write

$$T_p f(\nu) = y_\nu^{1/p} f(z_\nu) .$$

The operator T_p maps L^p into the space l^∞ of bounded sequences, because $\|P_\nu\|_q \leqslant c y_\nu^{-1/p}$, $q = p/(p-1)$. If for every l^p sequence (a_ν) the interpolation

$$y_\nu^{1/p} f(z_\nu) = a_\nu , \qquad \nu = 1, 2, \ldots \tag{1.1}$$

has solution within the class of harmonic functions $f(z)$ on U representable as Poisson integrals of L^p functions, then for brevity we write $T_p(L^p) \supset l^p$. Similarly, $T_p(H^p) \supset l^p$ means that (1.1) has solution $f(z) \in H^p$. By a theorem of Carleson [3], [7], $T_p(H^p) = l^p$ if and only if the points z_ν satisfy

$$\inf_\nu \prod_{\mu, \mu \neq \nu} \left| \frac{z_\nu - z_\mu}{z_\nu - \bar{z}_\mu} \right| = \delta > 0 . \tag{1.2}$$

Consequently a sequence satisfying (1.2) is called an *interpolating sequence.*

In [9] it was proved that $\{z_\nu\}$ is an interpolating sequence if and only if $T_\infty(L^\infty) = l^\infty$, and this result was refined in [4] and [13]. Here we extend the work of those papers to obtain (1.2) when $T_p(L^p) \supset l^p$, $1 < p$, or when $T_\infty(\mathrm{BMO}) \supset l^\infty$.

Condition (1.2) holds if and only if the following two geometric conditions both hold

$$|z_\nu - z_\mu| \geqslant \alpha y_\nu , \qquad \mu \neq \nu , \tag{S}$$
$$\sum_{z_\nu \in Q} y_\nu \leqslant B l(Q) , \tag{C}$$

for all squares $Q = \{a < x < a + l(Q), 0 < y < l(Q)\}$. See [10] or [9] for a proof of this well-known equivalence. Because of generalizations mentioned below we state our two theorems in terms of (S) and (C).

Theorem 1. — *If* $1 < p < \infty$ *and if*

$$T_p(L^p) \supset l^p, \tag{1.3}$$

or if $p = \infty$ *and if*

$$T_\infty(\mathrm{BMO}) \supset l^\infty \tag{1.4}$$

then (S) *and* (C) *hold.*

Corollary. — *The sequence* (z_ν) *is an interpolating sequence if and only if* (1.3) *or* (1.4) *holds.*

The other theorem draws the same conclusion from a weaker hypothesis, which is a version of (1.2) for harmonic functions from L^p or BMO.

THEOREM 2. — *If* $1 < p < \infty$ *and if there are* $f_\nu \in L^p$, $\nu = 1, 2, \ldots$ *such that* $\|f_\nu\|_p \leqslant 1$ *and*

$$\text{(1.5)} \qquad \begin{aligned} &T_p f_\nu(\mu) \leqslant 0, \qquad \mu \neq \nu \\ &\inf_\nu T_p f_\nu(\nu) = \delta > 0, \end{aligned}$$

then (S) *and* (C) *hold. If there are* $f_\nu \in \mathrm{BMO}$, $\nu = 1, 2, \ldots$ *such that* $\|f_\nu\|_{\mathrm{BMO}} \leqslant 1$ *and*

$$\text{(1.6)} \qquad \begin{aligned} &T_\infty f_\nu(\mu) \leqslant 0, \qquad \mu \neq \nu \\ &\inf_\nu T_\infty f_\nu(\nu) = \delta > 0 \end{aligned}$$

then (S) *and* (C) *hold.*

Conditions (S) and (C) have analogues in the upper half space $\mathbf{R}_+^{n+1}$, [4], and the two theorems stated here are true in $\mathbf{R}_+^{n+1}$ even when P_ν is replaced by

$$K_\nu(t) = \frac{1}{y_\nu^n} K\left(\frac{x_\nu - t}{y_\nu}\right)$$

where $K \geqslant 0$, $K \in L^1 \cap L^\infty$, $|\nabla K(t)| \leqslant C(1 + |t|)^{n+1}$, and $\int K\, dt = 1$. It is very likely that the proofs are valid in certain spaces of homogeneous type ([5], [6]), such as the unit ball of $\mathbf{C}^n$ with T_p defined using the Poisson-Szegö kernel ([10], [11]). See [1], in which a converse of Theorem 1 is proved in that generality. For $n > 1$ it is not known if (S) and (C) imply interpolation of l^∞ by bounded harmonic functions on $\mathbf{R}_+^{n+1}$, and we do not claim that the corollary to Theorem 1 generalizes to $\mathbf{R}^n$ or $\mathbf{C}^n$. To keep things simple we only prove the theorems for Poisson kernels on $\mathbf{R}^1$.

The methods here are all real analysis; the principle tool is the lemma from § 4 of [4].

In Section 2 we obtain the inequality needed to prove Theorem 1, we show that Theorem 1 is a corollary of Theorem 2, and we verify condition (S). We also include a proof, due to Varopoulos, of Theorem 1 for $p > 2$.

Theorem 2 is proved in Section 3. In Section 4 we show by example that (C) can fail when $T_1(L^1) = l_1$ or when

$$T_p(\mathrm{Re}\, H^p) = \mathrm{Re}\, l^p, \qquad 1/2 < p < 1,$$

'for $p < 1$, T_p must be defined by (1.1) with $f = \mathrm{Re}\, F$

$F \in H^p$). I suspect that $T_1(\mathrm{Re}\, H^1) = \mathrm{Re}\, l^1$ implies (C) but I have no proof.

I thank Eric Amar and Nicholas Varopoulos for useful correspondence and conversation.

The letters c and C stand for universal undetermined constants, the same letter denoting several constants.

2. In Theorem 1 it is not assumed that T_p is a bounded operator from L^p to l^p, or even that $T_p(L^p) \subset l^p$ (which is the same by the closed graph theorem). Indeed, if T_p were bounded then condition (C) would follow by the theorem on Carleson measures ([7] p. 193). Then, as noted in [9], $T_p(L^p) = l^p$ would trivially imply (S) and (C). However, there is an adequate substitute for boundedness.

LEMMA 2.1. — *If* $1 \leqslant p < \infty$ *and if* (1.3) *holds, then there is a constant* M *such that whenever* $\Sigma |a_\nu|^p \leqslant 1$, *the interpolation* $T_p f(\nu) = a_\nu$ *has solution with* $\|f\|_p^p \leqslant M$. *If* (1.4) *holds, there is a constant* M *such that whenever* $(a_\nu) \in l^\infty$, *the interpolation* $T_\infty f(\nu) = a_\nu$ *has solution with* $\|f\|_{\mathrm{BMO}} \leqslant M \sup |a_\nu|$.

Proof. — For $1 \leqslant p < \infty$, the set

$$E_N = l^p \cap T_p(\{f : \|f\|_p \leqslant N\})$$

is closed in l^p. With (1.3) category shows that some E_N has interior in l^p, so that some E_N then contains the unit ball of l^p.

For $p = \infty$, we use the fact that BMO is the dual of the real Banach space $\mathrm{Re}\, H^1$ [8], although a more elementary argument can be given in a few more words. Since

$$P_\nu - P_1 \in \mathrm{Re}\, H^1,$$

the set

$$\begin{aligned} E_N = \{(a_\nu) \in l^\infty : f(z_\nu) - f(z_1) &= a_\nu - a_1\,, \\ \nu = 1, 2, \ldots \ \|f\|_{\mathrm{BMO}} &\leqslant N\} \end{aligned}$$

is closed in l^∞. Since constant functions have zero BMO norm, (1.4) and category as above show interpolation is possible with $\|f\|_{\mathrm{BMO}} \leqslant M \sup_\nu |a_\nu|$.

Because of the lemma, Theorem 2 clearly implies Theorem 1.

In Theorem 2 (or Theorem 1), condition (S) is easy to verify. For $p < \infty$ there is $f \in L^p$, $\|f\|_p \leqslant 1$ such that $f(z_\mu) \leqslant 0$, $f(z_\nu) \geqslant \delta y_\nu^{-1/p}$. The harmonic function $f(z)$ satisfies $|\nabla f(z)| \leqslant c y^{-(1+1/p)}\|f\|_p \leqslant c y^{-(1+1/p)}$, so that

$$\delta y_\nu^{-1/p} \leqslant |f(z_\nu) - f(z_\mu)| \leqslant c y_\nu^{-(1+1/p)}|z_\nu - z_\mu|$$

if

$$|z_\nu - z_\mu| < y_\nu/2 .$$

Hence

$$\frac{|z_\nu - z_\mu|}{y_\nu} \geqslant \operatorname{Max}\left(\frac{1}{2}, \frac{\delta}{c}\right),$$

and we have verified (S). When $p = \infty$ there is $f \in \mathrm{BMO}$, $\|f\|_{\mathrm{BMO}} \leqslant 1$ such that $f(z_\mu) \leqslant 0$, $f(z_\nu) \geqslant \delta$. The elementary estimate $y|\nabla f(z)| \leqslant c\|f\|_{\mathrm{BMO}}$, then yields (S) just as in the case $p < \infty$ above.

N. Varopoulos has a simple proof of Theorem 1 for $p > 2$ which we now present. By the lemma, (1.3) has the dual formulation

$$\Sigma|\lambda_\nu|^q \leqslant M\|\Sigma\lambda_\nu y_\nu^{1/p} P_\nu\|_q^q \tag{2.1}$$

for all finite sequences (λ_ν), where $q = p/(p-1)$. To prove (C), fix a square Q with base I and let $\tilde{\mathrm{I}}$ be the interval concentric with I having length $|\tilde{\mathrm{I}}| = 3|\mathrm{I}|$. Let $z_1, z_2, \ldots, z_N$ be finitely many points from our sequence lying in Q. Let $\lambda_\nu = \pm\, y_\nu^{1/q}$, $\nu = 1, 2, \ldots, N$ with random $\pm$ sign. Taking expectations in (2.1) gives

$$\sum_1^N y_\nu \leqslant M \int |\Sigma y_\nu^2 P_\nu^2|^{q/2}\, dt = M\int_{\tilde{\mathrm{I}}} + M\int_{\mathbf{R}\setminus\tilde{\mathrm{I}}} .$$

Since $q/2 < 1$, Hölder's inequality gives

$$\begin{aligned}\int_{\tilde{\mathrm{I}}} |\Sigma y_\nu^2 P_\nu^2|^{q/2}\, dt &\leqslant |\tilde{\mathrm{I}}|^{1-\frac{q}{2}} \left(\int_{\tilde{\mathrm{I}}} |\Sigma y_\nu^2 P_\nu^2|\, dt\right)^{q/2} \\ &\leqslant 3^{1-\frac{q}{2}} |\mathrm{I}|^{1-\frac{q}{2}} \left(\sum_1^N y_\nu\right)^{q/2}.\end{aligned}$$

Fixing $x_0 \in \mathrm{I}$, we have $P_\nu^2(t) \leqslant c y_\nu^2/(t - x_0)^q$ if $z_\nu \in Q$ and

$t \notin \tilde{I}$, so that

$$\int_{R\setminus\tilde{I}} \left| \sum_1^N y_\nu^2 P_\nu^2 \right|^{q/2} dt \leqslant c\left(\sum_1^N y_\nu^4\right)^{q/2} \int_{(R\setminus\tilde{I})} \frac{dt}{|t - x_0|^{2q}}$$
$$\leqslant c\left(\sum_1^N y_\nu\right)^{q/2} \cdot |I|^{3q/2} \cdot \int_{|I|}^{\infty} \frac{ds}{s^{2q}}$$
$$\leqslant C_q |I|^{1-\frac{q}{2}} \left(\sum_1^N y_\nu\right)^{q/2}.$$

Hence

$$\left(\sum_1^N y_\nu\right)^{1-\frac{q}{2}} \leqslant C|I|^{1-\frac{q}{2}}$$

and condition (C) holds.

Varopoulos' argument can be modified to give the BMO case of Theorem 1 in this way. It is enough to verify (C) for a square Q whose upper half contains a point z_0 from the sequence. Let $z_1, \ldots, z_N$ be finitely many other points from the sequence and in Q. By the lemma and by duality, (1.4) gives

$$\sum_{j=1}^N |\lambda_j| \leqslant M \sup \left\{ \left| \sum_{j=1}^N \lambda_j f(z_j) \right| : \|f\|_{BMO} \leqslant 1,\ f(z_0) = 0 \right\}$$
$$\leqslant cM \left\| \sum_{j=1}^N \lambda_j (P_j - P_0) \right\|_{H^1}$$
$$= cM \left\| \sum_{j=1}^N \lambda_j \left(\frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right) \right\|_{L^1}.$$

We again set $\lambda_j = \pm\, y_j$ and take the expectation, getting

$$\sum_1^N y_j \leqslant cM \int_R \left\{ \Sigma y_j^2 \left| \frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right|^2 \right\}^{1/2} dt.$$

Now

$$\int_{\tilde{I}} \left\{ \sum_1^N y_j^2 \left| \frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right|^2 \right\}^{1/2} dt$$
$$\leqslant 3^{1/2} |I|^{1/2} \left\{ 2 \sum_1^N \int_{\tilde{I}} \frac{y_j^2}{|t - \bar{z}_j|^2} dt + 2 \sum_1^N y_j^2 \int_{\tilde{I}} \frac{dt}{|t - \bar{z}_0|^2} dt \right\}^{1/2}$$
$$\leqslant 3^{1/2} |I|^{1/2} \left\{ 2 \sum_1^N y_j \int_{\tilde{I}} \frac{y_j}{(t - x_j)^2 + y_j^2} dt + 2c\Sigma y_j^2/|\tilde{I}| \right\}^{1/2}$$
$$\leqslant C|I|^{1/2} \left(\sum_1^N y_j \right)^{1/2}.$$

For $t \notin \tilde{I}$,
$$\left|\frac{1}{t-\bar{z}_j} - \frac{1}{t-\bar{z}_0}\right|^2 \leqslant \frac{c|I|^2}{(t-x_0)^4},$$
so that
$$\int_{R\setminus\tilde{I}} \left\{ \sum_1^N y_j^2 \left|\frac{1}{t-\bar{z}_j} - \frac{1}{t-\bar{z}_0}\right|^2 \right\}^{1/2} dt$$
$$\leqslant \left(\sum_1^N y_j\right)^{1/2} |I|^{1/2} c \int_{R\setminus\tilde{I}} \frac{|I|}{(t-x_0)^2}\, dt$$
$$\leqslant C|I|^{1/2} (\Sigma y_j)^{1/2}.$$

Hence $(\Sigma_1^N y_j)^{1/2} \leqslant C|I|^{1/2}$ and (C) holds.

This reasoning does not apply to the case $p \leqslant 2$ nor to the situation in Theorem 2.

3. In proving Theorem 2 we can now assume the points satisfy (S)
$$|z_\nu - z_\mu| \geqslant \alpha y_\nu, \qquad \mu \neq \nu .$$

We prove (C) by contradiction. The idea is that if (C) fails with a large constant B then there are relations among the kernels P_ν which are inconsistent with (1.5) or (1.6). Our main tool is this lemma from [4].

LEMMA 3.1. — *For* $\varepsilon > 0$ *there is a constant* $B(\varepsilon,\alpha)$ *such that if*

$$\sum_{z_\nu \in Q} y_\nu \geqslant B(\varepsilon,\alpha) l(Q) \tag{3.1}$$

for some square $Q = \{a < x < a + l(Q), 0 < y < l(Q)\}$, *then there is a point* z_ν *in the sequence and there are weights* λ_μ *such that*

$$\lambda_\mu \geqslant 0, \qquad \Sigma\lambda_\mu = 1 \tag{3.2}$$
$$\lambda_\nu = 0 \tag{3.3}$$
$$\|P_\nu - \Sigma\lambda_\mu P_\mu\|_1 < \varepsilon \tag{3.4}$$
$$\sum_{z_\mu \in Q} \lambda_\mu \leqslant l(Q)\|P_\nu\|_\infty \leqslant \frac{l(Q)}{\pi y_\nu}, \qquad \text{for all} \qquad Q . \tag{3.5}$$

Except for (3.5) the lemma is proved in Section 4 (and Section 2) of [4], and (3.5) is implicit in that proof because the functions constructed there are non-negative. We refer to [4] for the details.

Suppose $1 < p < \infty$, let $\varepsilon > 0$ be determined later, and assume (3.1) holds. Write $G = P_\nu - \Sigma\lambda_\mu P_\mu$, where the λ_μ are given by Lemma 3.1.

LEMMA 3.2. — $\|G\|_{BMO} \leqslant c/y_\nu$.

Proof. — Fix an interval I with center t_0 and let

$$Q_n = \{z : y < 2^k|I|, |x - t_0| < 2^{n-1}|I|\}.$$

For $z_\mu \in Q_1$ we have trivially

$$\frac{1}{|I|}\int_I P_\mu \, dt \leqslant \frac{1}{|I|},$$

while for $z_\mu \in Q_n \setminus Q_{n-1}$, $n \geqslant 2$, we have

$$\frac{1}{|I|}\int_I |P_\mu - P_\mu(t_0)| \, dt \leqslant \frac{c}{|I|}\int_I \frac{|t-t_0|}{(x_\mu - t_0)^2 + y_\mu^2} \, dt$$
$$\leqslant \frac{c}{2^{2n}|I|}.$$

Letting $a = \Sigma_{z_\mu \notin Q_1} \lambda_\mu P_\mu(t_0)$, we then have

$$\frac{1}{|I|}\int_I |G - a| \, dt \leqslant \|P_\nu\|_\infty + \sum_{z_\mu \notin Q_1} \frac{\lambda_\mu}{|I|} + \sum_{n=2}^{\infty} \sum_{z_\mu \notin Q_n \setminus Q_{n-1}} \frac{c\lambda_\mu}{2^{2n}|I|}$$
$$\leqslant c/y_\nu$$

by (3.5), and the lemma is proved.

Now define $G^{\#}(x) = \sup_{x \in I} \frac{1}{|I|}\int_I |G - G_I| \, dt$, where G_I denotes the mean of G over I. By Lemma 3.2, $\|G^{\#}\|_\infty \leqslant c/y_\nu$, and by the Hardy-Littlewood maximal theorem and (3.4), $G^{\#}$ has small weak L^1 norm

$$m(\lambda) = |\{x : G^{\#}(x) > \lambda\}| \leqslant \frac{c\varepsilon}{\lambda}.$$

Consequently for $q = p/(p-1)$,

$$\|G^{\#}\|_q^q = q\int_0^\infty \lambda^{q-1} m(\lambda) \, d\lambda \leqslant Cq\varepsilon \int_0^{c/y_\nu} \lambda^{q-2} \, d\lambda$$

and

$$\|G^{\#}\|_q \leqslant C_q \varepsilon^{1/q} y_\nu^{-1/p}.$$

From Theorem 5 of [8] we conclude that $\|G\|_q \leqslant C'_q \varepsilon^{1/q} y_\nu^{-1/p}$. But then if $C'_q \varepsilon^{1/q} < \delta$, (1.5) and (3.2) give this contradiction:

$$\delta y_\nu^{-1/q} \leqslant \left| \int G f_\nu \, dt \right| \leqslant C'_q \varepsilon^{1/q} y_\nu^{-1/p}.$$

We conclude that (C) holds with constant $B\left(\frac{\delta^q}{C_q^q}, \alpha\right)$.

Now suppose $p = \infty$. Again if (C) fails we have a point z_ν and weights λ_μ such that (3.2), (3.3) and (3.4) hold for some $\varepsilon > 0$ to be determined. By (1.6) there is $f \in \mathrm{BMO}$ such that $\|f\|_{\mathrm{BMO}} \leqslant M = 1/\delta$, and

$$f(z_\nu) = 0, \qquad f(z_\mu) > 1, \qquad \mu \neq \nu. \tag{3.6}$$

If $f(z)$ were bounded, say $\|f\|_\infty \leqslant M$, (3.6) and (3.4) would be in contradiction as soon as $M\varepsilon > 1$. As we only have $\|f\|_{\mathrm{BMO}} \leqslant M$, more properties of the weights λ_μ must be used. From Section 4 of [4] it also follows that $\lambda_\mu = 0$ except when $y_\mu < y_\nu$ and $|x_\mu - x_\nu| < c y_\nu/\varepsilon^2$. Let

$$J = \{t : |t - x_\nu| < 3c y_\nu/\varepsilon^2\},$$

an interval containing all x_μ with $\lambda_\mu > 0$ in its middle third. For $t \notin J$, we then have

$$|G(t)| = |P_\nu(t) - \Sigma \lambda_\mu P_\mu(t)| \leqslant C P_\nu(t). \tag{3.7}$$

By (3.6) and the John-Nirenberg Theorem,

$$\int |f(t)|^4 P_\nu(t) \, dt \leqslant C M^4.$$

Hence by Hölder's inequality

$$\int_{\mathbf{R} \setminus J} |f(t)| P_\nu(t) \, dt \leqslant C M \varepsilon^{3/4}, \tag{3.8}$$

while trivially

$$\int_J |f(t)|^4 \, dt \leqslant C M^4 / y_\nu \varepsilon^2. \tag{3.9}$$

By (3.7) and (3.8), $\int_{\mathbf{R}\setminus J} |fG| \, dt \leqslant C M \varepsilon^{3/4}$. By (3.9), Hölder's inequality, and our estimate on $\|G\|_{4/3}$, we also have $\int_J |fG| \, dt \leqslant C M \varepsilon^{3/4 - 1/2}$. Since $\left| \int fG \, dt \right| \geqslant 1$ by (3.6), there is a contradiction if $C M \varepsilon^{1/4} < 1$.

This proof for $p = \infty$, due to Peter Jones, is much simpler than my original proof.

4. We give an example of a sequence $\{z_\nu\}$ for which (C) fails but for which $T_1(L^1) \supset l_1$. At the same time we show that (C) can fail for a sequence for which

$$T_p f(\nu) = y_\nu^{1/p} f(z_\nu) = a_\nu\,, \qquad \nu = 1\,, 2\,, \ldots$$

has solution $f \in \operatorname{Re} H^p$ whenever $\Sigma|a_\nu|^p < \infty$, provided $1/2 < p < 1$. Here $\operatorname{Re} H^p$ is the space of real parts of H^p functions with the quasinorm $\|\operatorname{Re} F\|_{H^p} = \|F\|_{H^p}$, $F \in H^p$.

LEMMA 4.1. — *Let* $0 < p \leqslant 1$, *and let* $\eta^p < 1/2$. *Suppose there are* $f_\nu \in H^p$ ($f_\nu \in L^1$ *when* $p = 1$) *such that*

$$\text{(4.1)} \qquad \|f_\nu\|_{H^p}^p \leqslant M \qquad \textit{if} \qquad p < 1$$

or

$$\|f_\nu\|_1 \leqslant M \qquad \textit{if} \qquad p = 1\,,$$

$$\text{(4.2)} \qquad |T_p f_\nu(\nu) - 1| < \eta\,,$$

$$\text{(4.3)} \qquad \sum_{\mu,\, \mu \neq \nu} |T_p f_\nu(\mu)|^p < \eta^p$$

for $\nu = 1, 2, \ldots$ *Then* $T_p(\operatorname{Re} H^p) \supset l^p$ *if* $p < 1$ *and* $T_1(L^1) \supset l_1$ *if* $p = 1$.

Proof. — If $\Sigma|a_\nu|^p < \infty$, let $F = \Sigma a_\nu f_\nu$. Then by (4.1) $\|F\|_{H^p}^p \leqslant M\Sigma|a_\nu|^p$ if $p < 1$, and $\|F\|_1 \leqslant M\Sigma|a_\nu|$ if $p = 1$. And by (4.2) and (4.3),

$$\sum_{\nu=1}^{\infty} |T^p F(\nu) - a_\nu|^p \leqslant 2\eta^p \Sigma |a_\nu|^p.$$

The lemma now follows by iteration.

For $z_0 = x_0 + iy_0$, and for $0 < \varepsilon < y_0$, let

$$f_{z_0,\varepsilon}(t) = \frac{\varepsilon}{\pi}\, y_0^{1-1/p}\left(\chi_{|t-x_0|<\varepsilon} - \chi_{|t-(x_0+y_0)|<\varepsilon}\right),$$

where χ_S is the characteristic function of S. Then

$$|y^{1/p} f_{z_0,\varepsilon}(z_0)| < 1 \qquad \text{and} \qquad y^{1/p} f_{z_0,\varepsilon}(z_0) \to 1 \quad (\varepsilon \to 0).$$

Also $\|f_{z_0,\varepsilon}\|_1 \leqslant 4\pi$ when $p = 1$.

LEMMA 4.2. — *For* $1/2 < p < 1$, $\|f_{z_0,\varepsilon}\|_{H^p}^p \leqslant M_p$.

Proof. — We have

$$\|f_{z_0,\varepsilon}\|_1 \leqslant Cy_0^{1-1/p} \tag{4.4}$$

and

$$\int f_{z_0,\varepsilon}(t)\, dt = 0 . \tag{4.5}$$

Also $f_{z_0\varepsilon}$ has support in $\{|t - x_0| < 2y_0\}$. This means that $f_{z_0,\varepsilon}$ is a $(p, 1)$ atom in the sense of [6], and the lemma follows from Theorem A of that paper. A well-known elementary argument can also be given for special case at hand. Recall the non-tangential maximal function f^* from § 3. We use the theorem that $f(z) \in \operatorname{Re} H^p$ if and only if $f^* \in L^p$, and that $\|f\|_{H^p} \sim \|f^*\|_p$. See [2] or [8].

When $|t - x_0| < 4y_0$, (4.4) and the Hardy-Littlewood theorem give us, for $f = f_{z_0,\varepsilon}$,

$$|\{t : |t - x_0| < 4y_0 , |f^*(t)|^p > \lambda\}| \leqslant \operatorname{Min}\left(8y_0 , \frac{Cy_0^{1-1/p}}{\lambda^{1/p}}\right).$$

Hence

$$\int_{|t-x_0|<4y_0} |f^*(t)|^p\, dt \leqslant 8y_0 \int_0^{c/y_0} d\lambda + \int_{c/y_0}^{\infty} \frac{Cy_0^{1-1/p}}{\lambda^{1/p}}\, d\lambda = M .$$

If $|t - x_0| > 4y_0$ and if $z \in \Gamma(t)$, then

$$\left|\frac{\partial}{\partial s} P_z(s)\right| \leqslant \frac{c}{|z - x_0|^2} \leqslant \frac{c}{|t - x_0|^2}$$

on the support of $f_{z_0,\varepsilon}$. Then (4.5) gives

$$|f(z)| \leqslant cy_0^{1-1/p} \frac{y_0}{|t - x_0|^2},$$

and so

$$\int_{|t-x_0|>4y_0} |f^*(t)|^p\, dt \leqslant cy_0^{2p-1} \int_{4y_0}^{\infty} u^{-2p}\, du \leqslant C_p$$

when $p > 1/2$.

Fix η with $\eta^p < 1/2$. Let $z_1 = \dfrac{1}{2} + i\delta$ where $\delta = \delta(\eta)$

is to be determined, and let ε_1 be so small that

$$|f_{z_1,\varepsilon_1}(z_1) - 1| < \eta .$$

Write $f_1 = f_{z_1,\varepsilon_1}$. From $I_1 = [0,1]$ delete the two intervals $|t - 1/2| < 2\varepsilon_1$, $|t - \delta - 1/2| < 2\varepsilon_1$ containing the support of f_1, and partition the remainder of I_1 into dyadic intervals $I_2, I_3, \ldots, I_{m_2}$ of length 2^{-n_2}. (We suppose ε_1 is a negative power of 2). Let x_ν be the center of I_ν, $2 \leqslant \nu \leqslant m_2$ and let $y_\nu = \delta 2^{-n_2}$. The points $z_\nu = x_\nu + iy_\nu$, $2 \leqslant \nu \leqslant m_2$, join z_1 in our sequence. Choose ε_2 and put $f_\nu = f_{z_0,\varepsilon_2}$, $2 \leqslant \nu \leqslant m_2$. When n_2 is fixed, ε_2 can be chosen so that (4.2) holds for $2 \leqslant \nu \leqslant m_2$. We claim that n_2 and ε_2 can be chosen so that (4.3) holds for the finite sequence $z_1, \ldots, z_{m_2}$. When $\nu = 1$ the left side of (4.3) is

$$\begin{aligned}\delta 2^{-n_2} \sum_{\mu=2}^{m_2} |f_1(z_\mu)|^p &\leqslant C 2^{-n_2} \sum_{k=\varepsilon_1 2^{n_2}}^{\infty} \left(\frac{\delta 2^{-n_2}}{k^2 2^{-2n_2} + \delta^2 2^{-2n_2}}\right)^p \\ &\leqslant C\delta^{1+p} 2^{-n_2(1-p)} (\varepsilon_1 2^{n_2})^{1-2p} \\ &= C\delta^{1+p} \frac{2^{-n_2 p}}{\varepsilon_1^{2p-1}},\end{aligned}$$

which is small if $n_2 > n_2(\varepsilon_1)$. For $\nu > 1$, one term in the left side of (4.3) is

$$\begin{aligned}|T_p f_\nu(1)|^p &\leqslant C y_\nu^{p-1} |P_1(x_\nu) - P_1(x_\nu + y_\nu)|^p \\ &\leqslant C\delta^{p-1} 2^{-n_2(2p-1)} \sup_{I_\nu} \left|\frac{\partial P_1}{\partial s}\right|^p,\end{aligned}$$

and since $p > 1/2$ this is small if 2^{-n_2} is small. For $\nu > 1$ we also have the sum

$$\begin{aligned}\sum_{\substack{\mu=2 \\ \mu\neq\nu}}^{m_2} |T_p f_\nu(\mu)|^p &\leqslant C(\delta 2^{-n_2})^p \sum_{\substack{\mu=2 \\ \mu\neq\nu}}^{m_2} |P_\mu(x_\nu) - P_\mu(x_\nu + y_\nu)|^p \\ &\leqslant C(\delta 2^{-n_2})^{2p} \sum_{k=1}^{\infty} \frac{1}{(k^2 2^{-2n_2} + \delta^2 2^{-2n_2})^p} \\ &\leqslant C\delta^{2p} < \eta/2\end{aligned}$$

if δ is chosen correctly.

From each I_ν, $2 \leqslant \nu \leqslant m_2$, delete the two intervals of length $4\varepsilon_2$ whose middle halves support f_ν. The remaining parts of $\bigcup_{2}^{m_2} I_\nu$ partition into dyadic intervals I_μ,

$$m_2 + 1 \leqslant \mu \leqslant m_3 ,$$

of length 2^{-n_3} and with centers x_μ. Let $z_\mu = x_\mu + i\delta 2^{-n_3}$, and let $f_\mu = f z_{\mu,\varepsilon_3}$, $m_2 + 1 \leqslant \mu \leqslant m_3$. Taking n_3 large and ε_3 small, we can use the above reasoning to obtain (4.2) and (4.3) for $1 \leqslant \nu \leqslant m_3$. This process can be continued to get an infinite sequence of points for which by Lemma 4.1, $T_p(\mathrm{Re}\, H^p) \supset l^p$ if $p < 1$ and $T_1(L^1) \supset l^1$ if $p = 1$.

The sequence lies in the unit square so that (C) will fail if $\Sigma y_\nu = \infty$. However

$$\frac{1}{\delta} \Sigma y_\nu = \Sigma |I_\nu| = 1 + (1 - 8\varepsilon_1) + (1 - 8\varepsilon_1)(1 - 8\varepsilon_2) + \cdots$$

and this sum diverges if $\Sigma \varepsilon_j < \infty$.

By using functions f_ν with several vanishing moments, one can obtain similar examples for $0 < p \leqslant 1/2$.

Added in Proof. — Peter Jones has proved $T_1(\mathrm{Re}\, H^1) = \mathrm{Re}\, l^1$ implies (C) by refining the proof of Lemma 3.1.

BIBLIOGRAPHY

[1] Eric AMAR, Interpolation L^p, to appear.
[2] D. BURKHOLDER, R. GUNDY and M. SILVERSTEIN, A maximal function characterization of the class H^p, *Trans. A.M.S.*, 157 (1971), 137-157.
[3] L. CARLESON, An interpolation problem for bounded analytic functions, *Amer. J. Math.*, 80 (1958), 921-930.
[4] L. CARLESON and J. GARNETT, Interpolating sequences and separation properties, *Jour. d'Analyse Math.*, 28 (1975), 273-299.
[5] R. COIFMAN, R. ROCHBERG and G. WEISS, Factorization theorems for Hardy spaces in several variables, *Ann. of Math.*, 103 (1976), 611-635.
[6] R. COIFMAN and G. WEISS, Extensions of Hardy spaces and their use in analysis, *Bull. A.M.S.*, 83 (1977), 569-645.
[7] P. R. DUREN, Theory of H^p Spaces, Academic Press, New York, 1970.
[8] C. FEFFERMAN and E. STEIN, H^p spaces of several variables, *Acta Math.*, 129 (1972), 137-193.
[9] J. GARNETT, Interpolating sequences for bounded harmonic functions, *Indiana U. Math. J.*, 21 (1971), 187-192.
[10] L. HÖRMANDER, L^p estimates for (pluri-) subharmonic functions, *Math. Scand.*, 20 (1967), 65-78.

[11] E. M. Stein, Boundary Behavior of Holomorphic Functions of Several Complex Variables, Princeton University Press, Princeton, 1972.
[12] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1970.
[13] N. Varopoulos, Sur un problème d'interpolation, *C.R. Acad. Sci. Paris*, Ser. A, 274 (1972), 1539-1542.

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