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ON THE ČECH BICOMPLEX ASSOCIATED WITH FOLIATED STRUCTURES

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1. We shall be in C^∞-category. Let M be a paracompact connected n-dimensional manifold with a foliation $\mathfrak{F}$ of codimension q, and let $\mathfrak{U} = \{U_\alpha\}$ be a simple covering of M such that each U_α is a flat neighborhood with respect to $\mathfrak{F}$. Then there exists a decomposable q-form $w = w^1 \wedge \ldots \wedge w^q$ on each U_α and, by Frobenius' theorem, there exists a 1-form η on each U_α satisfying $dw = w \wedge \eta$, where d denotes the exterior differentiation and $\wedge$ the exterior product. The 1-form η is an interesting object; it is well known that $\eta \wedge (d\eta)^q$ defines a de Rham class in $H^{2q+1}(M, R)$ ([1], [2], [3]). Our aim is to show that η itself defines a certain cohomology class, that is,

THEOREM A. – $((-1)^{q-1}/2\pi)\eta$ *defines a* D-*cohomology class in* $H^2(\check{C}^{(*)}(\mathfrak{U}; A^*(M)), D)$ *depending only on* $\mathfrak{F}$.

The above theorem was announced in [4], where it contained misstatements.

THEOREM B. – *Supposing* M *admits foliations* $\mathfrak{F}$, $\mathfrak{F}'$ *complementally transversal to each other,* η *defines a* D'*-cohomology class in* $H^1(\check{C}^{(*)}(\mathfrak{U}; A^*(M)), D')$ (Cf. [5], [6], [7]).

2. Since M has a foliation $\mathfrak{F}$ of codimension q, the tangent bundle TM of M has an integrable subbundle E with fibre dimension $n-q$. Let $Q = TM/E$ be a quotient bundle with fibre dimension q. Choosing a suitable Riemannian metric on TM, we obtain an isomorphism $TM \cong E \oplus Q$ (Whitney sum).

Then

$$d\overline{w}^i = \sum_j dt^i_j \wedge w^j + \sum_j t^i_j \, dw^j$$

$$= \sum_j \Big(dt^i_j - \sum_k t^i_k \, \varphi^k_j \Big) \wedge w^j ,$$

and on the other hand

$$d\overline{w}^i = \sum_k \overline{w}^k \wedge \overline{\varphi}^i_k$$

$$= \sum_j \Big(- \sum_k t^k_j \overline{\varphi}^i_k \Big) \wedge w^j .$$

Thus

$$- \sum_k t^k_j \overline{\varphi}^i_k = dt^i_j - \sum_k t^i_k \, \varphi^k_j + \sum_k f^i_{jk} \, w^k$$

where f^i_{jk} are functions on $U_{\alpha_0} \cap U_{\alpha_1}$.

Let $\begin{pmatrix} s^i_j & 0 \\ 0 & s^a_b \end{pmatrix}$ denote the inverse matrix of $\begin{pmatrix} t^i_j & 0 \\ 0 & t^a_b \end{pmatrix}$.

Then

$$\sum_{j,k} s^j_l \, t^k_j \overline{\varphi}^i_k = - \sum_j s^j_l \, dt^i_j + \sum_{j,k} s^j_l \, t^i_k \, \varphi^k_j - \sum_{j,k} s^j_l \, f^i_{jk} \, w^k$$

and we obtain

$$\sum_i \overline{\varphi}^i_i = - \sum_{i,j} s^j_i \, dt^i_j + \sum_i \varphi^i_i - \sum_{i,j,k} s^j_i \, f^i_{jk} \, w^k .$$

From (4), $dt^i_j = \sum_k t^i_{jk} \; w^k$.. Thus, by (3), we obtain $\sum_i \overline{\varphi}^i_i = \sum_i \varphi^i_i$. Therefore we obtain $\overline{\eta} = \eta$ on $U_{\alpha_0} \cap U_{\alpha_1}$ and $\eta \in \check{C}^{0,1}(\mathcal{U} ; A^*(M))$.
Q.E.D.

THEOREM B. – *Supposing* M *admits foliations* $\mathcal{F}, \mathcal{F}'$ *complementally transversal to each other,* η *defines a* D$'$*-cohomology class in* $H^1(\check{C}^{(*)}(\mathcal{U} \; ; A^*(M)), D')$ *where* η *is defined by* $\mathcal{F}$.

We may suppose that $\mathcal{U} = \{U_\alpha\}$ is a simple covering such that each U_α is a locally trivial neighborhood of the bundle $Q \longrightarrow M$.

Let ∇^α denote a local connection on $Q|_{U_\alpha}$, and let ω_α (resp. Ω_α) denote a connection form (resp. a curvature form) of ∇^α on U_α. Let Δ^p be a canonical p-simplex in R^{p+1} (with coordinates $(t_0, t_1, \ldots, t_p)$). We define a connection form $\omega_{\alpha_0 \ldots \alpha_p}$ on $U_{\alpha_0 \ldots \alpha_p} \times \Delta^p$ ($U_{\alpha_0 \ldots \alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_p}$) by

$$\begin{aligned}\omega_{\alpha_0 \ldots \alpha_p} &= t_0 \omega_{\alpha_0} + \ldots + t_p \omega_{\alpha_p} \\ &= (1 - t_1 - \ldots - t_p)\, \omega_{\alpha_0} + t_1 \omega_{\alpha_1} + \ldots + t_p \omega_{\alpha_p}\end{aligned}$$

and let $\Omega_{\alpha_0 \ldots \alpha_p}$ denote a corresponding curvature form on $U_{\alpha_0 \ldots \alpha_p} \times \Delta^p$.

For the set $A^k(U_{\alpha_0 \ldots \alpha_p} \times \Delta^p)$ of all k-forms on $U_{\alpha_0 \ldots \alpha_p} \times \Delta^p$, ${}_{\Delta^p}\!\int : A^k(U_{\alpha_0 \ldots \alpha_p} \times \Delta^p) \longrightarrow A^{k-p}(U_{\alpha_0 \ldots \alpha_p})$ denotes the integration along the fibre. Then we obtain Stokes' theorem

$${}_{\Delta^p}\!\int \circ\, d = (-1)^p\, d \circ {}_{\Delta^p}\!\int + {}_{\partial\Delta^p}\!\int \circ\, j^*$$

where $j : U_{\alpha_0 \ldots \alpha_p} \times \partial\Delta^p \longrightarrow U_{\alpha_0 \ldots \alpha_p} \times \Delta^p$ denotes the inclusion.

We consider Čech bicomplex $\check{C}^{(*)}(\mathcal{U}\,; A^*(M))$: Let $\check{C}^{p,q} = \prod_{\alpha_0 \ldots \alpha_p} A^q(U_{\alpha_0 \ldots \alpha_p})$, and let $D' : \check{C}^{p,q} \longrightarrow \check{C}^{p+1,q}$ denote the ordinary simplicial differential and $D'' = (-1)^p d : \check{C}^{p,q} \longrightarrow \check{C}^{p,q+1}$ the de Rham differential. A multiplication $\cdot\,; \check{C}^{p,q} \otimes \check{C}^{p',q'} \longrightarrow \check{C}^{p+p',q+q'}$ is defined by .

$$(\Phi \cdot \Phi')_{\alpha_0 \ldots \alpha_{p+q}} = (-1)^{qp'}\, \Phi_{\alpha_0 \ldots \alpha_p}|_{U_{\alpha_0 \ldots \alpha_{p+q}}} \wedge \Phi'_{\alpha_p \ldots \alpha_{p+q}}|_{U_{\alpha_0 \ldots \alpha_{p+q}}} .$$

For $\check{C}^{(k)}(\mathcal{U}\,; A^*(M)) = \sum_{p+q=k} \check{C}^{p,q}$ and $D = D' + D'' : \check{C}^{(k)} \longrightarrow \check{C}^{(k+1)}$, we obtain a graded algebra $(\check{C}^{(*)}(\mathcal{U}\,; A^*(M)), D, \cdot)$.

Let $I^*(\mathfrak{gl}_q)$ denote a graded algebra of invariant polynomials on a Lie algebra $\mathfrak{gl}_q$. A caracteristic homomorphism

$$\gamma : I^*(\mathfrak{gl}_q) \longrightarrow C^{(*)}(\mathcal{U}\,; A^*(M))$$

is defined by

$$\gamma\varphi = \sum_p (\gamma\varphi)^{p,2k-p} \qquad \varphi \in I^k(\mathfrak{gl}_q)$$

where

$$(\gamma\varphi)^{p,2k-p}_{\alpha_0\cdots\alpha_p} = \int_{\Delta^p} \varphi(\Omega_{\alpha_0\ldots\alpha_p}, \ldots, \Omega_{\alpha_0\ldots\alpha_p}).$$

Then we obtain,

LEMMA 1 (Cf. [8]). – *For* $\varphi \in \mathrm{I}^k(\mathfrak{gl}_q)$,

$$(-1)^{[p(p-1)]/2} \frac{(k-p)!}{k!} (\gamma\varphi)^{p,2k-p}_{\alpha_0\cdots\alpha_p}$$

$$= \begin{cases} \displaystyle\int_{\Delta^p} dt_1 \wedge \ldots \wedge dt_p \wedge \varphi(\omega_{\alpha_1} - \omega_{\alpha_0}, \omega_{\alpha_2} - \omega_{\alpha_0}, \ldots, \omega_{\alpha_p} - \omega_{\alpha_0}, \Gamma^{k-p}_{\alpha_0\ldots\alpha_p}) & p \leqslant k \\ 0 & p > k \end{cases}$$

where $\Gamma_{\alpha_0\cdots\alpha_p} = \Omega_{\alpha_0\cdots\alpha_p} - \sum_i dt_i \wedge (\omega_{\alpha_i} - \omega_{\alpha_0})$.

Remark. – γ induces the Chern-Weil homomorphism

$$\gamma^*: \ \mathrm{I}^*(\mathfrak{gl}_q) \longrightarrow \mathrm{H}^*(\check{\mathrm{C}}^{(*)}, \mathrm{D}) \xleftarrow{\cong} \mathrm{H}^*(\mathrm{M}).$$

The following lemma is easily proved.

LEMMA 2. – *Let* $w^1, \ldots, w^q$ *be 1-forms on* U_α *such that* $w^1 \wedge \ldots \wedge w^q \neq 0$ *on* U_α. *Put* $w = w^1 \wedge \ldots \wedge w^q$. *Then* (i) *and* (ii) *are equivalent:*

(i) *There exists a* (q,q)*-matrix* (φ^i_j) *of 1-forms on* U_α *such that* $dw^i = \sum_i w^j \wedge \varphi^i_j$.

(ii) *There exists a 1-form* η *on* U_α *such that* $dw = w \wedge \eta$.

Remark. – The existance of the matrix (φ^i_j) doesn't depend on choice of q 1-forms $w^1, \ldots, w^q$ on U_α.

Remark. – In the proof of this lemma, we obtain

$$\eta = (-1)^{q-1} \sum_i \varphi^i_i. \tag{1}$$

Let $\Gamma(\cdot)$ denote the space of all sections of bundle. The Bott connection $\widetilde{\nabla}: \Gamma(\mathrm{E}) \times \Gamma(\mathrm{Q}) \longrightarrow \Gamma(\mathrm{Q})$ is defined by

$$\widetilde{\nabla}_{\mathrm{X}} \mathrm{Z} = \pi_*([\mathrm{X}, \widetilde{\mathrm{Z}}]) \qquad \mathrm{X} \in \Gamma(\mathrm{E}), \mathrm{Z} \in \Gamma(\mathrm{Q})$$

where $\widetilde{\mathrm{Z}} \in \Gamma(\mathrm{TM})$ such that $\pi_*(\widetilde{\mathrm{Z}}) = \mathrm{Z}$ and $\pi: \mathrm{TM} \longrightarrow \mathrm{Q}$. Let

$\{e_i, e_a\}$ $(1 \leqslant i \leqslant q,\ q+1 \leqslant a \leqslant n)$ be a local basis dual to $\{w^i, w^a\}$ on U_α satisfying $e_i \in \Gamma(Q|_{U_\alpha})$ and $e_a \in \Gamma(E|_{U_\alpha})$ with respect to the isomorphism $TM \cong E \oplus Q$. Hereafter, we suppose that the indices run the following ranges : $1 \leqslant i, j, k, \ldots \leqslant q$, $q+1 \leqslant a, b, \ldots \leqslant n$. We define a connection ∇^α on U_α by

$$\nabla^\alpha_X Z = \widetilde{\nabla}_{X_E} Z + \sum_i X_Q(Z^i)\, e_i + \sum_{i,k} Z^i \varphi^k_i(X_Q) e_k \tag{2}$$

where $X = X_E + X_Q \in \Gamma(E|_{U_\alpha}) \oplus \Gamma(Q|_{U_\alpha})$ and $Z = \sum_i Z^i e_i \in \Gamma(Q|_{U_\alpha})$.

We put $\nabla^\alpha_X e_j = \sum_i \omega^i_{\alpha j}(X) e_i$, that is, $\omega^i_{\alpha j}$ denotes the connection form of ∇^α on U_α.

LEMMA 3. — $\omega^i_{\alpha j} = \varphi^i_j$ *on* U_α.

Proof. — We put $\widetilde{\nabla}_{X_E} e_j = \sum_i \widetilde{\omega}^i_j(X_E)\, e_i$, then $\omega^i_{\alpha j}(X) = \widetilde{\omega}^i_j(X_E) + \varphi^i_j(X_Q)$. Now we obtain

$$dw^i(e_a, e_j) = \frac{1}{2}\{e_a(w^i(e_j)) - e_j(w^i(e_a)) - w^i([e_a, e_j])\}$$
$$= -\frac{1}{2}\widetilde{\omega}^i_j(e_a).$$

On the other hand,

$$dw^i(e_a, e_j) = \Big(\sum_k w^k \wedge \varphi^i_k)(e_a, e_j)\Big)$$
$$= -\frac{1}{2}\varphi^i_j(e_a).$$

Thus we obtain $\widetilde{\omega}^i_j(X_E) = \varphi^i_j(X_E)$. Therefore, for any $X \in \Gamma(TM|_{U_\alpha})$, $\omega^i_{\alpha j}(X) = \varphi^i_j(X_E) + \varphi^i_j(X_Q) = \varphi^i_j(X)$. Q.E.D.

LEMMA 4. — *If we consider the connection* ∇^{α_0} *defined by (2) on* U_{α_0} *and a Riemannian connection* ∇^{α_1} *on* U_{α_1}, *then, for* $\varphi_1 \in I^1(\mathfrak{g}l_q)$, $(\gamma\varphi_1)^{1,1}_{\alpha_0\alpha_1} = ((-)^{q-1}/2\pi)\,\eta$.

Proof. – By Lemma 1,

$$(\gamma\varphi_1)^{1,1}_{\alpha_0\alpha_1} = \int_{\Delta^1} dt_1 \wedge \varphi_1(\omega_{\alpha_1} - \omega_{\alpha_0})$$

$$= \varphi_1(\omega_{\alpha_1} - \omega_{\alpha_0}) .$$

Since $\varphi_k \in I^k(\mathfrak{gl}_q)$ is defined by

$\det(\lambda I_q - (1/2\pi)X) = \sum_k \varphi_k(X)\lambda^{q-k}$, $X \in \mathfrak{gl}_q$ and $\mathrm{trace}(\omega^i_{\alpha_1 j}) = 0$,

we obtain $(\gamma\varphi_1)^{1,1}_{\alpha_0\alpha_1} = (1/2\pi)\,\mathrm{trace}\,(\omega_{\alpha_0 j})$. By (1) and Lemma 3, $(\gamma\varphi_1)^{1,1}_{\alpha_0\alpha_1} = ((-1)^{q-1}/2\pi)\,\eta$. Q.E.D.

From this lemma, we obtain

THEOREM A. – $((-1)^{q-1}/2\pi)\,\eta$ *defines a* D-*cohomology class in* $H^2(\check{C}^{(*)}(\mathfrak{U}\,;A^*(M)), D)$ *depending only on* $\mathfrak{F}$.

Proof. – If $\gamma\varphi_1 \in \check{C}^{(2)}(\mathfrak{U}\,;A^*(M))$ is D-closed, then particular object $(\gamma\varphi_1)^{1,1}_{\alpha_0\alpha_1}$ is D-closed and, by Lemma 4, $((-1)^{q-1}/2\pi)\,\eta \in \check{C}^{1,1}(\mathfrak{U}\,;A^*(M))$ define a D-cohomology class in $H^2(\check{C}^{(*)}(\mathfrak{U}\,;A^*(M)), D)$. Now we prove that $\gamma\varphi_1$ is D-closed.

$$\begin{aligned}(D(\gamma\varphi_1))^{1,2}_{\alpha_0\alpha_1} &= (D'(\gamma\varphi_1) + D''(\gamma\varphi_1))^{1,2}_{\alpha_0\alpha_1}\\ &= (D'(\gamma\varphi_1))^{1,2}_{\alpha_0\alpha_1} + (D''(\gamma\varphi_1))^{1,2}_{\alpha_0\alpha_1}\\ &= \{(\gamma\varphi_1)^{0,2}_{\alpha_1} - (\gamma\varphi_1)^{0,2}_{\alpha_0}\} + (-1)\,(d(\gamma\varphi_1))^{1,2}_{\alpha_0\alpha_1} .\end{aligned}$$

From Stokes' theorem,

$$(-1)\,d\cdot\int_{\Delta^1}\varphi_1\,(\Omega_{\alpha_0\alpha_1}) = \int_{\Delta^1}\circ\, d\varphi_1\,(\Omega_{\alpha_0\alpha_1}) - \int_{\partial\Delta^1}\circ j^*\varphi_1(\Omega_{\alpha_0\alpha_1})$$

and the left side of this is equal to $(-1)\,d\cdot(\gamma\varphi_1)^{1,1}_{\alpha_0\alpha_1}$, the first term of the right side vanishes and the second term of the right side is equal to $\varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})$. Thus $d\cdot(\gamma\varphi_1)^{1,1}_{\alpha_0\alpha_1} = \varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})$. From this and $(\gamma\varphi_1)^{0,2}_{\alpha_0} = \varphi_1(\Omega_{\alpha_0})$, we obtain

$$\begin{aligned}(D(\gamma\varphi_1))^{1,2}_{\alpha_0\alpha_1} &= \{\varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})\}\\ &\qquad + (-1)\,\{\varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})\}\\ &= 0 .\end{aligned}$$

From $(\gamma\varphi_1)^{0,2}_{\alpha_0} = \varphi_1(\Omega_{\alpha_0})$ and $d\circ\varphi_1(\Omega_{\alpha_0}) = 0$, we obtain

$$\begin{aligned}(D(\gamma\varphi_1))^{0,3}_{\alpha_0} &= (D''(\gamma\varphi_1))^{0,3}_{\alpha_0}\\ &= (-1)^0\,(d(\gamma\varphi_1))^{0,3}_{\alpha_0}\\ &= 0 .\end{aligned}$$

Now, from lemma 1, $(\gamma\varphi_1) \in \check{C}^{0,2} + \check{C}^{1,1}$. Thus we obtain

$$(D(\gamma\varphi_1))^{3,0}_{\alpha_0\alpha_1\alpha_2\alpha_3} = 0$$

and

$$\begin{aligned}(D(\gamma\varphi_1))^{2,1}_{\alpha_0\alpha_1\alpha_2} &= (D'(\gamma\varphi_1))^{2,1}_{\alpha_0\alpha_1\alpha_2} \\ &= \varphi_1(\omega_{\alpha_2}) - \varphi_1(\omega_{\alpha_1}) - \varphi_1(\omega_{\alpha_2}) + \varphi_1(\omega_{\alpha_0}) \\ &\qquad + \varphi_1(\omega_{\alpha_1}) - \varphi_1(\omega_{\alpha_0}) \\ &= 0 .\end{aligned}$$

Therefore we obtain $D(\gamma\varphi_1) = 0$. Q.E.D.

3. For q 1-forms $w^1, \ldots, w^q$ on U_α ($w^1 \wedge \ldots \wedge w^q \neq 0$) we may choose $n-q$ 1-forms $w^{q+1}, \ldots, w^n$ on U_α such that

$$w^1 \wedge \ldots \wedge w^q \wedge w^{q+1} \wedge \ldots \wedge w^n \neq 0 \quad \text{on} \quad U_\alpha .$$

Thus we obtain expressions

$$\varphi^i_j = \sum_k \varphi^i_{jk}\, w^k + \sum_a \varphi^i_{ja}\, w^a ,$$

$$\eta = \sum_k \eta_k\, w^k + \sum_a \eta_a\, w^a .$$

Using same letters φ^i_j, η to simplify, we put

$$\varphi^i_j = \sum_a \varphi^i_{ja}\, w^a \quad \text{and} \quad \eta = \sum_a \eta_a\, w^a \tag{3}$$

on U_α.

Hereafter, we suppose that the manifold M admits foliations $\mathcal{F}$, $\mathcal{F}'$ complementally transversal to each other and that $\mathcal{F}$ (resp. $\mathcal{F}'$) is of codimension q (resp. $n-q$). Then we may consider that $w^1, \ldots, w^q$ are defined by $\mathcal{F}$ and that $w^{q+1}, \ldots, w^n$ are defined by $\mathcal{F}'$.

Let 1-forms $\overline{w}^i$, $\overline{w}^a$, $\overline{\varphi}^i_j$, $\overline{\eta}$ on U_{α_1} correspond to 1-forms $w^i, w^a, \varphi^i_j, \eta$ on U_{α_0} respectively. Then we obtain

LEMMA 5. – *On* $U_{\alpha_0} \cap U_{\alpha_1}$ $(\neq \phi)$, $\overline{\eta} = \eta$.

Proof. – On $U_{\alpha_0} \cap U_{\alpha_1}$, we may put

$$\overline{w}^i = \sum_j t^i_j\, w^j , \quad \overline{w}^a = \sum_b t^a_b\, w^b . \tag{4}$$

Proof. – We take η_{α_0} (resp. η_{α_1}) for η on U_{α_0} (resp. $\overline{\eta}$ on U_{α_1}). Then we obtain

$$\eta_{\alpha_1} - \eta_{\alpha_0} = 0 \quad \text{on} \quad U_{\alpha_0} \cap U_{\alpha_1} ,$$

and

$$(D'\eta)^{1,1}_{\alpha_0\alpha_1} = \eta_{\alpha_1} - \eta_{\alpha_0} = 0 .$$

Thus η is D'-closed. Q.E.D.

BIBLIOGRAPHY

[1] R. Bott, Lectures on characteristic classes and foliations, *Lecture Notes in Math.*, Springer, 279 (1972), 1-94.

[2] C. Godbillon and J. Vey, Un invariant des feuilletages de codimension 1, *C.R. Acad. Sci.*, Paris, 273 (1971), A92-95.

[3] F.W. Kamber and P. Tondeur, Foliated bundles and characteristic classes, *Lecture Notes in Math.*, Springer, 493 (1975).

[4] H. Kitahara and S. Yorozu, Sur l'homomorphisme de Chern-Weil local et ses applications au feuilletage, *C.R. Acad. Sci.*, Paris, 281 (1975), A703-706.

[5] G. Reeb, Sur certaines propriétés topologiques des variétés feuilletées, Hermann (1952).

[6] Y. Shikata, On the spectral sequences associated to foliated structures, *Nagoya Math. J.*, 38 (1970), 53-61.

[7] Y. Shikata, On the cohomology of bigraded forms associated with foliated structures, *Bull. Soc. Math. Grèce*, 15 (1974), 68-76.

[8] A. So, J.C. Thomas and C. Watkiss, Sur la multiplicativité de l'homomorphisme de Chern-Weil local, *C.R. Acad. Sci.*, Paris, 280 (1975), A369-371.

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