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ON THE ČECH BICOMPLEX ASSOCIATED WITH FOLIATED STRUCTURES

by H. KITAHARA and S. YOROZU

1. We shall be in C^∞ -category. Let M be a paracompact connected n -dimensional manifold with a foliation \mathcal{F} of codimension q , and let $\mathcal{U} = \{U_\alpha\}$ be a simple covering of M such that each U_α is a flat neighborhood with respect to \mathcal{F} . Then there exists a decomposable q -form $w = w^1 \wedge \dots \wedge w^q$ on each U_α and, by Frobenius' theorem, there exists a 1-form η on each U_α satisfying $dw = w \wedge \eta$, where d denotes the exterior differentiation and \wedge the exterior product. The 1-form η is an interesting object; it is well known that $\eta \wedge (d\eta)^q$ defines a de Rham class in $H^{2q+1}(M, \mathbb{R})$ ([1], [2], [3]). Our aim is to show that η itself defines a certain cohomology class, that is,

THEOREM A. — $((-1)^{q-1}/2\pi)\eta$ defines a D -cohomology class in $H^2(\check{C}^{(*)}(\mathcal{U}; A^*(M)), D)$ depending only on \mathcal{F} .

The above theorem was announced in [4], where it contained misstatements.

THEOREM B. — Supposing M admits foliations $\mathcal{F}, \mathcal{F}'$ complementally transversal to each other, η defines a D' -cohomology class in $H^1(\check{C}^{(*)}(\mathcal{U}; A^*(M)), D')$ (Cf. [5], [6], [7]).

2. Since M has a foliation \mathcal{F} of codimension q , the tangent bundle TM of M has an integrable subbundle E with fibre dimension $n-q$. Let $Q = TM/E$ be a quotient bundle with fibre dimension q . Choosing a suitable Riemannian metric on TM , we obtain an isomorphism $TM \cong E \oplus Q$ (Whitney sum).

Then

$$\begin{aligned} d\bar{w}^i &= \sum_j dt_j^i \wedge w^j + \sum_j t_j^i dw^j \\ &= \sum_j \left(dt_j^i - \sum_k t_k^i \varphi_j^k \right) \wedge w^j, \end{aligned}$$

and on the other hand

$$\begin{aligned} d\bar{w}^i &= \sum_k \bar{w}^k \wedge \bar{\varphi}_k^i \\ &= \sum_j \left(- \sum_k t_j^k \bar{\varphi}_k^i \right) \wedge w^j. \end{aligned}$$

Thus

$$- \sum_k t_j^k \bar{\varphi}_k^i = dt_j^i - \sum_k t_k^i \varphi_j^k + \sum_k f_{jk}^i w^k$$

where f_{jk}^i are functions on $U_{\alpha_0} \cap U_{\alpha_1}$.

Let $\begin{pmatrix} s_j^i & 0 \\ 0 & s_b^a \end{pmatrix}$ denote the inverse matrix of $\begin{pmatrix} t_j^i & 0 \\ 0 & t_b^a \end{pmatrix}$.

Then

$$\sum_{i,k} s_l^i t_j^k \bar{\varphi}_k^i = - \sum_j s_l^j dt_j^i + \sum_{j,k} s_l^j t_k^i \varphi_j^k - \sum_{i,k} s_l^i f_{jk}^i w^k$$

and we obtain

$$\sum_i \bar{\varphi}_i^i = - \sum_{i,j} s_l^j dt_j^i + \sum_i \varphi_i^i - \sum_{i,j,k} s_l^i f_{jk}^i w^k.$$

From (4), $dt_j^i = \sum_k t_{jk}^i w^k$. Thus, by (3), we obtain $\sum_i \bar{\varphi}_i^i = \sum_i \varphi_i^i$.

Therefore we obtain $\bar{\eta} = \eta$ on $U_{\alpha_0} \cap U_{\alpha_1}$ and $\eta \in \check{C}^{0,1}(\mathcal{U}; A^*(M))$.
Q.E.D.

THEOREM B. — *Supposing M admits foliations $\mathcal{F}, \mathcal{F}'$ complementally transversal to each other, η defines a D' -cohomology class in $H^1(\check{C}^*(\mathcal{U}; A^*(M)), D')$ where η is defined by \mathcal{F} .*

We may suppose that $\mathcal{U} = \{U_\alpha\}$ is a simple covering such that each U_α is a locally trivial neighborhood of the bundle $Q \rightarrow M$.

Let ∇^α denote a local connection on $Q|_{U_\alpha}$, and let ω_α (resp. Ω_α) denote a connection form (resp. a curvature form) of ∇^α on U_α . Let Δ^p be a canonical p -simplex in R^{p+1} (with coordinates (t_0, t_1, \dots, t_p)). We define a connection form $\omega_{\alpha_0 \dots \alpha_p}$ on $U_{\alpha_0 \dots \alpha_p} \times \Delta^p$ ($U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_p}$) by

$$\begin{aligned} \omega_{\alpha_0 \dots \alpha_p} &= t_0 \omega_{\alpha_0} + \dots + t_p \omega_{\alpha_p} \\ &= (1 - t_1 - \dots - t_p) \omega_{\alpha_0} + t_1 \omega_{\alpha_1} + \dots + t_p \omega_{\alpha_p} \end{aligned}$$

and let $\Omega_{\alpha_0 \dots \alpha_p}$ denote a corresponding curvature form on $U_{\alpha_0 \dots \alpha_p} \times \Delta^p$.

For the set $A^k(U_{\alpha_0 \dots \alpha_p} \times \Delta^p)$ of all k -forms on $U_{\alpha_0 \dots \alpha_p} \times \Delta^p$, $\int_{\Delta^p} : A^k(U_{\alpha_0 \dots \alpha_p} \times \Delta^p) \rightarrow A^{k-p}(U_{\alpha_0 \dots \alpha_p})$ denotes the integration along the fibre. Then we obtain Stokes' theorem

$$\int_{\Delta^p} d \circ = (-1)^p d \circ \int_{\Delta^p} + \int_{\partial \Delta^p} \circ j^*$$

where $j : U_{\alpha_0 \dots \alpha_p} \times \partial \Delta^p \rightarrow U_{\alpha_0 \dots \alpha_p} \times \Delta^p$ denotes the inclusion.

We consider Čech bicomplex $\check{C}^{(*)}(\mathcal{U}; A^*(M))$: Let $\check{C}^{p,q} = \prod_{\alpha_0 \dots \alpha_p} A^q(U_{\alpha_0 \dots \alpha_p})$, and let $D' : \check{C}^{p,q} \rightarrow \check{C}^{p+1,q}$ denote the ordinary simplicial differential and $D'' = (-1)^p d : \check{C}^{p,q} \rightarrow \check{C}^{p,q+1}$ the de Rham differential. A multiplication; $\check{C}^{p,q} \otimes \check{C}^{p',q'} \rightarrow \check{C}^{p+p',q+q'}$ is defined by

$$(\Phi \cdot \Phi')_{\alpha_0 \dots \alpha_{p+q}} = (-1)^{qp'} \Phi_{\alpha_0 \dots \alpha_p} |_{U_{\alpha_0 \dots \alpha_{p+q}}} \wedge \Phi'_{\alpha_{p+1} \dots \alpha_{p+q}} |_{U_{\alpha_0 \dots \alpha_{p+q}}}$$

For $\check{C}^{(k)}(\mathcal{U}; A^*(M)) = \sum_{p+q=k} \check{C}^{p,q}$ and $D = D' + D'' : \check{C}^{(k)} \rightarrow \check{C}^{(k+1)}$, we obtain a graded algebra $(\check{C}^{(*)}(\mathcal{U}; A^*(M)), D, \cdot)$.

Let $I^*(\mathfrak{g}I_q)$ denote a graded algebra of invariant polynomials on a Lie algebra $\mathfrak{g}I_q$. A characteristic homomorphism

$$\gamma : I^*(\mathfrak{g}I_q) \rightarrow C^{(*)}(\mathcal{U}; A^*(M))$$

is defined by

$$\gamma\varphi = \sum_p (\gamma\varphi)^{p, 2k-p} \quad \varphi \in I^k(\mathfrak{g}I_q)$$

where

$$(\gamma\varphi)_{\alpha_0 \dots \alpha_p}^{p, 2k-p} = \int_{\Delta^p} \varphi(\Omega_{\alpha_0 \dots \alpha_p}, \dots, \Omega_{\alpha_0 \dots \alpha_p}).$$

Then we obtain,

LEMMA 1 (Cf. [8]). — For $\varphi \in \Gamma^k(\mathfrak{gl}_q)$,

$$(-1)^{[p(p-1)]/2} \frac{(k-p)!}{k!} (\gamma\varphi)_{\alpha_0 \dots \alpha_p}^{p, 2k-p} = \begin{cases} \int_{\Delta^p} dt_1 \wedge \dots \wedge dt_p \wedge \varphi(\omega_{\alpha_1} - \omega_{\alpha_0}, \omega_{\alpha_2} - \omega_{\alpha_0}, \dots, \omega_{\alpha_p} - \omega_{\alpha_0}, \Gamma_{\alpha_0 \dots \alpha_p}^{k-p}) & p \leq k \\ 0 & p > k \end{cases}$$

where $\Gamma_{\alpha_0 \dots \alpha_p} = \Omega_{\alpha_0 \dots \alpha_p} - \sum_i dt_i \wedge (\omega_{\alpha_i} - \omega_{\alpha_0})$.

Remark. — γ induces the Chern-Weil homomorphism

$$\gamma^*: I^*(\mathfrak{gl}_q) \longrightarrow H^*(\check{C}^*(\cdot), D) \xleftarrow{\cong} H^*(M).$$

The following lemma is easily proved.

LEMMA 2. — Let w^1, \dots, w^q be 1-forms on U_α such that $w^1 \wedge \dots \wedge w^q \neq 0$ on U_α . Put $w = w^1 \wedge \dots \wedge w^q$. Then (i) and (ii) are equivalent:

- (i) There exists a (q, q) -matrix (φ_j^i) of 1-forms on U_α such that $dw^i = \sum_j \varphi_j^i \wedge w_j^i$.
- (ii) There exists a 1-form η on U_α such that $dw = w \wedge \eta$.

Remark. — The existence of the matrix (φ_j^i) doesn't depend on choice of q 1-forms w^1, \dots, w^q on U_α .

Remark. — In the proof of this lemma, we obtain

$$\eta = (-1)^{q-1} \sum_i \varphi_i^i. \tag{1}$$

Let $\Gamma(\cdot)$ denote the space of all sections of bundle. The Bott connection $\tilde{\nabla}: \Gamma(E) \times \Gamma(Q) \longrightarrow \Gamma(Q)$ is defined by

$$\tilde{\nabla}_X Z = \pi_*([X, \tilde{Z}]) \quad X \in \Gamma(E), Z \in \Gamma(Q)$$

where $\tilde{Z} \in \Gamma(TM)$ such that $\pi_*(\tilde{Z}) = Z$ and $\pi: TM \longrightarrow Q$. Let

$\{e_i, e_a\}$ ($1 \leq i \leq q, q + 1 \leq a \leq n$) be a local basis dual to $\{w^i, w^a\}$ on U_α satisfying $e_i \in \Gamma(Q|_{U_\alpha})$ and $e_a \in \Gamma(E|_{U_\alpha})$ with respect to the isomorphism $TM \cong E \oplus Q$. Hereafter, we suppose that the indices run the following ranges : $1 \leq i, j, k, \dots \leq q, q + 1 \leq a, b, \dots \leq n$. We define a connection ∇^α on U_α by

$$\nabla_X^\alpha Z = \tilde{\nabla}_{X_E} Z + \sum_i X_Q(Z^i) e_i + \sum_{i,k} Z^i \varphi_i^k(X_Q) e_k \tag{2}$$

where $X = X_E + X_Q \in \Gamma(E|_{U_\alpha}) \oplus \Gamma(Q|_{U_\alpha})$ and $Z = \sum_i Z^i e_i \in \Gamma(Q|_{U_\alpha})$.

We put $\nabla_X^\alpha e_j = \sum_i \omega_{\alpha j}^i(X) e_i$, that is, $\omega_{\alpha j}^i$ denotes the connection form of ∇^α on U_α .

LEMMA 3. — $\omega_{\alpha j}^i = \varphi_j^i$ on U_α .

Proof. — We put $\tilde{\nabla}_{X_E} e_j = \sum_i \tilde{\omega}_j^i(X_E) e_i$, then

$\omega_{\alpha j}^i(X) = \tilde{\omega}_j^i(X_E) + \varphi_j^i(X_Q)$. Now we obtain

$$\begin{aligned} dw^i(e_a, e_j) &= \frac{1}{2} \{e_a(w^i(e_j)) - e_j(w^i(e_a)) - w^i([e_a, e_j])\} \\ &= -\frac{1}{2} \tilde{\omega}_j^i(e_a). \end{aligned}$$

On the other hand,

$$\begin{aligned} dw^i(e_a, e_j) &= \left(\sum_k w^k \wedge \varphi_k^i \right) (e_a, e_j) \\ &= -\frac{1}{2} \varphi_j^i(e_a). \end{aligned}$$

Thus we obtain $\tilde{\omega}_j^i(X_E) = \varphi_j^i(X_E)$. Therefore, for any $X \in \Gamma(TM|_{U_\alpha})$, $\omega_{\alpha j}^i(X) = \varphi_j^i(X_E) + \varphi_j^i(X_Q) = \varphi_j^i(X)$. Q.E.D.

LEMMA 4. — *If we consider the connection ∇^{α_0} defined by (2) on U_{α_0} and a Riemannian connection ∇^{α_1} on U_{α_1} , then, for $\varphi_1 \in I^1(\mathfrak{g}^I_q), (\gamma\varphi_1)_{\alpha_0\alpha_1}^{1,1} = ((-)^{q-1}/2\pi) \eta$.*

Proof. — By Lemma 1,

$$\begin{aligned}
 (\gamma\varphi_1)_{\alpha_0\alpha_1}^{1,1} &= \int_{\Delta^1} dt_1 \wedge \varphi_1(\omega_{\alpha_1} - \omega_{\alpha_0}) \\
 &= \varphi_1(\omega_{\alpha_1} - \omega_{\alpha_0}).
 \end{aligned}$$

Since $\varphi_k \in I^k(\mathfrak{g}I_q)$ is defined by

$$\det(\lambda I_q - (1/2\pi)X) = \sum_k \varphi_k(X) \lambda^{q-k}, \quad X \in \mathfrak{g}I_q \quad \text{and} \quad \text{trace}(\omega_{\alpha_{ij}}^i) = 0,$$

we obtain $(\gamma\varphi_1)_{\alpha_0\alpha_1}^{1,1} = (1/2\pi) \text{trace}(\omega_{\alpha_0j})$. By (1) and Lemma 3,

$$(\gamma\varphi_1)_{\alpha_0\alpha_1}^{1,1} = ((-1)^{q-1}/2\pi) \eta. \quad \text{Q.E.D.}$$

From this lemma, we obtain

THEOREM A. — $((-1)^{q-1}/2\pi) \eta$ defines a D-cohomology class in $H^2(\check{C}^{(*)}(\mathfrak{U}; A^*(M)), D)$ depending only on \mathfrak{F} .

Proof. — If $\gamma\varphi_1 \in \check{C}^{(2)}(\mathfrak{U}; A^*(M))$ is D-closed, then particular object $(\gamma\varphi_1)_{\alpha_0\alpha_1}^{1,1}$ is D-closed and, by Lemma 4,

$((-1)^{q-1}/2\pi) \eta \in \check{C}^{1,1}(\mathfrak{U}; A^*(M))$ define a D-cohomology class in $H^2(\check{C}^{(*)}(\mathfrak{U}; A^*(M)), D)$. Now we prove that $\gamma\varphi_1$ is D-closed.

$$\begin{aligned}
 (D(\gamma\varphi_1))_{\alpha_0\alpha_1}^{1,2} &= (D'(\gamma\varphi_1) + D''(\gamma\varphi_1))_{\alpha_0\alpha_1}^{1,2} \\
 &= (D'(\gamma\varphi_1))_{\alpha_0\alpha_1}^{1,2} + (D''(\gamma\varphi_1))_{\alpha_0\alpha_1}^{1,2} \\
 &= \{(\gamma\varphi_1)_{\alpha_1}^{0,2} - (\gamma\varphi_1)_{\alpha_0}^{0,2}\} + (-1)(d(\gamma\varphi_1))_{\alpha_0\alpha_1}^{1,2}.
 \end{aligned}$$

From Stokes' theorem,

$$(-1)d \cdot \int_{\Delta^1} \varphi_1(\Omega_{\alpha_0\alpha_1}) = \int_{\Delta^1} d \cdot \varphi_1(\Omega_{\alpha_0\alpha_1}) - \int_{\partial\Delta^1} j^* \varphi_1(\Omega_{\alpha_0\alpha_1})$$

and the left side of this is equal to $(-1)d \cdot (\gamma\varphi_1)_{\alpha_0\alpha_1}^{1,1}$, the first term of the right side vanishes and the second term of the right side is equal to $\varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})$. Thus $d \cdot (\gamma\varphi_1)_{\alpha_0\alpha_1}^{1,1} = \varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})$. From this and $(\gamma\varphi_1)_{\alpha_0}^{0,2} = \varphi_1(\Omega_{\alpha_0})$, we obtain

$$\begin{aligned}
 (D(\gamma\varphi_1))_{\alpha_0\alpha_1}^{1,2} &= \{\varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})\} \\
 &\quad + (-1)\{\varphi_1(\Omega_{\alpha_1}) - \varphi_1(\Omega_{\alpha_0})\} \\
 &= 0.
 \end{aligned}$$

From $(\gamma\varphi_1)_{\alpha_0}^{0,2} = \varphi_1(\Omega_{\alpha_0})$ and $d \circ \varphi_1(\Omega_{\alpha_0}) = 0$, we obtain

$$\begin{aligned}
 (D(\gamma\varphi_1))_{\alpha_0}^{0,3} &= (D''(\gamma\varphi_1))_{\alpha_0}^{0,3} \\
 &= (-1)^0 (d(\gamma\varphi_1))_{\alpha_0}^{0,3} \\
 &= 0.
 \end{aligned}$$

Now, from lemma 1, $(\gamma\varphi_1) \in \check{C}^{0,2} + \check{C}^{1,1}$. Thus we obtain

$$(D(\gamma\varphi_1))_{\alpha_0\alpha_1\alpha_2\alpha_3}^{3,0} = 0$$

and

$$\begin{aligned} (D(\gamma\varphi_1))_{\alpha_0\alpha_1\alpha_2}^{2,1} &= (D'(\gamma\varphi_1))_{\alpha_0\alpha_1\alpha_2}^{2,1} \\ &= \varphi_1(\omega_{\alpha_2}) - \varphi_1(\omega_{\alpha_1}) - \varphi_1(\omega_{\alpha_2}) + \varphi_1(\omega_{\alpha_0}) \\ &\quad + \varphi_1(\omega_{\alpha_1}) - \varphi_1(\omega_{\alpha_0}) \\ &= 0. \end{aligned}$$

Therefore we obtain $D(\gamma\varphi_1) = 0$. Q.E.D.

3. For q 1-forms w^1, \dots, w^q on U_α ($w^1 \wedge \dots \wedge w^q \neq 0$) we may choose $n-q$ 1-forms w^{q+1}, \dots, w^n on U_α such that

$$w^1 \wedge \dots \wedge w^q \wedge w^{q+1} \wedge \dots \wedge w^n \neq 0 \quad \text{on } U_\alpha.$$

Thus we obtain expressions

$$\varphi_j^i = \sum_k \varphi_{jk}^i w^k + \sum_a \varphi_{ja}^i w^a,$$

$$\eta = \sum_k \eta_k w^k + \sum_a \eta_a w^a.$$

Using same letters φ_j^i, η to simplify, we put

$$\varphi_j^i = \sum_a \varphi_{ja}^i w^a \quad \text{and} \quad \eta = \sum_a \eta_a w^a \quad (3)$$

on U_α .

Hereafter, we suppose that the manifold M admits foliations $\mathfrak{F}, \mathfrak{F}'$ complementally transversal to each other and that \mathfrak{F} (resp. \mathfrak{F}') is of codimension q (resp. $n-q$). Then we may consider that w^1, \dots, w^q are defined by \mathfrak{F} and that w^{q+1}, \dots, w^n are defined by \mathfrak{F}' .

Let 1-forms $\bar{w}^i, \bar{w}^a, \bar{\varphi}_j^i, \bar{\eta}$ on U_{α_1} correspond to 1-forms $w^i, w^a, \varphi_j^i, \eta$ on U_{α_0} respectively. Then we obtain

LEMMA 5. — On $U_{\alpha_0} \cap U_{\alpha_1} (\neq \emptyset)$, $\bar{\eta} = \eta$.

Proof. — On $U_{\alpha_0} \cap U_{\alpha_1}$, we may put

$$\bar{w}^i = \sum_j t_j^i w^j, \quad \bar{w}^a = \sum_b t_b^a w^b. \quad (4)$$

Proof. — We take η_{α_0} (resp. η_{α_1}) for η on U_{α_0} (resp. $\bar{\eta}$ on U_{α_1}). Then we obtain

$$\eta_{\alpha_1} - \eta_{\alpha_0} = 0 \quad \text{on } U_{\alpha_0} \cap U_{\alpha_1},$$

and

$$(D'\eta)_{\alpha_0\alpha_1}^{1,1} = \eta_{\alpha_1} - \eta_{\alpha_0} = 0.$$

Thus η is D' -closed.

Q.E.D.

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