Ann. Inst. Fourier, Grenoble
28, 3 (1978), 209-215

ON SOME ERGODIC PROPERTIES FOR CONTINUOUS AND AFFINE FUNCTIONS

by C.J.K. BATTY

1. Introduction.

Let X be a compact Hausdorff space, let C(X) denote the space of continuous real-valued functions on X, and let T be a positive linear operator of C(X) into itself. Choquet and Foias [1] have considered convergence properties of the iterates T^n of T and the associated arithmetic means $S_n = n^{-1} \sum_{r=0}^{n-1} T^r$. In particular, they obtained the following two results [1, Théorèmes 13, 1]:

THEOREM 1.1. – *If, for some non-negative function f in* C(X), $S_n f$ *converges pointwise to a continuous strictly positive function, then the convergence is uniform on* X.

THEOREM 1.2. – *If, for each x in* X, $\inf\{(T^n 1)(x) : n \geqslant 1\} < 1$, *then* $T^n 1$ *converges to* 0 *uniformly on* X.

Choquet and Foias showed that the condition that the limit in theorem 1.1 is strictly positive cannot be removed [1, Exemple 11]. They then raised the following problem:

PROBLEM 1. – *Suppose that* $S_n 1$ *converges pointwise to a continuous limit. Is the convergence necessarily uniform?*

If M(X) denotes the set of Radon measures on X, identified with C(X)*, and P(X) is the set of probability measures in M(X), then P(X) is weak*-compact and convex, its extreme boundary $\partial_e P(X)$ consists of the measures ϵ_x concentrated at one point x

of X, and there is an isometric order-isomorphism $f \longmapsto \hat{f}$ of C(X) onto the space A(P(X)) of continuous affine real-valued functions on P(X), given by $\hat{f}(\mu) = \int f\,d\mu$. This raises a second problem.

PROBLEM 2. – *Suppose that* K *is a compact convex subset of a locally convex space, and* T *is a positive linear operator on* A(K) *such that for each* x *in* $\partial_e K$, $\inf\{(T^n 1)(x) : n \geq 1\} < 1$. *Does it necessarily follow that* $\|T^n\| \longrightarrow 0$?

In § 2 we shall show (corollary 2.5) that the answer to problem 1 is affirmative, and in § 3 we shall give an example to show that the answer to problem 2 is negative, although it becomes affirmative if $\partial_e K$ is replaced by its closure $\overline{\partial_e K}$ in K.

2. Uniform convergence of arithmetic means.

Let T be a positive linear operator on C(X), and σ be a non-negative function in C(X). Let $F_\sigma = \sigma^{-1}(0)$ and G_σ be the complement of F_σ in X. For x in G_σ and $n \geq 1$ there is a bounded Radon measure $\mu^n_{x,\sigma}$ on G_σ such that

$$\int g\,d\mu^n_{x,\sigma} = \sigma(x)^{-1} T^n(g.\sigma)(x)$$

for all functions g in the space $C^b(G_\sigma)$ of continuous bounded real-valued functions on G_σ. For a Borel-measurable function f defined $\mu^n_{x,\sigma}$-a.e. in G_σ, put $(T^{(n)}_\sigma f)(x) = \int f\,d\mu^n_{x,\sigma}$ if the integral exists.

LEMMA 2.1. – *For* x *in* G_σ, $n \geq 1$ *and any bounded Borel function* f *on* G_σ. $T^{(n)}_\sigma (f.\sigma^{-1})(x) = \sigma(x)^{-1} T^{(n)}_1 (\chi_\sigma . f)(x)$, *where* χ_σ *is the characteristic function of* G_σ, *and both sides of the equality exist.*

Proof. – Suppose that f is continuous and non-negative. Let (g_λ) be an increasing net of continuous non-negative functions on X with support in G_σ and converging pointwise to χ_σ. Then $g_\lambda \cdot f \cdot \sigma^{-1} \in C^b(G_\sigma)$, and

$$\sigma(x)\int g_\lambda \cdot f \cdot \sigma^{-1}\,d\mu^n_{x,\sigma} = T^n(g_\lambda \cdot f)(x) = \int g_\lambda \,.\, f\,d\mu^n_{x,1}\,.$$

The right-hand integral increases to the finite integral $\int \chi_\sigma \cdot f\,d\mu^n_{x,1}$, so the result follows immediately in this special case.

The case when f is lower semi-continuous follows by approximating f from below by continuous functions, and the general case from the fact that the bounded Borel functions form the smallest linear space containing the lower semi-continuous functions and closed under bounded monotone sequential limits.

Now suppose that $T\sigma \leqslant \beta\sigma$ for some real number β. Then $T_\sigma^{(n)} 1 \leqslant \beta^n$, so $T_\sigma^{(n)}$ maps $C^b(G_\sigma)$ into itself. It follows immediately from the definitions that the following identity is valid for f in $C^b(G_\sigma)$: $T_\sigma^{(m)}(T_\sigma^{(n)} f)(x) = (T_\sigma^{(m+n)} f)(x)$. Elementary integration theory shows that this identity is valid for any Borel function f on G, in the sense that if either expression exists then so does the other and they are equal. We shall therefore write T_σ^n instead of $T_\sigma^{(n)}$. This discussion applies in particular to the case $\sigma = 1$ when it is consistent to write T instead of T_1.

For x in F_σ, $0 \leqslant (T^n\sigma)(x) \leqslant \beta^n\sigma(x) = 0$ so $\mu_{x,1}^n(G_\sigma) = 0$. Thus $T^n(\chi_\sigma \cdot f) = 0$ on F_σ. Note that this is consistent with lemma 2.1 which gives

$$\begin{aligned} T_\sigma^m(T_\sigma^n(f\cdot\sigma^{-1})) &= \sigma^{-1}T^m(\chi_\sigma \cdot T^n(\chi_\sigma \cdot f)) \\ T_\sigma^{m+n}(f.\,\sigma^{-1}) &= \sigma^{-1}T^{m+n}(\chi_\sigma \cdot f). \end{aligned}$$

LEMMA 2.2. – *Suppose that* $T\sigma \leqslant \sigma$ *and* $(T1)(x) < 1$ *for all* x *in* F_σ. *Then there is a real number* α *such that* $(T^n\chi_\sigma)(x) \leqslant \alpha$ *for all* $n \geqslant 1$ *and* x *in* G_σ.

Proof. – By continuity and compactness, there is a neighbourhood U of F_σ and real numbers $\beta_1 < 1$ and $\beta_2 \geqslant \beta_1(1-\beta_1)\|\sigma\|^{-1}$ such that

$$\begin{aligned} (T1)(x) &\leqslant \beta_1 & (x \in U) \\ (T1)(x) &\leqslant \beta_2\sigma(x) & (x \in K\backslash U). \end{aligned}$$

Let $\alpha = (1-\beta_1)^{-1}\beta_2\|\sigma\|$. Then $T1 \leqslant \alpha$ and $T1 \leqslant \beta_1 + \beta_2\sigma$. In particular, $T\chi_\sigma \leqslant T1 \leqslant \alpha$. Now suppose that $T^n\chi_\sigma \leqslant \alpha$ on G_σ, and take x in G_σ. Using lemma 2.1 and the fact that $T_\sigma 1 \leqslant 1$,

$$\begin{aligned} (T^{n+1}\chi_\sigma)(x) = T^n(T\chi_\sigma)(x) &= \sigma(x)\,T_\sigma^n(\sigma^{-1}\cdot T\chi_\sigma)(x) \\ &\leqslant \sigma(x)\,T_\sigma^n(\beta_1\sigma^{-1} + \beta_2)(x) \\ &\leqslant \beta_1(T^n\chi_\sigma)(x) + \beta_2\sigma(x) \\ &\leqslant \beta_1\alpha + \beta_2\sigma(x) \\ &\leqslant \alpha. \end{aligned}$$

LEMMA 2.3. – *Let* F *be a Borel subset of* X, χ *be the characteristic function of the complement of* F *in* X, *and*

$$\delta = \sup \{(\mathrm{T}1)(x) : x \in \mathrm{F}\}.$$

Then
$$\mathrm{T}^n 1 \leqslant \delta^n + \sum_{r=1}^{n} \delta^{r-1} \mathrm{T}^{n-r} (\chi \cdot \mathrm{T}1).$$

Proof. – It is trivial that $\mathrm{T}1 \leqslant \delta + \chi \cdot \mathrm{T}1$. Suppose the lemma holds for some integer n. Then since T is positive,

$$\mathrm{T}^{n+1}1 \leqslant \delta^n \mathrm{T}1 + \sum_{r=1}^{n} \delta^{r-1} \mathrm{T}^{n+1-r}(\chi \cdot \mathrm{T}1)$$
$$\leqslant \delta^{n+1} + \sum_{r=1}^{n+1} \delta^{r-1} \mathrm{T}^{n+1-r}(\chi \cdot \mathrm{T}1).$$

THEOREM 2.4. – *Let* T *be a positive linear operator on* C(X) *and suppose that there is a non-negative continuous function* σ *on* X *such that* $\mathrm{T}\sigma \leqslant \sigma$ *and* $(\mathrm{T}1)(x) < 1$ *whenever* $\sigma(x) = 0$. *Then* $\{\mathrm{T}^n 1 : n \geqslant 1\}$ *is uniformly bounded.*

Proof. – Take α as in lemma 2.2 and

$$\delta = \sup \{(\mathrm{T}1)(x) : x \in \mathrm{F}_\sigma\} < 1.$$

By lemma 2.3, for x in G_σ,

$$(\mathrm{T}^n 1)(x) \leqslant \delta^n + \alpha \|\mathrm{T}1\| \sum_{r=1}^{n} \delta^{r-1}$$
$$\leqslant \delta^n + (1-\delta)^{-1} \alpha \|\mathrm{T}1\|.$$

Also $\mathrm{T}^n 1 = \mathrm{T}((1-\chi_\sigma)\mathrm{T}^{n-1}1)$ on F_σ, so a simple inductive argument shows that $\mathrm{T}^n 1 \leqslant 1$ on F_σ.

COROLLARY 2.5. – *Suppose that* $\mathrm{S}_n 1$ *converges pointwise to a continuous limit* σ. *Then the convergence is uniform.*

Proof. – It is shown in the proof of [1, Lemme 12] that $\mathrm{T}\sigma \leqslant \sigma$. Hence $\mu^1_{x,1}(\mathrm{G}_\sigma) = 0$ for x in F_σ. so T induces a positive linear operator $\widetilde{\mathrm{T}}$ on $\mathrm{C}(\mathrm{F}_\sigma)$ given by

$$(\widetilde{\mathrm{T}}f)(x) = \int_{\mathrm{F}_\sigma} f \, d\mu^1_{x,1}.$$

Now $\widetilde{\mathrm{T}}^n 1$ is the restriction of $\mathrm{T}^n 1$ to $\mathrm{F}_\sigma = \sigma^{-1}(0)$, so $\inf\{\widetilde{\mathrm{T}}^n 1 : n \geqslant 1\} = 0$. By theorem 1.2 there is an integer m such that $\mathrm{T}^m 1 < 1$ on F_σ. Applying theorem 2.4 to T^m, it follows that $\{\mathrm{T}^{mn} : n \geqslant 1\}$ is uniformly bounded. Hence $\{\mathrm{T}^n 1 : n \geqslant 1\}$ is uniformly bounded. The result now follows from [1, Théorème 10].

3. Affine functions.

We shall now give an example to show that the answer to problem 2 is negative in general, even if K is a simplex.

Example 3.1. – Let N be the linear span in M[0,1] of $\lambda - \epsilon_0$, where λ is Lebesgue measure on [0,1], let $\pi: M[0,1] \longrightarrow M[0,1]/N$ be the quotient map, and let $K = \pi(P[0,1])$. Then K is a simplex with extreme boundary $\partial_e K = \{\pi(\epsilon_x) : x \in (0,1]\}$, and there is an isometric isomorphism Φ between A(K) and the space $C_0[0,1]$ of functions f in C[0,1] satisfying $f(0) = \int_0^1 f(x)\,dx$, given by $\Phi^{-1}(f) \circ \pi = \hat{f}$ $(f \in C_0[0,1])$. We shall identify these spaces.

Let g be any continuously differentiable function of [0,1] into itself (in the sense of one-sided derivatives at the end-points) such that

$$\begin{array}{lll} g(0) = 0, & g'(0) = 1 & \\ g(x) > x, & g'(x) \geqslant 0 & (x \in (0,1)) \\ g(1) = 1, & g'(1) = 0. & \end{array}$$

Define the operator T by $(Tf)(x) = g'(x)\, f(g(x))$. Then T is a positive linear operator of $C_0[0,1]$ into itself.

For any x in (0,1], let $x_0 = x$, $x_r = g(x_{r-1})$. Then x_r increases to the limit 1, so $g'(x_r) \longrightarrow 0$. Now

$$(T^n 1)(x) = \prod_{r=0}^{n-1} g'(x_r) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Thus T satisfies all the required properties. However

$$\|T^n\| \geqslant |(T^n 1)(0)| = 1.$$

It is noted in [1] that Mokobodzki has shown that problem 2 has an affirmative answer if $\partial_e K$ is closed. This is a special case of the following result, which deals with a general K, but assumes a strengthened condition on T. The proof is based on one of those given in [1].

THEOREM 3.2. – *Let* K *be a compact convex set, let* $\overline{\partial_e K}$ *be the closure of its extreme boundary, and let* T *be a positive linear operator on* A(K). *If, for each* x *in* $\overline{\partial_e K}$, $\inf\{(T^n 1)(x) : n \geqslant 1\} < 1$, *then* $\|T^n 1\| \longrightarrow 0$.

Proof. – For a bounded real-valued function g on K, and x in K, put $(\widetilde{T}g)(x) = \inf\{(Ta)(x) : a \in A(K),\ a \geqslant g \text{ on } \partial_e K\}$. Then $\widetilde{T}(\lambda g) = \lambda \widetilde{T}g$, $\widetilde{T}g_1 \leqslant \widetilde{T}g_2$ if $g_1 \leqslant g_2$ on $\partial_e K$, and $\widetilde{T}a = Ta$ for a in A(K).

By compactness of $\overline{\partial_e K}$, there is an integer r and constant $\alpha > 0$ such that if $g_0(x) = \min\{((T+\alpha)^n 1)(x) : 1 \leqslant n \leqslant r\}$, then $g_0 \leqslant 1$ on $\overline{\partial_e K}$. Then $(\widetilde{T} + \alpha) g_0 \leqslant (T+\alpha)1$ on $\partial_e K$. Also $g_0 \leqslant (T+\alpha)^n 1$, so $(\widetilde{T}+\alpha) g_0 \leqslant (T+\alpha)^{n+1} 1$ $(1 \leqslant n \leqslant r)$. Hence, on $\partial_e K$, $(\widetilde{T}+\alpha) g_0 \leqslant g_0$, so $\widetilde{T} g_0 \leqslant (1-\alpha) g_0$.

Now $g_0 \geqslant \alpha^r$, so $T^n 1 \leqslant \alpha^{-r} \widetilde{T}^n g_0 \leqslant \alpha^{-r}(1-\alpha)^n g_0$ on $\partial_e K$. The result now follows.

Similarly one may modify the proof of Théorème 2 of [1] to show that if, under the conditions of theorem 3.2,

$$\sup\{(T^n 1)(x) : n \geqslant 1\} > 1$$

for each x in $\overline{\partial_e K}$, then $\|T^n\| \longrightarrow \infty$.

Example 3.3. – Let $\mathcal{H}$ be a complex Hilbert space, and x be an operator on $\mathcal{H}$ such that $x - \alpha$ is compact for some scalar α with $|\alpha| < 1$. Suppose that for each unit vector ξ in $\mathcal{H}$, $\|x_\xi^n\| < 1$ for some n (possibly dependent on ξ). If x is self-adjoint, the spectral theorem may be used to deduce that $\|x\| < 1$. However it is easily verified for example that any non-self-adjoint operator x of rank 1 and norm 1 also satisfies $\|x^2\| < 1$.

Let A be the C*-algebra spanned by the identity and the compact operators on $\mathcal{H}$, and let K be its state space. It is well-known that the evaluation map is an isometric order-isomorphism of the self-adjoint part A^s of A onto A(K), and that $\partial_e K$ consists of the vector states ω_ξ ($\xi \in \mathcal{H}$, $\|\xi\| = 1$) given by $\omega_\xi(a) = \langle a\xi, \xi\rangle$ together with the unique state ϕ_0 annihilating the compacts [2, Corollaire 4.1.4]. Using the weak compactness of the unit ball of $\mathcal{H}$ it is easy to see that $\overline{\partial_e K}$ consists of states of the form $\beta\omega_\xi + (1-\beta)\phi_0$ $(\beta \in [0,1])$.

If x satisfies the above conditions, and T is defined by $Ta = x^* a x$ then T is a positive linear operator on A^s, and

$$(\beta\omega_\xi + (1-\beta)\phi_0)(T^n 1) = \beta\|x^n\xi\|^2 + (1-\beta)|\alpha|^{2n} < 1$$

for some n. Theorem 3.2 now shows that $\|T^n 1\| \longrightarrow 0$, so $\|x^n\| \longrightarrow 0$.

Acknowledgements.

I am grateful to Professor Choquet and to Dr G.F. Vincent-Smith for helpful discussions on this paper, and to the Science Research Council for their financial support.

BIBLIOGRAPHY

[1] G. CHOQUET and C. FOIAS, Solution d'un problème sur les itérés d'un opérateur positif sur $\mathscr{C}$ (K) et propriétés de moyennes associées, *Ann. Inst. Fourier* (Grenoble), 24, no. 3 & 4 (1975), 109-129.

[2] J. DIXMIER, Les C*-algèbres et leurs représentations, 2nd ed., Gauthier-Villars, Paris, 1969.

Manuscrit reçu le 21 septembre 1977
Proposé par G. Choquet.

C.J.K. BATTY,
Mathematical Institute
24-29 St Giles
Oxford OX1 2LB, Angleterre.