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A NOTE ON THE PAPER « THE POULSEN SIMPLEX » OF LINDENSTRAUSS, OLSEN AND STERNFELD

by Wolfgang LUSKY

It was shown in [5] that there is only one metrizable Poulsen simplex S (i.e. the extreme points ex S are dense in S) up to affine homeomorphism. Thus, S is universal in the following sense: Every metrizable simplex is affinely homeomorphic to a closed face of S ([5], [6]).

The Poulsen simplex can be regarded as the opposite simplex to the class of metrizable Bauer simplices ([5]). There is a certain analogy in the class of separable Lindenstrauss spaces (i.e. the preduals of L_1-spaces); the Gurarij space G is uniquely determined (up to isometric isomorphisms) by the following property: G *is separable and for any finite dimensional Banach spaces* $E \subset F$, *linear isometry* $T : E \to G$, $\varepsilon > 0$, *there is a linear extension* $\tilde{T} : F \to G$ *of* T *with* $(1 - \varepsilon)\|x\| \leqslant \|\tilde{T}(x)\| \leqslant (1 + \varepsilon)\|x\|$ *for all* $x \in F$. ([3], [7]).

G is universal: Any separable Lindenstrauss space X is isometrically isomorphic to a subspace $X \subset G$ with a contractive projection $P : G \to X$ ([9], [6]).

Furthermore G is opposite to the class of separable C(K)-spaces. There is another interesting property of G:

For any smooth points $x, y \in G$ there is a linear isometry T from G onto G with $T(x) = y$. ($x \in G$ is smooth point if $\|x\| = 1$ and there is only one $x^* \in G^*$ with

$$x^*(x) = 1 = \|x^*\|).$$

In their last remark the authors of [5] point out that here the analogy between G and $A(S) = \{f: S \to \mathbf{R} \mid f$ affine continuous$\}$ seems to break down.

The purpose of this note is to show that under the aspect of rotation properties there is still some kind of analogy between G and A(S).

Take $s_0 \in \operatorname{ex} S$ and consider

$$A_0(S; s_0) = \{f \in A(S) \mid f(s_0) = 0\},$$

for any normed space X let $B(X) = \{x \in X \mid \|x\| \leqslant 1\}$ and $\partial B(X) = \{x \in X \mid \|x\| = 1\}$. In particular

$$\partial B(A(S))_+ = \{f \in \partial B(A(S)) \mid f \geqslant 0\}$$

We show:

THEOREM.

(*a*) *Let* $f, g \in \partial B(A(S))_+$ *so that* $f, 1 - f, g, 1 - g$ *are smooth points of* A(S). *Then there is an isometric isomorphism* T *from* A(S) *onto* A(S) *with*

(i) $T(f) = g$

(ii) $T(A_0(S; s_0)) = A_0(S; s_1)$ where $f(s_0) = 0 = g(s_1)$

(iii) $T(1) = 1$

(*b*) *Let* $f \in \partial B(A_0(S; s_0))_+$ *and* $g \in \partial B(A_0(S; s_1))$ *so that neither* $g \leqslant 0$ *nor* $g \geqslant 0$ *hold. Then there is no isometric isomorphism* T *from* A(S) *onto* A(S) *with* $T(f) = g$.

(*c*) *The elements* $f \in A_0(S; s_0)$, *so that* $f, 1 - f$ *are smooth points of* A(S), *form a dense subset of* $\partial B(A_0(S; s_0))_+$.

The proof of the Theorem which is based on a method used in [5] and [7] is a consequence of the following lemmas and proposition 6. From now on let $s_0 \in \operatorname{ex} S$ be fixed and set $A_0(S) = A_0(S; s_0)$. We shall retain a notation of [5]:

By a peaked partition we mean positive elements $e_1, \dots, e_n \in A_0(S)$ so that $\left\| \sum_{i=1}^n \lambda_i e_i \right\| = \max_{i \leqslant n} |\lambda_i|$ for all $\lambda_i \in \mathbf{R}; i \leqslant n$. Notice that this definition just means « peaked partition of unity in A(S) » ([5]) if we add $e_0 = 1 - \sum_{i=1}^n e_i$. Call a l_∞^n-subspace $E \subset A_0(S)$ ([6]) positively generated if E is spanned by a peaked partition. If $l_\infty^{m+1} \cong \tilde{E} \subset A(S)$

is spanned by the peaked partition of unity $\{f_0, f_1, \ldots, f_m\}$ and contains $e_0, e_1, \ldots, e_n$ then we may arrange the indices $j = 0, 1, \ldots, m$ so that

$$(*) \qquad e_i = f_i + \sum_{j=1}^{m-n} k_j f_{j+n} \,; \qquad i = 0, 1, \ldots, n \,;$$

where $k_j \geqslant 0$ for all j and $\sum_{j=1}^{m-n} k_j \leqslant 1$ ([6] Lemma 1.3 (i)).

Lemma 1. — *Let* $E, F \subset A_0(S)$ *be finite dimensional subspaces so that* E *is a positively generated* l_∞^n*-space. For any* $\varepsilon > 0$ *there is a positively generated* l_∞^m*-space* $\hat{E} \subset A_0(S)$ *so that* $E \subset \hat{E}$ *and* $\inf\{\|x - y\| \mid y \in \hat{E}\} \leqslant \varepsilon\|x\|$ *for all* $x \in F$.

Proof. — We may assume without loss of generality that F is spanned by positive elements. Let $\{e_1, \ldots, e_n\}$ be the peaked partition which spans E. Add e_0 as above. By [3] Theorem 3.1. there is $l_\infty^m \cong \tilde{E} \subset A(S)$ with $E \subset \tilde{E}$ and $\inf\{\|x - y\| \mid y \in \tilde{E}\} \leqslant \varepsilon\|x\|$ for all $x \in F$. Hence $\tilde{E}$ is positively generated by a peaked partition of unity $\{f_0, f_1, \ldots, f_m\}$ By $(*)$ $f_i(s_0) = 0$; $1 \leqslant i \leqslant m$. Set $\hat{E} =$ linear span $\{f_1, \ldots, f_m\}$. $\square$

Lemma 2. — *Let* $l_\infty^n \cong E \subset F \cong l_\infty^m$ *be positively generated subspaces of* $A_0(S)$. *Let* $\Phi \in E^*$ *be positive. Then there is a positive extension* $\tilde{\Phi} \in F^*$ *of* Φ *with* $\|\tilde{\Phi}\| = \|\Phi\|$.

Proof. — Let $\{e_i \mid 1 \leqslant i \leqslant n\}$ and $\{f_j \mid 1 \leqslant j \leqslant m\}$ be peaked partitions spanning E and F respectively, so that $(*)$ holds. Define then $\tilde{\Phi}(f_i) = \Phi(e_i)$ for all $i = 1, \ldots, n$ and $\tilde{\Phi}(f_j) = 0$ for all $j = n + 1, \ldots, m$. $\square$

Lemma 3. — *Let* $\{e_{i,n} \in A_0(S) \mid 1 \leqslant i \leqslant n\}$ *be a peaked partition. Suppose that there is a positive* $\Phi \in \operatorname{ex} B(A_0(S)^*)$ *so that* $\sum_{i=1}^{n} \Phi(e_{i,n}) < 1$. *Then there is a peaked partition* $\{e_{i,n+1} \in A_0(S) \mid 1 \leqslant i \leqslant n + 1\}$ *with*

$$e_{i,n} = e_{i,n+1} + \Phi(e_{i,n}) e_{n+1,n+1}$$

for all $i = 1, \ldots, n$.

Proof. — Let $\Phi_0 \in \operatorname{ex} B(A(S)^*)$ be an element satisfying $\Phi_0(y) = 0$ for all $y \in A_0(S)$. Consider furthermore

$$\Phi_i \in \operatorname{ex} B(A(S)^*); \qquad i = 1, \ldots, n;$$

with

$$\Phi_i(e_{j,n}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}; \quad j = 1, \ldots, n.$$

Define the affine w^*-continuous function $f\colon H \to \mathbf{R}$ by $f(\pm \Phi_i) = 0$; $i = 0, 1, \ldots, n$; $f(\pm \Phi) = \pm 1$ where $H = \operatorname{conv}(\{\pm \Phi_i \mid i = 0,1, \ldots, n\} \cup \{\pm \Phi\})$. Set

$$h_1(y^*) = \min \left\{ \frac{1 - \sum_{i=1}^{n} \theta_i y^*(e_{i,n})}{1 - \sum_{i=1}^{n} \theta_i \Phi(e_{i,n})} \mid \theta_i = \pm 1;\, i = 1, \ldots, n \right\}$$

$$h_2(y^*) = \min \left\{ \frac{1 - y^*(e - e_{i,n})}{\Phi(e_{i,n})} \mid \Phi(e_{i,n}) > 0;\, i = 1, \ldots, n \right\}$$

and consider $g(y^*) = \min(h_1(y^*), h_2(y^*), 1 + y^*(e))$.

Hence $g\colon B(A(S)^*) \to \mathbf{R}$ is w^*-continuous, concave and nonnegative. In addition, $f(y^*) \leqslant g(y^*)$ holds for all $y^* \in H$.

By [3] Theorem 2.1. there is $e_{n+1,n+1} \in A(S)$ with

$$y^*(e_{n+1,n+1}) \leqslant g(y^*)$$

for all $y^* \in B(A(S)^*)$ and $y^*(e_{n+1,n+1}) = f(y^*)$ for all $y^* \in H$.

Hence, $\|e - [e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1}]\| \leqslant 1$ and

$$\|e - e_{n+1,n+1}\| \leqslant 1.$$

Thus $0 \leqslant e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1}$ and $0 \leqslant e_{n+1,n+1}$ for $i = 1, \ldots, n$. Furthermore $\Phi_0(e_{n+1,n+1}) = 0$, hence $e_{n+1,n+1} \in A_0(S)$. That means, $e_{n+1,n+1}$ and $e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1}$ are the elements of a peaked partition in $A_0(S)$. □

LEMMA 4. — *Let $r_1, \ldots, r_n > 0$ with $\sum_{i=1}^{n} r_i < 1$ and a peaked partition $\{e_{1,n}, \ldots, e_{n,n}\} \subset A_0(S)$ be given. Then there is a positive element $\Phi \in \operatorname{ex} B(A_0(S)^*)$ with $\Phi(e_{i,n}) = r_i$ for all $i \leqslant n$.*

Proof. — Let $\{x_n \mid n \in \mathbf{N}\}$ be dense in $A_0(S)$. Set linear span $\{e_{i,n} \mid i \leqslant n\} = E$. Define $\Phi|_E$ by $\Phi(e_{i,n}) = r_i$ for all i. Assume that we have defined Φ already on a positively generated l_∞^m-subspace $\tilde{E} \supset E$ of $A_0(S)$ so that $\|\Phi_{|\tilde{E}}\| < 1$. Then there is a basis $\{e_{i,m} \mid i \leqslant m\}$ of $\tilde{E}$ consisting of a peaked partition so that $\Phi(e_{i,m}) > 0$ for all $i = 1, \ldots, m$. Now, let $0 < \varepsilon < 1/2^{m+1}\left(1 - \sum_{i=1}^{m} \Phi(e_{i,m})\right)$. There is a positive linear extension $\Psi \in \operatorname{ex} B(A_0(S)^*)$ of Φ by Lemma 1 and Lemma 2. We derive from $\overline{\operatorname{ex} S} = S$ that $\operatorname{ex} B(A_0(S)^*)_+$ is w^*-dense in $B(A_0(S)^*)_+$. It follows that there is $\Omega \in \operatorname{ex} B(A_0(S)^*)_+$ with $\Phi(e_{i,m}) \geqslant \Omega(e_{i,m})$ for all $i = 1, \ldots, m$ and with $\sum_{i=1}^{m} |\Omega(e_{i,m}) - \Phi(e_{i,m})| \leqslant \varepsilon$. We infer from Lemma 3 that there is peaked partition

$$\{e_{i,m+1} \in A_0(S) \mid i = 1, \ldots, m+1\}$$

with $e_{i,m} = e_{i,m+1} + \Omega(e_{i,m})e_{m+1,m+1}$; $i = 1, \ldots, m$. Set $E_{m+1} = \operatorname{span}\{e_{i,m+1} \mid i \leqslant m+1\}$ and extend Φ linearly by defining $\Phi(e_{m+1,m+1}) = (1 + 2^{-m})^{-1}$. Hence $\|\Phi_{|E_{m+1}}\| < 1$. Find a positively generated l_∞^{m+1+k}-space $F \subset A_0(S)$ with $E_{m+1} \subset F$ and $\inf\{\|x_k - y\| \mid y \in F\} \leqslant (m+1)^{-1}\|x_k\|$ for all $k \leqslant m$. Continue this process with F instead of E. Finally we obtain an increasing sequence $E_m \subset A_0(S)$ of positively generated l_∞^m-spaces so that $A_0(S) = \overline{\cup E_m}$ where m runs through a subsequence of $\mathbf{N}$. Furthermore there are peaked partitions $\{e_{i,m} \in E_m \mid i \leqslant m\}$ so that $\lim_{m\to\infty} \Phi(e_{m,m}) = 1$. The latter condition implies that Φ is a positive extreme point of $B(A_0(S)^*)$. □

COROLLARY. — *Let $e_{i,n} \in A_0(S)$ be a peaked partition and let $0 < r_i$; $i = 1, \ldots, n$; be real numbers with $\sum_{i=1}^{n} r_i < 1$. Then there is a peaked partition $\{e_{j,n+1} \in A_0(S) \mid j = 1, \ldots, n+1\}$ with $e_{i,n} = e_{i,n+1} + r_i e_{n+1,n+1}$; $i = 1, \ldots, n$.*

Remark. — If we omit « $\sum_{i=1}^{n} r_i < 1$ » then the above corollary is no longer true (see [7], remark after the corollary of Lemma 2). The previous corollary does not hold either if we drop « $0 < r_i$ for all i ». This follows from the next lemma.

LEMMA 5. — *Let* $s_0 \in \operatorname{ex} S$ *be fixed. Then the set*

$$\Lambda(S, s_0) = \{f \in B(A_0(S, s_0)) \mid f$$

and $1 - f$ *are smooth points of* $A(S)\}$ *is dense in* $\partial B(A_0(S, s_0))_+$.

Proof. — Let $g \in \partial B(A_0(S, s_0))_+$ and $s_1 \in \operatorname{ex} S$ so that $g(s_1) = 1$. Set $F = \operatorname{conv}(\{s_0, s_1\})$. Let $\{x_n \mid n \in \mathbf{N}\}$ be dense in $\{x \in A_0(S, s_0) \mid \|x\| \leqslant 1; \ x|_F = 0\}$. Define the affine continuous function $h : F \to \mathbf{R}$ by $h(s_0) = 0$, $h(s_1) = 1$. Furthermore let $f_1(s) = 1 - 1/2 \sum_{n=1}^{\infty} 2^{-n}(x_n(s))^2$ and

$$f_2(s) = 1/2 \sum_{n=1}^{\infty} 2^{-n}(x_n(s))^2$$

for all $s \in S$. Then f_1 and f_2 are continuous; f_1 is concave, f_2 is convex. Furthermore $f_2(s) \leqslant h(s) \leqslant f_1(s)$ for all $s \in F$. Hence there is an affine, continuous extension $\tilde{h} : S \to \mathbf{R}$ of h with $f_2(s) \leqslant \tilde{h}(s) \leqslant f_1(s)$ for all $s \in S$ ([1], [2]).

Thus $\tilde{h}(s_0) = 0$, $\tilde{h}(s_1) = 1$, $0 < \tilde{h}(s) < 1$ for $s \neq s_0, s_1$. Then $\lim_{\varepsilon \to 0} \dfrac{(1-\varepsilon)g + \varepsilon\tilde{h}}{\|(1-\varepsilon)g + \varepsilon\tilde{h}\|} = g$. □

Now, if we take $e_{1,1} \in \Lambda(S, s_0)$ and suppose that there is $\Phi \in \operatorname{ex} B(A_0(S, s_0)^*)$ with $\Phi(e_{1,1}) = 0$ then there must be $s_1 \in \operatorname{ex} S$ with $s_1 \neq s_0$ so that $e_{1,1}(s_1) = 0$, which is a contradiction. This concludes our above remark.

PROPOSITION 6. — *Let* S *be the Poulsen simplex and* s, $\tilde{s} \in \operatorname{ex} S$. *Consider* $x \in \Lambda(S, s)$ *and* $y \in \Lambda(S, \tilde{s})$. *Then there is an isometric (linear and order-) isomorphism* T:

$$A_0(S, s) \to A_0(S, \tilde{s}) \ \textit{(onto) with} \ T(x) = y .$$

Proof. — In the following we set $X = A_0(S, s)$ and $Y = A_0(S, \tilde{s})$. We claim that there are peaked partitions

$$\{e_{i,n} \mid i \leqslant n\} \subset X, \qquad \{f_{i,n} \mid i \leqslant n\} \subset Y; \quad n \in \mathbf{N};$$

and real numbers $a_{i,n}$; $i \leqslant n$; $n \in \mathbf{N}$; with

$$(1) \qquad \begin{aligned} e_{i,n} &= e_{i,n+1} + a_{i,n} e_{n+1,n+1} \\ f_{i,n} &= f_{i,n+1} + a_{i,n} f_{n+1,n+1} \end{aligned}$$

$$0 < a_{i,n}; \quad i \leqslant n; \quad \sum_{i=1}^{n} a_{i,n} < 1; \quad n \in \mathbf{N};$$

$$e_{1,1} = x; \quad f_{1,1} = y .$$

For this purpose we construct peaked partitions

$$\{e_{i,n}^{(j)} \mid i \leqslant n\} \subset \mathrm{X}$$

$\{f_{i,n}^{(j)} \mid i \leqslant n\} \subset \mathrm{Y}$; $n \in \mathbf{N}$; $j \leqslant n$; such that

$$(2) \qquad e_{i,n}^{(j)} = e_{i,n+1}^{(j)} + a_{i,n} e_{n+1,n+1}^{(j)}$$

$$(2') \qquad f_{i,n}^{(j)} = f_{i,n+1}^{(j)} + a_{i,n} f_{n+1,n+1}^{(j)}$$

$$(3) \qquad \|e_{i,n}^{(j)} - e_{i,n}^{(j+1)}\| \leqslant 2^{-j}$$

$$(3') \qquad \|f_{i,n}^{(j)} - f_{i,n}^{(j+1)}\| \leqslant 2^{-j}.$$

We proceed by induction :

Let $\{x_n \mid n \in \mathbf{N}\}$ be dense in X and let $\{y_n \mid n \in \mathbf{N}\}$ be dense in Y. Assume that

$$\{e_{i,k}^{(p)} \mid i \leqslant k\}, \quad \{f_{i,k}^{(p)} \mid i \leqslant k\}$$

and $0 < a_{i,j}$; $j = 1, \ldots, n-1$; $k \leqslant p$; $k, p = 1, \ldots, n$; have been introduced already such that $e_{1,1}^{(n)} = x$ and $f_{1,1}^{(n)} = y$. Set $\mathrm{E}_n = \mathrm{Span}\ \{e_{i,n}^{(n)} \mid i \leqslant n\}$; $\mathrm{F}_n = \mathrm{Span}\ \{f_{i,n}^{(n)} \mid i \leqslant n\}$

(*) There are positively generated l_∞^k-subspaces $\mathrm{E}_k \subset \mathrm{X}$ with $\mathrm{E}_{k-1} \subset \mathrm{E}_k$; $k = n+1, \ldots, m$; so that

$$(4) \quad \inf\{\|x_j - x\| \mid x \in \mathrm{E}_m\} \leqslant 2^{-n}\|x_j\|; \quad j = 1, \ldots, n.$$

Consider a system of peaked partitions $\{e_{i,k}^{(k)} \mid i \leqslant k\}$ spanning E_k and real numbers $0 \leqslant b_{i,k}$ with

$$(5) \quad e_{i,k-1}^{(k-1)} = e_{i,k}^{(k)} + b_{i,k-1} e_{k,k}^{(k)}; \quad \sum_{i=1}^{k-1} b_{i,k-1} \leqslant 1; \quad k = n+1, \ldots, m.$$

Notice that (6) $0 < \sum_{i=1}^{k-1} b_{i,k-1}$ for all k.

Since otherwise there is $\Phi \in \operatorname{ex} \mathrm{B}(\mathrm{X}^*)$ with $\Phi|_{\mathrm{E}_{k-1}} = 0$ and $\Phi(e_{k,k}^{(k)}) = 1$. As $x \in \mathrm{E}_{k-1}$, there are two different s, $s_1 \in \operatorname{ex} \mathrm{S}$ with $x(s) = x(s_1) = 0$, a contradiction.

We first perturb $\{e_{i,n}^{(n)} \mid i \leqslant n\}$:

STEP $(n+1)$:

Consider

$$(7) \quad x = e_{1,1}^{(n)} = e_{1,n}^{(n)} + \sum_{j=2}^{n} k_j e_{j,n}^{(n)} = e_{1,n+1}^{(n+1)} + \sum_{j=2}^{n} k_j e_{j,n+1}^{(n+1)} + \left(b_{1,n} + \sum_{j=2}^{n} k_j b_{j,n}\right) e_{n+1,n+1}^{(n+1)}$$

where $0 \leqslant k_j \leqslant 1$; $2 \leqslant j \leqslant n$. Even $k_j < 1$ holds properly for all $j = 2, \ldots, n$; since otherwise there would be two different $s_1, s_2 \in \operatorname{ex} S$ with $x(s_1) = x(s_2) = 1$; which can be infered from (7) similarly as the proof of (6). Using the same kind of argument shows $0 < k_j$ for all $j = 2, \ldots, n$. In view of (6) there is some $b_{i,n} \neq 0$.

(*a*) Let $\sum_{i=1}^{n} b_{i,n} < 1$:

Let i_0 be an index with $b_{i_0,n} \neq 0$. Set $k_1 = 1$ and

$$\rho = \min\left(\left(1 - \sum_{i=1}^{n} b_{i,n}\right) |k_{i_0}(n-1) - \sum_{\substack{j=1 \\ j \neq i_0}}^{n} k_j|^{-1};\ 1/n\right).$$

Define

$$a_{i_0,n} = \left(1 - 2^{-2n}\rho \sum_{\substack{j=1 \\ j \neq i_0}}^{n} k_j\right) b_{i_0,n}$$

$$a_{i,n} = b_{i,n} + 2^{-2n}\rho k_{i_0} b_{i_0,n}; \quad i \neq i_0 .$$

(*b*) Assume now $\sum_{i=1}^{n} b_{i,n} = 1$.

From our assumption $x \in \Lambda(S,s)$ together with (7) it follows similarly as above that there is $i \geqslant 2$ with $b_{i,n} > 0$. Assume without loss of generality that $b_{n,n} > 0$.

Let $\rho = \min\left(\frac{1}{2}(1 - k_n) |k_n(n-1) - \sum_{j=1}^{n-1} k_j|^{-1};\ 1/n\right)$.

Define

$$a_{1,n} = b_{1,n} + 2^{-(2n+1)} k_n (1 + \rho) b_{n,n}$$

$$a_{i,n} = b_{i,n} + 2^{-(2n+1)} k_n \rho b_{n,n}; \quad 2 \leqslant i \leqslant n-1 \quad (\text{if } n > 2)$$

$$a_{n,n} = \left(1 - 2^{-(2n+1)} - 2^{-(2n+1)}\rho \sum_{j=1}^{n-1} k_j\right) b_{n,n} .$$

Hence in either case $0 < a_{i,n}$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^{n} a_{i,n} < 1$. Furthermore

$$|a_{i,n} - b_{i,n}| \leqslant 2^{-2n} \quad \text{for all} \quad i \leqslant n . \tag{8}$$

Define

$$\begin{aligned} e_{i,n}^{(n+1)} &= e_{i,n+1}^{(n+1)} + a_{i,n} e_{n+1,n+1}^{(n+1)} && i \leqslant n+1 \\ e_{i,n}^{(n+1)} &= e_{i,n}^{(n+1)} + a_{i,n-1} e_{n,n}^{(n+1)} && i \leqslant n \\ &\vdots \\ e_{1,1}^{(n+1)} &= e_{1,2}^{(n+1)} + a_{1,1} e_{2,2}^{(n+1)} . \end{aligned} \tag{9}$$

From (8) and (9) we derive easily $\|e_{i,k}^{(n+1)} - e_{i,k}^{(n)}\| \leqslant 2^{-n}$; $k = 1, \ldots, n+1$; $i \leqslant n$. Hence $(2)_{n+1}$ and $(3)_{n+1}$ are established.

Furthermore, because the elements k_j of (7) depend only on $a_{i,k}$; $i \leqslant k \leqslant n-1$; we obtain

$$\begin{aligned} e_{1,1}^{(n+1)} &= e_{1,n}^{(n+1)} + \sum_{j=2}^{n} k_j e_{j,n}^{(n+1)} \\ &= e_{1,n+1}^{(n+1)} + \sum_{j=2}^{n} k_j e_{j,n+1}^{(n+1)} + \left(a_{1,n} + \sum_{j=2}^{n} k_j a_{j,n}\right) e_{n+1,n+1}^{(n+1)} \\ &= e_{1,n+1}^{(n+1)} + \sum_{j=2}^{n} k_j e_{j,n+1}^{(n+1)} + \left(b_{1,n} + \sum_{j=2}^{n} k_j b_{j,n}\right) e_{n+1,n+1}^{(n+1)} \\ &= e_{1,1}^{(n)} = x . \end{aligned}$$

Now, in STEP $(n+2)$, repeat the procedure of STEP $(n+1)$ but replace E_{n+1} by E_{n+2} and $n+1$ by $n+2$. Then turn to STEP $(n+3)$, $\ldots$, STEP (m). We obtain $(2)_{n+1}, \ldots, (2)_m$ and $(3)_{n+1}, \ldots, (3)_m$.

Consider now F_n. Find positively generated l_∞^k subspaces $F_n \subset F_{n+1} \subset \ldots \subset F_m \subset Y$ and peaked partitions spanning F_k, $\{f_{i,k}^{(m)} \in F_k \mid i \leqslant k\}$ with

$$f_{i,k}^{(m)} = f_{i,k+1}^{(m)} + a_{i,k} f_{k+1,k+1}^{(m)} ; \quad k = n, \ldots, m-1$$

where we have set $f_{i,n}^{(m)} = f_{i,n}^{(n)}$; $i = 1, \ldots, n$. This is possible by the Corollary after Lemma 4. Define

$$\begin{aligned} f_{i,k}^{(j)} &= f_{i,k}^{(m)}; \quad i \leqslant k; \quad n+1 \leqslant k \leqslant m; \quad n+1 \leqslant j \leqslant m \\ f_{i,k}^{(j)} &= f_{i,k}^{(n)}; \quad i \leqslant k; \quad 1 \leqslant k \leqslant n; \quad n+1 \leqslant j \leqslant m. \end{aligned}$$

Find positively generated l_∞^k-subspaces F_k of Y with

$F_{k-1} \subset F_k$; $k = m + 1, \ldots, r$; such that

$$(10) \quad \inf \{\|y_j - x\| \mid x \in F_r\} \leqslant 2^{-m}\|y_j\|; \quad j = 1, \ldots, m.$$

Repeat (*) with r instead of m and F_r instead of E_m. This yields $(2')_{m+1}, \ldots, (2')_r$ and $(3')_{m+1}, \ldots, (3')_r$.

Then go back to E_m and find positively generated l^k_∞-subspaces $E_{m+1} \subset \ldots \subset E_r$ of X with $E_m \subset E_{m+1}$ and peaked partitions $\{e^{(r)}_{i,k} \mid i \leqslant k\}$ of E_k with

$$e^{(r)}_{i,k} = e^{(r)}_{i,k+1} + a_{i,k}e^{(r)}_{k+1,k+1}; \quad k = m, \ldots, r - 1.$$

(We have set $e^{(r)}_{i,m} = e^{(m)}_{i,m}$).

Define

$$e^{(j)}_{i,k} = e^{(r)}_{i,k}; \quad i \leqslant k; \quad m + 1 \leqslant k \leqslant r; \quad m + 1 \leqslant j \leqslant r;$$
$$e^{(j)}_{i,k} = e^{(m)}_{i,k}; \quad i \leqslant k; \quad 1 \leqslant k \leqslant m; \quad m + 1 \leqslant j \leqslant r.$$

Finally go back to (*) and repeat everything with E_r and F_r instead of E_n and F_n, respectively. By (3) and (3′) we obtain

$$e_{i,n} = \lim_{j \to \infty} e^{(j)}_{i,n}; \quad f_{i,n} = \lim_{j \to \infty} f^{(j)}_{i,n}; \quad i \leqslant n, \ n \in \mathbf{N};$$

which are elements of peaked partitions with

$$e_{i,n} = e_{i,n+1} + a_{i,n}e_{n+1,n+1}; \quad f_{i,n} = f_{i,n+1} + a_{i,n}f_{n+1,n+1}$$

$i \leqslant n$; $n \in \mathbf{N}$; $f_{1,1} = y$; $e_{1,1} = x$ ((2) and (2′)). From (4), (10) and (3), (3′) we infer that

closed span $\{f_{i,n} \mid i \leqslant n;\ n \in \mathbf{N}\} = Y$

and

closed span $\{e_{i,n} \mid i \leqslant n;\ n \in \mathbf{N}\} = X$.

We define an isometric isomorphism $T : A_0(S;s) \to A_0(S;\tilde{s})$ by $T(e_{i,n}) = f_{i,n}$; $i \leqslant n$; $n \in \mathbf{N}$. □

Proposition 6 establishes the assertion (*a*) of the Theorem if we extend T isometrically on A(S) by defining $T(1) = 1$.

Proof of (*b*):

Let $u, v \in \operatorname{ex} S$ so that $g(u) > 0$ and $g(v) < 0$. If there were an isometric isomorphism (onto) then in view of Lemma 5 there would be $\tilde{g} \in \partial B(A_0(S;s_1))$ with $\tilde{g}(u) > 0$ and $\tilde{g}(v) < 0$ so that $\tilde{g}(s) \neq 0$ for all $s \in S$; $s \neq s_1$. But then $s_1 = \lambda u + (1 - \lambda)v$ for suitable $\lambda; 0 < \lambda < 1$. Hence $u = v = s_1$, a contradiction.

(*c*) has been proved already by Lemma 5.

Concluding remarks. — The assertion (*a*) of the Theorem cannot be extended on any dense subset of $\partial B(A(S))_+$ since otherwise any element of $\partial B(A(S))_+$ would be extreme point of $B(A(S))$ which is certainly not true. This follows from the fact that for any $e \in \mathrm{ex}\, B(A(S))$,

$$\max (\|x + e\|, \|x - e\|) = 1 + \|x\|$$

holds for all $x \in A(S)$. (cf. [4] Theorem 4.7. and 4.8.).

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