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ON A THEOREM OF M. ITO (*)

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Let G be a locally compact abelian group and let N_0 be a convolution kernel on G (i.e. a positive measure on G) satisfying the principle of unicity of mass. A convolution kernel N on G is said to be a *divisor* of N_0 if there exists a convolution kernel N' on G such that $N_0 = N * N'$. If such an N' exists and is uniquely determined, it is denoted N_0/N ; this is the case if N' exists and is a Hunt kernel on G. The set of divisors of N_0 is denoted $D(N_0)$.

The following result [1] was recently proved by M. Itô, cf. Théorème 1 in [1].

THEOREM. — *The set*

$$C(N_0) = \{N \in D(N_0) \mid N_0/N \textit{ is a Hunt kernel on } G\}$$

is a convex cone.

The proof in [1] of this result is somewhat complicated, and it is the purpose of this note to give a simpler proof, mainly using the ideas from [1].

LEMMA 1 ([1], p. 292-293). — *Let N be a non-zero convolution kernel. In order that there exist a Hunt kernel N' and an N'-invariant convolution kernel η such that $N_0 = N * N' + \eta$, it is necessary and sufficient that $N + pN_0 \in D(N_0)$ for all $p > 0$. In the affirmative case N' and η are uniquely deter-*

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(1) The statement of Théorème 1 in [1] is not true; the argument uses Corollaire 2 of [1] which is false. What the proof actually shows is what is formulated in the Theorem.

mined as the vague limits

$$N' = \lim_{p\to 0} (N_0/(N + pN_0)) \qquad \text{and} \qquad \eta = \lim_{t\to\infty} N_0 * \alpha'_t ,$$

where $(\alpha_t)_{t\geqslant 0}$ *is the vaguely continuous convolution semigroup (vccs) on* G *associated with* N'.

Lemma 2 (cf. [1], p. 295-299). — *Let* N_i *for* $i = 1, 2$ *be a Hunt kernel with associated vccs* $(\alpha^i_t)_{t\geqslant 0}$. *Suppose that the convolution* $N_1 * N_2$ *exists. Then* $(\alpha^1_t * \alpha^2_t)_{t\geqslant 0}$ *defines a vccs and*

$$N = \int_0^\infty \alpha^1_t * \alpha^2_t \, dt ,$$

defines a Hunt kernel satisfying

$$N_1 * N_2 = N * (N_1 + N_2) . \qquad (*)$$

Proof. — For $s, t \geqslant 0$ we have $\alpha^1_s * N_1 \leqslant N_1$ and $\alpha^2_t * N_2 \leqslant N_2$ which implies that the convolution $\alpha^1_s * \alpha^2_t$ exists. By the convergence Lemma in [1] p. 295, it is rather easy to see that the mapping $(s,t) \longmapsto \alpha^1_s * \alpha^2_t$ is vaguely continuous on $[0, \infty[\times [0, \infty[$, and in particular $(\alpha^1_t * \alpha^2_t)_{t\geqslant 0}$ is a vccs. For $n > 0$ we have (vaguely)

$$\begin{aligned}\left(\int_0^n \alpha^1_t * \alpha^2_t \, dt\right) * (N_1 + N_2) &= \left(\int_0^n \alpha^1_t * \alpha^2_t \, dt\right) * \left(\int_0^\infty (\alpha^1_s + \alpha^2_s) \, ds\right) \\ &= \int_0^\infty \left(\int_0^n \alpha^1_s * \alpha^1_t * \alpha^2_t \, dt\right) ds + \int_0^\infty \left(\int_0^n \alpha^2_s * \alpha^1_t * \alpha^2_t \, dt\right) ds \\ &\leqslant \int_0^\infty \left(\int_t^\infty \alpha^1_u * \alpha^2_t \, du\right) dt + \int_0^\infty \left(\int_t^\infty \alpha^1_t * \alpha^2_u \, du\right) dt \\ &= \int_0^\infty \int_0^\infty \alpha^1_s * \alpha^2_t \, ds \, dt = N_1 * N_2 .\end{aligned}$$

This shows that $\int_0^\infty \alpha^1_t * \alpha^2_t \, dt$ converges vaguely, and equation (*) follows by a monotone convergence argument.|.

Proof of the theorem. — We shall see that $C(N_0)$ is convex. Let $N_1, N_2 \in C(N_0)$ and let $(\alpha^i_t)_{t\geqslant 0}$, $i = 1, 2$, be the vccs associated with the Hunt kernel $N'_i = N_0/N_i$, $i = 1, 2$. Let $(N'_{i,p})_{p>0}$, $i = 1, 2$, be the resolvent for N'_i,

$$N'_{i,p} = \int_0^\infty e^{-pt} \alpha^i_t \, dt \quad \text{for} \quad p > 0 \quad \text{and} \quad i = 1, 2 .$$

For $p > 0$ and $i = 1, 2$ we have $N'_{i,p} * (N_i + pN_0) = N_0$ which shows that for $p > 0$

$$N'_{1,p} * N'_{2,p} * N_0 \leqslant N'_{1,p} * \frac{1}{p} N_0 \leqslant \left(\frac{1}{p}\right)^2 N_0 .$$

It follows that the convolution $N'_{1,p} * N'_{2,p}$ exists, and by Lemma 2 the convolution kernel

$$\tilde{N}_p = \int_0^\infty e^{-2pt} \alpha_t^1 * \alpha_t^2 \, dt$$

is well-defined and satisfies

$$N'_{1,p} * N'_{2,p} = \tilde{N}_p * (N'_{1,p} + N'_{2,p}) .$$

Moreover we have

$$(N_1 + N_2 + 2pN_0) * \tilde{N}_p = N_0 . \qquad (**)$$

In fact,

$$\begin{aligned}(N_1 + N_2 + 2pN_0) * \tilde{N}_p * N'_{1,p} * N'_{2,p} \\ = (N_1 + pN_0) * N'_{1,p} * N'_{2,p} * \tilde{N}_p + (N_2 + pN_0) * N'_{2,p} * N'_{1,p} * \tilde{N}_p \\ = N_0 * N'_{2,p} * \tilde{N}_p + N_0 * N'_{1,p} * \tilde{N}_p = N_0 * N'_{1,p} * N'_{2,p} ,\end{aligned}$$

where all the convolutions exist, and this implies (**) because $N'_{1,p} * N'_{2,p}$ satisfies the principle of unicity of mass.

By Lemma 1 the Hunt kernel

$$\tilde{N} = \lim_{p \to 0} \tilde{N}_p = \int_0^\infty \alpha_t^1 * \alpha_t^2 \, dt$$

satisfies $\tilde{N} * (N_1 + N_2) + \eta = N_0$ for some $\tilde{N}$-invariant convolution kernel η, and since $N_0 = N_1 * N'_1 = N_2 * N'_2$ we have (as in [1], p. 302)

$$\eta = \lim_{t \to \infty} N_0 * \alpha_t^1 * \alpha_t^2 \leqslant \lim_{t \to \infty} N_0 * \alpha_t^2 = 0 ,$$

which shows that $\tilde{N} * (N_1 + N_2) = N_0$, i.e.

$$N_1 + N_2 \in C(N_0). \mid .$$

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