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ON THE GREEN TYPE KERNELS
ON THE HALF SPACE IN $\mathbb{R}^n$

by Masayuki ITÔ

1. Let $\mathbb{R}^n$ be the $n(\geq 2)$-dimensional Euclidian space and $D$ be the half space \( \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; x_1 > 0 \} \). For a point \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), we write

\[
\bar{x} = (-x_1, x_2, \ldots, x_n) \quad \text{and} \quad |x| = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2}.
\]

When $n \geq 3$, we put $G_2(x, y) = |x - y|^{2-n} - |x - \bar{y}|^{2-n}$ in $D \times D$. Then $G_2$ is the Green kernel on $D$. Analogously we set, for a number $\alpha$ with $0 < \alpha < n$,

\[
G_{\alpha}(x, y) = |x - y|^{\alpha-n} - |x - \bar{y}|^{\alpha-n}
\]

in $D \times D$, and we call it the Green type kernel of order $\alpha$ on $D$. The following question was proposed to me in a letter by H. L. Jackson: Does $G_{\alpha}$ also satisfy the domination principle provided that $0 < \alpha < 2$?

This paper is inspired by this question. Let $C_c(D)$ and $C(D)$ be the usual topological vector space of real-valued continuous functions in $D$ with compact support and the usual topological vector space of real-valued continuous functions in $D$, respectively. We set

\[
C_c^+(D) = \{ f \in C_c(D); f \geq 0 \}
\]

and $C^+(D) = \{ f \in C(D); f \geq 0 \}$. For a given Hunt convo-
olution kernel \( x \) on \( \mathbb{R}^n \), we define the linear operator
\[
V_x : C_c(D) \ni f \mapsto (x \ast f - x \ast \overline{f})_D \in C(D),
\]
where \( \overline{f} \) is the reflection of \( f \) about the boundary \( \partial D \) of \( D \) and where \( (x \ast f - x \ast \overline{f})_D \) is the restriction of
\[
x \ast f - x \ast \overline{f}
\]
to \( D \). If \( V_x \) is positive (that is, \( f \geq 0 \implies V_x f \geq 0 \)), we say that \( V_x \) is the Green type kernel associated with \( x \).

The purpose of this paper is to show the following two theorems.

**Theorem 1.** — Let \( x \) be a Hunt convolution kernel on \( \mathbb{R}^n \) and \( (x_p)_{p \geq 0} \) be the resolvent associated with \( x \). Suppose that \( x \) is symmetric with respect to \( \partial D \). Then the following two conditions are equivalent:

1. \( V_x \) is a Hunt kernel on \( D \).
2. For each \( p > 0 \), \( \frac{\partial}{\partial x_1} x_p \leq 0 \) in the sense of distributions in \( D \).

**Theorem 2.** — Let \( x \) be a Dirichlet convolution kernel on \( \mathbb{R}^n \) and \( \alpha \) be the singular measure (the Lévy measure) associated with \( x \). Suppose that \( x \) is also symmetric with respect to \( \partial D \). Then the following two conditions are equivalent:

1. \( V_x \) is a Dirichlet kernel on \( D \).
2. \( \frac{\partial}{\partial x_1} \alpha \leq 0 \) in the sense of distributions in \( D \).

This theorem gives immediately that the question raised by H. L. Jackson is affirmatively solved.

2. Let \( x \) be a convolution kernel on \( \mathbb{R}^n \). Similarly we define \( V_x \). When \( V_x \) is positive, we set
\[
\mathcal{D}^+(V_x) = \{ f \in C^+(D) ; V_x f \in C^+(D) \},
\]
where
\[
V_x f(x) = \sup \{ V_x g(x) ; g \in C^+_c(D), g \leq f \}
\]

(1) An \( f \in C^+_c(D) \) may be considered as a finite continuous function in \( \mathbb{R}^n \) with compact support \( \subset D \).

(2) In potential theory, a convolution kernel means a positive measure.
in D. Put $\mathcal{D}(V_x) = \{f \in C(D) : f^+, f^- \in \mathcal{D}^+(V_x)\}$ and, for an $f \in \mathcal{D}(V_x)$, $V_x f = V_x f^+ - V_x f^-$. Then $V_x$ is a linear operator from $\mathcal{D}(V_x)$ into $C(D)$.

**Lemma 3.** Let $x$ and $x'$ be two convolution kernels on $\mathbb{R}^n$. Suppose that $x$ and $x'$ are symmetric with respect to $\partial D$ and that the convolution $x \ast x'$ is defined. If $V_x$ is positive, then, for any $f \in C_c(D)$, $V_x f \in \mathcal{D}(V_x)$ and

$$V_x(V_x f) = (x \ast x' \ast f - x \ast x' \ast \overline{f})_D.$$  

**Proof.** We may assume that $f \geq 0$. Since $x \ast x'$ is defined and $|V_x f| \leq x' \ast f + x' \ast \overline{f}$, we have $V_x f \in \mathcal{D}(V_x)$. Our convolution kernels $x$ and $x'$ being symmetric with respect to $\partial D$, $x \ast \overline{f}(x) = x \ast f(x)$ and

$$x' \ast \overline{f}(x) = x' \ast f(x).$$

For the sake of simplicity, we write $h(x) = V_x f(x)$ in $D$ and $h(x) = 0$ on $\mathbb{R}^n - D$. Then, for a $g \in C_c^+(D)$, we have

$$\int \mathcal{D}(V_x f)(x) g(x) \, dx \, dx$$

$$= \int (x \ast h(x) - x \ast \overline{h}(x)) g(x) \, dx$$

$$= \int h(x) \check{x} \ast g(x) \, dx - \int \overline{h}(x) \check{x} \ast g(x) \, dx$$

$$= \int_D (x' \ast f(x) - x' \ast \overline{f}(x)) \check{x} \ast g(x) \, dx$$

$$- \int_{\mathbb{R}^n - D} (x' \ast \overline{f}(x) - x' \ast f(x)) \check{x} \ast g(x) \, dx$$

$$= \int x' \ast f(x) \check{x} \ast g(x) \, dx - \int x' \ast \overline{f}(x) \check{x} \ast g(x) \, dx$$

$$= \int x \ast x' \ast (f - \overline{f})(x) g(x) \, dx,$$

where $\check{x}$ is the adjoint convolution kernel of $x$; that is, $\check{x}(E) = x(\{-x; \, x \in E\})$ for any Borel set $E$. Since $g$ is arbitrary, we obtain the required equality.

**Remark 4.** In the above lemma, we have $V_x f \in \mathcal{D}(V_x)$ and $V_x(V_x f) = V_x(V_x f)$ provided that $V_x$ is also positive.

**Lemma 5.** Let $x$ be a convolution kernel on $\mathbb{R}^n$. Suppose that $x$ is symmetric with respect to $\partial D$. Then $V_x$ is positive if and only if $\frac{\partial}{\partial x_1} x \leq 0$ in the sense of distributions in $D$. 

Proof. — First we shall show the « if » part. For a $t \in (0, \infty)$, put $H_t = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; x_1 = t\}$ and

$$D' = \left\{ x = (x_1, x_2, \ldots, x_n) \in D; \int_{H_t} dx = 0 \right\}.$$

It suffices to prove that, for any $f \in C_c^\infty(D)$ and any $x \in D'$, $x \ast f(x) \geq x \ast f(\bar{x})$, because $\int_{D-D'} dx = 0$ and

$$x \ast f(\bar{x}) = x \ast f(x).$$

We choose a sequence $(\varphi_k)_{k=1}^\infty$ of non-negative, spherically symmetric and infinitely differentiable functions such that $\int \varphi_k dx = 1$ and that the support of $\varphi_k$, supp $(\varphi_k)$, is contained in $\{x \in \mathbb{R}^n; |x| < 1/k\}$. Then $x \ast \varphi_k$ is symmetric with respect to $\partial D$ and $\frac{\partial}{\partial x_1} x \ast \varphi_k(x) \leq 0$ in

$$\{x \in \mathbb{R}^n; x_1 \geq 1/k\}.$$

Let $f \in C_c^\infty(D)$ and $x = (x_1, x_2, \ldots, x_n) \in D'$. Then

$$\int_{|x_1-x_1| \geq 1/m} f(y) x \ast \varphi_k(x-y) dy \geq \int_{|y_1-x_1| \geq 1/m} f(y) x \ast \varphi_k(\bar{x}-y) dy$$

provided with $0 < m \leq k$. By letting $k \to \infty$ and $m \to \infty$, we obtain that

$$x \ast f(x) = \int f(y) d\hat{x} \ast \varepsilon_x(y)$$

$$\geq \int_{\mathbb{R}^n-H_x} f(y) d\hat{x} \ast \varepsilon_x(y)$$

$$\geq \int_{\mathbb{R}^n-H_x} f(y) d\hat{x} \ast \varepsilon_x(y)$$

$$\geq x \ast f(\bar{x}) - \left( \sup_{z \in \mathbb{R}^n} |f(z)| \right) \int_{H_x} dx = x \ast f(\bar{x})$$

where $\varepsilon_x$ denote the unit measure at $x$. Since $f$ and $x$ are arbitrary, the « if » part is true.

Next we shall show the « only if » part. Suppose that the « only if » part is false. Then there exist a number $t > 0$, a point $x = (x_1, x_2, \ldots, x_n) \in D$ with $x_1 > t$ and a non-negative, spherically symmetric and infinitely differentiable function $\varphi$ in $\mathbb{R}^n$ with supp $(\varphi) \subseteq \{x \in \mathbb{R}^n; |x| < t\}$ such that $\frac{\partial}{\partial x_1} x \ast \varphi(x) > 0$. Hence we can choose a number
s > 0 such that s < x_1 - t and that, for every \( y \in D \) with \( |y| < s \), \( x * \varphi(x - y) < x * \varphi(x - y) \). Since

\[
x * \varphi(x - y) = x * \varphi(x - y),
\]
we have, for an \( f \neq 0 \in C_\infty^+(D) \) satisfying

\[
\text{supp} (f) \subset \{ y \in \mathbb{R}^n; |y| < s \},
\]
\[
x * f * \varphi(x) < x * f * \varphi(x) = x * \bar{f} * \varphi(x).
\]
But this contradicts the inequality \( x * f \geq x * \bar{f} \) in \( D \).
Thus we see that the "only if" part is true.

In the same manner as above, we obtain the following

**Lemma 6.** — Let \( \alpha \) be a positive measure in \( \mathbb{R}^n - \{0\} \).
Suppose that \( \alpha \) is symmetric with respect to \( \partial D \). If \( \frac{\partial}{\partial x_1} \alpha \leq 0 \)
in the sense of distributions in \( D \), then, for any \( f \in C_\infty^+(D) \),

\[
\int f(x - y) \, d\alpha(y) \geq \int \bar{f}(x - y) \, d\alpha(y)
\]
in \( D \cap C \text{ supp} (f) \).

3. We say that a convolution kernel \( x \) on \( \mathbb{R}^n \) is a Hunt convolution kernel if \( x = \int_0^\infty x_t \, dt \), where \( (x_t)_{t \geq 0} \) is a vaguely continuous semi-group of positive measures in \( \mathbb{R}^n \); that is,

\[
\alpha_0 = \delta (\text{the Dirac measure}), \quad \alpha_t * \alpha_s = \alpha_{t+s} (\forall t \geq 0, \forall s \geq 0)
\]
and the application \( \mathbb{R}^+ = [0, \infty) \ni t \to \alpha_t \) is vaguely continuous. In this case, \( (x_t)_{t \geq 0} \) is uniquely determined (see, for example, [3]) and called the vaguely continuous semi-group associated with \( x \). For a \( p \in \mathbb{R}^+ \), put

\[
x_p = \int_0^\infty \exp (-pt) x_t \, dt;
\]
then \( (x_p)_{p \geq 0} \) is called the resolvent associated with \( x \). This is characterized by a family \( (x_p)_{p \geq 0} \) of convolution kernels on \( \mathbb{R}^n \) satisfying

\[
x_p - x_q = (q - p) x_p * x_q (\forall p \geq 0, \forall q > 0)
\]
and \( \lim_{p \to 0} x_p = x_0 = x \) (vaguely).
Lemma 7 (see [3] or Theorem 5 in [6]). — Let \( x, (x_t)_{t \geq 0} \) and \((x_p)_{p \geq 0}\) be the same as above. For a \( p > 0 \) and a \( t > 0 \), put
\[
\alpha_{p, t} = \exp (- pt) \sum_{k=0}^{\infty} \frac{p^k t^k}{k!} (px_p)^k \quad \text{and} \quad \alpha_{p, 0} = \varepsilon;
\]
then \((x_{p, t})_{t \geq 0}\) is a vaguely continuous semi-group of positive measures and we have
\[
x + \frac{1}{p} \varepsilon = \int_0^\infty x_{p, t} \, dt \quad \text{and} \quad \lim_{p \to \infty} x_{p, t} = x_t \quad \text{(vaguely)} \quad (t \geq 0).
\]

Lemma 8. — Let \( x = \int_0^\infty x_t \, dt \) be a Hunt convolution kernel on \( \mathbb{R}^n \) and \((x_p)_{p \geq 0}\) be the resolvent associated with \( x \). If \( x \) is symmetric with respect to \( \partial D \), then, for any \( p \) and any \( t \), \( x_p \) and \( x_t \) are also symmetric with respect to \( \partial D \).

Proof. — For a \( p > 0 \), we denote by \( \bar{x}_p \) the reflection of \( x_p \) about \( \partial D \). Evidently \((\bar{x}_p)_{p \geq 0}\) is the resolvent associated with \( \bar{x} \). By using \( \bar{x} = \bar{x}_0 \) and the unicity of the resolvent associated with \( x \), we have, for each \( p > 0 \), \( x_p = \bar{x}_p \). This means that \( x_p \) is symmetric with respect to \( \partial D \). This gives also that, for any \( f \in C_c(D) \),
\[
\int_0^\infty \exp (- pt) f \, dx_t \, dt = \int_0^\infty \exp (- pt) \bar{f} \, dx_t \, dt \quad (\forall p \geq 0).
\]
The Laplace transformation being injective, we have, for each \( t \geq 0 \), \( \int f \, dx_t = \int \bar{f} \, dx_t \). Hence, \( f \) being arbitrary, we see that \( x_t \) is symmetric with respect to \( \partial D \).

Similarly we have the following

Remark 9. — If \( x \) is symmetric with respect to the origin 0 (resp. spherically symmetric), then \( x_p \) and \( x_t \) are also symmetric with respect to 0 (resp. spherically symmetric).

Let \( x \) be a convolution kernel on \( \mathbb{R}^n \). We say that \( x \) is a Dirichlet convolution kernel if the (generalised) Fourier transformation \( \hat{x} \) of \( x \) is defined and equal to \( \frac{1}{\psi} \), where \( \psi \) is a real-valued negative definite function in \( \mathbb{R}^n \) such that \( \frac{1}{\psi} \)
is locally summable. By virtue of the Lévy-Khinchine theorem, we have, for any \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \),

\[
\psi(x) = c + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j + \int (1 - \cos (2\pi x \cdot y)) \, d\alpha(y),
\]

where \( c \) is a non-negative constant, \( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j \) is a positive semi-definite form, \( x \cdot y \) is the inner product in \( \mathbb{R}^n \) and where \( \alpha \) is a positive measure in \( \mathbb{R}^n \) — {0} symmetric with respect to 0 and satisfying \( \int |x|^2/(1 + |x|^2) \, dx(x) < \infty \).

It is well-known that the above decomposition of \( \psi \) is unique. The positive measure \( \alpha \) in \( \mathbb{R}^n \) — {0} is called the singular measure associated with \( x \). Since, for each \( t \geq 0 \), \( \exp (-t\psi) \) is of positive type in \( \mathbb{R}^n \), there exists a positive measure \( \alpha_t \) in \( \mathbb{R}^n \) such that \( \alpha_t = \exp (-t\psi) \). Evidently \( (\alpha_t)_{t>0} \) is a vaguely continuous semi-group of positive measures and \( \alpha = \int_0^\infty \alpha_t \, dt \). Hence a Dirichlet convolution kernel is a Hunt convolution kernel and symmetric with respect to 0.

4. A positive linear operator \( V : C_c(D) \to C(D) \) is called a continuous kernel on \( D \) (Evidently \( V \) is continuous). Similarly as in the section 2, we define \( \mathcal{D}(V) \) and \( \mathcal{D}(V) \).

We say that \( V \) is a Hunt kernel on \( D \) if \( V = \int_0^\infty \tilde{V}_t \, dt \) (that is, for any \( f \in C_c(D) \), \( Vf(x) = \int_0^\infty \tilde{V}_t f(x) \, dt \) in \( D \)), where \( (\tilde{V}_t)_{t>0} \) is a continuous semi-group of continuous kernels on \( D \); that is, \( \tilde{V}_0 = I \) (the identity), for any \( t \geq 0 \), \( s \geq 0 \) and any \( f \in C_c(D) \), \( \tilde{V}_s \tilde{V}_t f = \tilde{V}_{t+s} f \) and the application \( \mathbb{R}^+ \ni t \to \tilde{V}_t f \) is continuous in \( C(D) \). Similarly as in [3], we see that \( (\tilde{V}_t)_{t>0} \) is uniquely determined, and we call it the continuous semi-group associated with \( V \).

For a \( p \geq 0 \), put \( V_p = \int_0^\infty \exp (-pt) \tilde{V}_t \, dt \); then we call \( (V_p)_{p>0} \) the resolvent associated with \( V \). It is known that, for any \( p \geq 0 \), \( q > 0 \) and any \( f \in C_c(D) \), \( V_pf \in \mathcal{D}(V_q) \), \( V_qf \in \mathcal{D}(V_p) \),

\[
V_pf - V_qf = (q - p)V_q(V_pf) = (q - p)V_p(V_qf)
\]

(the resolvent equation) and \( \lim_{p \to 0} V_pf = V_qf = Vf \) in \( C(D) \).
Let $V_1$ and $V_2$ two continuous kernels on $D$. If, for any $f \in C_c(D)$, $V_2 f \in \mathcal{D}(V_1)$, the application $C_c(D) \ni f \mapsto V_1(V_2 f)$ is positive linear, we denote it by $V_1 \cdot V_2$.

**Remark 10** (see [2]). — A Hunt kernel $V$ on $D$ satisfies the domination principle; that is, for two $f, g \in C_c^+(D)$, $V f \leq V g$ on $\text{supp}(f)$ implies the same inequality on $D$.

5. We shall show Theorem 1 mentioned in the section 1.

(1) $\Rightarrow$ (2). By Lemmas 5 and 8, it suffices to prove that, for each $p > 0$, $V_x^p$ is positive. Let $(V_p^x)_{p \geq 0}$ be the resolvent associated with $V_x$. Then, for an $f \in C_c^+(D)$ and a $p > 0$, $V_x f = (pV_x + I)(V_p f)$. On the other hand, Lemmas 3 and 8 give the $V_x f \in \mathcal{D}(V_x)$ and

$$V_x f = (x * (f - \overline{f}))_D = ((px + \varepsilon) * x_p * (f - \overline{f}))_D = (pV_x + I)(V_x f).$$

By using the resolvent equation, we have

$$V_p f - V_x p f = (I - pV_p)((pV_x + I)(V_p f - V_x f)) = 0.$$

The function $f$ being arbitrary, we have $V_p = V_x^p$, and hence $V_x^p$ is positive.

(2) $\Rightarrow$ (1). By Lemma 5, $V_x^p$ is positive ($\forall p > 0$). Let $\alpha_p$, be the positive measure defined in Lemma 7 ($\forall p > 0, \forall t > 0$) and $(\alpha_t)_{t \geq 0}$ be the vaguely continuous semi-group associated with $x$. By Lemmas 3 and 7,

$$V_{x_p}^t = \exp(-pt) \sum_{k=0}^{\infty} \frac{p^{k} k!}{(pV_{x_p})^k}.$$

where $(pV_{x_p})^0 = I$, $(pV_{x_p})^1 = pV_{x_p}$ and

$$(pV_{x_p})^{n+1} = (pV_{x_p})^n \cdot (pV_{x_p}).$$

Therefore $V_{x_p}^t$ is positive. From Lemma 7, it follows that, for any $f \in C_c(D)$, $\lim_{p \to \infty} V_{x_p}^t f = V_x f$ in $C(D)$ ($\forall t > 0$). Hence $V_x$ is positive. By using Lemma 3, we see that $(V_x^t)_{t \geq 0}$ is a continuous semi-group of continuous kernels on $D$ and that $V_x = \int_{0}^{\infty} V_{x_t} dt$. Consequently $V_x$ is a Hunt kernel on $D$. This completes the proof.
**Question 11.** — Let \( x \) be a Hunt convolution kernel on \( \mathbb{R}^n \) satisfying \( x = \overline{x} \). Is it true that \( V_x \) is a Hunt kernel on \( D \) provided that \( V_x \) is positive?

**Remark 12.** — Let \( k(x) \) be a non-negative continuous function in the wide sense in \( \mathbb{R}^n \) satisfying \( k(x) = k(\overline{x}) \). Suppose that \( \lambda = k(x) \, dx \) is a Hunt convolution kernel and that \( V_x \) is also a Hunt kernel on \( D \). Put

\[
G(x,y) = k(x-y) - k(x-y) \quad \text{in} \quad D \times D.
\]

If the function kernel \( k(x-y) \) satisfies the continuity principle (3), then \( G \) satisfies the domination principle; that is, for two positive measures \( \mu \) and \( \nu \) in \( D \) with compact support and with \( \int G \mu \, d\mu < \infty \), then \( G\mu \leq G\nu \) on \( \text{supp} (\mu) \) implies the same inequality in \( D \), where

\[
G\mu(x) = \int G(x,y) \, d\mu(y).
\]

It is known that \( k(x-y) \) satisfies the continuity principle when \( x \) is a Dirichlet convolution kernel (see [4]).

We show this remark. We see that \( G \) also satisfies the continuity principle. Therefore it suffices to prove that, for a positive measure \( \mu \) in \( D \) with compact support and an \( x \in D \), \( G\mu \leq G\varepsilon_x \) in \( D \) provided that \( G\mu \leq G\varepsilon_x \) on \( \text{supp} (\mu) \) and that \( G\mu \) is finite continuous (see [8]). Since \( V_x \) is a Hunt kernel, there exists \( f \in C_c^\infty(D) \) such that \( V_xf = Gf = 1 \) on \( \text{supp} (\mu) \), where \( Gf(x,z) = \int G(y,z)f(z) \, dz \). Here we remark that \( \mu \) is considered as a positive measure in \( \mathbb{R}^n \). For a given positive number \( \delta \), there exists a neighborhood \( U \) of \( 0 \) such that, for any finite continuous function \( \varphi \geq 0 \) in \( \mathbb{R}^n \) with \( \text{supp} (\varphi) \subset U \) with \( \int \varphi \, dx = 1 \), \( \mu * \varphi, \varepsilon_x * \varphi \in C_c^\infty(D) \) and \( G(\mu * \varphi) \leq G(\varepsilon_x * \varphi) + \delta Gf \) on \( \text{supp} (\mu * \varphi) \). By letting \( \varphi \rightarrow \varepsilon \) (vaguely) and \( \delta \downarrow 0 \), we have \( G\mu \leq G\varepsilon_x \).

(3) This means that, for a positive measure \( \mu \) in \( \mathbb{R}^n \) with compact support, the function \( \int k(x-y) \, d\mu(y) \) of \( x \) is finite continuous provided that its restriction to \( \text{supp} (\mu) \) is finite continuous.
6. Theorem 1 gives the following

**Corollary 13.** — Let \( x = \int_0^\infty \alpha_t \, dt \) be a Hunt convolution kernel on \( \mathbb{R}^n \). Then \( x \) is symmetric with respect to \( \partial D \) and \( V_x \) is a Hunt kernel on \( D \) if and only if, for each \( t \geq 0 \), \( \alpha_t \) is symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} \alpha_t \leq 0 \) in the sense of distribution in \( D \).

**Corollary 14.** — Let \( x = \int_0^\infty \alpha_t \, dt \) be a Hunt convolution kernel on \( \mathbb{R}^n \) and \( \mu \) be a Hunt convolution kernel on \( \mathbb{R}^1 \) supported by \( \mathbb{R}^+ \). Suppose that \( x_\mu = \int_0^\infty \alpha_t \, d\mu(t) \) is defined (in the sense of measures) and that \( x \) is symmetric with respect to \( \partial D \). If \( V_x \) is a Hunt kernel on \( D \), then \( V_{x_\mu} \) is also a Hunt kernel on \( D \).

**Proof.** — We denote by \( (\mu_p)_{p \geq 0} \) the resolvent associated with \( \mu \). Since \( \mu_p \leq \mu \), \( x_{\mu, p} = \int \alpha_t \, d\mu_p(t) \) is defined \( (\forall p \geq 0) \). It is known that \( x_{\mu, p} \) is a Hunt convolution kernel on \( \mathbb{R}^n \) and that \( (x_{\mu, p})_{p \geq 0} \) is the resolvent associated with \( x_\mu \) (see Theorem 1 in [5]). By Theorem 1 and Corollary 13, \( \alpha_t \) is symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} \alpha_t \leq 0 \) in the sense of distributions in \( D \). Hence \( x_\mu \) is also symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} x_{\mu, p} \leq 0 \) in the sense of distributions in \( D \) \( (\forall p \geq 0) \). Consequently Theorem 1 gives this corollary.

In the same manner as above, we have the following

**Corollary 15.** — Let \( (\alpha_t)_{t \geq 0} \) be a vaguely continuous semi-group of positive measures in \( \mathbb{R}^n \) and \( \mu \) be a Hunt convolution kernel on \( \mathbb{R}^1 \) supported by \( \mathbb{R}^+ \). Suppose that \( \int_0^\infty \alpha_t \, d\mu(t) \) is defined and that, for each \( t \geq 0 \), \( \alpha_t \) is symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} \alpha_t \leq 0 \) in the sense of distributions in \( D \). Then \( V_{x_\mu} \) is a Hunt kernel on \( D \), where \( x_\mu = \int_0^\infty \alpha_t \, d\mu(t) \).
We shall show that the question raised by H. L. Jackson is affirmatively solved.

**Remark 16.** — Let \( \nu \) be a positive measure in \((0, 2)\) such that \( \int_0^2 \frac{1}{x} \, d\nu(x) < \infty \) and \( c_0, c_1 \) be non-negative constants. Put

\[
\varphi = \begin{cases} 
   c_0 x + \left( \int |x|^{\alpha-n} \, d\nu(x) \right) dx & \text{if } n = 2 \\
   c_0 x + \left( \int |x|^{\alpha-n} \, d\nu(x) + c_1 |x|^{2-n} \right) dx & \text{if } n \geq 3.
\end{cases}
\]

Then \( \varphi \) is a Hunt kernel.

In fact, we have, with a positive constant \( c(\alpha) \),

\[
|x|^{\alpha-n} = c(\alpha) \int_0^\infty \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{2t} \right) t^{\alpha-1} \, dt
\]

\((0 < \alpha < 2 \text{ if } n = 2, 0 < \alpha \leq 2 \text{ if } n \geq 3)\). Evidently the function \( c(\alpha) \) of \( \alpha \) is finite continuous. Put

\[
\mu = \begin{cases} 
   c_0 x + \left( \int c(x) t^{2/2-1} \, d\nu(x) \right) dt & \text{if } n = 2 \\
   c_0 x + \left( \int c(x) t^{2/2-1} \, d\nu(x) + c_1 c(2) \right) dt & \text{if } n \geq 3
\end{cases}
\]

in \( \mathbb{R}^1 \). Since \( \int_0^2 \frac{1}{x} \, d\nu(x) < \infty \), \( \varphi \) is a convolution kernel on \( \mathbb{R}^n \) and

\[
\varphi = \left( \int \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{2t} \right) d\mu(t) \right) dx.
\]

Hence \( \mu \) is a convolution kernel on \( \mathbb{R}^1 \) supported by \( \mathbb{R}^+ \). Then \( \mu \) is a Hunt convolution kernel on \( \mathbb{R}^1 \) (cf. [5]), and Corollary 14 gives our remark.

Let \( G_x \) be the Green type kernel of order \( \alpha \) in \( D \). Put

\[
G(x,y) = \begin{cases} 
   \int G_x(x,y) \, d\nu(x) & \text{if } n = 2 \\
   \int G_x(x,y) \, d\nu(x) + c_1 G_x(x,y) & \text{if } n \geq 3.
\end{cases}
\]

Then Remarks 12 and 16 give that \( G \) satisfies the domination principle.

7. Let \( L_{\text{loc}}(D) \) be the usual Fréchet space of real-valued locally summable functions in \( D \). A Hilbert space \( H(D) \)
 contained in $L_{\text{loc}}(D)$ is called a Dirichlet space on $D$ if the following three conditions are satisfied:

1. For each compact set $K$ in $D$, there exists a constant $A(K) > 0$ such that, for any $u \in D$, $\int_K |u| \, dx \leq A(K) \|u\|$.

2. $C_c(D) \cap H(D)$ is dense both in $C_c(D)$ and in $H(D)$.

3. For any normalized contraction $T$ on $R^1$ and any $u \in H(D)$, $T \cdot u \in H(D)$ and $\|T \cdot u\| \leq \|u\|$.

This is the definition by A. Beurling and J. Deny (see [1]). Here we denote by $\|\cdot\|$ and by $(\cdot, \cdot)$ the norm in $H(D)$ and the associated inner product, respectively. For an $f \in C_c(D)$, (1) gives that there exists uniquely $u^*_f \in H(D)$ such that, for any $u \in H(D)$, $(u^*_f, u) = \int uf \, dx$.

Let $V$ be a linear operator from $C_c(D)$ into $L_{\text{loc}}(D)$. We say that $V$ is a Dirichlet kernel on $D$ if there exists a Dirichlet space $H(D; V)$ on $D$ such that, for any $f \in C_c(D)$, $Vf = u^*_f$.

Evidently $H(D; V)$ is uniquely determined. We call $H(D; V)$ the Dirichlet space associated with $V$ and $V$ the kernel of $H(D; V)$. For a Dirichlet kernel $V$ on $D$, we set

$$\mathcal{D}(V) = \left\{ f \in L_{\text{loc}}(D); \sup_{u \neq 0 \in C_c(D) \cap H(D; V)} \left\{ \frac{\int uf \, dx}{\|u\|} \right\} < \infty \right\}$$

and $\mathcal{D}^+(V) = \{ f \in \mathcal{D}(V); f \geq 0 \}$, where $\|\cdot\|$ denote the norm in $H(D; V)$. By virtue of (2), for an $f \in \mathcal{D}(V)$, there exists uniquely $Vf \in H(D; V)$ such that, for any $u \in C_c(D) \cap H(D; V)$, $(Vf, u) = \int uf \, dx$.

where $(\cdot, \cdot)$ denote the inner product in $H(D; V)$. Thus $V$ may be considered as a linear operator from $\mathcal{D}(V)$ into $H(D; V)$. It is known that $V$ is positive (that is),

$$f \in \mathcal{D}^+(V) \implies Vf \geq 0 \text{ a.e.} \quad (\text{see [1]}) .$$

(4) This means that $T$ is an application: $R^1 \to R^1$ such that $T(0) = 0$ and $|Ta - Tb| \leq |a - b|$ $(\forall a, b \in R^1)$. 
Lemma 17. — Let $x$ be a Hunt convolution kernel on $\mathbb{R}^n$ satisfying $x = \overline{x}$. If $V_x$ is a Dirichlet kernel on $D$, then $V_x$ is a Hunt kernel.

Proof. — For the sake of simplicity, we write $H = H(D; V_x)$. Denote by $\| \cdot \|$ and by $(\cdot, \cdot)$ the norm in $H$ and the inner product in $H$, respectively. Let $L^2(D)$ be the Hilbert space of real-valued square summable functions in $D$. For a $p \geq 0$, $H_p$ denotes the Hilbert space associated to the norm $\| u \|_p = (p \int |u|^2 \, dx + \| u \|^2)^{1/2}$ on $H \cap L^2(D)$. Evidently $H_p$ is a Dirichlet space on $D$. Let $f \in C_c(D)$. For any $u \in C_c(D) \cap H$, we have

$$\int V_p f(x) u(x) \, dx = \frac{1}{p} \left( (V_p f, u) - (V_p f, u) \right)$$

$$= \frac{1}{p} \left( (V_x f, u) - (V_p f, u) \right)$$

$$\leq \frac{1}{p} \left( \| V_x f \| + \| V_p f \| \right) \| u \|,$$

where $V_p$ is the kernel of $H_p$ and where $(\cdot, \cdot)_p$ is the inner product in $H_p$. Hence $V_p f \in D(V)$. Since, for any $u \in C_c(D) \cap H$,

$$p(V_x(V_p f), u) = p \int u(x) V_p f(x) \, dx$$

$$= (V_p f, u) - (V_p f, u) = (V_x f - V_p f, u),$$

(2) gives $V_x f - V_p f = p V_x(V_p f)$ a.e. in $D$. Let $(x_p)_{p \geq 0}$ be the resolvent associated with $x$. By Lemmas 3 and 8, we have $V_x f - V_{x_p} f = p V_x(V_{x_p} f)$. In the same manner as in the proof of Theorem 1, we have $V_p f = V_{x_p} f$ a.e. in $D$, and hence $V_{x_p}$ is positive ($\forall p > 0$). By Theorem 1 and Lemma 5, we see that $V_x$ is a Hunt kernel.

We shall prove Theorem 2 mentioned in the section 1.

$(1) \implies (2)$. Let $(x_p)_{p \geq 0}$ be the resolvent associated with $x$. Then it is known that $p^2 x_p \to x$ vaguely in $\mathbb{R}^n - \{0\}$ as $p \to \infty$ (see [1]), and hence theorem 1 and Lemma 17 give that $\frac{\partial}{\partial x_1} x \leq 0$ in the sense of distributions in $D$.

$(2) \implies (1)$. Since $p^2 x_p \to x$ vaguely in $\mathbb{R}^n - \{0\}$ as $p \to \infty$, Lemma 8 gives that $x$ is symmetric with respect to $\partial D$. Let $A$ be the diagonal set of $D \times D$ and $\beta$ be the
positive measure in $D \times D - A$ defined by 
\[ \iint f(x)g(y) \, d\beta(x,y) = \iint (f(x-y) - \bar{f}(x-y))g(x) \, dx(x) \, dy(y) \]
for any couple $f, g \in C_c(D)$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$ (see Lemma 6). For any $p, \alpha$ being symmetric with respect to the origin, we have $\alpha = \bar{\alpha}$, and hence $\beta$ is symmetric with respect to $A$. Let $C_c^\infty(D)$ be the topological vector space of real-valued and infinitely differentiable functions in $D$ with compact support (we identify an element of $C_c^\infty(D)$ and an infinitely differentiable function in $\mathbb{R}^n$ with compact support in $D$).

Let $f \in C_c^\infty(D)$. Consider the approximation of the function $|f(x) - f(y)|^2$ of $(x,y)$ by the functions of form $\sum_i \varphi_i(x)\psi_i(y)$ in $D \times D$, where $\varphi_i \in C_c^\infty(D)$ and $\psi_i \in C_c^\infty(D)$ with 
\[ \text{supp}(\varphi_i) \cap \text{supp}(\psi_i) = \emptyset. \]

Then we see that 
\[ 0 \leq \iint |f(x) - f(y)|^2 \, d\beta(x,y) + \int |f(x)|^2 \, a(x) \, dx 
= \iint |f(x - y) - f(x)|^2 \, dx(x) \, dy(y) 
- \iint (\bar{f}(x-y) - \bar{f}(x))(f(x-y) - f(x)) \, dx(x) \, dy(y) < \infty \]
where, for $x = (x_1, x_2, \ldots, x_n) \in D$, $a(x) = 2 \int_{y_i \geq x_i} dx(y)$.

Let $\tilde{H}$ be the specialized Dirichlet space with the kernel $x$ (see [1]). We denote by $||| \cdot |||$ and by $\langle (\cdot, \cdot) \rangle$ the norm in $\tilde{H}$ and the associated inner product. For a couple $f, g \in C_c^\infty(D)$, we put 
\[ (f,g) = \int fg \left( \frac{a}{2} + c \right) \, dx + \frac{1}{4\pi^2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \, dx \]
\[ + \frac{1}{2} \int \int (f(x) - f(y))(g(x) - g(y)) \, d\beta(x,y) = \langle (f - \bar{f}, g) \rangle = \frac{1}{2} \langle (f - \bar{f}, g - \bar{g}) \rangle, \]

(6) The author would like to express his hearty thanks to Prof. F. Hirsch for the correction of this formula.
where $\hat{x} = \left( c + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j} + \int (1 - \cos (2\pi x \cdot y)) \, dx(y) \right)^{-1}$. Then $(\cdot, \cdot)$ is an inner product in $C^\infty_c(D)$. For a compact set $K$ in $D$, we have

$$\sup_{u \in C^\infty_c(D), u \neq 0} \frac{\int_k |u| \, dx}{\|u\|} = \sup_{u \in C^\infty_c(D), u \neq 0} \frac{\sqrt{2} \int_k |u - \bar{u}| \, dx}{\|u - \bar{u}\|} < \infty,$$

where $\|u\| = (u, u)^{1/2}$. Hence the completion $H$ of $C^\infty_c(D)$ by $\| \cdot \|$ is contained in $L^1(D)$. Evidently, for any $u \in C^\infty_c(D)$ and any normalized contraction $T$ on $\mathbb{R}^1$, $T \cdot u \in H$ and $\|T \cdot u\| \leq \|u\|$. For a $u \in H$, we choose a sequence $(u_k)_{k=1}^\infty \subseteq C^\infty_c(D)$ such that

$$\lim_{k \to \infty} \|u_k - u\| = 0.$$ 

Since $(T \cdot u_k)_{k=1}^\infty$ converges weakly to $T \cdot u$ in $H$ as $k \to \infty$ (see [1]), we have $T \cdot u \in H$ and $\|T \cdot u\| \leq \|u\|$. Hence $H$ is a Dirichlet space on $D$. We shall show that $V_x$ is the kernel of $H$. For an integer $m \geq 1$, let $T_m$ denote the projection from $\mathbb{R}^1$ into $\left[ -\frac{1}{m}, \frac{1}{m} \right]$. Let $f \in C_c(D)$; then $x \ast (f - \bar{f}) - T_m \cdot x \ast (f - \bar{f}) \in H$ and

$$V_x f - T_m \cdot V_x f \in C_c(D),$$

because $x \ast (f - \bar{f}) = 0$ on $\partial D$ and $\lim_{|x| \to \infty} x \ast (f - \bar{f})(x) = 0$. Therefore there exists a neighborhood $V_m$ of the origin such that, for any non-negative, spherically symmetric and infinitely differentiable function $\varphi$ in $\mathbb{R}^1$ with $\text{supp} \, (\varphi) \subseteq V_m$ and $\int \varphi \, dx = 1$, $f \ast \varphi \in C^\infty_c(D)$ and

$$(V_x f - T_m \cdot V_x f) \ast \varphi \in C^\infty_c(D).$$

Since

$$(x \ast (f - \bar{f}) - T_m \cdot x \ast (f - \bar{f})) \ast \varphi = (V_x f - T_m \cdot V_x f) \ast \varphi - (V_x f - T_m \cdot V_x f) \ast \varphi$$

and, for a $u \in \tilde{H}$,

$$\|u \ast \varphi\|^2 = \iint ((u \ast \varepsilon_x, u \ast \varepsilon_y)) \varphi(x) \varphi(y) \, dx \, dy \leq \|u\|^2,$$
we have
\[ \| (V_x f - T_m \cdot V_x f) \star \varphi \|^2 \]
\[ \leq \frac{1}{2} \| | x \star (f - \bar{f}) - T_m \cdot x \star (f - \bar{f}) | |^2 \leq 2 \| | x \star (f - \bar{f}) | |^2. \]

By letting \( \varphi \, dx \to \varepsilon \) (vaguely) and \( m \to \infty \), we see that \( V_x f \in H \) and, for any \( u \in C_c^\infty (D) \),
\[ (V_x f, u) = (x \star (f - \bar{f}), u) = \int u(f - \bar{f}) \, dx = \int uf \, dx. \]
This implies immediately that, for any \( u \in H \),
\[ (V_x f, u) = \int uf \, dx. \]
Consequently \( V_x \) is the kernel of the Dirichlet space \( H \). This completes the proof.

Theorem 2 gives also that the question raised by H. L. Jackson is affirmatively solved. In fact, the singular measure associated with the convolution kernel \( r^{2-\alpha} \) is equal to \( c_\alpha |x|^{-\alpha} \, dx \) provided that \( 0 < \alpha < 2 \), where \( c_\alpha \) is a positive constant, where \( |x|^{2-\alpha} \, dx \) is symbolically denoted by \( r^{2-\alpha} \) \((0 < \alpha < n)\).

We denote now by \( \Delta \) the laplacian on \( \mathbb{R}^n \). We say that a convolution kernel \( \kappa \) on \( \mathbb{R}^n \) is a Frostman-Kunugui kernel if \( \kappa \) is spherically symmetric, vanishes at infinity \((^6)\), and if \( \Delta \kappa \geq 0 \) in the sense of distributions outside the origin \( 0 \). Theorem 2 and Theorem 1 in [7] give the following

**Corollary 18.** — Suppose \( n \geq 3 \). Then the following two statements hold.

(1) For a Frostman-Kunugui kernel \( \kappa \neq 0 \) on \( \mathbb{R}^n \) satisfying
\[ \frac{\partial}{\partial x_1} \Delta \kappa \leq 0 \] in the sense of distributions in \( D \), there exists uniquely a spherically symmetric Dirichlet convolution kernel \( \kappa' \) on \( \mathbb{R}^n \) such that \( V_{\kappa'} \) is a Dirichlet kernel on \( D \) and that, for any \( f \in C_c (D) \),
\[ V_x (V_{\kappa'} f)(x) = V_x (V_\kappa f)(x) = G_\kappa f(x) \] in \( D \).

(2) For a spherically symmetric Dirichlet kernel \( \kappa \) on \( \mathbb{R}^n \) such that \( V_\kappa \) is a Dirichlet kernel on \( D \), there exists uniquely

\(^6\) This means that, for any finite continuous function \( f \) in \( \mathbb{R}^n \) with compact support, \( \kappa \star f(x) \to 0 \) as \( |x| \to \infty \).
a Frostman-Kunugui kernel $x'$ on $\mathbb{R}^n$ such that $\frac{\partial}{\partial x_1} \Delta x \leq 0$
in the sense of distributions in $D$ and that, for any $f \in C_c(D)$, $V_x(V_x f)(x) = V_x(V_x f)(x) = G_x f(x)$ in $D$.

Proof. — First we shall show (1). By Theorem 1 in [7], there exists uniquely a spherically symmetric Dirichlet kernel $x'$ on $\mathbb{R}^n$ such that $x \ast x' = r^{2-n}$. We have, with a positive constant $c$, $(\Delta x) \ast x' = -c \varepsilon$ in the sense of distributions in $\mathbb{R}^n$. This implies that the singular measure associated with $x$ is equal to $\frac{1}{c} \Delta x$ outside 0. Theorem 2 and our assumption give that $V_x f$ is a Dirichlet kernel on $D$. Since $\Delta x \geq 0$ in the sense of distributions in $\mathbb{R}^n - \{0\}$ and $x$ vanishes at infinity, $\frac{\partial}{\partial x_1} x \leq 0$ in the sense of distributions in $D$. By Lemma 5, $V_x$ is positive, and by Lemma 3 and Remark 4, we obtain the required equality. Let's show the uniqueness of $x'$. Let $x''$ be a Dirichlet convolution kernel on $\mathbb{R}^n$ which is possessed of the same properties as of $x$. Since $x$ is injective (see Theorem 1 in [7]) (7) and

$$x \ast (V_x f - V_x f) = x \ast (V_x f - V_x f)$$
in $\mathbb{R}^n$ (8), we have $V_x f = V_x f$ ($\forall f \in C_c(D)$). This implies that, for any $f \in C_c(D)$, $(x' - x'') f = (x' - x'') \ast f$. In the same manner as in Lemma 5, we have $\frac{\partial}{\partial x_1} (x' - x'') = 0$ in the sense of distributions in $D$. Since $x' - x''$ is spherically symmetric and vanishes at the infinity, we have $x' = x''$. Thus we see that (1) holds.

Next we shall show (2). By Theorem 1 in [7], there exists uniquely a Frostman-Kunugui kernel $x'$ on $\mathbb{R}^n$ such that $x \ast x' = r^{2-n}$. Since the singular measure associated with $x$ is equal to $\frac{1}{c} \Delta x'$ outside 0, Theorem 2 gives that

$$\frac{\partial}{\partial x_1} \Delta x' \leq 0$$
in the sense of distributions in $D$. Similarly as

(7) This means that, for an $f \in C(D)$, $f = 0$ provided that $x \ast |f|$ is defined and that $x \ast f = 0$.

(8) We may assume that $V_x f$ is a continuous function in $\mathbb{R}^n$ with support $\subset \bar{D}$. 
above, we see that $V_{\alpha'}$ is positive and the required equality holds. Since $\alpha$ is also injective (see, for example, [1]), we can similarly show the uniqueness of $\alpha'$.

Remember the Riesz decomposition formula

$$r^{a-n} \ast r^{(2-a)-n} = a_\alpha r^{2-n} \quad (0 < \alpha < 2),$$

where $a_\alpha$ is a positive constant (see [9]). Then, by this corollary, we see that $G_\alpha$ satisfies the domination principle provided with $n \geq 3$ and $0 < \alpha < 2$.

**Remark 19.** — For a spherically symmetric convolution kernel $\alpha$ on $\mathbb{R}^n$, $\frac{\partial}{\partial x_i} \alpha \leq 0$ in the sense of distributions in $D$ if and only if $\frac{\partial}{\partial r} \alpha \leq 0$ in the sense of distributions in $\mathbb{R}^n - \{0\}$, where $r = |x|$. In this case, $\alpha$ is absolutely continuous outside $0$.

By using Theorem 1, Corollary 13 and this remark 19, we have the following

**Remark 20.** — Let $\alpha = \int_0^\infty \alpha_t dt$ be a spherically symmetric Dirichlet kernel on $\mathbb{R}^n$. Then $V_\alpha$ is a Dirichlet kernel on $D$ if and only if, for any $t \geq 0$, $\alpha_t$ is of form

$$\alpha_t = c_t e + k_t(|x|) dx,$$

where $c_t$ is a non-negative constant and $k_t$ is a non-negative decreasing (in the wide sense) function on $\mathbb{R}^+$.

8. First we shall show that the inverse of the question raised by H. L. Jackson is also affirmative.

**Proposition 21.** — If the Green type kernel $G_\alpha (0 < \alpha < n)$ on $D$ satisfies the domination principle, then $0 < \alpha \leq 2$.

**Proof.** — Since $G_\alpha$ satisfies the domination principle, $G_\alpha$ also satisfies the balayage principle (see, for example, [8]); that is, for a positive measure $\mu$ in $D$ with compact support and a compact set $F$ in $D$, there exists a positive measure $\mu_F^\prime$ supported by $F$ such that $G_\alpha \mu \geq G_\alpha \mu_F^\prime$ in $D$ and
G_\alpha \mu = G_\alpha \mu_F \text{ G}_\alpha \text{-n.e. on } F \quad (\ast). \text{ Let } \mu \neq 0 \text{ and } F \text{ be a closed ball contained in } D \text{ such that } \text{supp } (\mu) \cap F = \emptyset. \text{ Suppose that } \alpha > 2. \text{ Let } t \text{ be positive integer satisfying } 0 < \alpha - 2t \leq 2 \text{ and } \beta = \alpha - 2t. \text{ Then}

\begin{align*}
G_\alpha(x,y) &= \int G_{2t}(x,z)G_\beta(z,y) \, dz \\
(\text{see Lemma 3}). \text{ Since } G_{2t}(G_\beta \mu) = G_{2t}(G_\beta \mu_F) \text{ a.e. on } F, \text{ we have } G_\beta \mu = G_\beta \mu_F \text{ a.e. on } F, \text{ because } \\
\Delta^t(G_{2t}(G_\beta \mu) - G_{2t}(G_\beta \mu_F)) &= (-c)^t(G_\beta \mu - G_\beta \mu_F)
\end{align*}

in the sense of distributions in } D, \text{ where } c \text{ is the positive constant satisfying } \Delta r^{2-n} = -c \varepsilon. \text{ Since } G_\beta \mu \text{ is continuous on } F \text{ and } G_\beta \mu_F \text{ is lower semi-continuous, we have } G_\beta \mu \geq G_\beta \mu_F \text{ on } F, \text{ and so } \int G_\beta \mu_F \, d\mu_F < \infty. \text{ The function kernel } G_\beta \text{ satisfying the domination principle, we have } G_\beta \mu \geq G_\beta \mu_F \text{ in } D. \text{ By virtue of the injectivity of } G_\beta, \text{ we have } G_\beta \mu \neq G_\beta \mu_F. \text{ But this contradicts the equality } G_{2t}(G_\beta \mu) = G_{2t}(G_\beta \mu_F) \text{ G}_\alpha \text{-n.e. on } F. \text{ Thus we achieve the proof.}

We raise a question.

**Question 22.** — Let \( x \) be a convolution kernel on \( \mathbb{R}^n \) satisfying \( x = \overline{x} \). Suppose that \( V_x \) is a Hunt kernel on \( D \). Then is it true that \( x \) is the sum of a Hunt convolution kernel and of a non-negative constant ?

The following proposition shows that the answer is «yes» in a special case.

**Proposition 23.** — Let \( x \) be a convolution kernel on \( \mathbb{R}^n \) satisfying \( x = \overline{x} \). Suppose that \( V_x \) is a Hunt kernel on \( D \). If \( \int dx < \infty \) and \( x \) is absolutely continuous outside 0, then \( x \) is a Hunt convolution kernel.

**Proof.** — We may assume that \( \int dx < 1 \). For a \( p \in (0,1] \), we put

\[ x_p = \sum_{k=0}^{\infty} (-p)^k(x)^{k+1}; \]

(\ast) We write \( G_\lambda \mu = G_\lambda \mu_F \text{ G}_\lambda \text{-n.e. on } F \) if, for any positive measure \( \nu \) in \( D \) with \( \text{supp } (\nu) \subset F \) and \( \int G_\lambda \nu \, d\nu < \infty, \int G_\lambda \mu \, d\nu = \int G_\lambda \mu_F \, d\nu. \)
then \( x_p \) is a real measure in \( \mathbb{R}^n \), absolutely continuous outside 0, \( x_p = \bar{x}_p \) and \( \int d|\bar{x}_p| < \infty \), where \( |x_p| \) denote the total variation of \( x_p \). Since \((p\lambda + \varepsilon) * x_p = \lambda \), Lemma 3 gives that, for any \( f \in C_c(D) \), \((pV_x + 1)(V_{x_p}f) = V_xf \). Let \((V_{x_p})_{p \geq 0} \) the resolvent associated with \( V_x \). In the same manner as in Theorem 1, we have, for any \( f \in C_c(D) \), \( V_{x_p}f = V_xf \) in \( D \). Hence \( V_{x_p} \) is positive. In the same manner as in Lemma 5, we have \( \frac{d}{dx_1} x_p \leq 0 \) in the sense of distributions in \( D \). We show that \( x_p \) is a convolution kernel. It suffices to prove that, for any \( f \in C_c(D) \), \( \int_D f(x) \bar{x}_p(x) \geq 0 \), because

\[
\mathcal{X}_p(\{0\}) = \frac{x(\{0\})}{1 + px(\{0\})} \geq 0, \quad x_p = \bar{x}_p
\]

and \( x_p \) is absolutely continuous outside 0. For each integer \( k \geq 1 \), we choose a non-negative, spherically symmetric and infinitely differentiable function \( \varphi_k \) in \( \mathbb{R}^n \) such that \( \int \varphi_k \, dx = 1 \) and \( \text{supp} \, (\varphi_k) \subset \{ x \in \mathbb{R}^n; \, |x| < \frac{1}{k} \} \). Since

\[
\frac{d}{dx_1} x_p * \varphi_k(x) \leq 0 \quad \text{in the set}
\]

\[
\left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; \, x_1 \geq \frac{1}{k} \right\}
\]

and \( \lim_{|x| \to \infty} x_p * \varphi_k(x) = 0 \), we have \( x_p * \varphi_k(x) \geq 0 \) in the above set. Hence, for any \( f \in C_c(D) \),

\[
\int_D f(x) \, dx_p = \lim_{k \to \infty} \int_{x_1 \geq \frac{1}{k}} f(x) \left( x_p * \varphi_k(x) \right) \, dx \geq 0.
\]

Consequently \( x_p \) is a convolution kernel \( (\forall p \in (0,1]) \). Since \( x - x_p = p* x_p \), \( x \geq x_p \). For a \( p \in (1, 2] \), we put

\[
x_p = \sum_{k=0}^{\infty} (1 - p)^k (x_1)^{k+1};
\]

then \( x_p \) is also a real measure in \( \mathbb{R}^n \), absolutely continuous outside 0, \( x_p = \bar{x}_p \), \( \int d|\bar{x}_p| < \infty \) and \( x - x_p = p* x_p \). In the same manner as above, \( x_p \) is a convolution kernel. Inductively we obtain a family \( (x_p)_{p \geq 0} \) of convolution ker-
nels satisfying \( x - x_p = px \ast x_p \) and \( \lim_{p \to 0} x_p = x \) (vaguely). By Lemma 3.2 in [6], we obtain that, for each \( p \geq 0 \) and \( q > 0 \), \( x_p - x_q = (q - p)x_p \ast x_q \) and \( \lim_{p \to 0} x_p = x \) (vaguely), where \( x_0 = x \). Since \( V_x \) is a Hunt kernel on \( D, \ x \neq 0 \), and hence, for any \( x \neq 0 \in \mathbb{R}^n, x \neq x \ast \varepsilon_{x} \), because
\[
\lim_{|x| \to \infty} x \ast f(x) = 0
\]
for any finite continuous function \( f \) in \( \mathbb{R}^n \) with compact support. Hence, by Corollary 1 of Theorem 5 in [6], \( x \) is a Hunt convolution kernel. This completes the proof.

**Remark 24.** — In the above proposition, if \( x \) is spherically symmetric, the same conclusion holds without the assumption that \( x \) is absolutely continuous outside \( 0 \). See Remark 19.

**BIBLIOGRAPHY**


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