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THE LEVI PROBLEM FOR DOMAINS SPREAD OVER LOCALLY CONVEX SPACES WITH A FINITE DIMENSIONAL SCHAUDER DECOMPOSITION (*)

by Martin SCHOTTENLOHER

Introduction.

In this article it is shown that for certain locally convex Hausdorff spaces E over **C** with a finite dimensional Schauder decomposition (for example for Fréchet spaces with a Schauder basis) the Levi problem has a solution, i.e. every pseudoconvex domain spread over E is a domain of existence.

The Levi problem for infinite dimensional spaces has previously been investigated for less general situations by several authors (cf. list of references). Their common method is the following: For a suitable sequence (x_n) in a pseudoconvex domain $\Omega \subset E$ with the property that the cluster points of (x_n) are dense in the boundary $\partial\Omega$ of Ω, an analytic function $f: \Omega \to \mathbf{C}$ is constructed which is unbounded on sufficiently many subsequences of (x_n). Non-schlicht domains have been studied in a similar manner (cf. [10]).

A different method is presented in this paper. We show that a pseudoconvex domain Ω spread over a metrizable, locally convex space E with an equicontinuous finite dimensional Schauder decomposition (cf. Section 2 for the definition) has a strong convexity property which is expressed in terms

*) This article constitutes a part of the author's « Habilitationsschrift », cf. [31].

of suitable subalgebras (namely « regular classes », cf. [26]) of the algebra $\mathcal{O}(\Omega)$ of all analytic functions on Ω. A characterization given in [26] implies that Ω is a domain of existence. Similar to [5], where only schlicht domains are considered, the result extends to non-metrizable spaces E which satisfy certain countability conditions, for instance it extends to hereditary Lindelöf spaces with an equicontinuous finite dimensional Schauder decomposition. In particular, a pseudoconvex domain spread over a Silva space (called « LS-Raum » in [6]) with a finite dimensional Schauder decomposition is a domain of existence.

The method presented here also provides a tool to prove an approximation theorem of the Oka-Weil type for pseudoconvex domains, thereby generalizing and strengthening results of Noverraz [20]. Moreover, we prove that a pseudoconvex domain spread over a Fréchet space (resp. a Silva space) with a finite dimensional Schauder decomposition is holomorphically convex.

The Levi problem remains unanswered for arbitrary separable, locally convex spaces since a Banach space with a finite dimensional Schauder decomposition has the bounded approximation property, and not every separable Banach space has the bounded approximation property (cf. [7]). For the non-separable case counterexamples are known (cf. [13]).

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0. Notations and Preliminaries.

Throughout this paper let E be a locally convex Hausdorff space over the field $\mathbf{C}$ of complex numbers. The set of nontrivial continuous seminorms on E will be denoted by cs(E). For $\alpha \in \mathrm{cs(E)}$, $y \in \mathrm{E}$ and $r > 0$ the *α-ball* with *radius* r and *center* y is $\mathrm{B}_{\mathrm{E}}^{\alpha}(y, r) = \{x \in \mathrm{E} | \alpha(x-y) < r\}$, and the *line segment* of *length* r in the *direction* of $a \in \mathrm{E}$ is $\mathrm{D}_{\mathrm{E}}(y; a, r) = \{y + \lambda a | \lambda \in \mathbf{C}, |\lambda| < r\}$.

A *domain spread over* E is a pair (Ω, p) where Ω is a connected Hausdorff space and $p : \Omega \rightarrow \mathrm{E}$ is a local homeomorphism. If p is injective, the domain (Ω, p) is called a *schlicht domain.* For short, we often write Ω instead of (Ω, p) for a domain spread over E although p is the essential part of the pair (Ω, p) and many properties depend on the particular projection $p : \Omega \to \mathrm{E}$.

For a fixed domain (Ω, p) spread over E the *distance functions* $d_\Omega^\alpha : \Omega \to [0, \infty]$, $\alpha \in \mathrm{cs(E)}$, and

$$\delta_\Omega : \Omega \times \mathrm{E} \to]0, \infty]$$

are useful to describe local and global geometric properties of (Ω, p). They are defined as follows:

$d_\Omega^\alpha(x) = \sup \{r > 0|$ There exists a connected neighborhood U of x such that $p|\mathrm{U} : \mathrm{U} \to \mathrm{B}_\mathrm{E}^\alpha(px, r)$ is a homeomorphism$\} \cup \{0\}$ for $x \in \Omega$, and

$\delta_\Omega(x, a) = \sup \{r > 0|$ There is a connected $\mathrm{D} \subset \Omega$ with $x \in \mathrm{D}$ such that $p|\mathrm{D} : \mathrm{D} \to \mathrm{D}_\mathrm{E}(px; a, r)$ is a homeomorphism$\}$

for $(x, a) \in \Omega \times \mathrm{E}$. d_Ω^α is continuous while δ_Ω is in general only lower semicontinuous. For $0 < r \leqslant d_\Omega^\alpha(x)$ (resp. $0 < r \leqslant \delta_\Omega(x, a)$) the *$\alpha$-ball* $\mathrm{B}_\Omega^\alpha(x, r)$ in Ω (resp. the *line segment* $\mathrm{D}_\Omega(x; a, r)$ in Ω) is defined to be that component of $p^{-1}(\mathrm{B}_\mathrm{E}^\alpha(px, r))$ (resp. of $p^{-1}(\mathrm{D}_\mathrm{E}(px; a, r))$) which contains x. For $\mathrm{V} \subset \Omega$ we put

$$d_\Omega^\alpha(\mathrm{V}) = \inf \{d_\Omega^\alpha(x) | x \in \mathrm{V}\},$$

and $\mathrm{V}_s^\alpha = \cup \{\mathrm{B}_\Omega^\alpha(x, s) | x \in \mathrm{V}\}$ if $0 < s < d_\Omega^\alpha(\mathrm{V})$. d_Ω^α and δ_Ω are related by the following formula:

$$d_\Omega^\alpha(x) = \inf \{\delta_\Omega(x, a) | a \in \mathrm{E},\ \alpha(a) \leqslant 1\}$$

for $x \in \Omega$ and $\alpha \in \mathrm{cs(E)}$.

An upper semicontinuous function $\varphi : \Omega \to [-\infty, \infty[$ on a domain (Ω, p) spread over E is called *plurisubharmonic* (*psh*) if all restrictions of φ to line segments in Ω are subharmonic. (Ω, p) is called *pseudoconvex* if $-\log \delta_\Omega$ is psh on $(\Omega \times \mathrm{E}, p \times \mathrm{id_E})$.

A continuous function $f : \Omega \to \mathbf{C}$ on a domain (Ω, p) spread over E is called *analytic* if all restrictions of f to line segments in Ω are analytic. $\mathcal{O}(\Omega)$ denotes the algebra of analytic functions on Ω. A function $f : \Omega \to \mathbf{C}$ is analytic if and only if for every $x \in \Omega$ there exist $\alpha \in cs(E)$, $r > 0$ with $r < d_\Omega^\alpha(x)$, and a sequence of continuous n-homogeneous polynomials $P^n f(x) : E \to \mathbf{C}$ such that

$$f(y) = \sum_{0 \leqslant n} P^n f(x) . (py - px) \quad \text{uniformly for} \quad y \in B_\Omega^\alpha(x, r).$$

A *simultaneous analytic continuation* (s.a.c.) of a collection A of analytic functions on a domain (Ω, p) is a morphism $j : (\Omega, p) \to (\tilde\Omega, \tilde p)$ (i.e. a continuous map $j : \Omega \to \tilde\Omega$ into another domain $(\tilde\Omega, \tilde p)$ with $p = \tilde p \circ j$) such that

$$A \subset \{g \circ j | g \in \mathcal{O}(\tilde\Omega)\}.$$

An s.a.c. j of $A \subset \mathcal{O}(\Omega)$ is called *maximal* if for every s.a.c. j' of A there is a unique morphism $\tilde j$ with $j = \tilde j \circ j'$. There exists always a maximal s.a.c. of A, and it is unique up to isomorphisms. When $A = \mathcal{O}(\Omega)$ it is called the *envelope of holomorphy*. (Ω, p) is said to be an A-*domain of holomorphy* if $id_\Omega : (\Omega, p) \to (\Omega, p)$ is a maximal s.a.c. of A. (Ω, p) is a *domain of existence* if (Ω, p) is an $\{f\}$-domain of holomorphy for a suitable $f \in \mathcal{O}(\Omega)$, and a *domain of holomorphy* if it is an $\mathcal{O}(\Omega)$-domain of holomorphy. A domain of holomorphy is always pseudoconvex.

Finally, the A-*hull* $\hat V_A$ of $V \subset \Omega$ for $A \subset \mathcal{O}(\Omega)$ is defined by $\hat V_A = (V)\hat{}_A = \{x \in \Omega | \, |f(x)| \leqslant \|f\|_V \text{ for all } f \in A\}$, where

$$\|f\|_V = \sup \{|f(x)| \, | x \in V\}.$$

1. Permanence Properties.

In this section we first show that for a separable, metrizable, locally convex space E the property that every pseudoconvex domain spread over E is a domain of existence is inherited by complemented subspaces of E. Then a dual situation is investigated (following Dineen [3], [4] who considers only schlicht domains) : Let $\varphi : E \to F$ be a linear, continuous, open surjection of locally convex Hausdorff spaces. Then the Levi problem for domains spread over E is studied under the assumption that it can be solved for domains spread over F.

The first permanence property is proved with the aid of admissible coverings and regular classes (cf. [26], [30]). The definitions and some results which are needed also in Section 3 are briefly recalled:

1.1 Definition. — *A covering* $\mathfrak{V}$ *of a domain* (Ω, p) *spread over* E *is called admissible if* $\Omega = \bigcup \{\mathring{V} \mid V \in \mathfrak{V}\}$, *where* $\mathring{V}$ *is the interior of* V, *and*

1° *For* $U, V \in \mathfrak{V}$ *there exists always* $W \in \mathfrak{V}$ *with*

$$U \cup V \subset W.$$

2° *For every* $U \in \mathfrak{V}$ *there exist* $\alpha \in cs(E)$, $s > 0$ *and* $V \in \mathfrak{V}$ *such that* $d_\Omega^\alpha(U) > s$ *and* $U_s^\alpha \subset V$.

For an admissible covering $\mathfrak{V}$ of Ω we define

$$A_{\mathfrak{V}} = \{f \in \mathcal{O}(\Omega) \mid \|f\|_U < \infty \quad \text{for all} \quad U \in \mathfrak{V}\}.$$

Then $A_{\mathfrak{V}}$ is a regular class of analytic functions in the following sense: A collection $A \subset \mathcal{O}(\Omega)$, $A \neq \emptyset$, of analytic functions on Ω is said to be a *regular class* if

3° $\lambda f^n \in A$ and $P_a^n f \in A$ for all $\lambda \in \mathbf{C}$, $f \in A$, $n \in \mathbf{N}$ and $a \in E$, where $P_a^n f \colon \Omega \to \mathbf{C}$ is given by $x \longmapsto P^n f(x).a$, $x \in \Omega$.

4° For each $x \in \Omega$ there is a neighborhood U of x with $\|f\|_U < \infty$ for all $f \in A$.

Note that according to a result of Josefson [14] it follows from condition 4° that $\mathcal{O}(\Omega)$ is a regular class if and only if E is finite dimensional.

For an admissible covering $\mathfrak{V}$ of Ω the regular class $A_{\mathfrak{V}}$ is endowed with a natural topology, the topology of uniform convergence on all sets $U \in \mathfrak{V}$. $A_{\mathfrak{V}}$ is a Fréchet algebra when $\mathfrak{V}$ is countable.

1.2 Définition. — *Let* $A \subset \mathcal{O}(\Omega)$. *Then* (Ω, p) *is called* A*-separated if* A *separates the fibers of* p, i.e. *if for* $x, y \in \Omega$ *with* $x \neq y$ *and* $px = py$ *there exists* $f \in A$ *with* $f(x) \neq f(y)$. *When* $\mathfrak{V}$ *is an admissible covering of* Ω *then* (Ω, p) *is called* $A_{\mathfrak{V}}$*-convex if for every* $U \in \mathfrak{V}$ *there exists* $\alpha \in cs(E)$ *and* $s > 0$ *with* $d_\Omega^\alpha(\hat{U}_{A_{\mathfrak{V}}}) > 0$.

1.3. Proposition (cf. [26], [30]). — *Let* $\mathfrak{V}$ *be a countable, admissible covering of a domain* (Ω, p) *spread over* E. *Then* Ω *is an* $A_{\mathfrak{V}}$*-domain of holomorphy if and only if* Ω *is* $A_{\mathfrak{V}}$*-convex and* $A_{\mathfrak{V}}$*-separated.*

1.4. Proposition (cf. [26], [30]; see also [17]). — *For a domain* (Ω, p) *spread over a separable, metrizable, locally convex space* E *the following properties are equivalent:*

1° Ω *is a domain of existence.*

2° Ω *is an* A*-domain of holomorphy for a suitable regular class* A.

3° *There exists an admissible covering* $\mathfrak{V}$ *of* Ω *such that* Ω *is* $A_{\mathfrak{V}}$*-convex and* $A_{\mathfrak{V}}$*-separated.*

4° *There exists a countable, admissible covering* $\mathfrak{V}$ *of* Ω *such that* $R = \{f \in A_{\mathfrak{V}} | \ \Omega$ *is an* $\{f\}$*-domain of holomorphy*$\}$ *contains a countable intersection of open and dense subsets of* $A_{\mathfrak{V}}$. *In particular,* R *is dense in* $A_{\mathfrak{V}}$.

Recall that a vector subspace F of a locally convex Hausdorff space E is said to be *complemented* if there exists a linear, continuous projection $\varphi : E \to E$ with $\varphi(E) = F$. E is then isomorphic to $F \times \varphi^{-1}(0)$.

1.5. Proposition. — *Let* (Ω, p) *be a domain spread over the complemented subspace* F *of* $E = F \times G$ (G *a locally convex Hausdorff space over* **C**). *Suppose there is an admissible covering* $\mathfrak{V}$ *of* $(\Omega \times G, p \times \mathrm{id}_G)$ *such that* $\Omega \times G$ *is* $A_{\mathfrak{V}}$*-convex (resp.* $A_{\mathfrak{V}}$*-separated). Then there exists an admissible covering* $\mathfrak{W}$ *of* Ω *such that* Ω *is* $A_{\mathfrak{W}}$*-convex (resp.* $A_{\mathfrak{W}}$*-separated).* $\mathfrak{W}$ *can be chosen to be countable whenever* $\mathfrak{V}$ *is countable.*

Proof. — For $U \in \mathfrak{V}$ let U_0 be defined by

$$U_0 = \{x \in \Omega | (x, 0) \in U\}.$$

It can easily be checked that $\mathfrak{W} = (U_0)_{U \in \mathfrak{V}}$ is an admissible covering of Ω. Let $\Omega \times G$ be $A_{\mathfrak{V}}$-convex. For every $U \in \mathfrak{V}$ there are $\alpha \in cs(E)$ and $s > 0$ with $d^{\alpha}_{\Omega \times G}(\hat{U}_{A_{\mathfrak{V}}}) > s$. Let $x \in \Omega$ with $d^{\alpha}_{\Omega}(x) \leqslant s$. There exists $f \in A_{\mathfrak{V}}$ with

$$\|f\|_U < |f(x, 0)|$$

since $d^{\alpha}_{\Omega\times G}(x, 0) \leqslant d^{\alpha}_{\Omega}(x) \leqslant s$. Put $f_0(x) = f(x, 0)$ for $x \in \Omega$. Then $f_0 \in A_{\mathfrak{W}}$ and $\|f_0\|_{U_0} \leqslant \|f\|_U < |f_0(x)|$ which implies $x \notin (U_0)\hat{}_{A_{\mathfrak{W}}}$. Hence $d^{\alpha}_{\Omega}((U_0)\hat{}_{A_{\mathfrak{W}}}) \geqslant s$, and Ω is $A_{\mathfrak{W}}$-convex. The separation property can be shown similarly.

1.6. Corollary [1]. — *Let* F *be a complemented subspace of a metrizable, locally convex space* E *such that every pseudoconvex domain spread over* E *is a domain of existence. Then every pseudoconvex domain* Ω *spread over* F *is a domain of holomorphy (resp. a domain of existence when* E *is separable).*

Proof. — Let (Ω, p) be a pseudoconvex domain spread over F. Then $(\Omega \times G, p \times id_G)$ is pseudoconvex over $E = F \times G$, and thus an $\{f\}$-domain of holomorphy for a suitable $f \in \mathcal{O}(\Omega \times G)$. Since E is metrizable there is a countable, admissible covering $\mathfrak{V}$ of $\Omega \times G$ such that $f \in A_{\mathfrak{V}}$. The corollary now follows from 1.5, 1.3 and 1.4.

The following generalization of 1.5 would be a useful result: Instead of a product $E = F \times G$ and the projection $\varphi : E \to F$ consider a linear, continuous, open surjection $\varphi : E \to F$. To a domain (Ω, p) spread over F there corresponds the pull-back (Ω^*, p^*), a domain spread over E which is defined by $\Omega^* = \{(x, a) \in \Omega \times E | p(x) = \varphi(a)\}$ and $p^*(x, a) = a$, $(x, a) \in \Omega^*$.

Conjecture. — If Ω^* is $A_{\mathfrak{V}}$-convex (resp. $A_{\mathfrak{V}}$-separated) for an admissible covering $\mathfrak{V}$ of Ω^* then Ω is $A_{\mathfrak{W}}$-convex (resp. $A_{\mathfrak{W}}$-separated) for a suitable admissible covering $\mathfrak{W}$ of Ω.

A confirmation of the above conjecture for separable Banach spaces E would answer the Levi problem positively for all separable Banach spaces, since for every such space F there exists a linear, continuous surjection $\varphi : L_1 \to F$ and since L_1 has a Schauder basis.

Instead of the above conjecture we can deduce certain continuation properties of Ω^* from corresponding properties

[1] G. Katz has proved a similar result in his thesis, Rochester, N.Y., 1974.

of Ω(2) which is done already by Dineen in [3] for schlicht domains (see also Nachbin [18] for more special situations, and Aurich [1] for the case of $E = \mathbf{C}^{\Lambda}$).

We first need a lemma:

1.7. Lemma. — *Let* (Ω, p) *be a pseudoconvex domain spread over* E. *Suppose there are* $x_0 \in \Omega$ *and* $\alpha \in \mathrm{cs}(E)$ *with*

$$d_{\Omega}^{\alpha}(x_0) > 0.$$

Then $\delta_{\Omega}(x, a) = \infty$ *holds for all* $x \in \Omega$ *and* $a \in \alpha^{-1}(0)$.

Proof. — Fix $a \in \alpha^{-1}(0)$ and let Z denote the interior of $\{x \in \Omega |\ \delta_{\Omega}(x, a) = \infty\}$. $Z \neq \emptyset$ since $x_0 \in Z$. To prove the lemma it is enough to show $Z = \Omega$ which follows from

(*) $D_{\Omega}(z; b, t) \subset Z$ for all $z \in Z$, $b \in E$ and $t < \delta_{\Omega}(z, b)$.

To prove (*) we choose $\beta \in \mathrm{cs}(E)$ and $s > 0$ so that $\beta(b) = 1$, $B_{\Omega}^{\beta}(z, 2s) \subset Z$ and $d_{\Omega}^{\beta}(y) > s$ for all

$$y \in D_{\Omega}(z; b, t).$$

Let $z' \in B_{\Omega}^{\beta}(z, s)$ and $x \in D_{\Omega}(z'; a, \infty)$. Then $\delta_{\Omega}(x, b) \geqslant s$ (by [29, 1.7] for instance). Hence $x \longmapsto -\log \delta_{\Omega}(x, b)$, $x \in D_{\Omega}(z'; a, \infty)$, is bounded from above and thus a constant ($-\log \delta_{\Omega}$ is psh, and a subharmonic function on $\mathbf{C}$ which is bounded from above is a constant). Consequently, $\delta_{\Omega}(x, b) = \delta_{\Omega}(z', b)$ for all $x \in D_{\Omega}(z'; a, \infty)$, and it follows (again by [29, 1.7]) that $\delta_{\Omega}(y', a) = \infty$ for all

$$y' \in \bigcup \{B_{\Omega}^{\beta}(y, s) | y \in D_{\Omega}(z; b, t)\}.$$

Hence $D_{\Omega}(z; b, t) \subset Z$ which completes the proof.

1.8. Proposition. — *Let* $\varphi : E \to F$ *be a linear, continuous, open surjection of locally convex Hausdorff spaces over* $\mathbf{C}$, *and let* (Ω, p) *be a pseudoconvex domain spread over* E. *Assume further that there are* $\beta \in \mathrm{cs}(E)$ *and* $x_0 \in \Omega$ *with* $d_{\Omega}^{\beta \circ \varphi}(x_0) > 0$.

1° *There exists a pseudoconvex domain* $(\Omega_{\varphi}, p_{\varphi})$ *spread over* F *and a continuous open surjection* $\varphi_{\Omega} : \Omega \to \Omega_{\varphi}$ *with* $p = p_{\varphi} \circ \varphi_{\Omega}$ *such that* φ_{Ω} *separates the fibers of* p, *and the following universal property is satisfied: For every continuous*

(2) We have recently learned that P. Berner has obtained some related results in his thesis, Rochester, N.Y., 1974.

map $\psi : \Omega \to \Omega'$ *into a domain* (Ω', p') *spread over* F *with* $p = p' \circ \psi$ *and* $\psi(x) = \psi(y)$ *for* $x, y \in \Omega$ *whenever* $\varphi_\Omega(x) = \varphi_\Omega(y)$, *there exists a unique morphism of domains* $j : \Omega_\varphi \to \Omega'$ *with* $j \circ \varphi_\Omega = \psi$. *Hence* $(\Omega_\varphi, p_\varphi)$ *is a quotient in a suitable category. Also*: (Ω, p) *is isomorphic to the pull-back of* $(\Omega_\varphi, p_\varphi)$.

2° *For every morphism* $j : \Omega \to \Sigma$ *into a pseudoconvex domain* (Σ, q) *spread over* E *there exists a unique morphism* $j_\varphi : (\Omega_\varphi, p_\varphi) \to (\Sigma_\varphi, q_\varphi)$ *with* $\varphi_\Sigma \circ j = j_\varphi \circ \varphi_\Omega$. *Hence the following diagram is commutative*

$$\begin{tikzcd}
& \Sigma \arrow[rr, "\varphi_\Sigma"] \arrow[dd] & & \Sigma_\varphi \arrow[dd] \\
\Omega \arrow[ur, "j"] \arrow[rr, "\varphi_\Omega"] \arrow[dr] & & \Omega_\varphi \arrow[ur, "j_\varphi"] \arrow[dr] & \\
& \mathrm{E} \arrow[rr, "\varphi"] & & \mathrm{F}
\end{tikzcd}$$

Moreover, j *is an isomorphism of domains whenever* j_φ *is.*

3° *Suppose* Ω_φ *is an A-domain of holomorphy for a collection* A *of analytic functions on* Ω_φ. *Then* Ω *is an* $\mathrm{A} \circ \varphi$-*domain of holomorphy, where* $\mathrm{A} \circ \varphi = \{f \circ \varphi_\Omega | f \in \mathrm{A}\}$. *In particular,* Ω *is a domain of existence (resp. a domain of holomorphy) whenever* Ω_φ *is.*

4° *Suppose* Ω_φ *is* $\mathrm{A}_{\mathfrak{V}}$-*convex (resp.* $\mathrm{A}_{\mathfrak{V}}$-*separated) for a (countable) admissible covering* $\mathfrak{V}$ *of* Ω_φ. *Then* Ω *is* $\mathrm{A}_{\mathfrak{W}}$-*convex (resp.* $\mathrm{A}_{\mathfrak{W}}$-*separated) for a suitable (countable) admissible covering* $\mathfrak{W}$ *of* Ω.

Proof. — 1° According to 1.7, $\delta_\Omega(x, a) = \infty$ for all $x \in \Omega$ and all $a \in (\beta \circ \varphi)^{-1}(0)$, in particular for $a \in \varphi^{-1}(0)$. Hence the relation $\sim$ defined by

$$x \sim y \iff \text{There is } a \in \varphi^{-1}(0) \text{ with } y \in \mathrm{D}_\Omega(x; a, \infty),$$

for $x, y \in \Omega$, is an equivalence relation on Ω. It is not difficult to see that $\sim$ is an open equivalence relation and that the graph of $\sim$ is closed in $\Omega \times \Omega$. Therefore, $\Omega_\varphi = \Omega/\sim$ endowed with the quotient topology is a connected Hausdorff space, the natural projection $\varphi_\Omega : \Omega \to \Omega_\varphi$ is a continuous, open surjection, and $p_\varphi : \Omega_\varphi \to \mathrm{F}$, $\varphi_\Omega(x) \longmapsto \varphi(px)$ for

$x \in \Omega$, is a local homeomorphism. Clearly $\varphi \circ p = p_\varphi \circ \varphi_\Omega$. For $x, y \in \Omega$ with $x \neq y$ and $px = py$ the definition of $\sim$ yields $x \not\sim y$ immediately, hence $\varphi_\Omega(x) \neq \varphi_\Omega(y)$. The universal property of $\varphi_\Omega : \Omega \to \Omega_\varphi$ follows directly from the corresponding property of Ω_φ as a quotient in the category of topological spaces.

It remains to show that $(\Omega_\varphi, p_\varphi)$ is pseudoconvex. From the construction of Ω_φ it follows that

$$(*) \qquad \delta_\Omega(x, a) = \delta_{\Omega_\varphi}(\varphi_\Omega(x), \varphi(a)) \quad \text{for} \quad x \in \Omega,\ a \in \mathrm{E},$$

since, in a suitable neighborhood U of $\mathrm{D}_\Omega(x; a, \delta_\Omega(x, a))$,

$$(p|\mathrm{U})^{-1}(px + \lambda a) = (p_\varphi|\varphi_\Omega(\mathrm{U}))^{-1}(\varphi(px) + \lambda\varphi(a)),$$
$$|\lambda| < \delta_\Omega(x, a).$$

Now $(*)$ implies that Ω_φ is pseudoconvex because Ω was supposed to be pseudoconvex.

2° Let $j : \Omega \to \Sigma$ be a morphism of pseudoconvex domains spread over E. Because of $d_\Sigma^{\beta\circ\varphi}(j(x_0)) \geqslant d_\Omega^{\beta\circ\varphi}(x_0) > 0$, there is a quotient map $\varphi_\Sigma : \Sigma \to \Sigma_\varphi$ as in 1°. The universal property applied to $\psi = \varphi_\Sigma \circ j$ then yields a morphism

$$j_\varphi : \Omega_\varphi \to \Sigma_\varphi \quad \text{with} \quad \varphi_\Sigma \circ j = j_\varphi \circ \varphi_\Omega.$$

Suppose now that j_φ is an isomorphism. Let $x, y \in \Omega$ with $x \neq y$. When $px = py$, then $\varphi_\Omega(x) \neq \varphi_\Omega(y)$ by 1°. Since j_φ is injective, $j_\varphi \circ \varphi_\Omega(x) \neq j_\varphi \circ \varphi_\Omega(y)$ and therefore $j(x) \neq j(y)$. When $px \neq py$, then $j(x) \neq j((y)$ since j is a morphism. Thus j is injective. It remains to show that j is surjective. Let $y \in \Sigma$. Because j_φ and φ_Ω are surjective there is $x \in \Omega$ with $j_\varphi \circ \varphi_\Omega(x) = \varphi_\Sigma(y)$. Now $\varphi_\Sigma \circ j(x) = \varphi_\Sigma(y)$ implies $j(x) \sim y$, and hence there exists a point $x' \in \Omega$, $x' \sim x$, with $j(x') = y$: Take $a = qy - q \circ j(x) \in \varphi^{-1}(0)$ and $x' = (p|\mathrm{D})^{-1}(px + a)$, where $\mathrm{D} = \mathrm{D}_\Omega(x; a, \infty)$.

3° Let Ω_φ be an A-domain of holomorphy, $\mathrm{A} \subset \mathcal{O}(\Omega_\varphi)$, and let $j : \Omega \to \Sigma$ be the maximal s.a.c. of $\mathrm{A} \circ \varphi$. Then Σ is in particular a domain of holomorphy and therefore pseudoconvex. According to 2° there is a canonical morphism $j_\varphi : \Omega_\varphi \to \Sigma_\varphi$, and it is easy to see that j_φ is an s.a.c. of A. Therefore j_φ is an isomorphism and so is j due to 2°, which means that Ω is an $\mathrm{A} \circ \varphi$-domain of holomorphy.

4° $\mathfrak{W} = (\varphi_\Omega^{-1}(\mathrm{U}))_{\mathrm{U}\in\mathfrak{V}}$ has the required properties.

The consequences of Proposition 1.8 can be formulated conveniently by the following notion: A collection Φ of continuous, linear surjections $\varphi : \mathrm{E} \to \mathrm{E}_\varphi$, $\varphi \in \Phi$, is called a *basic system* of E (and E is then said to be a *surjective limit* of the spaces E_φ, $\varphi \in \Phi$, cf. [4], [15] if

$$\Gamma = \{\beta \circ \varphi | \varphi \in \Phi,\ \beta \in \mathrm{cs}(\mathrm{E}_\varphi)\}$$

generates the topology of E and if for all $\alpha, \beta \in \Gamma$ there is $\gamma \in \Gamma$ with $\sup\{\alpha, \beta\} \leqslant \gamma$. Φ is called *open* (and E is said to be an *open surjective limit*) when all $\varphi \in \Phi$ are open mappings. Examples can be found in [3], [4], [15], [18].

1.9 Corollary. — *Let* E *be an open surjective limit of spaces* E_φ, $\varphi \in \Phi$, *such that every pseudoconvex domain spread over* E_φ, $\varphi \in \Phi$, *is a domain of existence (resp. a domain of holomorphy,* $\mathrm{A}_{\mathfrak{W}}$*-convex,* $\mathrm{A}_{\mathfrak{W}}$*-separated). Then every pseudoconvex domain spread over* E *is a domain of existence (resp. a domain of existence,* $\mathrm{A}_{\mathfrak{W}}$*-convex,* $\mathrm{A}_{\mathfrak{W}}$*-separated).*

Proof. — Let (Ω, p) be a pseudoconvex domain spread over E. For $x_0 \in \Omega$ there exists $\alpha \in \mathrm{cs}(\mathrm{E})$ with

$$d_\Omega^\alpha(x_0) > 0,$$

and one can find $\varphi \in \Phi$ and $\beta \in \mathrm{cs}(\mathrm{E}_\varphi)$ so that $\alpha \leqslant \beta \circ \varphi$, hence $d_\Omega^{\beta\circ\varphi}(x_0) \geqslant d_\Omega^\alpha(x_0) > 0$. The assertion now follows from Proposition 1.8.

2. Schauder decompositions.

Let E be again a locally convex Hausdorff space over **C**.

2.1. Definition. — *A sequence* (π_n) *of linear, continuous projections* $\pi_n : \mathrm{E} \to \mathrm{E}$ *with* $\dim_{\mathbf{C}} \pi_n(\mathrm{E}) < \infty$, $n \in \mathbf{N}$, *is called a finite dimensional Schauder decomposition (for short f.d. decomposition) of* E *if* $\pi_{n+1} \circ \pi_n = \pi_n \circ \pi_{n+1} = \pi_n$ *for all* $n \in \mathbf{N}$ *and* $\lim_{n\to\infty} \pi_n(x) = x$ *for all* $x \in \mathrm{E}$.

This definition differs slightly from the definition of a Schauder decomposition in [16, ch. VII], but coincides in the case of a Fréchet space. An f.d. decomposition is called *finite dimensional expansion of identity* in [23].

Let (π_n) be an f. d. decomposition of E. Then (π_n) is said to be α-*monotone* for a seminorm $\alpha \in \mathrm{cs(E)}$ if the following (obviously equivalent) conditions are satisfied:

1° $\alpha \circ \pi_n \leqslant \alpha \circ \pi_{n+1}$ for all $n \in \mathbf{N}$.

2° $\alpha \circ \pi_n \leqslant \alpha$ for all $n \in \mathbf{N}$.

3° $\alpha = \sup \{\alpha \circ \pi_n | n \in \mathbf{N}\}$.

The straightforward proof of the following lemma is omitted.

2.2. Lemma. — *For an f.d. decomposition* (π_n) *of* E *the following properties are equivalent:*

1° (π_n) *converges locally uniformly to* id_{E} *in the following sense: For all* $x_0 \in \mathrm{E}$, $\alpha \in \mathrm{cs(E)}$ *and* $\varepsilon > 0$ *there are* $\mathrm{N} \in \mathbf{N}$, $\beta \in \mathrm{cs(E)}$ *and* $\delta > 0$ *such that* $\alpha(\pi_n(x) - x) < \varepsilon$ *for all* $x \in \mathrm{B}_{\mathrm{E}}^{\beta}(x_0, \delta)$ *and* $n \geqslant \mathrm{N}$.

2° *For each* $\alpha \in \mathrm{cs(E)}$, *the seminorm*

$$\tilde{\alpha} = \sup \{\alpha \circ \pi_n | n \in \mathbf{N}\}$$

is continuous.

3° *The topology of* E *is generated by* $\{\alpha \in \mathrm{cs(E)} | (\pi_n)$ *is* α-*monotone*$\}$.

4° (π_n) *is equicontinuous.*

Note that an f.d. decomposition of a barrelled space E is already equicontinuous, since for every $\alpha \in \mathrm{cs(E)}$,

$$\mathrm{T} = \{x \in \mathrm{E} | \alpha \circ \pi_n(x) \leqslant 1$$

for all $n \in \mathbf{N}\} = \{x \in \mathrm{E} | \tilde{\alpha}(x) \leqslant 1\}$ is a barrel, and thus $\tilde{\alpha}$ is continuous.

A locally convex space with an f.d. decomposition is obviously separable. However, not every separable, locally convex space has an f.d. decomposition, since a Banach space with an f.d. decomposition has the approximation property, and there are separable Banach spaces without the approximation property (cf. [7]). General examples of locally convex spaces with an f.d. decomposition are the spaces with a Schauder basis (cf. [16]). An example of a space with an equicontinuous Schauder decomposition which occurs in Complex

Analysis is the Fréchet space $\mathcal{O}(\mathrm{R})$ of analytic functions on a Reinhardt domain $\mathrm{R} \subset \mathbf{C}^n$ with $0 \in \mathrm{R}$. Each $f \in \mathcal{O}(\mathrm{R})$ has a locally uniformly convergent power series expansion

$$f(z) = \Sigma \mathrm{P}^n f(0).z, \ z \in \mathrm{R},$$

hence $\lim_{m\to\infty} \sum_{n=0}^{m} \mathrm{P}^n f(0) = f$ in $\mathcal{O}(\mathrm{R})$. Therefore, the projections $\pi_m : \mathcal{O}(\mathrm{R}) \to \mathcal{O}(\mathrm{R})$, defined by $\pi_m(f) = \sum_{n=0}^{m} \mathrm{P}^n f(0)$, $f \in \mathcal{O}(\mathrm{R})$, form an f.d. decomposition which is equicontinuous since $\mathcal{O}(\mathrm{R})$ is barrelled.

The next proposition, due to Dineen [3], allows to reduce our study of the Levi problem to spaces which have a continuous norm.

2.3. Proposition. — *Let (π_n) be an equicontinuous f. d. decomposition of* E. *Then* E *is an open surjective limit of spaces* E_φ, $\varphi \in \Phi$, *where each* E_φ *has a continuous norm and an equicontinuous f. d. decomposition.*

The *proof* is essentially the same as for spaces E with a Schauder basis (cf. [3]).

3. The Levi problem.

3.1. Theorem. — *Let (Ω, p) be a pseudoconvex domain spread over a locally convex Hausdorff space* E *with an equicontinuous f.d. decomposition such that there is a sequence (α_ν) of seminorms $\alpha_\nu \in \mathrm{cs(E)}$ with $\Omega = \bigcup_{\nu \in \mathrm{N}} \Omega^{\alpha_\nu}$, where $\Omega^\alpha = \{x \in \Omega | d_\Omega^\alpha(x) > 0\}$ for $\alpha \in \mathrm{cs(E)}$. Then there exists a countable, admissible covering $\mathfrak{V}$ of Ω such that Ω is $\mathrm{A}_{\mathfrak{V}}$-convex and $\mathrm{A}_{\mathfrak{V}}$-separated.*

The rest of this section will be devoted to proving 3.1. However, we first want to present the main consequences:

3.2. Corollary. — *Let* E *be a metrizable, locally convex space with an equicontinuous f.d. decomposition. The following properties of a domain Ω spread over* E *are equivalent:*

1° Ω *is pseudoconvex.*

2° Ω *is a domain of existence.*

3° *There exists an admissible covering* $\mathfrak{V}$ *of* Ω *such that* Ω *is* $A_{\mathfrak{V}}$*-convex.*

4° *The subset* $\{f \in \mathcal{O}(\Omega) | \Omega$ is an $\{f\}$-domain of holomorphy$\}$ *of* $\mathcal{O}(\Omega)$ *is sequentially dense in* $\mathcal{O}(\Omega)$, *when* $\mathcal{O}(\Omega)$ *is endowed with a locally convex Hausdorff topology with the same bounded sets as the compact open topology.*

Proof. — The implications $4° \Rightarrow 2° \Rightarrow 3° \Rightarrow 1°$ are obvious. To show $1° \Rightarrow 4°$ let $f \in \mathcal{O}(\Omega)$ and let $\mathfrak{V}$ be as in 3.1. Since E is metrizable there is a countable, admissible refinement $\mathfrak{W}$ of $\mathfrak{V}$ with $f \in A_{\mathfrak{W}}$. Now 4° follows from 1.4.4° and the fact that the injection $A_{\mathfrak{W}} \to \mathcal{O}(\Omega)$ is continuous for the bornological topology associated with the compact open topology on $\mathcal{O}(\Omega)$ (cf. [26], [30]).

The following are examples of locally convex topologies on $\mathcal{O}(\Omega)$ with the same bounded sets as the compact open topology: The topology τ_n of uniform convergence on compact subsets of Ω of analytic functions and their derivatives up to the order n, for $n \in \mathbf{N}$; the topology

$$\tau_\infty = \bigcup_{n \in \mathbf{N}} \tau_n;$$

the Nachbin topology τ_ω (cf., for instance, [3]); the ε-topology, that is the finest locally convex topology on $\mathcal{O}(\Omega)$ which agrees on the equicontinuous sets with the simple topology.

3.3. Corollary. — *Let* E *be a locally convex Hausdorff space with an equicontinuous f.d. decomposition, and suppose that* E *is hereditary Lindelöf. The following properties of a domain* Ω *spread over* E *are equivalent:*

1° Ω *is pseudoconvex.*

2° *There exists an admissible covering* $\mathfrak{V}$ *of* Ω *such that* Ω *is* $A_{\mathfrak{V}}$*-convex.*

3° Ω *is a domain of holomorphy.*

Proof. — It is enough to observe that Ω is hereditary Lindelöf, and to apply 3.1 and 1.3.

3.4. Corollary. — *Every pseudoconvex domain spread over a Fréchet space* F *with the bounded approximation property is a domain of existence.*

Proof. — According to [23] a Fréchet space with the bounded approximation property is isomorphic to a complemented subspace of a Fréchet space with an f.d. decomposition. Therefore the corollary follows from 1.5 and 3.2.

Note that 3.2 contains the results of [9], [10] and [5] while 3.4 is a generalization of [19, 6.6]. The Levi problem is unsolved for the more general (cf. [7]) separable spaces with the approximation property, and for separable Banach spaces. For non-separable Banach spaces counterexamples are known (cf. Josefson [13]).

Theorem 3.1 also yields a solution of the Levi problem for domains spread over a Silva space with an f.d. decomposition, thus generalizing [21], [24], [25]. A *Silva space* is the strong dual of a Fréchet Schwartz space (or an « LS-Raum » in the terminology of [6]), and hence barrelled and hereditary Lindelöf. Therefore, a pseudoconvex domain Ω spread over a Silva space with an f.d. decomposition is $A_{\mathfrak{B}}$-convex and $A_{\mathfrak{B}}$-separated. Similar to a reasoning in [21], where the schlicht case is investigated, it can be shown that Ω is a domain of existence (details are in [31]).

Moreover, following [5] Corollary 3.2 gives a positive answer to the Levi problem for other non-metrizable locally convex spaces with certain countability conditions.

The second permanence property of Section 1, in particular Proposition 1.9, can be applied to obtain a solution of the Levi problem for various other locally convex spaces : For instance for an arbitrary product of suitable spaces, in particular for $E = \mathbf{C}^{\Lambda}$ (cf. [1]); for locally convex spaces with the weak topology; for the spaces $E = \mathscr{C}(M)$ of continuous $\mathbf{C}$-valued functions on a metrizable space M with the compact open topology; for the space $E = \mathcal{O}(F)$, where F is a locally convex Hausdorff space over $\mathbf{C}$ and $\mathcal{O}(F)$ is endowed with the topology of uniform convergence on all compact subsets of F with finite dimensional span.

The next corollary generalizes results in [2], [10], [11], [27], [30].

3.5. Corollary. — *Let* (Σ, q) *be a domain spread over a metrizable, locally convex space* F, *and let* (Ω, p) *be a domain spread over a metrizable, holomorphically complete (cf., for instance,* [30]) *locally convex space with an equicontinuous f.d. decomposition. Then every analytic mapping* $f\colon \Sigma \to \Omega$ *has an analytic continuation* $\mathcal{E}(f) : \mathcal{E}(\Sigma) \to \mathcal{E}(\Omega)$ *with*

$$\mathcal{E}(f) \circ j_\Sigma = j_\Omega \circ f,$$

where $j_\Sigma : \Sigma \to \mathcal{E}(\Sigma)$ (resp. $j_\Omega\colon \Omega \to \mathcal{E}(\Omega)$) *denotes the envelope of holomorphy of* Σ *(resp.* Ω*) (i.e.* $\mathcal{E}$ *is functorial).*

Proof. — In [30] it is shown that $p \circ f\colon \Sigma \to$ E has an analytic continuation to $\mathcal{E}(\Sigma)$ since E is holomorphically complete. The corollary now follows as in [27] from 3.1 and the representation of $\mathcal{E}(\Sigma)$ (resp. $\mathcal{E}(\Omega)$) as a subset of the spectrum of $\mathcal{O}(\Sigma)$ (resp. $\mathcal{O}(\Omega)$) (cf. [26], [30]).

Corollary 3.5 could also be regarded as a general « Kontinuitätssatz » :

3.6. Corollary. — *Let* E *be a metrizable, holomorphically complete, locally convex space with an equicontinuous f.d. decomposition. Then a domain* Ω *spread over* E *is pseudoconvex if and only if for every domain* Σ *spread over any metrizable, locally convex space and for every analytic* $f\colon \Sigma \to \Omega$ *there exists an analytic continuation* $\mathcal{E}(f) : \mathcal{E}(\Sigma) \to \Omega$ *of* f *with* $f = \mathcal{E}(f) \circ j_\Sigma$.

The assertion follows immediately from 3.5 and the classical Kontinuitätssatz.

Proof of the theorem. — To prove 3.1 we can assume that E has a continuous norm. The general case then follows from 2.3 and 1.9.

For the remainder of this section let (Ω, p) be a pseudoconvex domain spread over the locally convex Hausdorff space E with an f.d. decomposition (π_n), and let (α_ν) be an increasing sequence of continuous norms on E such that $\Omega = \bigcup_{\nu \in \mathbf{N}} \Omega^{\alpha_\nu}$, and such that (π_n) is α_ν-monotone for every $\nu \in \mathbf{N}$ (See 2.2). Furthermore, let Ω_n denote the pseudoconvex manifold $\Omega_n = \Omega \cap p^{-1}(\pi_n(\mathrm{E}))$ spread over the finite dimensional space $\mathrm{E}_n = \pi_n(\mathrm{E})$.

The main step of the proof of 3.1 will be applied also in the next section. Therefore we want to present it in a general form. In order to do this, let us assume we have determined an open covering $(X_n)_{n\in\mathbf{N}}$, $X_n \subset X_{n+1}$, of Ω, and continuous maps $\tau_n : X_n \to \Omega_n$, $n \in \mathbf{N}$, such that for all $n \in \mathbf{N}$: $\Omega_n \subset X_n$, $\tau_n|\Omega_n = \mathrm{id}_{\Omega_n}$, $p \circ \tau_n = \pi_n \circ p|X_n$, and

$$\tau_n \circ \tau_{n+1}(x) = \tau_n(x)$$

for all $x \in X_n$ with $\tau_{n+1}(x) \in X_n$. (For a schlicht domain $\Omega \subset E$ put $X_n = \pi_n^{-1}(\Omega \cap E_n) \cap \Omega$ and $\tau_n = \pi_n|X_n$.)

3.7. Definition. — A *countable covering* $(V_\nu)_{\nu\in\mathbf{N}}$ *of* Ω *will be called compatible with* (τ_n) *if the following holds: There exists a strictly decreasing null sequence* (r_ν) *of real numbers such that*

1° $V_\nu \subset V_{\nu+1}$, $\Omega = \bigcup_{\nu\in\mathbf{N}} \mathring{V}_\nu$, $d_\Omega^{\alpha_\nu}(V_\nu) \geqslant r_\nu$ *and* $V_\nu \subset X_\nu$ *for* $\nu \in \mathbf{N}$.

2° $\tau_\mu(V_\nu)$ *is relatively compact in* Ω_μ, *and* $\tau_\mu(V_\nu) = V_\nu \cap \Omega_\mu$ *for all* $\mu, \nu \in \mathbf{N}$, $\mu \geqslant \nu$.

3° $(V_\nu \cap \Omega_\mu)^{\wedge}_{\mathcal{O}(\Omega_\mu)} \subset \{x \in \Omega_\mu | d_\Omega^{\alpha_\nu}(x) \geqslant r_\nu\}$ *for all* $\mu \geqslant \nu$.

4° $K_\nu = (V_\nu \cap \Omega_\nu)^{\wedge}_{\mathcal{O}(\Omega_\nu)}$ *satisfies* $K_\nu \subset X_{\nu-1}$ *for all* $\nu > 1$.

The following lemma is related to the key result of Gruman and Kiselman [9, lemme].

3.8. Main Lemma. — *Let* (V_ν) *be a compatible covering of* Ω, *and let* $f \in \mathcal{O}(\Omega_n)$. *For every* $\varepsilon > 0$ *there exists*

$$g \in \mathcal{O}(\Omega)$$

satisfying

1° $\|g - f \circ \tau_n\|_{V_n} < \varepsilon$ *and*

2° $\|g\|_{V_\nu} < \infty$ *for all* $\nu \in \mathbf{N}$.

Proof of the lemma. — By induction, a sequence $(f_\mu)_{\mu\geqslant n}$, $f_\mu \in \mathcal{O}(\Omega_\mu)$, will be defined satisfying

$$(*) \quad \|f_{\mu+1} - f_\mu \circ \tau_\mu\|_{K_{\mu+1}} < \varepsilon . 2^{-(\mu+1)} \quad \text{for} \quad \mu \geqslant n.$$

Put $f_n = f$ and assume that $f_n, \ldots, f_\mu$ are already defined. $X_\mu \cap \Omega_{\mu+1}$ is an open neighborhood of $K_{\mu+1}$ according to 3.7.4°. Hence, by the classical Oka-Weil approximation theorem (cf. [12, p. 116]) for the Stein manifold $\Omega_{\mu+1}$ there exists $f_{\mu+1} \in \mathcal{O}(\Omega_{\mu+1})$ with $(*)$. (Note that every Ω_n has at most countably many components which are pseudoconvex domains spread over the finite dimensional space E_n. Therefore, Ω_n is a Stein manifold according to [12, 5.4.6]).

Now, for $\nu \leqslant k \leqslant m$:

$$\begin{aligned}\|f_m \circ \tau_m - f_k \circ \tau_k\|_{V_\nu} &\leqslant \sum_{\mu=k}^{m-1} \|f_{\mu+1} \circ \tau_{\mu+1} - f_\mu \circ \tau_\mu\|_{V_\nu} \\ &\leqslant \sum_{\mu=k}^{m-1} \|f_{\mu+1} - f_\mu \circ \tau_\mu\|_{\tau_{\mu+1}(V_\nu)} \qquad (\tau_\mu \circ \tau_{\mu+1} = \tau_\mu \text{ on } V_\nu) \\ &\leqslant \sum_{\mu=k}^{m-1} \|f_{\mu+1} - f_\mu \circ \tau_\mu\|_{K_{\mu+1}} \qquad (\text{due to } 3.7.2^{\circ}) \\ &\leqslant \varepsilon \cdot \sum_{\mu=k}^{m-1} 2^{-(\mu+1)} \leqslant \varepsilon . 2^{-k} \qquad (\text{by } (*)).\end{aligned}$$

Therefore, $(f_\mu \circ \tau_\mu)_{\mu \geqslant \nu}$ converges uniformly on $\mathring{V}_\nu$ and defines an analytic function $g \in \mathcal{O}(\Omega)$ with

$$\|g - f_\nu \circ \tau_\nu\|_{V_\nu} \leqslant \varepsilon . 2^{-\nu} < \varepsilon$$

for all $\nu \in \mathbf{N}$. It follows 1° and

$$\|g\|_{V_\nu} < \varepsilon + \|f_\nu \circ \tau_\nu\|_{V_\nu} \leqslant \varepsilon + \|f_\nu\|_{K_\nu} < \infty,$$

hence 2°.

A countable, admissible covering $\mathfrak{V} = (U_\nu)$ of Ω (cf. 1.1) will be called *compatible* with (π_n) if there exist a compatible covering (V_ν) of Ω and a strictly decreasing null sequence (s_ν) of real numbers with $(U_\nu)^{\alpha_\nu}_{s_\nu} \subset V_\nu$ for all $\nu \in \mathbf{N}$. To a given compatible covering (V_ν) of Ω with (r_ν) as in 3.7 there always corresponds a compatible, admissible covering (U_ν) defined as follows:

$$U_\nu = \{x \in V_\nu | B^{\alpha_\nu}_\Omega(x, r_\nu) \subset V_\nu\} \quad \text{for} \quad \nu \in \mathbf{N}.$$

In fact, $\mathfrak{V} = (U_\nu)$ is admissible since $(U_\nu)^{\alpha_{\nu+1}}_{r_\nu - r_{\nu+1}} \subset U_{\nu+1}$ for all $\nu \in \mathbf{N}$.

3.9. Lemma. — *Let $\mathfrak{V}$ be a compatible, admissible covering of Ω. Then Ω is $A_{\mathfrak{V}}$-convex and $A_{\mathfrak{V}}$-separated.*

Proof of the lemma. — Let (V_ν), (r_ν) be as in 3.7 and (s_ν) with $(U_\nu)^{\alpha_\nu}_{s_\nu} \subset V_\nu$. We show that Ω is $A_{\mathfrak{V}}$-convex by proving $d^{\alpha_\nu}_\Omega((U_\nu)^\wedge_A) \geqslant r_\nu$ for all $\nu \in \mathbf{N}$ $(A = A_{\mathfrak{V}}$ to simplify the notation). Let x be a point of Ω with

$$d^{\alpha_\nu}_\Omega(x) < r_\nu.$$

When $x \in \Omega_n \cap V_n$ for a suitable $n \geqslant \nu$, there exists $f \in \mathcal{O}(\Omega_n)$ with

$$|f(x)| > \|f\|_{V_\nu \cap \Omega_n},$$

since $(V_\nu \cap \Omega_n)^\wedge_{\mathcal{O}(\Omega_n)} \subset \{x \in \Omega_n | d^{\alpha_\nu}_\Omega(x) \geqslant r_\nu\}$ (cf. 3.7.3°). Thus, according to 3.8 there exists $g \in A$ with $|g(x)| > \|g\|_{V_\nu}$ (take $\varepsilon = \frac{1}{2}(|f(x)| - \|f\|_{V_\nu \cap \Omega_n})$). This implies $x \notin (V_\nu)^\wedge_A$ and therefore $x \notin (U_\nu)^\wedge_A$.

When $x \notin \bigcup_{n \in \mathbf{N}} \Omega_n$ there are $s > 0$, $\alpha \in \mathrm{cs}(E)$ and $W \in \mathfrak{V}$ with $\alpha_\nu \leqslant \alpha$, $s_\nu \leqslant 2s \leqslant d^\alpha_\Omega(x)$, $s + d^{\alpha_\nu}_\Omega(x) < r_\nu$, $d^\alpha_\Omega(U_\nu) > 2s$ and $(U_\nu)^\alpha_{2s} \subset W$. Assume $x \in (U_\nu)^\wedge_A$. Then $|P^n_a f(x)| \leqslant \|P^n_a f\|_{U_\nu}$ for all $f \in A$, $n \in \mathbf{N}$ and $a \in E$, because A is a regular class. Hence, by the Cauchy inequalities, for $a \in B^\alpha_E(0, 1)$:

$$|P^n_a f(x)| \leqslant (2s)^{-n}\|f\|_{(U_\nu)^\alpha_{2s}} \leqslant (2s)^{-n}\|f\|_{V_\nu \cap W},$$

since $(U_\nu)^\alpha_{2s} \subset (U_\nu)^{\alpha_\nu}_{s_\nu} \subset V_\nu$. Consequently, for every

$$y \in B^\alpha_\Omega(x, s) \cap \Big(\bigcup_{n \in \mathbf{N}} \Omega_n\Big)$$
$$d^{\alpha_\nu}_\Omega(y) \leqslant d^{\alpha_\nu}_\Omega(x) + s < r_\nu$$

and

$$|f(y)| \leqslant \sum_{n \geqslant 0} |P^n f(x) . (py - px)| \leqslant \sum_{n \geqslant 0} 2^{-n}\|f\|_{V_\nu \cap W} = 2\|f\|_{V_\nu \cap W}.$$

This inequality is also true for all powers of $f \in A$, and it follows that $|f(y)| \leqslant \|f\|_{V_\nu \cap W}$, $f \in A$. Thus $y \in (V_\nu)^\wedge_A$ in contradiction to the first part of the proof. Hence $x \notin (U_\nu)^\wedge_A$ for every $x \in \Omega$ with $d^{\alpha_\nu}_\Omega(x) < r_\nu$ which implies

$$d^{\alpha_\nu}_\Omega((U_\nu)^\wedge_A) \geqslant r_\nu.$$

It remains to show that Ω is $A_{\mathfrak{V}}$-separated. Let $x, y \in \Omega$ with $x \neq y$ and $px = py$. When $px \in \bigcup_{n \in \mathbf{N}} E_n$, then there exists $n \in \mathbf{N}$ with $x, y \in \Omega_n \cap V_n$. Since Ω_n is in particular a Stein manifold, there is $f \in \mathcal{O}(\Omega_n)$ with $f(x) \neq f(y)$, and 3.8 implies $g(x) \neq g(y)$ for a suitable $g \in A$. When $px \notin \bigcup_{n\in\mathbf{N}} E_n$, the assumption $g(x) = g(y)$ for all $g \in A$ would imply $g(x') = g(y')$ for all

$$(x', y') \in B_{\Omega}^{\alpha}(x, s) \times B_{\Omega}^{\alpha}(y, s),\ px' = py',$$

for a suitable choice of $\alpha \in cs(E)$ and $s > 0$. Because of $B_E^{\alpha}(px, s) \cap \left(\bigcup_{n\in\mathbf{N}} E_n\right) \neq \emptyset$, this is a contradiction to what we have just proved.

To complete the proof of the theorem it remains now to construct a compatible covering (V_ν) of Ω. To define the covering (X_ν) and the mappings τ_ν (see 3.7) let us introduce the functions $\eta_\nu : \Omega \to]0, \infty]$, $\nu \in \mathbf{N}$, given by

$$\eta_\nu(x) = \inf \{\delta_\Omega(x, \pi_\mu(px) - px) \mid \mu \geqslant \nu\},\ x \in \Omega.$$

Since $\pi_\mu(px) \to px$ for all $x \in \Omega$ it is clear that $\eta_\nu > 0$ for every $\nu \in \mathbf{N}$. Moreover, each function η_ν is lower semicontinuous: Let $x \in \Omega$ with $\eta_\nu(x) > c$, where $c \in \mathbf{R}$. For every r, $c < r < \eta_\nu(x)$, the set

$$L = \bigcup \{\overline{D_\Omega(x; \pi_\mu(px) - px, r)} \mid \mu \geqslant \nu\}$$

is well-defined and compact. Hence, there is a seminorm $\alpha \in cs(E)$ with $\alpha = \tilde{\alpha}$ (cf. 2.2) and $d_\Omega^\alpha(L) > 0$. Choose $s > 0$ so that $s(2r + 1) < d_\Omega^\alpha(L)$. Then, for all

$$y \in B_\Omega^\alpha(x, s),\ |\lambda| \leqslant r \quad\text{and}\quad \mu \geqslant \nu$$

$$\begin{aligned}\alpha(py + \lambda(\pi_\mu(py) - py) - px - \lambda(\pi_\mu(px) - px))\\ &\leqslant (1 + |\lambda|)\alpha(py - px) + |\lambda|\alpha(\pi_\mu(py - px))\\ &< (1 + r)s + rs < d_\Omega^\alpha(L).\end{aligned}$$

Consequently, $\delta_\Omega(y, \pi_\mu(py) - py) \geqslant r$ and hence

$$c < r \leqslant \eta_\nu(y)$$

for all $y \in B_\Omega^\alpha(x, s)$.

Since Ω is pseudoconvex, each function

$$-\log \eta_\nu : \Omega \to [-\infty, \infty[$$

is psh as an upper semicontinuous supremum of the plurisubharmonic functions $x \longmapsto -\log \delta_\Omega(x, \pi_\mu(px) - px)$, $\mu \geqslant \nu$.

Now let $X_\nu = \{x \in \Omega | \eta_\nu(x) > 1\}$ for $\nu \in \mathbf{N}$. According to the definition of η_ν there is a canonical continuous map $\tau_\nu : X_\nu \to \Omega_\nu$ such that $\tau_\nu | \Omega_\nu = \mathrm{id}_{\Omega_\nu}$ and $\pi_\nu \circ p = p \circ \tau_\nu$: Put $\tau_\nu(x) = (p|D_x)^{-1}(\pi_\nu(px))$ for $x \in X_\nu$, where

$$D_x = D_\Omega(x; \pi_\nu(px) - px, \eta_\nu(x)).$$

Note that $X_\nu \subset X_{\nu+1}$, $\tau_{\nu+1}(X_\nu) \subset X_\nu$ and $\tau_\nu \circ \tau_{\nu+1} | X_\nu = \tau_\nu$ for all $\nu \in \mathbf{N}$.

To define (V_ν) we need another real-valued function. For $\nu \in \mathbf{N}$ put $\Omega'_\nu = \left\{x \in \Omega_\nu | d_\Omega^{\alpha_\nu}(x) \geqslant \frac{1}{\nu}\right\}$. Without loss of generality we can assume that $\Omega'_1 \neq \emptyset$. Fix a point $x_1 \in \Omega'_1$. On Ω'_ν the distance functions $\gamma_\nu : \Omega'_\nu \to [0, \infty]$ are defined as follows: Let $\Gamma_\nu(x)$ be the set of all finite sequences

$$(x_1, \ldots, x_k, x_{k+1})$$

in Ω'_ν with $\bigcup_{\varkappa=1}^{k} D_\Omega(x_\varkappa; px_{\varkappa+1} - px_\varkappa, 1) \subset \Omega'_\nu$ and $x = x_{k+1}$. Put

$$\gamma_\nu(x) = \begin{cases} \inf \left\{ \sum_{\varkappa=1}^{k} \alpha_1(px_{\varkappa+1} - px_\varkappa) | (x_1, \ldots, x_{k+1}) \in \Gamma_\nu(x) \right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad \Gamma_\nu(x) \neq \emptyset, \\ \infty, \quad \text{if} \quad \Gamma_\nu(x) = \emptyset. \end{cases}$$

The functions γ_ν are continuous and satisfy $\gamma_{\nu+1} | \Omega'_\nu \leqslant \gamma_\nu$ for all $\nu \in \mathbf{N}$.

3.10. Lemma. — *$L_\nu = \{x \in \Omega'_\nu | \gamma_\nu(x) \leqslant \nu\}$ is compact in Ω_ν, and for every $x \in \Omega$ there exist $n \in \mathbf{N}$ and a neighborhood U of x with $U \subset X_n$ and $\tau_\mu(U) \subset L_\mu$ for all $\mu \geqslant n$.*

Proof of the lemma. — The restriction of the norms α_1 and α_ν are equivalent on E_ν, hence there is $C > 0$ with

$$\alpha_\nu | E_\nu \leqslant C\alpha_1 | E_\nu.$$

Therefore,

$$\frac{1}{\nu} \leqslant d_\Omega^{\alpha_\nu}(x) \leqslant d_{\Omega_\nu}^{\alpha_\nu}(x) \leqslant d_{\Omega_\nu}^{C\alpha_1}(x) = C d_{\Omega_\nu}^{\alpha_1}(x) \quad \text{for} \quad x \in \Omega_\nu.$$

Choose $t > 0$ with $Ct < \frac{1}{\nu}$. Put $T_1 = \Omega'_\nu \cap \overline{B^{\alpha_\nu}_{\Omega_\nu}(x_1, t)}$ and $T_{m+1} = \Omega'_\nu \cap (\bigcup \{\overline{B^{\alpha_\nu}_{\Omega_\nu}(x, t)} | x \in T_m\})$. Then T_m is compact for $m \in \mathbf{N}$. Furthermore, by induction it can be shown that $\{x \in \Omega'_\nu | \gamma_\nu(x) < mt\} \subset T_m$. It follows that L_ν is compact since $L_\nu \subset T_m$ for all $m \in \mathbf{N}$ with $\nu < mt$.

To show the second property of the lemma let $x \in \Omega$. Since Ω is pathwise connected there is a compact, connected subset W of Ω with $x, x_1 \in W$. There exists $\nu \in \mathbf{N}$ with $W \subset X_\nu$, and ν can be chosen in such a way that

$$d^{\alpha_\nu}_\Omega(W) > 4s + \frac{1}{\nu}$$

for a suitable $s > 0$. Because of the uniform convergence of (π_n) in the sense of 2.2 there is $m \in \mathbf{N}$, $m \geqslant \nu$, with

$$\alpha_\nu(\pi_\mu(pw) - pw) < s \quad \text{for all} \quad w \in W \quad \text{and} \quad \mu \geqslant m.$$

It follows $d^{\alpha_\nu}_\Omega(\tau_\mu(W)) > 3s + \frac{1}{\nu}$, hence $\tau_\mu(W) \subset \Omega'_\mu$ and $\gamma_\mu \circ \tau_\mu(x) < \infty$ for all $\mu \geqslant m$.

Now let $U = B^{\alpha_\nu}_\Omega(x, s)$. For $y \in U$ and $\mu \geqslant m$

$$\alpha_\nu(\pi_\mu(py) - py) \leqslant \alpha_\nu(\pi_\mu(py - px)) + \alpha_\nu(\pi_\mu(px) - px) + \alpha_\nu(px - py) < 3s,$$

and

$$d^{\alpha_\nu}_\Omega(y) \geqslant d^{\alpha_\nu}_\Omega(x) - s > 3s + \frac{1}{\nu}.$$

From these inequalities we deduce $\eta_m(y) > 1$, thus $U \subset X_m$. Furthermore,

$$d^{\alpha_\nu}_\Omega(\tau_\mu(y)) \geqslant d^{\alpha_\nu}_\Omega(y) - 3s > \frac{1}{\nu},$$

hence $\tau_\mu(U) \subset \Omega'_\mu$ for all $\mu \geqslant m$. From

$$\alpha_\nu(\pi_m(px) - \pi_\mu(py)) < 2s$$

and $d^{\alpha_\nu}_\Omega(\tau_m(x)) > 3s + \frac{1}{\nu}$ we get

$$\delta_\Omega(\tau_m(x), \pi_\mu(py) - \pi_m(px)) > 1,$$

and thus

$$\gamma_\mu \circ \tau_\mu(y) \leqslant \gamma_\mu \circ \tau_m(x) + \alpha_1(\pi_m(px) - \pi_\mu(py)) \leqslant \gamma_m \circ \tau_m(x) + 2s.$$

Therefore, U and $n \geqslant \sup \{m, \gamma_m \circ \tau_m(x) + 2s\}$ have the required properties. This completes the proof of the lemma.

After these preparations we now come to the *construction of a compatible covering* (V_ν) of Ω. For $\nu \in \mathbf{N}$

$$V_\nu = \{x \in X_{\nu-1} | \tau_\mu(x) \in L_\mu, \quad d_\Omega^{\alpha_\nu}(\tau_\mu(x)) \geqslant \frac{1}{\nu}$$

and $$\eta_{\nu-1} \circ \tau_\mu(x) \geqslant 1 + \frac{1}{\nu} \quad \text{for all} \quad \mu \geqslant \nu\}.$$

(V_1 is well-defined when $X_0 = X_1$ and $\eta_0 = \eta_1$.) We have to check the conditions 3.7.1° — 4° with $\left(\frac{1}{\nu}\right)$ instead of (r_ν):

1° Obviously, $V_\nu \subset V_{\nu+1} \subset X_{\nu+1}$ and $d_\Omega^{\alpha_\nu}(V_\nu) \leqslant \frac{1}{\nu}$ for $\nu \in \mathbf{N}$. To show $\Omega = \bigcup_{\nu \in \mathbf{N}} \mathring{V}_\nu$ let $x \in \Omega$. We can find $\nu \in \mathbf{N}$, $m \geqslant \nu$ and $s > \frac{1}{\nu}$ so that $d_\Omega^{\alpha_\nu}(x) > 7s$ and

$$\alpha_\nu(px - \pi_\mu(px)) < s$$

for all $\mu \geqslant m$. Put $V = B_\Omega^{\alpha_\nu}(x, s)$. For $y \in V$ and $\mu, \varkappa \geqslant m$

$$\alpha_\nu(\pi_\mu(py) - \pi_\varkappa(py)) < 4s,$$

and

$$d_\Omega^{\alpha_\nu}(\tau_\mu(y)) \geqslant d_\Omega^{\alpha_\nu}(x) - \alpha_\nu(p \circ \tau_\mu(y)) - px) > 7s - 2s = 5s > \frac{1}{\nu},$$

hence

$$\delta_\Omega(\tau_\mu(y), \ \pi_\varkappa(p \circ \tau_\mu(y)) - p \circ \tau_\mu(y)) \geqslant \frac{5}{4}.$$

It follows that $\eta_m \circ \tau_\mu(y) \geqslant \frac{1}{m} + 1$ for all $\mu \geqslant m$ and $y \in V$. With U and n as in Lemma 3.10 we obtain

$$U \cap V \subset V_\mu$$

for all $\mu \geqslant \sup \{n, m + 1\}$, hence $x \in \bigcup_{\nu \in \mathbf{N}} \mathring{V}_\nu$.

2° $\tau_\mu(V_\nu) = V_\nu \cap \Omega_\mu$ is an immediate consequence of the definition of V_ν, and $\tau_\mu(V_\nu)$ is relatively compact for $\mu \geqslant \nu$, since $\tau_\mu(V_\nu) \subset L_\mu$.

3° Let $x \in (V_\nu \cap \Omega_\mu)\hat{}_{\mathcal{O}(\Omega_\mu)}$. For every $a \in E$

$$-\log \delta_\Omega(x, a) \leqslant \sup \{-\log \delta_\Omega(y, a) | y \in V_\nu \cap \Omega_\mu\},$$

since Ω_μ is a Stein manifold and $z \longmapsto -\log \delta_\Omega(z, a)$ is psh on Ω_μ (cf. [12, p. 116]). Therefore

$$d_\Omega^{\alpha_\nu}(x) = \inf \{\delta_\Omega(x, a) | a \in E,\ \alpha_\nu(a) \leqslant 1\} \geqslant d_\Omega^{\alpha_\nu}(V_\nu \cap \Omega_\mu) \geqslant \frac{1}{\nu}.$$

4° $-\log \eta_{\nu-1}$ is psh. Hence, as in 3°, for $x \in K_\nu$

$$-\log \eta_{\nu-1}(x) \leqslant \sup \{-\log \eta_{\nu-1}(y) | y \in V_\nu \cap \Omega_\mu\} \leqslant -\log\left(1 + \frac{1}{\nu}\right).$$

It follows that $K_\nu \subset X_{\nu-1}$.

4. Approximation.

In this section some results of Noverraz [20] on the approximation of analytic functions on a pseudoconvex domain are generalized and strengthened using the methods of Section 3.

First, we investigate whether a pseudoconvex domain Ω is holomorphically convex (i.e. $\hat{K}_{\mathcal{O}(\Omega)}$ is precompact for all compact $K \subset \Omega$). For schlicht domains satisfying the conditions of 3.1 this follows immediately from the fact that the convex hull of a compact subset of a locally convex Hausdorff space is precompact. However, for non-schlicht domains a separate consideration is necessary. The above question is closely related to the Levi problem. In fact, Oka showed in [22] that a pseudoconvex domain spread over $\mathbf{C}^n$ is holomorphically convex, to deduce then that it is a domain of existence.

4.1. Proposition. — *Let* (Ω, p) *be a domain spread over a sequentially complete, locally convex space* E *with an equicontinuous Schauder decomposition, and suppose there are*

$\alpha_\nu \in \mathrm{cs}(E)$, $\nu \in \mathbf{N}$, *with* $\Omega = \bigcup_{\nu \in \mathbf{N}} \Omega^{\alpha_\nu}$. *Then for every sequence* (x_n) *in* Ω *without an accumulation point there exists an analytic function* $f \in \mathcal{O}(\Omega)$ *with* $\sup |f(x_n)| = \infty$.

Proof. — When (px_n) contains no Cauchy sequence in E, then there exists $g \in \mathcal{O}(E)$ with $\sup |g(px_n)| = \infty$ according to Lemma 4.2. Hence, $\sup |f(x_n)| = \infty$, where

$$f = g \circ p \in \mathcal{O}(\Omega).$$

When (px_n) has a Cauchy subsequence we can assume $px_n \to a$ for a suitable $a \in E$. Assume that $\sup |f(x_n)| < \infty$ for all $f \in \mathcal{O}(\Omega)$. According to 3.1 there is a countable, admissible covering $\mathfrak{V}$ so that Ω is $A_{\mathfrak{V}}$-convex. The seminorm $\gamma : A_{\mathfrak{V}} \to \mathbf{R}$, $f \longmapsto \sup |f(x_n)|$, is continuous since

$$\{f \in A_{\mathfrak{V}} | \, \gamma(f) \leqslant 1\}$$

is a barrel in $A_{\mathfrak{V}}$ and $A_{\mathfrak{V}}$ is barrelled as a Fréchet space. Consequently there are a constant $C > 0$ and $U \in \mathfrak{V}$ with $\gamma(f) \leqslant C\|f\|_U$ for all $f \in A_{\mathfrak{V}}$. Since this inequality also holds for all powers of a given $f \in A_{\mathfrak{V}}$ ($A_{\mathfrak{V}}$ is a regular class) we can assume $C = 1$. Since Ω is $A_{\mathfrak{V}}$-convex there are $\alpha \in \mathrm{cs}(E)$ and $s > 0$ with $d_\Omega^\alpha(\hat{U}_{A_{\mathfrak{V}}}) > s$, hence $d_\Omega^\alpha(x_n) > s$ for all $n \in \mathbf{N}$. Moreover, α and s can be chosen in such a way that there exists $V \in \mathfrak{V}$ with $U_s^\alpha \subset V$. For all $f \in A_{\mathfrak{V}}$, $a \in E$ and $m, n \in \mathbf{N}$

$$|P_a^n f(x_m)| \leqslant \|P_a^n f\|_U,$$

since $x_m \in \hat{U}_{A_{\mathfrak{V}}}$ and $A_{\mathfrak{V}}$ is a regular class. Therefore, by the Cauchy inequalities

$$|P_a^n f(x_m)| \leqslant s^{-n}\|f\|_{U_s^\alpha} \leqslant s^{-n}\|f\|_V \quad \text{for} \quad \alpha(a) \leqslant 1.$$

It follows $|f(y)| \leqslant \|f\|_V \sum_{n=0}^{\infty} \left(\frac{s}{t}\right)^{-n}$ for $y \in B_\Omega^\alpha(x_m, t)$, $0 < t < s$, and $f \in A_{\mathfrak{V}}$ which implies $B_\Omega^\alpha(x_m, s) \subset \hat{V}_{A_{\mathfrak{V}}}$ for all $m \in \mathbf{N}$.

Now choose $N \in \mathbf{N}$ with $\alpha(px_m - a) < \frac{s}{2}$ for all $m \geqslant N$ and $z \in E$ with $a(z - a) < \frac{s}{2}$ and $z \in E_n$ for a suitable $n \in \mathbf{N}$. For every $m \geqslant N$ there exists a uniquely determined $z_m \in B^{\alpha}_{\Omega}(x_m, s)$ with $pz_m = z$, and $(z_m)_{m \geqslant N}$ has no accumulation point.

Let us suppose that E has a continuous norm. Then, according to the proof of 3.1, $\mathfrak{V} = (U_\nu)$ can be chosen to be compatible. Applying 3.8 we obtain

$$\Omega_n \cap (U_\nu)\hat{}_{A_{\mathfrak{V}}} \subset (\Omega_n \cap V_\nu)\hat{}_{\mathcal{O}(\Omega_n)}.$$

Hence, $\Omega_n \cap (U_\nu)\hat{}_{A_{\mathfrak{V}}}$ is compact in Ω_n. For a suitable subsequence (z_{m_μ}) of (z_m) with $z_{m_\mu} \in (U_{m_{\mu+1}})\hat{}_{A_{\mathfrak{V}}} \setminus (U_{m_\mu})\hat{}_{A_{\mathfrak{V}}}$ there are $f_\mu \in A_{\mathfrak{V}}$ with

$$\|f_\mu\|_{U_{m_\mu}} < 2^{-\mu} \quad \text{and} \quad |f_\mu(z_{m_\mu})| \geqslant \mu + 1 + \sum_{\nu=1}^{\mu-1} |f_\nu(z_{m_\mu})|.$$

Now $f = \Sigma f_\mu \in A_{\mathfrak{V}}$ satisfies $|f(z_{m_\mu})| \geqslant \mu$, i.e.

$$\sup |f(z_m)| = \infty.$$

This is a contradiction to $B^{\alpha}_{\Omega}(x_m, s) \subset \hat{V}_{A_{\mathfrak{V}}}$ for all $m \in \mathbf{N}$.

When E has no continuous norm, there exists a quotient Ω_φ of Ω spread over a space E_φ with an equicontinuous Schauder decomposition and a continuous norm (cf. 2.3 and 1.8). From $px_n \to a$ it follows that $(\varphi_\Omega(x_n))$ has no accumulation point, since φ_Ω separates the fibers of p. Moreover, $p \circ \varphi_\Omega(x_n) \to \varphi(a) \in E_\varphi$. Thus, as above, there is an analytic $g \in \mathcal{O}(\Omega_\varphi)$ with $\sup |g \circ \varphi_\Omega(x_n)| = \infty$ and

$$f = g \circ \varphi_\Omega \in \mathcal{O}(\Omega)$$

satisfies $\sup |f(x_n)| = \infty$. This completes the proof.

4.2. Lemma. — *Let* E *be a separable, locally convex Hausdorff space over* **C**, *and let* (a_n) *be a sequence in* E *with no Cauchy subsequence. Then there exists an analytic function* $g \in \mathcal{O}(E)$ *with* $\sup |g(a_n)| = \infty$.

The proof given in [28] for separable Banach spaces can be transferred.

4.3. Corollary. — *A pseudoconvex domain* Ω *spread over a Fréchet space (resp. over a Silva space) with an f.d. decomposition is holomorphically convex:* $\hat{K}_{\mathcal{O}(\Omega)}$ *is compact for every compact subset* K *of* Ω.

For the next proposition we need the following notations: $\mathscr{P}(\Omega)$ (resp. $\mathscr{P}_c(\Omega)$) denotes the set of plurisubharmonic (resp. continuous plurisubharmonic) functions on a domain Ω. For $V \subset \Omega$ and $Q \subset \mathscr{P}(\Omega)$, $\hat{V}_Q$ is given by

$$\hat{V}_Q = \{x \in \Omega | \varphi(x) \leqslant \sup_{y \in V} \varphi(y) \quad \text{for all} \quad \varphi \in Q\}.$$

4.4. Proposition. — *Let* E *be a locally convex Hausdorff space with an equicontinuous Schauder decomposition, and let* Ω *be a pseudoconvex domain spread over* E *with*

$$\Omega = \bigcup_{\nu \in \mathbf{N}} \Omega^{\alpha_\nu}$$

for a sequence (α_ν) *of continuous seminorms on* E. *Moreover, suppose there exists a compatible, admissible covering* $\mathfrak{V}$ *of* Ω. *(Such a covering exists when* E *has a continuous norm, cf. Section* 3.) *Then*

$$\hat{K}_{A_{\mathfrak{V}}} = \hat{K}_{\mathcal{O}(\Omega)} = \hat{K}_{\mathscr{P}_c(\Omega)}$$

for all compact subsets K *of* Ω.

Proof. — Let $K \subset \Omega$ be compact. Obviously,

$$\hat{K}_{\mathscr{P}_c(\Omega)} \subset \hat{K}_{\mathcal{O}(\Omega)} \subset \hat{K}_{A_{\mathfrak{V}}}$$

since $|f|$ is psh for every $f \in \mathcal{O}(\Omega)$. To show the reverse inclusion let $x \in \hat{K}_{A_{\mathfrak{V}}}$. There is $N \in \mathbf{N}$ with $\{x\} \cup K \subset \mathring{V}_N$, where (V_ν) is the compatible covering of Ω defining the admissible covering $\mathfrak{V} = (U_\nu)$. It follows that

$$\tau_n(x) \in (\tau_n(K))\hat{}_{\mathcal{O}(\Omega_n)} \quad \text{for all} \quad n \geqslant N.$$

Otherwise, $|f \circ \tau_n(x)| > \|f\|_{\tau_n(K)}$ for a suitable $f \in \mathcal{O}(\Omega_n)$, and according to 3.8 $\left(\text{with} \quad \varepsilon = \frac{1}{2}(|f \circ \tau_n(x)| - \|f \circ \tau_n\|_K)\right)$ there would exist $g \in A_{\mathfrak{V}}$ with $|g(x)| > \|g\|_K$ in contradiction to $x \in \hat{K}_{A_{\mathfrak{V}}}$. Because of $(\tau_n(K))\hat{}_{\mathcal{O}(\Omega_n)} = (\tau_n(K))\hat{}_{\mathscr{P}(\Omega_n)}$ (cf. [12, p. 116]) every $\varphi \in \mathscr{P}(\Omega)$ satisfies

$$\varphi \circ \tau_n(x) \leqslant \sup\{\varphi(y) | y \in \tau_n(K)\} = \sup\{\varphi \circ \tau_n(y) | y \in K\}.$$

Since $\tau_n \to \mathrm{id}_\Omega$ uniformly on compact subsets of Ω (cf. 2.2), this implies $\varphi(x) \leqslant \sup_{y \in \mathrm{K}} \varphi(y)$ for all $\varphi \in \mathscr{P}_c(\Omega)$, hence

$$x \in \hat{\mathrm{K}}_{\mathscr{P}_c(\Omega)}.$$

4.5. Proposition *(Oka-Weil). — Let* E, Ω, *and* $\mathfrak{V}$ *be as in* 4.4. *Furthermore, suppose that* E *is sequentially complete. If* K *is a compact subset of* Ω *with* $\mathrm{K} = \hat{\mathrm{K}}_{\mathcal{O}(\Omega)}$, *then every function which is analytic in a neighborhood of* K *can be approximated uniformly on* K *by functions in* $\mathrm{A}_{\mathfrak{V}}$.

Proof. — Let $\mathrm{V} \subset \Omega$ be an open neighborhood of K and let $f \in \mathcal{O}(\mathrm{V})$. There exist $\alpha \in \mathrm{cs}(\mathrm{E})$, $s > 0$ and $\mathrm{N} \in \mathbf{N}$ with $\mathrm{K}_s^\alpha \subset \mathrm{V}$ and $\mathrm{K}_s^\alpha \subset \mathring{\mathrm{V}}_\mathrm{N}$, where (V_ν) is again the compatible covering belonging to $\mathfrak{V}$. We show that there is $m \geqslant \mathrm{N}$ such that

$$(*) \qquad \tau_n(\mathrm{K})\hat{}_{\mathcal{O}(\Omega_n)} \subset \mathrm{K}_s^\alpha \cap \Omega_n \quad \text{for all} \quad n \geqslant m.$$

Otherwise there were $x_\nu \in (\tau_{n_\nu}(\mathrm{K}))\hat{}_{\mathcal{O}(\Omega_{n_\nu})}, \nu \in \mathbf{N}$, with $x_\nu \notin \mathrm{K}_s^\alpha$. For all $g \in \mathcal{O}(\Omega)$

$$|g(x_\nu)| \leqslant \|g\|_{\tau_{n_\nu}(\mathrm{K})} = \|g \circ \tau_{n_\nu}\|_\mathrm{K} \to \|g\|_\mathrm{K},$$

hence $\sup |g(x_\nu)| < \infty$. According to 4.1 we can assume that (x_ν) converges to a point $x_0 \in \Omega$. Now $x_0 \in \mathrm{K}$, since $|g(x_0)| = \lim |g(x_\nu)| \leqslant \lim \|g \circ \tau_{n_\nu}\|_\mathrm{K} = \|g\|_\mathrm{K}$ (cf. 2.2). This contradicts $x_\nu \notin \mathrm{K}_s^\alpha$.

Now let $\varepsilon > 0$. Choose $n \geqslant m$ such that

$$\|f \circ \tau_n - f\|_\mathrm{K} < \frac{\varepsilon}{3}.$$

According to the classical Oka-Weil Theorem for the Stein manifold Ω_n there exists $h \in \mathcal{O}(\Omega_n)$ with $\|h - f\|_{\tau_n(\mathrm{K})} < \frac{\varepsilon}{3}$, since f is analytic in a neighborhood of $(\tau_n(\mathrm{K}))\hat{}_{\mathcal{O}(\Omega_n)}$ (*). Due to 3.8 there is $g \in \mathrm{A}_{\mathfrak{V}}$ with $\|g - h \circ \tau_n\|_{\mathrm{V}_n} < \frac{\varepsilon}{3}$. Therefore, $\|g - f\|_\mathrm{K} < \varepsilon$ which completes the proof.

From 4.5 one can deduce a result on Runge pairs. (The proof is the same as for finite dimensional E, cf. [12, Th. 4.3.3].)

4.6. Corollary. — *Let* E, Ω *and* $\mathfrak{B}$ *be as in* 4.5, *and let* $\Sigma \subset \Omega$ *be a pseudoconvex subdomain of* Ω. *The following properties are equivalent*:

1° *Every function in* $\mathcal{O}(\Sigma)$ *can be approximated by functions in* $A_{\mathfrak{B}}$ *uniformly on every compact subset of* Σ.

2° $\hat{K}_{A_{\mathfrak{B}}} = \hat{K}_{\mathcal{O}(\Sigma)}$ *for every compact subset* K *of* Σ.

3° $\hat{K}_{A_{\mathfrak{B}}} \cap \Sigma = \hat{K}_{\mathcal{O}(\Sigma)}$ *for every compact subset* K *of* Σ.

4° *The closure of* $\hat{K}_{A_{\mathfrak{B}}} \cap \Sigma$ *is compact in* Σ *for every compact subset* K *of* Σ.

Lemma 3.8 implies that the restriction mappings

$$A_{\mathfrak{B}} \to \mathcal{O}(\Omega_n)$$

have dense image whenever $\mathfrak{B}$ is a compatible, admissible covering of Ω. With the aid of 4.5 this can be generalized to arbitrary complemented subspaces F of E:

4.7. Proposition. — *Let* E, Ω *and* $\mathfrak{B}$ *be as in* 4.5, *and let* F *be a complemented subspace of* E. *Then the restriction map* $A_{\mathfrak{B}} \to \mathcal{O}(\Omega \cap p^{-1}(F))$ *has a dense image when*

$$\mathcal{O}(\Omega \cap p^{-1}(F))$$

is endowed with the compact open topology.

Proof. — Let $\pi : E \to E$ be a linear, continuous projection with $\pi(E) = F$. There exist an open neighborhood X of $\Omega \cap p^{-1}(F)$ and a continuous mapping

$$\tau : X \to \Omega \cap p^{-1}(F)$$

with $\pi \circ p|X = p \circ \tau$ and $\tau|\Omega \cap p^{-1}(F) = \mathrm{id}_{\Omega \cap p^{-1}(F)}$. Let $f \in \mathcal{O}(\Omega \cap p^{-1}(F))$. Then $f \circ \tau$ is analytic in a neighborhood of every compact subset of $\Omega \cap p^{-1}(F)$, hence the assertion follows from 4.5.

It is an open question whether or not the above restriction map is onto. The restriction map $A_{\mathfrak{B}} \to \mathcal{O}(\Omega_n)$ is onto when $\dim_{\mathbf{C}} E_{n+1} = \dim_{\mathbf{C}} E_n + 1$ for all $n \in \mathbf{N}$.

Final remark. — Certain results of this section can be generalized with the aid of the methods given in Section 1.

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