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AN INVERSION FORMULA AND A NOTE ON THE RIESZ KERNEL

by Andrejs DUNKELS

0. Introduction.

For a certain class of kernels K every compact set $E \subset \mathbf{R}^m$ has an equilibrium distribution Q, whose potential

$$U_K^Q = K * Q$$

is constant, equal to 1, on Int E (see [2, p. 126], cf. also [1], [6]). If the kernel is the Riesz kernel, $K(r) = r^{\alpha-m}$, of order α with $0 < \alpha \leqslant 2$, then Q is identical with the capacitary measure λ of E (see [2, p. 144]). When $\alpha > 2$ then Q no longer coincides with λ; the capacitary potential U_α^λ is then strictly greater than 1 on Int E; moreover the support of λ lies on the boundary of E, which is not the case for Q unless α is an even integer. (See [2, p. 125], [5, pp. 103-104], cf. also [4, pp. 10, 30-38].)

Wallin [9] has studied the regularity properties of Q when $0 < \alpha < \min(2, m)$ and proved, among other things, that the restriction of Q to Int E is an absolutely continuous measure with analytic density which may be expressed by an explicit formula. In this paper we shall extend Wallin's result to $0 < \alpha < m$ (Theorem 2). We shall use a kind of inversion formula (Theorem 1) by which it is possible to express Q in terms of the potential and an elementary solution D_K for K, i.e. a distribution such that $D_K * K = \delta$. The method employed is based on Fourier transforms of distributions. One of the difficulties encountered due to the fact that one is led to dealing with convolutions of distributions with non-compact support. The method in Wallin [9] is not applicable for $\alpha \geqslant 2$ (cf. [9, p. 75]).

A result similar to Theorem 2 holds for the Bessel kernel of order α (see [4, p. 7]).

I wish to thank Professor Hans Wallin for suggesting the topic of this paper and for his kind interest, advice and guidance during the work with it.

1. Statement of main results.

In connection with distributions we have adopted the notation of Schwartz [8]. For details of definitions and notations the reader is referred to [4]. The kernels considered will be such that they satisfy Deny's Condition (A), see [2, p. 119], which is as follows. (Cf. also [3].)

Condition (A). — *A distribution* K *satisfies Condition* (A) *if it is of positive type,* $\hat{K}$ *is a positive function in* $L^1_{Loc}(\mathbf{R}^m)$, *and there exists an integer* $q \geqslant 0$ *such that*

$$\int \frac{\hat{K}(\xi)}{(1+|\xi|^2)^q}\,d\xi < \infty \qquad \text{and} \qquad \int \frac{1}{\hat{K}(\xi)(1+|\xi|^2)^q}\,d\xi < \infty.$$

Here and elsewhere the integrations are to be extended over the whole space $\mathbf{R}^m$, when no limits of integration are indicated.

Our first theorem is a kind of inversion formula for potentials $U_K^T = K * T$ (see [2, p. 118]). We shall actually use the formula for the Riesz kernel only but we present it in a more general setting since the proof in the special case is no simpler.

Theorem 1. *(Inversion formula) — Let* K *be a distribution satisfying Condition* (A). *Assume that* $K \in (\mathscr{D}'_{L^q})$, $1 \leqslant q \leqslant \infty$, *that* D_K *exists and* $D_K \in (\mathscr{D}'_{L^1})$. *If* $T \in (\mathscr{E}')$ *then for every* $\varphi \in (\mathscr{D})$ *we have*

$$(1) \qquad T(\varphi) = U_K^T(\check{D}_K * \varphi).$$

COROLLARY 1. — *If, in addition,* T *has finite energy* (see [2], [3]) $I_K(T) = \int K(\xi)|T(\xi)|^2\, d\xi$, K *is a positive distribution, and* $U_K^T . \nu \in L^1(\mathbf{R}^m)$, *where* $\nu = \check{D}_K * \varphi$, *then*

$$(2) \qquad T(\varphi) = \int U_K^T(x)\nu(x)\, dx.$$

COROLLARY 2. — *If, in addition to the assumptions of Cor.* 1, $E \subset \mathbf{R}^m$ *is compact and* $T = Q$ *is the equilibrium distribution of* E *with respect to* K *then*

$$(3) \quad Q(\varphi) = \hat{\check{D}}_K(0) \int \varphi(x)\, dx + \int (U_K^Q(x) - 1)(\check{D}_K * \varphi)(x)\, dx.$$

THEOREM 2. — *Assume that* $E \subset \mathbf{R}^m$ *is compact and has non-empty interior. Let* Q *be the equilibrium distribution of* E *with respect to the Riesz kernel* $r^{\alpha-m}$ *of order,* α, $0 < \alpha < m$. *Then* $Q|\,\mathrm{Int}\, E$ *is an absolutely continuous measure. Its density* f_α *is, after perhaps modification on a set of Lebesgue measure zero, an analytic function on* $\mathrm{Int}\, E$ *defined by*

$$(4) \quad \begin{cases} f_\alpha(x) = B(\alpha, m) \int (U_\alpha^Q(y) - 1)|x-y|^{-(\alpha+m)}\, dy, & \text{if} \quad \alpha \neq 2k, \\ f_{2k}(x) = 0, \end{cases}$$

where k *is a positive integer and* $B(\alpha, m)$ *a constant given by*

$$B(\alpha, m) = -\frac{\alpha}{2} \cdot \pi^{-(m+1)} \sin\frac{\alpha\pi}{2}\, \Gamma\left(\frac{m-\alpha}{2}\right) \Gamma\left(\frac{m+\alpha}{2}\right).$$

2. Some lemmas.

In this section we present lemmas needed to prove our results. Some of the lemmas might be of intrinsic interest.

LEMMA 1. — (Schwartz [8, p. 201]) *Assume that* $1 \leqslant p \leqslant \infty$ *and* $\psi \in (\mathcal{D})$. *Then* $T \in (\mathcal{D}'_{L^p})$ *if and only if* $T * \psi \in L^p(\mathbf{R}^m)$.

LEMMA 2. — (Schwartz [8, p. 270]) If $S \in (\mathcal{D}'_{L^p})$, $1 \leqslant p \leqslant 2$, *and* $T \in (\mathcal{D}'_{L^q})$, $1 \leqslant q \leqslant 2$, *then*

$$\mathcal{F}(S * T) = \mathcal{F}(S) . \mathcal{F}(T).$$

We proceed with two lemmas concerning the behaviour for large $|x|$ of certain potentials. The first one is of « Weyl type » and the second is well-known for measures rather than distributions.

Lemma 3. — *Let* E *and* F *be compact subsets of* $\mathbf{R}^m$. *If* $\mathrm{T} \in (\mathscr{E}')$, *with* $\operatorname{supp} \mathrm{T} = \mathrm{E}$, *and* K *is a kernel such that* $\mathrm{K} \in \mathrm{C}^\infty(\mathbf{R}^m \setminus \mathrm{F})$, *then* $\mathrm{U}_\mathrm{K}^\mathrm{T} \in \mathrm{C}^\infty((\mathrm{E} + \mathrm{F})^c)$.

The straight-forward proof is omitted here. (See [4, p. 11].)

Lemma 4. — *Assume that* $\mathrm{F} \subset \mathbf{R}^m$ *is compact and* K *a kernel such that* $\mathrm{K} \in \mathrm{C}^\infty(\mathbf{R}^m \setminus \mathrm{F})$ *and, for each multiindex* ν,

$$\partial^\nu \mathrm{K}(x) \to 0, \qquad \text{as} \qquad |x| \to \infty.$$

Then for any $\mathrm{T} \in (\mathscr{E}')$ *we have*

$$\mathrm{U}_\mathrm{K}^\mathrm{T}(x) \to 0, \qquad \text{as} \qquad |x| \to \infty.$$

The proof uses (see [4, pp. 12-13]) Lemma 3 and a representation (see [8, p. 91]) of T as a finite sum of derivatives, in the distribution sense, of continuous functions with compact support; the differentiations are shifted from these functions to the kernel.

Corollary. — *Let* F *and* K *be as in Lemma 4. Then for any multiindex* τ *and any* $\mathrm{T} \in (\mathscr{E}')$ *we have*

$$\partial^\tau \mathrm{U}_\mathrm{K}^\mathrm{T}(x) \to 0, \qquad \text{as} \qquad |x| \to \infty.$$

The corollary is proved by shifting all the differentiations to T, thereby producing another distribution in $(\mathscr{E}')$ to which Lemma 4 is applied.

Our next lemma is a companion of Lemma 2 above. Lemma 2 does not contain our lemma but is used in the proof. The reason why Lemma 5 cannot be proved exactly as Lemma 2 is that, although each distribution in $(\mathscr{D}'_{\mathrm{L}^p})$ has a representation as a finite sum of derivatives, in the distribution sense, of functions in $\mathrm{L}^p(\mathbf{R}^m)$ (see [8, p. 201]), their Fourier transforms are not functions when $p > 2$ (cf. [8, p. 270]).

Lemma 5. — *Assume that*

$$\mathrm{S} \in (\mathscr{D}'_{\mathrm{L}^q}), \quad 1 \leqslant q \leqslant \infty, \quad \mathscr{F}(\mathrm{S}) \in \mathrm{L}^1_{\mathrm{Loc}}(\mathbf{R}^m),$$

and $T \in (\mathscr{D}'_{L^1})$. *Then*

$$\mathscr{F}(S^*T) = \mathscr{F}(S) . \mathscr{F}(T). \tag{5}$$

The proof uses a sequence of distributions to which Lemma 2 is applicable and which converges to S in an appropriate way. A complete proof is found in [4, pp. 14-16].

3. Proof of the inversion formula.

Choose $\varphi \in (\mathscr{D})$. From $U_K^T = K * T$ one *formally* obtains

$$D_K * U_K^T = T, \tag{6}$$

and so, again formally,

$$T(\varphi) = \text{Tr. } T * \check{\varphi} = \text{Tr. } U_K^T * (D_K * \check{\varphi}) = U_K^T(\check{D}_K * \varphi), \tag{7}$$

which gives (1).

If $\operatorname{supp} K$ is non-compact then so is $\operatorname{supp} U_K^T$ as well as $\operatorname{supp} D_K$, unless D_K is a pointmass (cf. [8, p. 211]), and so the above formal uses of the associative law for convolutions need justification.

Furthermore it is necessary to make certain that the right hand side of (1) is well defined, even though $\check{D}_K * \varphi$ is not a testfunction. That this is indeed the case is readily checked (see [4, p. 17]) with the aid of Lemma 1 and the fact that $(\mathscr{D}_{L^1}) \subseteq (\mathscr{D}_{L^p})$ for every $p \geqslant 1$, since $\check{D}_K * \varphi \in (\mathscr{D}_{L^1})$ (see below).

It remains thus to justify (6) and the second equality of (7).

Since $D_K \in (\mathscr{D}'_{L^1})$ and $U_K^T \in (\mathscr{D}'_{L^q})$ their convolution $D_K * U_K^T$ is a well defined distribution in $(\mathscr{D}'_{L^q})$ ([8, p. 203]). The kernel K is assumed to satisfy Condition (A), hence $\mathscr{F}(K) \in L^1_{\text{Loc}}(\mathbf{R}^m)$. Therefore also

$$\mathscr{F}(U_K^T) = \mathscr{F}(K) . \mathscr{F}(T) \in L^1_{\text{Loc}}(\mathbf{R}^m).$$

Furthermore Lemma 1 gives $D_K * \check{\varphi} \in L^1(\mathbf{R}^m)$, and since for any multiindex $\varkappa$ we have $\partial^\varkappa(D_K * \check{\varphi}) = D_K * (\partial^\varkappa \check{\varphi})$, we conclude, by Lemma 1, that in fact $D_K * \check{\varphi}$ is in $(\mathscr{D}_{L^1})$, hence considered as a distribution, in $(\mathscr{D}'_{L^1})$. Therefore also $U_K^T * (D_K * \check{\varphi})$

is well defined. Thus we can use Lemma 5 to obtain

$$\begin{aligned}\mathscr{F}(\mathrm{D_K} * \mathrm{U_K^T}) &= \mathscr{F}(\mathrm{D_K}) . \mathscr{F}(\mathrm{U_K^T}) \\ &= \mathscr{F}(\mathrm{D_K}) . \mathscr{F}(\mathrm{K}) . \mathscr{F}(\mathrm{T}) = \mathscr{F}(\mathrm{T}),\end{aligned}$$

which establishes (6); and

$$\mathscr{F}(\mathrm{U_K^T} * (\mathrm{D_K} * \check{\varphi})) = \mathscr{F}(\mathrm{U_K^T}) . \mathscr{F}(\mathrm{D_K} * \check{\varphi}) = \mathscr{F}(\mathrm{T}) . \mathscr{F}(\check{\varphi}),$$

which shows that we have

$$\mathrm{U_K^T} * (\mathrm{D_K} * \check{\varphi}) = \mathrm{T} * \check{\varphi},$$

and so the second step of (7) is justified, and the proof complete.

Proof of Corollary 1. — Take $(\psi_j)_{j=1}^{\infty} \subset (\mathscr{D})$ so that, as $j \uparrow \infty$, $0 \leqslant \psi_j(x) \uparrow 1$ for every $x \in \mathbf{R}^m$, and, for each multiindex ν, $\partial^\nu \psi_j(x) \to 0$ uniformly on $\mathbf{R}^m$.

With $\rho = \check{\mathrm{D}}_\mathrm{K} * \varphi$ we conclude, as before, that $\rho \in (\mathscr{D}_{\mathrm{L}^1})$.

Since T has finite energy and K is positive, $\mathrm{U_K^T}$ may be interpreted as a function in $\mathrm{L^1_{Loc}}(\mathbf{R}^m)$ (see [2, p. 138]). Introducing the testfunctions $\rho_j = \psi_j . \rho$ we have

$$\mathrm{U_K^T}(\rho_j) = \int \mathrm{U_K^T}(x) \rho_j(x)\, dx. \tag{8}$$

On the right hand side of (8) the integrand tends to $\mathrm{U_K^T} . \rho \in \mathrm{L}^1(\mathbf{R}^m)$ for every $x \in \mathbf{R}^m$, and so by Lebesgue's dominated convergence theorem the right hand side tends to $\int \mathrm{U_K^T}(x)\rho(x)\, dx$, as $j \to \infty$. Moreover,

$$\psi_j . \rho \to \rho \qquad \text{in} \qquad (\mathscr{D}_{\mathrm{L}^1}), \qquad \text{as} \qquad j \to \infty,$$

hence in $(\mathscr{D}_{\mathrm{L}^{q'}})$, where $1/q + 1/q' = 1$, and so

$$\mathrm{U_K^T}(\rho_j) \to \mathrm{U_K^T}(\rho) \qquad \text{in} \qquad (\mathscr{D}'_{\mathrm{L}^q}), \qquad \text{as} \qquad j \to \infty.$$

This completes the proof.

Remark. — If the distribution T is a measure $\mu \geqslant 0$ (defined at least on the Borel sets of $\mathbf{R}^m$) and $\mathrm{K} = r^{\alpha-m}$, $0 < \alpha < m$, then our formula (2) reduces to the inversion formula (3.5) derived by Wallin [10, p. 155].

Proof of Corollary 2. — We now have $U_K^Q = 1$ on Int E (see [2, p. 127]), hence (2) gives

$$\begin{aligned} Q(\varphi) &= \int_{\mathrm{Int}\,E} 1 \,.\, \psi(x)\, dx + \int_{(\mathrm{Int}\,E)^c} U_K^Q(x)\psi(x)\, dx \\ &= \hat{\psi}(0) + \int (U_K^Q(x) - 1)\psi(x)\, dx, \end{aligned}$$

where the last integral is actually extended over $(\mathrm{Int}\,E)^c$ only.

Since $\hat{\psi} = (\check{D}_K * \varphi)^\wedge = \hat{\check{D}}_K \,.\, \hat{\varphi}$, (3) follows, and the proof is complete.

4. Proof of Theorem 2.

If $\alpha \neq 2k$ (k integer) then a fundamental solution, D_α, of the kernel under consideration, $r^{\alpha-m}$, $0 < \alpha < m$, is known to be (cf. [2, p. 153], [4, pp. 20-21], [7])

$$D_\alpha = B(\alpha, m) \,.\, \mathrm{Pf}\,.\, r^{-(\alpha+m)},$$

where Pf. denotes Hadamard's « partie finie » and $B(\alpha, m)$ is the constant in (4).

With the aid of Lemma 1 we now get (for details see [4, p. 21])

$$D_\alpha \in (\mathscr{D}'_{L^1}) \quad \text{and} \quad r^{\alpha-m} \in (\mathscr{D}'_{L^q}), \quad \text{with} \quad q = \frac{m+1}{m-\alpha}.$$

Assume that $E \subset \mathbf{R}^m$ is compact and has non-empty interior. Let Q be the equilibrium distribution of E. Choose $\varphi \in (\mathscr{D})$ with $\mathrm{supp}\,\varphi \subset \mathrm{Int}\,E$, and set $\psi_\alpha = D_\alpha * \varphi$. Then for every $x \in (\mathrm{Int}\,E)^c$ we obtain

$$\begin{aligned} \psi_\alpha(x) &= D_\alpha(\tau_x \check{\varphi}) \\ &= B(\alpha, m) \,.\, \lim_{\varepsilon \to 0} \Big(\int |t|^{-(\alpha+m)} (\tau_x \check{\varphi})(t)\, dt \\ &\quad + \sum_k H(k, m)\, \Delta^k(\tau_x \check{\varphi})(0)\, \frac{\varepsilon^{2k-\alpha}}{2k - \alpha} \Big) \\ &= B(\alpha, m) \int |x - t|^{-(\alpha+m)} \varphi(t)\, dt, \end{aligned}$$

where $H(k, m)$ are explicit constants emerging from the definition of Pf. (see e.g. [8, p. 44]).

The Riesz kernel satisfies Condition (A). Furthermore it is a positive distribution and Q has finite energy, hence

$U_\alpha^Q \in L^1_{\mathrm{Loc}}(\mathbf{R}^m)$ (cf. Cor. 1 above). For large $|x|$, according to Lemma 4, U_α^Q is a bounded function, and, according to Lemma 1, $\nu_\alpha \in (\mathcal{D}_{L^1})$. Hence $U_\alpha^Q . \nu_\alpha \in L^1(\mathbf{R}^m)$. Thus (3) is applicable, and since we have $\check{D}_\alpha = D_\alpha$ and $\hat{D}_\alpha(0) = 0$, we obtain

$$\begin{aligned} Q(\varphi) &= \int (U_\alpha^Q(x) - 1)(D_\alpha * \varphi)(x)\, dx \\ &= \int (U_\alpha^Q(x) - 1)\left(B(\alpha, m) \int |x - t|^{-(\alpha+m)} \varphi(t)\, dt\right) dx. \end{aligned}$$

The order of integration may be changed by virtue of Fubini's theorem giving

$$Q(\varphi) = \int \varphi(t) \left(B(\alpha, m) \int (U_\alpha^Q(x) - 1)|x - t|^{-(\alpha+m)}\, dx\right) dt.$$

A straight forward calculation shows that the inmost integral defines an analytic function on Int E. This completes the proof when $\alpha \neq 2k$.

Finally for the polyharmonic case $\alpha = 2k$, k integer, we have (see [7, p. 4], [2, p. 153], [4, p. 23])

$$D_{2k} = b(2k, m) . \Delta^k \delta, \quad \text{where} \quad b(2k, m) = \frac{(-1)^k \Gamma\left(\frac{m}{2} - k\right)}{2^{2k} \pi^{\frac{m}{2}} (k-1)\,!},$$

and so $\operatorname{supp}(D_{2k} * \varphi) \subset \operatorname{Int} E$ for any $\varphi \in (\mathcal{D})$ with $\operatorname{supp} \varphi \subset \operatorname{Int} E$. Hence, since the integration in (3) is in fact extended over $(\operatorname{Int} E)^c$ only, we get

$$Q(\varphi) = \int (U_{2k}^Q(x) - 1)(D_{2k} * \varphi)(x)\, dx = 0,$$

whence $f_{2k}(x) = 0$, and so the proof is complete.

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