ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 26, nº 1 (1976), p. 225-237
<http://www.numdam.org/item?id=AIF_1976__26_1_225_0>

Ann. Inst. Fourier, Grenoble
26, 1 (1976), 225-237

EXOTIC CHARACTERISTIC CLASSES AND SUBFOLIATIONS

by Luis A. CORDERO and P.M. GADEA (*)

1. Introduction.

The development in the last years of the study of topological invariants associated to a foliated structure on a differentiable manifold(**) (usually called exotic characteristic classes of the foliation) has been well known.

Within the general context of this study, the following problem appears in a canonical way : let M be a differentiable manifold on which two foliations F_1 and F_2 are defined, and such that $F_1 \subset F_2$, that is, every leaf of F_2 is, itself, foliated by leaves of F_1 ; briefly, F_1 is said to be subfoliation of F_2 ; in fact, this geometrical structure on M can be described as a special type of multifoliate structure (in the sense of Kodaira-Spencer ({8})) ; now, we present two questions : 1) does a relation exist between exotic classes of F_1 and F_2 ?, and 2) : is it possible to give a topological obstruction to the existence of such a structure on M ?.

In this paper we give the answer to these questions, by studying the problem through a more general situation and using Lehmann's techniques ({9}, {10}). For this purpose, we consider the following situation : let Q_i , $i = 1, 2$, be two G_i-principal fibre bundles over M, and let $\Pi : Q_1 \rightarrow Q_2$ be a morphism of principal fibre bundles (over the identity of M) ; by an appropiate choice of connections on these fibre bundles we point out a relation between the images of Lehmann's exoticism associated to those connections (Theorem 4.5) ; in the special case of F_1 and F_2, two foliations as above, that relation gives the answer to our questions : every exotic characteristic class of F_2 is also an

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(*) This author's work has been supported by a fellowship of C.S.I.C. (Spain).

(**) Always manifolds will mean paracompact differentiable manifold of class C^∞.

exotic characteristic class of F_1 ; in fact, this result can be expressed as a topological obstruction F_1 to be a subfoliation of F_2.

2. Notations and basic concepts.

Let M be a differentiable manifold. We shall denote $\mathfrak{X}(M)$ the Lie algebra of vector fields and $A^*(M)$ the exterior algebra of differential forms on M.

Given a G-principal fibre bundle $E \to M$, G being the structural Lie group, ω indistinctly denotes an (infinitesimal) connection on the bundle or the 1-form of that connection ; I(G) is the algebra of invariant polynomials over the Lie algebra $\underline{G}$ of G ; I(G) is a graded algebra, $I(G) = \underset{k \geq 0}{\oplus} I^k(G)$ and $I^+(G)$ denotes its maximal ideal

$$I^+(G) = \underset{k \geq 1}{\oplus} I^k(G).$$

Denote by $\lambda_\omega : I(G) \to A^*(M)$ the Chern-Weil homomorphism, defined by $\lambda_\omega(f) = f(\Omega)$, for $f \in I(G)$ and Ω being the curvature form of ω. If $I = [0, 1]$ is the unit interval, $\int_0^1 : A^k(M \times I) \to A^{k-1}(M)$ denotes the integration along the fibre of $M \times I \to M$. If ω' is another connection on E, we write $[\overrightarrow{\omega, \omega'}]$ the connection on $E \times I \to M \times I$ defined by

$$[\overrightarrow{\omega, \omega'}]\left(\frac{\partial}{\partial t}\right) = 0, \quad [\overrightarrow{\omega, \omega'}]|_{E \times \{t\}} = t\omega' + (1-t)\omega$$

and by $\Delta_{\omega,\omega'} : I^k(G) \to A^{2k-1}(M)$ the composition $\int_0^1 . \lambda_{[\overrightarrow{\omega,\omega'}]}$.

As it is well known, λ_ω induces an homomorphism λ :

$$I(G) \to H^{\text{even}}(M, \mathbb{R})$$

which is independent of ω.

Let $J \subset I(G)$ be a homogeneous ideal ; ω is said a J-connection if $\lambda_\omega(f) = 0$ for every $f \in J$. If P denotes a property of the degree of elements of I(G), J(P) denotes the homogeneous ideal generated by the elements satisfying the property P. For example, if $\dim M = n$, every connection on E is a $J\left(> \left[\frac{n}{2}\right]\right)$-connection.

If $G = Gl(q, \mathbb{R})$, it is $I(G) = \mathbb{R}[c_1, \ldots, c_q]$, where $c_1, \ldots, c_q$ are the usual generators given by

$$\det(I + tA) = 1 + \sum_{i=1}^{q} c_i(A)t^i \text{ , for every } A \epsilon gl\,(q, \mathbb{R})$$

If $Q \to M$ is a vector bundle, ∇ denotes the derivation law of a linear connection on Q. Thus, every metric connection on Q is a J(odd)-connection.

If $Q \to M$ is the normal bundle of a foliation on M, of codimension q, and ∇ is a basic connection on Q (in the sense of Bott ({1})), then ∇ is a $J(>q)$-connection.

3.The Lehmann's exoticism (({9}), ({10})).

Let E be a G-principal fibre bundle on M. Consider J, J′ homogeneous ideales of I(G) ; if $f \epsilon I^k(G)$, we write

$$\bar{f} = f\,(\text{mod } J)\ , \bar{\bar{f}} = f\,(\text{mod } J')$$

and introduce a graduation on the quotient algebras $I(G)/J$, $I(G)/J'$ by $\deg \bar{f} = \deg \bar{\bar{f}} = 2k$, for every $f \epsilon I^k(G)$; also, we shall denote $\Lambda(I^+(G))$ the exterior algebra over $\mathbb{R}$ generated by the elements of $I^+(G)$ and define a graduation on $\Lambda(I^+(G))$ by $\deg f = 2k - 1$, for every $f \epsilon I^k(G)$, $k > 0$. Then, consider the graded algebra

$$\hat{W}(J, J') = I(G)/J \otimes_{\mathbb{R}} I(G)/J' \otimes_{\mathbb{R}} \Lambda(I^+(G))$$

and $I(G)/J$, $I(G)/J'$, $\Lambda(I^+(G))$ are canonically identified to subalgebras of $\hat{W}(J, J')$; $I^+(G)$ can be identified to one part of $\Lambda(I^+(G)) \subset \hat{W}(J, J')$ by the isomorphism

$$h : I^+(G) \to \Lambda^1(I^+(G))$$

and, if $G = Gl(q, \mathbb{R})$, we write $h_i = h(c_i)$.

$\hat{W}(J, J')$ is endowed with a structure of graded differential algebra by defining a differential (of degree 1)

$$d(\bar{f}) = d(\bar{\bar{f}}) = 0, \quad \text{for } f \epsilon I(G)$$

$$d(f) = \bar{f} - \bar{\bar{f}}, \quad \text{for } f \epsilon I^+(G)$$

and, clearly, $d^2 = 0$.

If ω is a J-connection and ω' is a J'-connection on E, a homomorphism of graded algebras $\rho_{\omega\omega'} : \hat{W}(J, J') \to A^*(M)$ is defined by

$$\rho_{\omega\omega'}(\bar{f}) = \lambda_\omega(f)$$

$$\rho_{\omega\omega'}(\bar{\bar{f}}) = \lambda_{\omega'}(f)$$

$$\rho_{\omega\omega'}(f_1 \wedge \ldots \wedge f_r) = \Delta_{\omega,\omega'}(f_1) \wedge \ldots \wedge \Delta_{\omega,\omega'}(f_r) \ , \text{ for } f_i \in I^+(G)$$

and, in cohomology, $\rho_{\omega\omega'}$ induces a homomorphism of graded algebras

$$\rho^*_{\omega\omega'} : H^*(\hat{W}(J, J')) \to H^*(M, \mathbb{R})$$

The elements of Im $\rho^*_{\omega\omega'}$ are said to be the exotic characteristic classes associated to J, J', ω and ω'.

Let $J \subset I(G)$ be a homogeneous ideal and ω_0, ω_1 two J-connections on E ; ω_0 and ω_1 are said to be differentiably J-homotopic if there does exist a J-connection $\tilde{\omega}$ on $E \times I \to M \times I$ such that

$$\tilde{\omega}|_{E \times \{0\}} = \omega_0 \ , \ \tilde{\omega}|_{E \times \{1\}} = \omega_1$$

and, in a more general form, ω_0 and ω_1 are said to be J-homotopic if there does exist a finite sequence $\omega_0 = \omega_{s_0}, \omega_{s_1}, \ldots, \omega_{s_k} = \omega_1$ of J-connections such that, for every $i = 0, 1, \ldots, k-1$, ω_{s_i} and $\omega_{s_{i+1}}$ are differentiably J-homotopic. A set C of connections on E is said to be J-connected if it is not-empty and any two connections in C are J-homotopic.

PROPOSITION 3.1. – *Im $\rho^*_{\omega\omega'}$ depends only on the J-connected component of ω and the J'-connected component of ω'.*

In particular, if C is the set of basic connections on the transversal bundle Q of a q-codimensional foliation on M and C' is the set of metric connections on Q, Lehmann shows that C is $J(>q)$-connected and C' is J(odd)-connected ; moreover, in this case $\hat{W}(J(>q), J(\text{odd}))$ has the same cohomology that its subalgebra

$$WO_q = \mathbb{R}[c_1, \ldots, c_q] / J(>q) \otimes_{\mathbb{R}} \Lambda(h_1, h_3, \ldots, h_{(q)})$$

where (q) denotes the largest odd integer $\leqslant q$ and $h_i = h(c_i)$. Therefore

PROPOSITION 3.2. – *The homomorphism* $\rho^*_{\nabla\nabla'}$:

$$H^*(WO_q) \to H^*(M, \mathbb{R})$$

does not depend on the choice of $\nabla \in C$ *and* $\nabla' \in C'$.

4. Homomorphism of principal fibre bundles and the Lehmann's exoticism.

In this paragraph we shall consider the following situation : let $Q_i \to M$ be a G_i-principal fibre bundle ($i = 1,2$) and let

$$\begin{tikzcd} Q_1 \arrow[rr, "\Pi"] \arrow[dr] & & Q_2 \arrow[dl] \\ & M & \end{tikzcd}$$

a homomorphism of principal fibre bundles ; also, denote

$$\Pi : G_1 \longrightarrow G_2$$

the corresponding homomorphism of Lie groups and assume that Π is surjective but not a submersion in general, e.g.

$$d\Pi : \underline{G}_1 \longrightarrow \underline{G}_2$$

is not of maximal rank in general ; the linear mapping $d\Pi$ permits to define :

DEFINITION 4.1. – *If* $f \in I^k(G_2)$, $i(f)$ *is defined by*

$$i(f)(X_1 \otimes \ldots \otimes X_k) = f(d\Pi(X_1) \otimes \ldots \otimes d\Pi(X_k)),$$

for every $X_j \in \underline{G}_1$, $j = 1, 2 \ldots k$

A direct application of this definition shows

PROPOSITION 4.2. – *For every* $f \in I(G_2)$, $i(f) \in I(G_1)$ *and*

$$i : I(G_2) \to I(G_1)$$

is a homomorphism of graded algebras. Moreover, if $d\Pi$ is of maximal rank, then i is injective.

Let $J_2 \subset I(G_2)$ be an homogeneous ideal and J_1 an arbitrary homogeneous ideal of $I(G_1)$, such that $J_1 \supseteq i(J_2)$ (in particular, J_1 could be thought as the homogeneous ideal generated by the elements of $i(J_2)$).

THEOREM 4.3. – *Let* ω_1 be a *connection in* Q_1, *and* Ω_1 *its curvature form. Then :*

a) there is a unique connection ω_2 *in* Q_2 *such that the horizontal subspaces of* ω_1 *are mapped into horizontal subspaces of* ω_2 by Π.

b) if Ω_2 *is the curvature form of* ω_2, *then*

$$\Pi^*\omega_2 = d\Pi \,.\, \omega_1$$

$$\Pi^*\Omega_2 = d\Pi \,.\, \Omega_1$$

c) if ω_1 *is a* J_1*-connection, then* ω_2 *is a* J_2*-connection.*

Proof. – a) and b) are well-known results (see Kobayashi-Nomizu, vol I ({7}), p. 79).

In order to prove c), we have to show that, if $f \epsilon J_2$ with $\deg f = k$, then $\lambda_{\omega_2}(f) = 0$, e.g.

$$f(\Omega_2)\,(Y_1 \otimes \ldots \otimes Y_{2k}) = 0, \quad \text{for} \quad Y_1, \ldots, Y_{2k} \in \mathfrak{X}(Q_2)$$

But it suffices to show that when Y_i, $i = 1, \ldots, 2k$, is horizontal with respect to ω_2 and, in this case, there exist $X_1, \ldots, X_{2k} \in \mathfrak{X}(Q_1)$ such that $d\Pi(X_i) = Y_i$ for every $i = 1,2, \ldots, 2k$. But $i(f) \in J_1$, then

$$0 = i(f)\,(\Omega_1)\,(X_1 \otimes \ldots \otimes X_{2k}) =$$

$$= \frac{1}{(2k)!} \sum_\sigma \epsilon_\sigma\, i(f)\,(\Omega_1(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \ldots \otimes \Omega_1(X_{\sigma(2k-1)}, X_{\sigma(2k)})) =$$

$$= \frac{1}{(2k)!} \sum_\sigma \epsilon_\sigma\, f(d\Pi\,(\Omega_1(X_{\sigma(1)}, X_{\sigma(2)})) \otimes \ldots \otimes d\Pi\,(\Omega_1(X_{\sigma(2k-1)}, X_{\sigma(2k)}))) =$$

$$= \frac{1}{(2k)!} \sum_\sigma \epsilon_\sigma\, f((\Pi^*\Omega_2)\,(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \ldots \otimes (\Pi^*\Omega_2)\,(X_{\sigma(2k-1)}, X_{\sigma(2k)})) =$$

$$= \frac{1}{(2k)!} \sum_\sigma \epsilon_\sigma\, f(\Omega_2(d\Pi\,(X_{\sigma(1)}), d\Pi\,(X_{\sigma(2)})) \otimes \ldots \otimes \Omega_2(d\Pi\,(X_{\sigma(2k-1)}), d\Pi\,(X_{\sigma(2k)}))) =$$

$$= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_\sigma f(\Omega_2(Y_{\sigma(1)}, Y_{\sigma(2)}) \otimes \ldots \otimes \Omega_2(Y_{\sigma(2k-1)}, Y_{\sigma(2k)})) =$$

$$= f(\Omega_2)\ (Y_1 \otimes \ldots \otimes Y_{2k})$$

Remark. – Note that, if $\overline{J}_2$ is another homogeneous ideal of $I(G_2)$ with $\overline{J}_2 \supset J_2$, it might happen that ω_2 be, in fact, a $\overline{J}_2$-connection.

Now, let J_2, J_2' (respect. J_1, J_1') homogeneous ideales of $I(G_2)$ (respect. $I(G_1)$) such that

$$J_1 \supseteq i(J_2)\ ,\ J_1' \supseteq i(J_2')$$

By virtue of Theorem 4.3, given ω_1 a J_1-connection and ω_1' a J_1'-connection, there exist ω_2 a J_2-connection and ω_2' a J_2'-connection satisfying the condition b) in the Theorem. Then, consider the graded differential algebras

$$\hat{W}_1(J_1, J_1') = I(G_1)/J_1 \otimes_{\mathbb{R}} I(G_1)/J_1' \otimes_{\mathbb{R}} \Lambda(I^+(G_1))$$

$$\hat{W}_2(J_2, J_2') = I(G_2)/J_2 \otimes_{\mathbb{R}} I(G_2)/J_2' \otimes_{\mathbb{R}} \Lambda(I^+(G_2))$$

The homomorphism $i : I(G_2) \to I(G_1)$ induces canonically a new homomorphism of graded algebras

$$\overline{i} : \hat{W}_2(J_2, J_2') \to \hat{W}_1(J_1, J_1')$$

PROPOSITION 4.4. – *The following diagram is commutative*

$$\begin{tikzcd} \hat{W}_1(J_1, J_1') \arrow[dr, "\rho_{\omega_1\omega_1'}"'] & & \hat{W}_2(J_2, J_2') \arrow[ll, "\overline{i}"'] \arrow[dl, "\rho_{\omega_2\omega_2'}"] \\ & A^*(M) & \end{tikzcd} \tag{4.1}$$

Proof. – It suffices to prove the commutativity for $\overline{f} = f \pmod{J_2}$, $\overline{\overline{f}} = f \pmod{J_2'}$ with $f \in I(G_2)$, and $\Delta_{\omega_2,\omega_2'} = \Delta_{\omega_1,\omega_1'} . \overline{i}$.

If $\overline{i} : I(G_2)/J_2 \to I(G_1)/J_1$ denotes, once more, the mapping given by $\overline{i}(\overline{f}) = \overline{i(f)}$, we have

$$\rho_{\omega_2\omega_2'}(\overline{f}) = \lambda_{\omega_2}(f) = f(\Omega_2)$$

and

$$\rho_{\omega_1 \omega'_1}(\overline{i(\bar{f})}) = \rho_{\omega_1 \omega'_1}(\overline{i(f)}) = i(f)\,(\Omega_1)$$

and it is clear that $f(\Omega_2)$ and $i(f)\,(\Omega_1)$ define the same element of $A^*(M)$. In a similar way, the commutativity is proved for $\bar{\bar{f}}$.

Now, we consider

$$\begin{tikzcd} Q_1 \times I \arrow[rr, "\pi \times id"] \arrow[dr] & & Q_2 \times I \arrow[dl] \\ & M \times I & \end{tikzcd}$$

where $[\overrightarrow{\omega_2, \omega'_2}]$ is the unique connection in $Q_2 \times I$ which might be obtained form $[\overrightarrow{\omega_1, \omega'_1}]$ in $Q_1 \times I$ through Theorem 4.3 ; hence, the following diagram is commutative

$$\begin{tikzcd} I(G_2) \arrow[rr, "i"] \arrow[dr, "\lambda_{[\overrightarrow{\omega_2,\omega'_2}]}"] \arrow[ddr, "\Delta_{\omega_2\omega'_2}"'] & & I(G_1) \arrow[dl, "\lambda_{[\overrightarrow{\omega_1,\omega'_1}]}"'] \arrow[ddl, "\Delta_{\omega_1\omega'_1}"] \\ & A^*(M \times I) \arrow[d, "\int_0^1"] & \\ & A^*(M) & \end{tikzcd}$$

Remark. – If $\bar{J}_2$ and $\bar{J}'_2$ are homogeneous ideales of $I(G_2)$ such that $\bar{J}_2 \supset J_2$, $\bar{J}'_2 \supset J'_2$, and the connections ω_2, ω'_2 are not only J_2-and J'_2-connections but $\bar{J}_2$- and $\bar{J}'_2$-connections, respectively, and if

$$\eta : \hat{W}_2(J_2, J'_2) \to \hat{W}_2(\bar{J}_2, \bar{J}'_2)$$

is the canonical projection, (4.1) can be enlarged to a new commutative diagram

$$\begin{tikzcd}
\hat{W}_1(J_1, J'_1) \arrow[d, "\rho_{\omega_1\omega'_1}"'] & & \hat{W}_2(J_2, J'_2) \arrow[ll, "\bar{\imath}"'] \arrow[dll, "\rho_{\omega_2\omega'_2}"] \arrow[d, "\eta"] \\
A^*(M) & & \hat{W}_2(\bar{J}_2, \bar{J}'_2) \arrow[ll, "\rho_{\omega_2\omega'_2}"']
\end{tikzcd} \tag{4.2}$$

THEOREM 4.5. – *Diagram (4.1) induces, in cohomology, a new commutative diagram*

$$\begin{tikzcd}
H^*(\hat{W}_1(J_1, J'_1)) \arrow[dr, "\rho^*_{\omega_1\omega'_1}"'] & & H^*(\hat{W}_2(J'_2, J'_2)) \arrow[ll, "\bar{\imath}^*"'] \arrow[dl, "\rho^*_{\omega_2\omega'_2}"] \\
& H^*(M, \mathbb{R}) &
\end{tikzcd}$$

Hence

$$\mathrm{Im}\, \rho^*_{\omega_2\omega'_2} \subset \mathrm{Im}\, \rho^*_{\omega_1\omega'_1} \tag{4.3}$$

Moreover, $\mathrm{Im}\, \rho^*_{\omega_2\omega'_2}$ *does not change when* ω_1 *(respect.* ω'_1*) runs over its* J_1*-connected component (respect.* J'_1*-connected component).*

Proof. – The commutativity of this diagram is evident from that of (4.1), and this fact implies trivially (4.3).

In order to prove the last assertion, it suffices to show that if ω_1 (respect. ω'_1) runs over its J_1-connected (respect. J'_1-connected) component, then ω_2 (respect. ω'_2) does it over its J_2-connected (respect. J'_2-connected) component.

For that, let $\bar{\omega}_1$ be a connection in Q_1 differentiably J_1-homotopic to ω_1 and let $\bar{\omega}_2$ be the connection in Q_2 corresponding to $\bar{\omega}_1$ through Theorem 4.3 ; $\bar{\omega}_2$ is a J_2-connection. Now, consider the connection $\tilde{\omega}$ in $Q_1 \times I \to M \times I$ which defines the J_1-homotopy between ω_1 and $\bar{\omega}_1$; $\tilde{\omega}$ is also a J_1-connection and its corresponding connection in $Q_2 \times I$ through Theorem 4.3 is a J_2-connection which defines a J_2-homotopy between ω_2 and $\bar{\omega}_2$. All these facts can be easily checked by a direct calculation.

5. Application to subfoliations.

The geometric situation which we have described in § 1 is a particular case of multifoliate structure on the manifold M and is defined as follows : consider the set $P = \{1, 2, 3\}$ with the usual order, $1 < 2 < 3$, and suppose $\dim M = n$. Now, we define a mapping

$$\alpha = \{1, 2, \ldots, n\} \to P$$

and, thus, $\{\alpha\}$ is P-multifoliate and we have determined the subgroup $G_P \subset Gl(n, \mathbb{R})$ of matrices

$$\left(\begin{array}{c|c|c} * & * & * \\ \hline 0 & * & * \\ \hline 0 & 0 & * \end{array}\right) \begin{array}{l} P_1 \\ \\ \qquad P_2 \end{array}$$

Let us suppose given an integrable G_P-structure on M ; then, on M, there exist two foliations F_1, F_2 of dimensions p_1, p_2, respectively, and such that every leaf of F_2 is, itself, foliated by leaves of F_1. This fact is equivalent to the existence of two vector subbundles $E_i \subset TM$, $i = 1,2$, and an injective morphism

$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] & & E_2 \arrow[dl] \\ & M & \end{tikzcd}$$

If $N_i = TM/E_i$, $i = 1,2$, is the normal bundle of F_i, there is canonically defined a surjective morphism

$$\begin{tikzcd} N_1 \arrow[rr, "p"] \arrow[dr] & & N_2 \arrow[dl] \\ & M & \end{tikzcd}$$

Denote $q_i = n - p_i = \operatorname{codim} F_i$, $i = 1, 2$; it is possible to choose a covering $\{U\}$ of M which trivializes simultaneously N_1 and N_2, and a local basis of sections of N_1

$$\omega^1, \ldots, \omega^{q_2}, \omega^{q_2+1}, \ldots, \omega^{q_1}$$

in such form that $\omega^1, \ldots, \omega^{q_2}$ is a local basis of sections of N_2 ; it is clear that this choice can be done compatibly with $p : N_1 \to N_2$. Moreover, as E_1 and E_2 are completely integrable

$$d\omega^i = \theta^i_j \wedge \omega^j \quad , \; i, j = 1, 2, \ldots, q_2$$

$$d\omega^a = \theta^a_j \wedge \omega^j + \theta^a_b \wedge \omega^b \quad , \; a, b = q_2 + 1, \ldots, q_1$$

and the matrix of 1-forms

$$\theta = \begin{pmatrix} \theta^i_j & 0 \\ \theta^a_j & \theta^a_b \end{pmatrix}$$

is the 1-form of a connection in N_1, which is basic with respect to F_1, and

$$\theta' = (\theta^i_j)$$

is the 1-form of a connection in N_2, basic with respect to F_2. If ∇ (respect. ∇') denotes to derivation law associated to θ (respect. θ'), the following diagram commutes

$$\begin{array}{ccc} \Gamma(N_1) & \xrightarrow{\quad\nabla\quad} & \Gamma(T^*M \otimes N_1) \\ {\scriptstyle p}\downarrow & & \downarrow {\scriptstyle 1 \otimes p} \\ \Gamma(N_2) & \xrightarrow{\quad\nabla'\quad} & \Gamma(T^*M \otimes N_2) \end{array} \qquad (5.1)$$

Similarly, if we consider a weakly-compatible Riemannian metric (see Vaisman ({11})) on the multifoliate manifold M, it is possible to define two metric connections $\widetilde{\nabla}$ and ∇' on N_1 and N_2 respectively, which permit to write a new commutative diagram like (5.1) (in particular, by using the techniques introduced in ({4 }), it is possible to write the global expression of these connections).

By another part, consider the Lie groups G_1 and G_2 given as follows : $G_1 \subset Gl(q_1, \mathbb{R})$ is the group of all matrices

$$m = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with $A \in Gl(q_2, \mathbb{R})$, and $G_2 = Gl(q_2, \mathbb{R})$, and the homomorphism

$$\begin{aligned} G_1 &\longrightarrow G_2 \\ m &\longmapsto A \end{aligned}$$

Next, consider $I(G_1)$ and $I(G_2)$ and their homogeneous ideales given by

Ideales of $I(G_1)$: $J_1 = J(>q_1)$, $J'_1 = J(\text{odd})$

Ideales of $I(G_2)$: $J_2 = J(>q_1)$, $J'_2 = J(\text{odd})$, $\bar{J}_2 = J(>q_2)$

Clearly, ∇ (respect. ∇') is a J_1-connection (respect. J_2-connection) and $\tilde{\nabla}$ (respect. $\tilde{\nabla}'$) is a J'_1-connection (respect. J'_2-connection) ; in fact, ∇' is a $\bar{J}_2$-connection.

Under these assumptions, we can use the results of § 4 and state

PROPOSITION 5.1. – *The following diagram commutes*

$$\begin{tikzcd} H^*(\hat{W}_1(J_1, J'_1)) \arrow[d, "\rho^*_{\nabla\tilde{\nabla}}"'] & & H^*(\hat{W}_2(J_2, J'_2)) \arrow[ll, "\bar{i}^*"'] \arrow[dll, "\rho^*_{\nabla'\tilde{\nabla}'}"'] \arrow[d, "\eta^*"] \\ H^*(M, R) & & H^*(\hat{W}_2(\bar{J}_2, J'_2)) \arrow[ll, "\rho^*_{\nabla'\tilde{\nabla}'}"'] \end{tikzcd}$$

Hence, Im $\rho^*_{\nabla'\nabla'} \subset \operatorname{Im} \rho^*_{\nabla\nabla}$, *e.g. the set of exotic classes of* F_2 is *a subset of the set of exotic classes of* F_1.

This result permits us to give a topological obstruction to F_1 be a subfoliation of F_2, as follows :

COROLLARY 5.2. – *A necessary condition for* F_1 *be a subfoliation of* F_2 *is that every exotic class of* F_2 *be also an exotic class of* F_1.

At last, note that if F_2 is given by $E_2 = TM$, e.g. if F_2 has the manifold M as unique leaf, that obstruction is trivial.

BIBLIOGRAPHY

[1] R. BOTT, Lectures on characteristic classes and foliations, *Lecture Notes on Math.*, n° 279 (1972), Springer-Verlag, Berlin.

[2] S.S. CHERN and J. SIMONS, Some cohomology classes in principal fibre bundles and their applications to Riemannian geometry, *Proc. Nat. Acad. Sci. U.S.A.*, 66 (1971), 791-794.

[3] S.S. CHERN and J. SIMONS, Characteristic forms and transgression, I, Preprint.

[4] Luis A CORDERO, Special connections on almost-multifoliate Riemannian manifolds, Preprint.

[5] Luis A. CORDERO, Sobre la geometria de las variedades multifoliadas, *III Jornadas Mat. Hispano-Lusitanas,* Sevilla 1974.

[6] C. GODBILLON, Conferencias sobre cohomologia de Gelfand-Fuks y foliaciones, *Public. Dep. Geometria y Topologia,* Univ. Santiago de Compostela, n° 22 (1973), notas de P.M. Gadea.

[7] S. KOBAYASHI and K. NOMIZU, Foundations of Differential Geometry, I, II. Interscience Publ. (1963-69).

[8] K. KODAIRA and D.C. SPENCER, Multifoliate structure, *Ann. of Math.,* 74 (1961), 52-99.

[9] D. LEHMANN, J-homotopie dans les espaces de connexions et classes exotiques de Chern-Simons, *C.R. Acad. Sc.* Paris, 275 A (1972), 835-838.

[10] D. LEHMANN, Classes caractéristiques exotiques et J-connexité des espaces de connexions, Preprint.

[11] I. VAISMAN, Almost-multifoliate Riemannian manifolds, *Ann. Sc. Univ. "Al. I. Cuza"*, sect. I, vol. XVI (1970), 97-104.

Manuscrit reçu le 26 juillet 1974
Accepté par G. Reeb.

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