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EXOTIC CHARACTERISTIC CLASSES AND SUBFOLIATIONS

by Luis A. CORDERO and P.M. GADEA (*)

1. Introduction.

The development in the last years of the study of topological invariants associated to a foliated structure on a differentiable manifold(**) (usually called exotic characteristic classes of the foliation) has been well known.

Within the general context of this study, the following problem appears in a canonical way : let M be a differentiable manifold on which two foliations F_1 and F_2 are defined, and such that $F_1 \subset F_2$, that is, every leaf of F_2 is, itself, foliated by leaves of F_1 ; briefly, F_1 is said to be subfoliation of F_2 ; in fact, this geometrical structure on M can be described as a special type of multifoliate structure (in the sense of Kodaira-Spencer ({8})) ; now, we present two questions : 1) does a relation exist between exotic classes of F_1 and F_2 ?, and 2) : is it possible to give a topological obstruction to the existence of such a structure on M ?.

In this paper we give the answer to these questions, by studying the problem through a more general situation and using Lehmann's techniques ({9}, {10}). For this purpose, we consider the following situation : let $Q_i, i = 1, 2$, be two G_i -principal fibre bundles over M , and let $\Pi : Q_1 \rightarrow Q_2$ be a morphism of principal fibre bundles (over the identity of M) ; by an appropriate choice of connections on these fibre bundles we point out a relation between the images of Lehmann's exoticism associated to those connections (Theorem 4.5) ; in the special case of F_1 and F_2 , two foliations as above, that relation gives the answer to our questions : every exotic characteristic class of F_2 is also an

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(**) Always manifolds will mean paracompact differentiable manifold of class C^∞ .

exotic characteristic class of F_1 ; in fact, this result can be expressed as a topological obstruction F_1 to be a subfoliation of F_2 .

2. Notations and basic concepts.

Let M be a differentiable manifold. We shall denote $\mathfrak{X}(M)$ the Lie algebra of vector fields and $A^*(M)$ the exterior algebra of differential forms on M .

Given a G -principal fibre bundle $E \rightarrow M$, G being the structural Lie group, ω indistinctly denotes an (infinitesimal) connection on the bundle or the 1-form of that connection ; $I(G)$ is the algebra of invariant polynomials over the Lie algebra \underline{G} of G ; $I(G)$ is a graded algebra, $I(G) = \bigoplus_{k \geq 0} I^k(G)$ and $I^+(G)$ denotes its maximal ideal

$$I^+(G) = \bigoplus_{k \geq 1} I^k(G).$$

Denote by $\lambda_\omega : I(G) \rightarrow A^*(M)$ the Chern-Weil homomorphism, defined by $\lambda_\omega(f) = f(\Omega)$, for $f \in I(G)$ and Ω being the curvature form of ω . If $I = [0, 1]$ is the unit interval, $\int_0^1 : A^k(M \times I) \rightarrow A^{k-1}(M)$ denotes the integration along the fibre of $M \times I \rightarrow M$. If ω' is another connection on E , we write $[\overrightarrow{\omega}, \overrightarrow{\omega}']$ the connection on $E \times I \rightarrow M \times I$ defined by

$$[\overrightarrow{\omega}, \overrightarrow{\omega}']\left(\frac{\partial}{\partial t}\right) = 0, [\overrightarrow{\omega}, \overrightarrow{\omega}']|_{E \times \{t\}} = t \omega' + (1 - t) \omega$$

and by $\Delta_{\omega, \omega'} : I^k(G) \rightarrow A^{2k-1}(M)$ the composition $\int_0^1 \cdot \lambda_{[\overrightarrow{\omega}, \overrightarrow{\omega}']}$.

As it is well known, λ_ω induces an homomorphism $\lambda :$

$$I(G) \rightarrow H^{\text{even}}(M, \mathbb{R})$$

which is independent of ω .

Let $J \subset I(G)$ be a homogeneous ideal ; ω is said a J -connection if $\lambda_\omega(f) = 0$ for every $f \in J$. If P denotes a property of the degree of elements of $I(G)$, $J(P)$ denotes the homogeneous ideal generated by the elements satisfying the property P . For example, if $\dim M = n$, every connection on E is a $J\left(> \left[\frac{n}{2}\right]\right)$ -connection.

If $G = \text{Gl}(q, \mathbb{R})$, it is $I(G) = \mathbb{R}[c_1, \dots, c_q]$, where c_1, \dots, c_q are the usual generators given by

$$\det(I + tA) = 1 + \sum_{i=1}^q c_i(A)t^i, \text{ for every } A \in \text{gl}(q, \mathbb{R})$$

If $Q \rightarrow M$ is a vector bundle, ∇ denotes the derivation law of a linear connection on Q . Thus, every metric connection on Q is a $J(\text{odd})$ -connection.

If $Q \rightarrow M$ is the normal bundle of a foliation on M , of codimension q , and ∇ is a basic connection on Q (in the sense of Bott ($\{1\}$)), then ∇ is a $J(> q)$ -connection.

3. The Lehmann's exoticism ($\{9\}$), ($\{10\}$)).

Let E be a G -principal fibre bundle on M . Consider J, J' homogeneous ideals of $I(G)$; if $f \in I^k(G)$, we write

$$\bar{f} = f \pmod{J}, \quad \overline{\overline{f}} = f \pmod{J'}$$

and introduce a graduation on the quotient algebras $I(G)/J, I(G)/J'$ by $\deg \bar{f} = \deg \overline{\overline{f}} = 2k$, for every $f \in I^k(G)$; also, we shall denote $\Lambda(I^+(G))$ the exterior algebra over \mathbb{R} generated by the elements of $I^+(G)$ and define a graduation on $\Lambda(I^+(G))$ by $\deg f = 2k - 1$, for every $f \in I^k(G), k > 0$. Then, consider the graded algebra

$$\hat{W}(J, J') = I(G)/J \otimes_{\mathbb{R}} I(G)/J' \otimes_{\mathbb{R}} \Lambda(I^+(G))$$

and $I(G)/J, I(G)/J', \Lambda(I^+(G))$ are canonically identified to subalgebras of $\hat{W}(J, J')$; $I^+(G)$ can be identified to one part of $\Lambda(I^+(G)) \subset \hat{W}(J, J')$ by the isomorphism

$$h : I^+(G) \rightarrow \Lambda^1(I^+(G))$$

and, if $G = \text{Gl}(q, \mathbb{R})$, we write $h_i = h(c_i)$.

$\hat{W}(J, J')$ is endowed with a structure of graded differential algebra by defining a differential (of degree 1)

$$\begin{aligned} d(\bar{f}) &= d(\overline{\overline{f}}) = 0, \quad \text{for } f \in I(G) \\ d(f) &= \bar{f} - \overline{\overline{f}}, \quad \text{for } f \in I^+(G) \end{aligned}$$

and, clearly, $d^2 = 0$.

If ω is a J-connection and ω' is a J'-connection on E, a homomorphism of graded algebras $\rho_{\omega\omega'} : \hat{W}(J, J') \rightarrow A^*(M)$ is defined by

$$\rho_{\omega\omega'}(\bar{f}) = \lambda_{\omega}(f)$$

$$\rho_{\omega\omega'}(\bar{\bar{f}}) = \lambda_{\omega'}(f)$$

$$\rho_{\omega\omega'}(f_1 \wedge \dots \wedge f_r) = \Delta_{\omega, \omega'}(f_1) \wedge \dots \wedge \Delta_{\omega, \omega'}(f_r), \text{ for } f_i \in I^+(G)$$

and, in cohomology, $\rho_{\omega\omega'}$ induces a homomorphism of graded algebras

$$\rho_{\omega\omega'}^* : H^*(\hat{W}(J, J')) \rightarrow H^*(M, \mathbb{R})$$

The elements of $\text{Im } \rho_{\omega\omega'}^*$ are said to be the exotic characteristic classes associated to J, J', ω and ω' .

Let $J \subset I(G)$ be a homogeneous ideal and ω_0, ω_1 two J-connections on E ; ω_0 and ω_1 are said to be differentiably J-homotopic if there does exist a J-connection $\tilde{\omega}$ on $E \times I \rightarrow M \times I$ such that

$$\tilde{\omega}|_{E \times \{0\}} = \omega_0, \quad \tilde{\omega}|_{E \times \{1\}} = \omega_1$$

and, in a more general form, ω_0 and ω_1 are said to be J-homotopic if there does exist a finite sequence $\omega_0 = \omega_{s_0}, \omega_{s_1}, \dots, \omega_{s_k} = \omega_1$ of J-connections such that, for every $i = 0, 1, \dots, k-1$, ω_{s_i} and $\omega_{s_{i+1}}$ are differentiably J-homotopic. A set C of connections on E is said to be J-connected if it is not-empty and any two connections in C are J-homotopic.

PROPOSITION 3.1. — *Im $\rho_{\omega\omega'}^*$ depends only on the J-connected component of ω and the J'-connected component of ω' .*

In particular, if C is the set of basic connections on the transversal bundle Q of a q-codimensional foliation on M and C' is the set of metric connections on Q, Lehmann shows that C is $J(> q)$ -connected and C' is J(odd)-connected ; moreover, in this case $\hat{W}(J(> q), J(\text{odd}))$ has the same cohomology that its subalgebra

$$WO_q = \mathbb{R} [c_1, \dots, c_q] / J(> q) \otimes_{\mathbb{R}} \Lambda(h_1, h_3, \dots, h_{(q)})$$

where (q) denotes the largest odd integer $\leq q$ and $h_i = h(c_i)$. Therefore

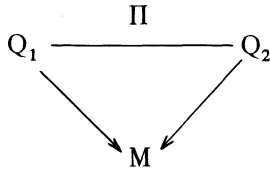
PROPOSITION 3.2. — *The homomorphism $\rho_{\nabla\nabla'}^*$:*

$$H^*(W O_q) \rightarrow H^*(M, \mathbb{R})$$

does not depend on the choice of $\nabla \in C$ and $\nabla' \in C'$.

4. Homomorphism of principal fibre bundles and the Lehmann's exoticism.

In this paragraph we shall consider the following situation : let $Q_i \rightarrow M$ be a G_i -principal fibre bundle ($i = 1, 2$) and let



a homomorphism of principal fibre bundles ; also, denote

$$\Pi : G_1 \longrightarrow G_2$$

the corresponding homomorphism of Lie groups and assume that Π is surjective but not a submersion in general, e.g.

$$d\Pi : \underline{G}_1 \longrightarrow \underline{G}_2$$

is not of maximal rank in general ; the linear mapping $d\Pi$ permits to define :

DEFINITION 4.1. — *If $f \in I^k(G_2)$, $i(f)$ is defined by*

$$i(f) (X_1 \otimes \dots \otimes X_k) = f(d\Pi(X_1) \otimes \dots \otimes d\Pi(X_k)),$$

for every $X_j \in \underline{G}_1$, $j = 1, 2, \dots, k$

A direct application of this definition shows

PROPOSITION 4.2. — *For every $f \in I(G_2)$, $i(f) \in I(G_1)$ and*

$$i : I(G_2) \rightarrow I(G_1)$$

is a homomorphism of graded algebras. Moreover, if $d\Pi$ is of maximal rank, then i is injective.

Let $J_2 \subset I(G_2)$ be an homogeneous ideal and J_1 an arbitrary homogeneous ideal of $I(G_1)$, such that $J_1 \supseteq i(J_2)$ (in particular, J_1 could be thought as the homogeneous ideal generated by the elements of $i(J_2)$).

THEOREM 4.3. — *Let ω_1 be a connection in Q_1 , and Ω_1 its curvature form. Then :*

- a) *there is a unique connection ω_2 in Q_2 such that the horizontal subspaces of ω_1 are mapped into horizontal subspaces of ω_2 by Π .*
- b) *if Ω_2 is the curvature form of ω_2 , then*

$$\Pi^*\omega_2 = d\Pi \cdot \omega_1$$

$$\Pi^*\Omega_2 = d\Pi \cdot \Omega_1$$

- c) *if ω_1 is a J_1 -connection, then ω_2 is a J_2 -connection.*

Proof. — a) and b) are well-known results (see Kobayashi-Nomizu, vol I ({7}), p. 79).

In order to prove c), we have to show that, if $f \in J_2$ with $\deg f = k$, then $\lambda_{\omega_2}(f) = 0$, e.g.

$$f(\Omega_2)(Y_1 \otimes \dots \otimes Y_{2k}) = 0, \text{ for } Y_1, \dots, Y_{2k} \in \mathfrak{X}(Q_2)$$

But it suffices to show that when $Y_i, i = 1, \dots, 2k$, is horizontal with respect to ω_2 and, in this case, there exist $X_1, \dots, X_{2k} \in \mathfrak{X}(Q_1)$ such that $d\Pi(X_i) = Y_i$ for every $i = 1, 2, \dots, 2k$. But $i(f) \in J_1$, then

$$\begin{aligned} 0 &= i(f)(\Omega_1)(X_1 \otimes \dots \otimes X_{2k}) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} i(f)(\Omega_1(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \dots \otimes \Omega_1(X_{\sigma(2k-1)}, X_{\sigma(2k)})) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(d\Pi(\Omega_1(X_{\sigma(1)}, X_{\sigma(2)})) \otimes \dots \otimes d\Pi(\Omega_1(X_{\sigma(2k-1)}, X_{\sigma(2k)}))) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f((\Pi^*\Omega_2)(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \dots \otimes (\Pi^*\Omega_2)(X_{\sigma(2k-1)}, X_{\sigma(2k)})) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(\Omega_2(d\Pi(X_{\sigma(1)}), d\Pi(X_{\sigma(2)})) \otimes \dots \otimes \Omega_2(d\Pi(X_{\sigma(2k-1)}), d\Pi(X_{\sigma(2k)}))) = \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(\Omega_2(Y_{\sigma(1)}, Y_{\sigma(2)}) \otimes \dots \otimes \Omega_2(Y_{\sigma(2k-1)}, Y_{\sigma(2k)})) = \\
 &= f(\Omega_2)(Y_1 \otimes \dots \otimes Y_{2k})
 \end{aligned}$$

Remark. – Note that, if \bar{J}_2 is another homogeneous ideal of $I(G_2)$ with $\bar{J}_2 \supset J_2$, it might happen that ω_2 be, in fact, a \bar{J}_2 -connection.

Now, let J_2, J'_2 (respect. J_1, J'_1) homogeneous ideales of $I(G_2)$ (respect. $I(G_1)$) such that

$$J_1 \supseteq i(J_2), J'_1 \supseteq i(J'_2)$$

By virtue of Theorem 4.3, given ω_1 a J_1 -connection and ω'_1 a J'_1 -connection, there exist ω_2 a J_2 -connection and ω'_2 a J'_2 -connection satisfying the condition b) in the Theorem. Then, consider the graded differential algebras

$$\hat{W}_1(J_1, J'_1) = I(G_1)/J_1 \otimes_{\mathbb{R}} I(G_1)/J'_1 \otimes_{\mathbb{R}} \Lambda(I^+(G_1))$$

$$\hat{W}_2(J_2, J'_2) = I(G_2)/J_2 \otimes_{\mathbb{R}} I(G_2)/J'_2 \otimes_{\mathbb{R}} \Lambda(I^+(G_2))$$

The homomorphism $i : I(G_2) \rightarrow I(G_1)$ induces canonically a new homomorphism of graded algebras

$$\bar{i} : \hat{W}_2(J_2, J'_2) \rightarrow \hat{W}_1(J_1, J'_1)$$

PROPOSITION 4.4. – *The following diagram is commutative*

$$\begin{array}{ccc}
 \hat{W}_1(J_1, J'_1) & \xleftarrow{\bar{i}} & \hat{W}_2(J_2, J'_2) \\
 \rho_{\omega_1 \omega'_1} \searrow & & \swarrow \rho_{\omega_2 \omega'_2} \\
 & A^*(M) &
 \end{array} \tag{4.1}$$

Proof. – It suffices to prove the commutativity for $\bar{f} = f(\text{mod } J_2)$, $\bar{f}' = f(\text{mod } J'_2)$ with $f \in I(G_2)$, and $\Delta_{\omega_2, \omega'_2} = \Delta_{\omega_1, \omega'_1} \cdot \bar{i}$.

If $\bar{i} : I(G_2)/J_2 \rightarrow I(G_1)/J_1$ denotes, once more, the mapping given by $\bar{i}(\bar{f}) = \bar{i}(f)$, we have

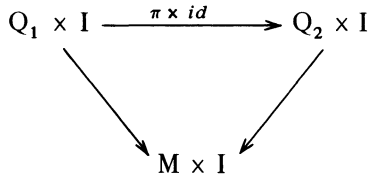
$$\rho_{\omega_2 \omega'_2}(\bar{f}) = \lambda_{\omega_2}(f) = f(\Omega_2)$$

and

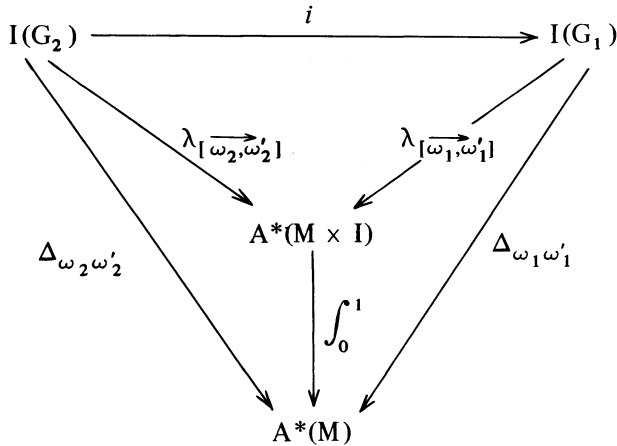
$$\rho_{\omega_1, \omega'_1}(\overline{i(f)}) = \rho_{\omega_1, \omega'_1}(\overline{i(f)}) = i(f) (\Omega_1)$$

and it is clear that $f(\Omega_2)$ and $i(f) (\Omega_1)$ define the same element of $A^*(M)$. In a similar way, the commutativity is proved for \overline{f} .

Now, we consider



where $\overrightarrow{[\omega_2, \omega'_2]}$ is the unique connection in $Q_2 \times I$ which might be obtained from $\overrightarrow{[\omega_1, \omega'_1]}$ in $Q_1 \times I$ through Theorem 4.3 ; hence, the following diagram is commutative



Remark. – If \overline{J}_2 and \overline{J}'_2 are homogeneous ideales of $I(G_2)$ such that $\overline{J}_2 \supset J_2, \overline{J}'_2 \supset J'_2$, and the connections ω_2, ω'_2 are not only J_2 - and J'_2 -connections but \overline{J}_2 - and \overline{J}'_2 -connections, respectively, and if

$$\eta : \widehat{W}_2(J_2, J'_2) \rightarrow \widehat{W}_2(\overline{J}_2, \overline{J}'_2)$$

is the canonical projection, (4.1) can be enlarged to a new commutative diagram

$$\begin{array}{ccc}
 \hat{W}_1(J_1, J'_1) & \xleftarrow{\bar{i}} & \hat{W}_2(J_2, J'_2) \\
 \downarrow \rho_{\omega_1 \omega'_1} & \nearrow \rho_{\omega_2 \omega'_2} & \downarrow \eta \\
 A^*(M) & \xleftarrow{\rho_{\omega_2 \omega'_2}} & \hat{W}_2(\bar{J}_2, \bar{J}'_2)
 \end{array} \tag{4.2}$$

THEOREM 4.5. — *Diagram (4.1) induces, in cohomology, a new commutative diagram*

$$\begin{array}{ccc}
 H^*(\hat{W}_1(J_1, J'_1)) & \xleftarrow{\bar{i}^*} & H^*(\hat{W}_2(J'_2, J_2)) \\
 \searrow \rho_{\omega_1 \omega'_1}^* & & \swarrow \rho_{\omega_2 \omega'_2}^* \\
 & H^*(M, \mathbb{R}) &
 \end{array}$$

Hence

$$\text{Im } \rho_{\omega_2 \omega'_2}^* \subset \text{Im } \rho_{\omega_1 \omega'_1}^* \tag{4.3}$$

Moreover, $\text{Im } \rho_{\omega_2 \omega'_2}^*$ does not change when ω_1 (respect. ω'_1) runs over its J_1 -connected component (respect. J'_1 -connected component).

Proof. — The commutativity of this diagram is evident from that of (4.1), and this fact implies trivially (4.3).

In order to prove the last assertion, it suffices to show that if ω_1 (respect. ω'_1) runs over its J_1 -connected (respect. J'_1 -connected) component, then ω_2 (respect. ω'_2) does it over its J_2 -connected (respect. J'_2 -connected) component.

For that, let $\bar{\omega}_1$ be a connection in Q_1 differentially J_1 -homotopic to ω_1 and let $\bar{\omega}_2$ be the connection in Q_2 corresponding to $\bar{\omega}_1$ through Theorem 4.3 ; $\bar{\omega}_2$ is a J_2 -connection. Now, consider the connection $\tilde{\omega}$ in $Q_1 \times I \rightarrow M \times I$ which defines the J_1 -homotopy between ω_1 and $\bar{\omega}_1$; $\tilde{\omega}$ is also a J_1 -connection and its corresponding connection in $Q_2 \times I$ through Theorem 4.3 is a J_2 -connection which

defines a J_2 -homotopy between ω_2 and $\bar{\omega}_2$. All these facts can be easily checked by a direct calculation.

5. Application to subfoliations.

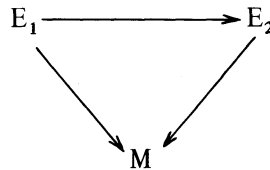
The geometric situation which we have described in § 1 is a particular case of multifoliate structure on the manifold M and is defined as follows : consider the set $P = \{1, 2, 3\}$ with the usual order, $1 < 2 < 3$, and suppose $\dim M = n$. Now, we define a mapping

$$\alpha = \{1, 2, \dots, n\} \rightarrow P$$

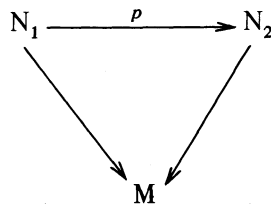
and, thus, $\{\alpha\}$ is P -multifoliate and we have determined the subgroup $G_P \subset Gl(n, \mathbb{R})$ of matrices

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \begin{matrix} \text{---} P_1 \\ \text{---} P_2 \end{matrix}$$

Let us suppose given an integrable G_P -structure on M ; then, on M , there exist two foliations F_1, F_2 of dimensions p_1, p_2 , respectively, and such that every leaf of F_2 is, itself, foliated by leaves of F_1 . This fact is equivalent to the existence of two vector subbundles $E_i \subset TM$, $i = 1, 2$, and an injective morphism



If $N_i = TM/E_i$, $i = 1, 2$, is the normal bundle of F_i , there is canonically defined a surjective morphism



Denote $q_i = n - p_i = \text{codim } F_i, i = 1, 2$; it is possible to choose a covering $\{U\}$ of M which trivializes simultaneously N_1 and N_2 , and a local basis of sections of N_1

$$\omega^1, \dots, \omega^{q_2}, \omega^{q_2+1}, \dots, \omega^{q_1}$$

in such form that $\omega^1, \dots, \omega^{q_2}$ is a local basis of sections of N_2 ; it is clear that this choice can be done compatibly with $p : N_1 \rightarrow N_2$. Moreover, as E_1 and E_2 are completely integrable

$$d\omega^i = \theta_j^i \wedge \omega^j, \quad i, j = 1, 2, \dots, q_2$$

$$d\omega^a = \theta_j^a \wedge \omega^j + \theta_b^a \wedge \omega^b, \quad a, b = q_2 + 1, \dots, q_1$$

and the matrix of 1-forms

$$\theta = \begin{pmatrix} \theta_j^i & 0 \\ \theta_j^a & \theta_b^a \end{pmatrix}$$

is the 1-form of a connection in N_1 , which is basic with respect to F_1 , and

$$\theta' = (\theta_j^i)$$

is the 1-form of a connection in N_2 , basic with respect to F_2 . If ∇ (respect. ∇') denotes to derivation law associated to θ (respect. θ'), the following diagram commutes

$$\begin{CD} \Gamma(N_1) @>\nabla>> \Gamma(T^*M \otimes N_1) \\ @VpVV @VV1 \otimes pV \\ \Gamma(N_2) @>\nabla'>> \Gamma(T^*M \otimes N_2) \end{CD} \tag{5.1}$$

Similarly, if we consider a weakly-compatible Riemannian metric (see Vaisman ({11})) on the multifoliate manifold M , it is possible to define two metric connections $\tilde{\nabla}$ and ∇' on N_1 and N_2 respectively, which permit to write a new commutative diagram like (5.1) (in particular, by using the techniques introduced in ({4}), it is possible to write the global expression of these connections).

By another part, consider the Lie groups G_1 and G_2 given as follows : $G_1 \subset Gl(q_1, \mathbb{R})$ is the group of all matrices

$$m = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with $A \in Gl(q_2, \mathbb{R})$, and $G_2 = Gl(q_2, \mathbb{R})$, and the homomorphism

$$\begin{array}{ccc} G_1 & \longrightarrow & G_2 \\ m & \longmapsto & A \end{array}$$

Next, consider $I(G_1)$ and $I(G_2)$ and their homogeneous ideales given by

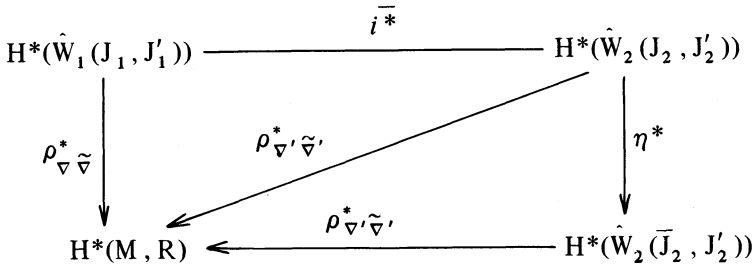
Ideales of $I(G_1)$: $J_1 = J(> q_1)$, $J'_1 = J(\text{odd})$

Ideales of $I(G_2)$: $J_2 = J(> q_1)$, $J'_2 = J(\text{odd})$, $\bar{J}_2 = J(> q_2)$

Clearly, ∇ (respect. ∇') is a J_1 -connection (respect. J_2 -connection) and $\tilde{\nabla}$ (respect. $\tilde{\nabla}'$) is a J'_1 -connection (respect. J'_2 -connection) ; in fact, ∇' is a \bar{J}_2 -connection.

Under these assumptions, we can use the results of § 4 and state

PROPOSITION 5.1. – *The following diagram commutes*



Hence, $Im \rho_{\nabla' \tilde{\nabla}'}^* \subset Im \rho_{\nabla \tilde{\nabla}}^*$, e.g. the set of exotic classes of F_2 is a subset of the set of exotic classes of F_1 .

This result permits us to give a topological obstruction to F_1 be a subfoliation of F_2 , as follows :

COROLLARY 5.2. – *A necessary condition for F_1 be a subfoliation of F_2 is that every exotic class of F_2 be also an exotic class of F_1 .*

At last, note that if F_2 is given by $E_2 = TM$, e.g. if F_2 has the manifold M as unique leaf, that obstruction is trivial.

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