JOHANNES SJÖSTRAND

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PROPAGATION OF SINGULARITIES FOR OPERATORS WITH MULTIPLE INVOLUTIVE CHARACTERISTICS

by Johannes SJOSTRAND (*)

0. Introduction.

The purpose of this paper is to give a result on propagation of singularities, which generalizes some results of Duistermaat-Hörmander [6] and Chazarain [4]. In the category of hyperfunctions such a result has already been obtained by Bony-Schapira [1] for the propagation of the analytic wavefront set. The main difference with their result is that we will have to impose a condition on the lower order symbols of the operator.

Let X be a paracompact C^∞ manifold of dimension n and let $\Sigma \subset T^*X\backslash 0$ be a closed conic submanifold of codimension d. We shall assume that Σ is involutive. This means that if Σ is locally given by $q_1(x, \xi) = \cdots = q_d(x, \xi) = 0$, where q_j are smooth, real valued, with linearly independent differentials, then all the Poisson brackets $\{q_j, q_k\}$ vanish on Σ. The Hamilton fields $H_{q_1}, \ldots, H_{q_d}$ are then tangential to Σ and form an integrable Frobenius system on Σ. If $\rho \in \Sigma$, we denote by $\Gamma_\rho \subset \Sigma$; the set obtained by integrating successively all such Hamilton fields, starting at the point ρ. Locally this "flow out" is a d-dimensional submanifold of Σ, but globally Γ_ρ may be more complicated.

We shall assume :

(0.1) For all choices of $q_1, \ldots, q_d$ as above, the cone axis direction and $H_{q_1}, \ldots, H_{q_d}$ are linearly independent at Σ.

(*) This article was prepared, while visiting the Mittag-Leffler Institute.

This condition means that it is possible to construct locally a homogeneous canonical transformation $\kappa : T^*X\backslash 0 \to T^*\mathbf{R}^n\backslash 0$ which maps Σ into $\widetilde{\Sigma} = \{(x, \xi) \in T^*\mathbf{R}^n\backslash 0 \; ; \; \xi'' = 0\}$. See [6]. Here we use the notation $\xi = (\xi', \xi'')$, $\xi' \in \mathbf{R}^{n-d}$, $\xi'' \in \mathbf{R}^d$ for arbitrary vectors $\xi \in \mathbf{R}^n$.

We now consider a classical properly supported pseudodifferential operator $P \in L^{m+k}(X)$, where m is a positive integer and $k \in \mathbf{R}$. Let p_{m+k} be the principal symbol and assume that p_{m+k} vanishes at Σ to order m and that $p_{m+k}(x, \xi) \neq 0$ outside Σ. Considering the Taylor expansion of p_{m+k} at any point $\rho \in \Sigma$, we get a homogeneous polynomial $a_\rho(t)$, $t \in T_\rho(T^*X)/T_\rho(\Sigma) = F_\rho$ of degree m. We assume that

(0.2) $\quad a_\rho(t) \neq 0 \quad$ when $\quad 0 \neq t \in F_\rho \quad$ for all $\quad \rho \in \Sigma$.

In the proof below we shall work with Carleman estimates. We therefore need the following condition, that will permit us to apply the estimates in the proof of Calderon's uniqueness theorem for the Cauchy problem :

(0.3) Let $F = T_\Sigma(T^*X)/T(\Sigma)$ and let $F \times_\Sigma F$ be the product bundle over Σ. Let $z_1(s, t), \dots, z_m(s, t) \in \mathbf{C}$ be the roots of the equation $a_\rho(s + zt) = 0$, s, $t \in F_\rho$. Then when (s, t) varies in $\{(s, t) \in F \times_\Sigma F \; ; \; s, t$ are linearly independent$\}$ the roots $z_j(s, t)$ are either simple or double and of constant multiplicity.

The condition on the lower order symbols, that we shall need, is the natural extension of the Levi condition in the case when $\operatorname{codim}(\Sigma) = 1$. C.f. Chazarain [4].

(0.4) Let Σ be given locally by $q_1(x, \xi) = \cdots = q_d(x, \xi) = 0$, where $q_j \in C^\infty(T^*X\backslash 0)$ are real valued, positively homogeneous of degree 1 with linearly independent differentials. Then, if Q_j are classical pseudo-differential operators with principal symbol $= q_j$, there exist classical pseudodifferential operators A_α *of order* k such that

$$P \equiv \sum_{|\alpha| \leqslant m} A_\alpha Q_1^{\alpha_1} \dots Q_d^{\alpha_d}$$

microlocally.

When $m = 2$ this condition means precisely that the subprincipal symbol of P vanishes on Σ.

The result of this paper is now

THEOREM 0.1. – *Suppose that* P *satisfies* (0.1) – (0.4). *Then if* $u \in \mathcal{D}'(\mathrm{X})$, $\mathrm{P}u \in \mathrm{C}^\infty(\mathrm{X})$ *and* ρ *is a point in* $\mathrm{WF}(u)$ (*so that* $\rho \in \Sigma$), *we have* $\Gamma_\rho \subset \mathrm{WF}(u)$.

When $m = 1$ we have $d = 1$ or 2 and the theorem gives some of the wellknown results of Duistermaat-Hörmander [6]. When $d = 1$ and m arbitrary we find a result of Chazarain [4].

When $m = 2$ and $d \geqslant 3$ it is easy to see that (0.3) is always verified. The same is true when $(m, d) = (2,2)$ if we assume that the argument variation of p_{m+k} along small closed curves in $(\mathrm{T}^*\mathrm{X}\backslash 0)$ is always 0. (C.f. Sjöstrand [12]).

The condition (0.4) is very essential for our results. In fact, when $m = 2$, Boutet de Monvel [2] has constructed a pseudodifferential parametrix under the assumptions (0.1), (0.2) and the assumption that the subprincipal symbol of P avoids the values of $-a_\rho$ at every point $\rho \in \Sigma$. For other cases with non-vanishing subprincipal symbol, Boutet de Monvel [3] and Lascar [9] have shown that there is a propagation of singularities, but not along the whole leaves Γ_ρ in general.

The condition (0.3) is also important. Let $\mathrm{P} = \mathrm{P}(x'', \mathrm{D}'')$ be an elliptic operator in $\mathbf{R}^d$ of order m, such that $\mathrm{P}\varphi = 0$ for some $\varphi \in \mathrm{C}_0^\infty(\mathbf{R}^d)$, $\varphi \not\equiv 0$. Considering P as an operator in $\mathbf{R}^n$ we have (0.1), (0.2), (0.4) satisfied (with Σ given by $\xi'' = 0$) but the conclusion of Theorem 0.1 is false, for we have $\mathrm{P}(\varphi(x'') \otimes \delta(x')) = 0$.

In the sections 1 and 2 we are going to make some preparations for the main part of the proof, which is given in section 3. As mentioned above, the proof is based on the use of Carleman estimates in the spirit of Hörmander [7, section 8.8], Unterberger [13, 14] and Duistermaat [5].

Bony has communicated to us that the methods of [1] most certainly can be modified to give our Theorem 0.1 in the case when the condition (0.3) is replaced by the assumption that P is a differential operator with analytic coefficients.

1. Some preparations.

Using Fourier integral operators we can transform the operator P microlocally into an operator $\widetilde{\mathrm{P}}$ in $\mathbf{R}^n$ with characteristic variety $\widetilde{\Sigma} = \{(x, \xi) \in \mathrm{T}^*\mathbf{R}^n \backslash 0 ; \xi'' = 0\}$. The conditions (0.1) – (0.4) are preserved, so we can assume that

$$\widetilde{\mathrm{P}} = \sum_{|\alpha| \leqslant m} a_\alpha(x, \mathrm{D}) \mathrm{D}_{x''}^\alpha , \tag{1.1}$$

where a_α are classical pseudodifferential operators *of order* 0. The sets Γ_ρ are now of the form $(x', \xi') = \text{const.}$, $\xi'' = 0$. We shall prove

THEOREM 1.1. – *Let* $\widetilde{\mathrm{P}}$ *satisfy* (0.1) – (0.4) *with* Σ *replaced by* $\widetilde{\Sigma}$. *Let* $\xi'_0 \in \mathbf{R}^{n-d} \backslash \{0\}$ *and suppose that* $u \in \mathcal{D}'(\mathbf{R}^n)$ *satisfies* $(0, (\xi'_0, 0)) \notin \mathrm{WF}(u) \subset \widetilde{\Sigma}$ *and* $\{(0, x'', \xi'_0, 0) ; |x''| \leqslant 1\} \cap \mathrm{WF}(\mathrm{P}u) = \phi$. *Then* $\{(0, x'', \xi'_0, 0) ; |x''| < 1\} \cap \mathrm{WF}(u) = \phi$.

Theorem 0.1 follows from Theorem 1.1, for in the situation of Theorem 0.1 let us assume that $\rho \in \mathrm{WF}(u)$, $\rho' \notin \mathrm{WF}(u)$, where $\rho' \in \Gamma_\rho$. Then take a continuous curve γ in Γ_ρ joining ρ to ρ' and let ρ'' be the last point of γ belonging to $\mathrm{WF}(u)$. Near ρ'' we can apply Fourier integral operators and then apply Theorem 1.1 to get a contradiction.

From now on we work in $\mathbf{R}^n$ so we shall drop all the superscripts $\sim$ in the notations of Theorem 1.1. Let $a_\alpha^0(x, \xi)$ be the homogeneous principal symbol of a_α of degree 0. In the proof below, we shall have to approximate $\mathrm{P}(x, \mathrm{D})$ by $\mathrm{P}_0(x, \mathrm{D}) = \sum_{|\alpha| \leqslant m} a_\alpha^0(x, \mathrm{D}', 0) \mathrm{D}_{x''}^\alpha$. Let us first establish a simple a priori estimate.

LEMMA 1.2. – *For all* $\mathrm{K} \subset\subset \mathbf{R}^n$ *and* $s \in \mathbf{R}$ *there is a constant* C *such that*

$$\sum_{|\alpha| \leqslant m} \| \mathrm{D}_{x''}^\alpha u \|_s \leqslant \mathrm{C}(\| \mathrm{P}u \|_s + \| u \|_s) , \quad u \in \mathrm{C}_0^\infty(\mathrm{K}) .$$

Here $\| \ \|_s$ is the norm in the usual Sobolev space $\mathrm{H}^s(\mathbf{R}^n)$.

Proof – Let $(x_0, (\xi'_0, 0)) \in \Sigma$ and let $\chi(x, \mathrm{D})$ be a properly supported pseudodifferential operator of degree 0 with $\mathrm{WF}(\chi)$ in a small conic neighbourhood of $(x_0, (\xi'_0, 0))$ and such that the principal symbol takes its values in $[0, 1]$. Then

$$\|[P, \chi]u\|_s \leq C_1 \sum_{|\alpha| \leq m-1} \|D^\alpha_{x''} u\|_s \ , \quad u \in C_0^\infty(K) \ , \tag{1.2}$$

where C_1 depends on K, s, χ. Put $Q = \sum_{|\alpha| \leq m} a^0_\alpha(x_0, \xi'_0, 0) D^\alpha_{x''}$. Then by Fourier transformation we get

$$\sum_{|\alpha| \leq m} \|D^\alpha_{x''} v\|_s \leq C(\|Qv\|_s + \|v\|_s) \ , \tag{1.3}$$

for all $v \in C_0^\infty(\mathbf{R}^n)$ and all $s \in \mathbf{R}$. Moreover from the wellknown continuity properties of pseudodifferential operators we know that for all $\epsilon > 0$ and $s \in \mathbf{R}$, we have

$$\|(Q - P)\chi u\|_s \leq \epsilon \sum_{|\alpha| \leq m} \|D^\alpha_{x''} \chi u\|_s + C_2 \|u\|_s \ , \quad u \in C_0^\infty(K) \ , \tag{1.4}$$

if $WF(\chi)$ is sufficiently close to the half ray through $(x_0, \xi'_0, 0)$. The constant C_2 depends on ϵ, s, χ, K.

Using (1.2) – (1.4) we get for $u \in C_0^\infty(K)$:

$$\sum_{|\alpha| \leq m} \|D^\alpha_{x''} \chi u\|_s \leq C(\|Q\chi u\|_s + \|\chi u\|_s) \leq$$

$$\leq C(\|\chi P u\|_s + \|[P, \chi]u\|_s + \|(Q - P)\chi u\|_s + \|\chi u\|_s) \leq$$

$$\leq C(\|\chi P u\|_s + \epsilon \sum_{|\alpha| \leq m} \|D^\alpha_{x''} \chi u\|_s + C_3 \sum_{|\alpha| \leq m-1} \|D^\alpha_{x''} u\|_s) .$$

Choose $\epsilon < 1/2C$. Then we get

$$\sum_{|\alpha| \leq m} \|D^\alpha_{x''} \chi u\|_s \leq 2C(\|\chi P u\|_s + C_3 \sum_{|\alpha| \leq m-1} \|D^\alpha_{x''} u\|_s) \ , \quad u \in C_0^\infty(K). \tag{1.5}$$

If $WF(\chi)$ does not intersect Σ, we still have (1.5) because of the ellipticity of P outside Σ. By a pseudodifferential partition of unity we get

$$\sum_{|\alpha| \leq m} \|D^\alpha_{x''} u\|_s \leq C_4(\|Pu\|_s + \sum_{|\alpha| \leq m-1} \|D^\alpha_{x''} u\|_s) \ , \quad u \in C_0^\infty(K) \ . \tag{1.6}$$

Then the lemma follows by using the inequality

$$\sum_{|\alpha| \leqslant m-1} \| D_{x''}^{\alpha} u \|_s \leqslant \epsilon \sum_{|\alpha| \leqslant m} \| D_{x''}^{\alpha} u \|_s + C_\epsilon \| u \|_s \ , \ u \in C_0^\infty(\mathbf{R}^n)$$

and choosing $\epsilon < 1/2C_4$.

Remark. – From Lemma 1.2 it easily follows that if $D_{x''}^{\alpha} u \in H_{\text{comp}}^s(\mathbf{R}^n)$ for $|\alpha| \leqslant m-1$ and $Pu \in H^s(\mathbf{R}^n)$, then $D_{x''}^{\alpha} u \in H^s(\mathbf{R}^n)$ for $|\alpha| \leqslant m$.

If $u \in \mathcal{D}'(\mathbf{R}^n)$ and $(x, \xi) \in T^*\mathbf{R}^n \backslash 0$ we put

$$S_u(x, \xi) = \sup \{ s \in \mathbf{R} \ ; u \in H^s \text{ microlocally near } (x, \xi) \} .$$

Then S_u is a lower half continuous function on $T^*\mathbf{R}^n \backslash 0$, positively homogeneous of degree 0.

LEMMA 1.3. – *If $u \in \mathcal{D}'(\mathbf{R}^n)$, $(x_0, \xi_0) \notin WF(Pu)$, then*

$$S_{D_{x''}^{\beta} u}(x_0, \xi_0) \geqslant \min_{|\alpha| \leqslant m-1} S_{D_{x''}^{\alpha} u}(x_0, \xi_0) \tag{1.7}$$

for all multiindices β.

Proof. – (1.7) is trivial when $(x_0, \xi_0) \notin \Sigma$ so we assume that $(x_0, \xi_0) = (x_0, \xi_0', 0) \in \Sigma$. Let $\chi \in L^0(\mathbf{R}^n)$ be a properly supported operator with $WF(\chi)$ close to (x_0, ξ_0) and such that $\chi \equiv I$ near (x_0, ξ_0). If γ is a multiindex we have

$$P\chi D_{x''}^{\gamma} u = [P, \chi D_{x''}^{\gamma}] u = \sum_{|\beta| \leqslant m + |\gamma| - 1} b_\beta(x, D) D_{x''}^{\beta} u \bmod C^\infty ,$$

where b_β are of order 0 with $WF(b_\beta)$ close to (x_0, ξ_0). It follows from Lemma 1.2 and the remark above, that $D_{x''}^{\alpha} D_{x''}^{\gamma} u \in H^s$ near (x_0, ξ_0) for all α with $|\alpha| \leqslant m$, if $D_{x''}^{\beta} u \in H^s$ near (x_0, ξ_0) for all $|\beta|$ with $|\beta| \leqslant m + |\gamma| - 1$. The lemma then follows by induction.

Now suppose that $u \in \mathcal{D}'(\mathbf{R}^n)$ and $WF(u) \subset \Sigma$. By Taylor's formula we can write

$$(a_\alpha(x, D) - a_\alpha^0(x, D', 0)) u =$$

$$= \Big(\sum_{|\gamma| = 1} b_\gamma(x, D) D_{x''}^{\gamma} + c(x, D) \Big) u \bmod C^\infty , \tag{1.8}$$

where b_γ and c are of order -1. (We refer to the appendix in [11] for some results about "pseudodifferential" operators of the type $S(x, D')$. In (1.8) we assume that $a^0_\alpha(x, D', 0)$ has been suitably modified by adding an operator of order $-\infty$, in order to become properly supported). From (1.8) and Lemma 1.3 we get

LEMMA 1.4. – *Let* $u \in \mathcal{D}'(\mathbf{R}^n)$, $\mathrm{WF}(u) \subset \Sigma$, $(x_0, \xi_0) \notin \mathrm{WF}(\mathrm{P}u)$. *Then*

$$S_{(P-P_0)u}(x_0, \xi_0) \geq \min_{|\alpha| \leq m-1} S_{D^\alpha_{x''}u}(x_0, \xi_0) + 1 .$$

2. Localization in the (x', ξ')-space.

Our localization method will be essentially the same as the one introduced by Hörmander [8]. We denote by $K^m(\mathbf{R}^n \times \mathbf{R}^{n-d} \times \mathbf{R}^{n-d})$ the space of symbols $\chi(x, y', \xi') \in S^{m+(n-d)/4}_{1,1/2}((\mathbf{R}^n \times \mathbf{R}^{n-d}) \times \mathbf{R}^{n-d})$ having their support in a set of the form $|x' - y'| \leq$ const. and being of class $S^{-\infty}$ outside a set of the form $|x' - y'| \leq (\text{const.})\, |\xi'|^{-1/2}$. For instance if $\psi \in C^\infty_0(\mathbf{R}^{n-d})$, we can take

$$\chi(x, y', \xi') = \psi((x' - y')\,|\xi'|^{1/2})\,|\xi'|^{(n-d)/4} \quad \text{for} \quad |\xi'| \geq 1 \tag{2.1}$$

and extend this definition suitably for small ξ'. Then $\chi \in K^0$.

If $u \in \mathcal{D}'(\mathbf{R}^n)$ and $\mathrm{WF}(u)$ does not meet the normals of the planes $x'' =$ const., we can define

$$T_\chi u(x, \xi') = \int \chi(x, y', \xi')\, e^{i\langle x'-y', \xi'\rangle}\, u(y', x'')\, dy' . \tag{2.2}$$

It is easy to verify that $T_\chi u$ is a C^∞ function of $(x, \xi') \in \mathbf{R}^n \times \mathbf{R}^{n-d}$.

LEMMA 2.1. – *Let* $\chi \in K^0$ *and* $a(x, \xi') \in S^0_{1,0}(\mathbf{R}^n \times \mathbf{R}^{n-d})$. *Suppose that the distribution kernel of* $a(x, D')$ *has its support in a set of the form* $|x' - y'| \leq$ *const.. Then we have*

$$T_\chi(a(x, D')\,u)\,(x, \xi') = a(x, \xi')\,T_\chi u(x, \xi') + T_{\chi_a} u(x, \xi') , \tag{2.3}$$

where $\chi_a \in K^{-1/2}(\mathbf{R}^n \times \mathbf{R}^{n-d} \times \mathbf{R}^{n-d})$. *If* $V \subset \mathbf{R}^n \times (\mathbf{R}^{n-d} \backslash \{0\})$ *is an open cone where* $a(x, \xi')$ *or* $1 - a(x, \xi')$ *belongs to* $S^{-\infty}$, *then* χ_a *is of class* $S^{-\infty}$ *in the set* $\{(x, y', \xi') ; (x, \xi') \in V, y' \in \mathbf{R}^{n-d}\}$.

The proof is straight forward, we write the formula (2.2) for u replaced by $a(x\ , \mathrm{D}')\, u$. After a partial integration in the integral, we apply the wellknown asymptotic formula to

$$ta(y'\ , x''\ , \mathrm{D}_{y'})\, (\chi(x\ , y'\ , \xi')\, e^{i\langle x' - y', \xi'\rangle})$$

The leading term is

$$a(y'\ , x''\ , \xi')\, \chi(x\ , y'\ , \xi')\, e^{i\langle x' - y', \xi'\rangle}$$

and modulo $\mathrm{K}^{-1/2}\, e^{i\langle x' - y', \xi'\rangle}$ this is congruent to

$$a(x\ , \xi')\, \chi(x\ , y'\ , \xi')\, e^{i\langle x' - y', \xi'\rangle}$$

In fact, we can Taylor expand $a(x''\ , y'\ , \xi')$ with respect to y' at the point $y' = x'$ and we only have to note that $(x_j' - y_j')\, \chi(x\ , y'\ , \xi') \in \mathrm{K}^{-1/2}$ since χ is of class $\mathrm{S}^{-\infty}$ outside a set of the form

$$|x' - y'| \leqslant (\text{const.})\, |\xi'|^{-1/2}\ .$$

We omit the details.

Lemma 2.2. – *Let* $\chi \in \mathrm{K}^0(\mathbf{R}^n \times \mathbf{R}^{n-d} \times \mathbf{R}^{n-d})$ *and let* $u \in \mathcal{D}'(\mathbf{R}^n)$ *with* $\mathrm{WF}(u) \subset \Sigma = \{(x\ , \xi) \in \mathrm{T}^*\mathbf{R}^n \backslash 0\ ;\ \xi'' = 0\}$. *If* $u \in \mathrm{H}^s$ *microlocally near a point* $(x_0\ , \xi_0'\ , 0)$, *then there is a conic neighbourhood* $\mathrm{V} \subset \mathbf{R}^n \times (\mathbf{R}^{n-d}\backslash\{0\})$ of $(x_0\ , \xi_0')$ *such that*

$$(1 + |\xi'|)^s\, \mathrm{T}_\chi u(x\ , \xi') \in \mathrm{L}^2(\mathrm{V}).$$

When χ *is of the form* (2.1) *and* $\psi \not\equiv 0$, *the converse implication is also true.*

Proof. – If $v \in \mathrm{C}_0^\infty(\mathbf{R}^n)$, then $\mathrm{T}_\chi v(x\ , \xi')$ is rapidly decreasing as a function of ξ' and

$$\iint (1 + |\xi'|^2)^s\, |\mathrm{T}_\chi v(x\ , \xi')|^2\, dx\, d\xi' = (\mathrm{B}_{2s} v\ , v)_{\mathrm{L}^2(\mathbf{R}^n)} \tag{2.4}$$

where

$$\mathrm{B}_{2s} v(x) = \iint b_{2s}(x\ , y'\ , \xi')\, e^{i\langle x' - y', \xi'\rangle}\, v(y'\ , x'')\, dy'\, d\xi'$$

and b_{2s} is given by

$$b_{2s}(x\ , y', \xi') = (1 + |\xi'|^2)^s \int \chi((z'\ , x'')\ , y'\ , \xi')\, \overline{\chi}((z'\ , x'')\ , x'\ , \xi')\, dz'. \tag{2.5}$$

It is easy to see that $b_{2s} \in S^{2s}_{1,1/2}(\mathbf{R}^n \times \mathbf{R}^{n-d} \times \mathbf{R}^{n-d})$, so B_{2s} is a pseudodifferential operator of order $2s$ in the tangential variables x'. We therefore have

$$(B_{2s} v , v) \leqslant C_K \| v \|^2_{(0,s)} \ , \ v \in C_0^\infty(K) \ , \tag{2.6}$$

for all $K \subset\subset \mathbf{R}^n$, where $\| v \|_{(0,s)}$ is the norm in $H^{(0,s)}(\mathbf{R}^n)$, given by

$$\| v \|^2_{(0,s)} = \int (1 + |\xi'|)^{2s} |\hat{v}(\xi)|^2 \, d\xi \ .$$

Now let $u \in \mathcal{D}'(\mathbf{R}^n)$ be such that $WF(u) \subset \Sigma$ and $u \in H^s$ near $(x_0 , \xi_0' , 0)$. Choose $a(x , \xi') \in S^0_{1,0}(\mathbf{R}^n \times \mathbf{R}^{n-d})$ such that the distribution kernel of $a(x , D')$ has compact support and such that $a(x , \xi')$ (and $1 - a(x , \xi')$) belong to $S^{-\infty}$ outside (respectively inside) a small conic neighbourhood of (x_0 , ξ_0').

Then $a(x , D')u \in H^{(0,s)}_{\text{comp}}(\mathbf{R}^n)$ and we let $v_j \in C_0^\infty(\mathbf{R}^n)$, $j = 1 , 2 , \ldots$ be a sequence converging to $a(x , D')u$ in $H^{(0,s)}_{\text{comp}}$. Then

$$T_\chi v_j(x , \xi') \to T_\chi a(x , D) u(x , \xi')$$

locally uniformly, so combining (2.4), (2.6) with v replaced by v_j we see that

$$\iint (1 + |\xi'|^2)^s \, |T_\chi(a(x , D')u)(x , \xi')|^2 \, dx \, d\xi' < \infty \ .$$

By Lemma 2.1 we have

$$T_\chi a(x , D') u(x , \xi') = T_\chi u(x , \xi') + \mathcal{O}(|\xi'|^{-N})$$

for (x , ξ') in a small conic neighbourhood of (x_0 , ξ_0') and for all $N > 0$. Then the first part of Lemma 2.2 follows.

We now prove the second part, so we assume that χ is of the form (2.1). Then from (2.5) it follows that

$$b_{2s}(x , x' , \xi') = \| \psi \|^2_{L^2}(1 + |\xi'|^2)^s \quad \text{for} \quad |\xi'| \geqslant 1 \ .$$

This means that B_{2s} is an elliptic operator in the tangential variables, so we have the Gårding inequality for all $s' \in \mathbf{R}$ and $K \subset\subset \mathbf{R}^n$:

$$(B_{2s} v , v) \geqslant (1/C) \| v \|^2_{(0,s)} - C \| v \|^2_{(0,s')} \tag{2.7}$$

$$v \in C_0^\infty(K) \ .$$

Here C is a positive constant depending on K, s, s'.

Let $u \in \mathcal{D}'(\mathbf{R}^n)$ with $\mathrm{WF}(u) \subset \Sigma$ and assume that

$$(1 + |\xi'|)^s \, \mathrm{T}_\chi u(x, \xi') \in \mathrm{L}^2(\mathrm{V}) ,$$

where V is a small conic neighbourhood of (x_0, ξ'_0). Let $t \in \mathbf{R}$ be such that $u \in \mathrm{H}^t$ near $(x_0, \xi'_0, 0)$. If $t \geqslant s$ there is nothing to prove, so we can assume that $t < s$. Take $a(x, \mathrm{D}')$ as above. Then we have

$$\mathrm{T}_\chi(a(x, \mathrm{D}')\, u)(x, \xi') = a(x, \xi')\, \mathrm{T}_\chi u(x, \xi') + \mathrm{T}_{\chi_a} u(x, \xi') ,$$

where $\chi_a \in \mathrm{K}^{-1/2}$. Applying the part of the lemma, which has already been proved, we conclude that

$$(1 + |\xi'|)^{t'} \, \mathrm{T}_\chi(a(x, \mathrm{D}')\, u)(x, \xi') \in \mathrm{L}^2(\mathbf{R}^n \times \mathbf{R}^{n-d}) ,$$

where $t' = \min(t + 1/2, s)$.

Combining (2.4), (2.7) for s replaced by t' and using an easy regularization argument, we conclude that $a(x, \mathrm{D}')\, u \in \mathrm{H}^{(0, t')}(\mathbf{R}^n)$ and therefore that $u \in \mathrm{H}^{t'}$ near $(x_0, \xi'_0, 0)$ (because $\mathrm{WF}(u) \in \Sigma$). Repeating this argument we finally get that $u \in \mathrm{H}^s$ near $(x_0, \xi'_0, 0)$ and the proof of Lemma 2.2 is complete.

Note that the first part of the proof also shows that $\mathrm{T}_\chi u(x, \xi')$ is rapidly decreasing in a conic neighbourhood of (x_0, ξ'_0) if $(x_0, \xi'_0, 0) \notin \mathrm{WF}(u) \subset \Sigma$.

3. Proof of Theorem 1.1.

Let $\mathrm{P}(x, \mathrm{D})$ $(= \widetilde{\mathrm{P}}(x, \mathrm{D}))$ be the operator in Theorem 1.1 and let P_0 be the operator, introduced in section 1. The polynomial $a_\rho(t)$ in the introduction is just the principal symbol of $\mathrm{P}_0(x, \xi', \mathrm{D}_{x''})$: $\mathrm{C}^\infty(\mathbf{R}^d) \to \mathrm{C}^\infty(\mathbf{R}^d)$. The condition (0.3) then implies that $\mathrm{P}_0(x, \xi', \mathrm{D}_{x''})$ satisfies the conditions of Calderón's uniqueness theorem with respect to any hypersurface in $\mathbf{R}^d$. We therefore have the following Carleman type estimate, which follows from Nirenberg [10, inequality (6.1)] by a partition of unity :

For all $(x_0, \xi'_0) \in \mathbf{R}^n \times (\mathbf{R}^{n-d} \backslash \{0\})$ and $r > 0$, there are numbers $\tau_0 > 0$, $\mathrm{R}_0 > r$ and C such that for all $\tau > \tau_0$ and $v \in \mathrm{C}_0^\infty(\mathbf{R}^d)$ with support in $r < |x''| < \mathrm{R} \leqslant \mathrm{R}_0$, we have

$$\sum_{|\alpha| \leqslant m-1} \| e^{\tau \varphi_R(x'' - x''_0)} D^\alpha_{x''} v(x'') \|_{L^2(R^d)} \leqslant$$

$$\leqslant C \| e^{\tau \varphi_R(x'' - x''_0)} P_0(x'_0, x'', \xi'_0, D_{x''}) v(x'') \|_{L^2(R^d)} . \qquad (3.1)$$

Here $\varphi_R(x'') = (R - |x''|)^2 - ((R - r)/2)^2$ so that $\varphi_R(x'')$ is $\gtrless 0$ for $|x''| \lessgtr (R + r)/2$. In fact, the coordinates (t, x) in [10] correspond here to polar coordinates (ρ, θ), $\rho \in R_+$, $\theta \in S^{d-1}$ in $\mathbf{R}^d$, centered at x''_0. The inequality (6.1) in [10] then implies that (3.1) is true for $v \in C_0^\infty(\mathbf{R}^d)$ as above with the supplementary condition that θ belongs to a small open subset of S^{d-1} when (ρ, θ) is in the support of v. This subset is independent of τ and R, and we also know from [10] that the constant C can be chosen arbitrarily small when $R - r$ and $1/\tau$ are small enough. Then we can obtain (3.1) by a partition of unity in the θ-variables, for the "bad" terms coming from the commutators between P_0 and the cut off functions can easily be eliminated.

It also follows from [10] that for given r, we can choose R_0, C τ_0 independent of (x_0, ξ'_0) when x_0 varies in a compact subset of $\mathbf{R}^n$ and ξ'_0 varies in $\mathbf{R}^{n-d} \backslash \{0\}$.

Now assume that $u \in \mathcal{D}'(\mathbf{R}^n)$, $Pu = w$, where

$$(0, \xi'_0, 0) \notin WF(u) \subset \Sigma$$

and

$$\{(0, x'', \xi'_0, 0) \; ; \; |x''| \leqslant 1\} \cap WF(w) = \phi .$$

If $r > 0$ is small enough we have

$$\{(0, x'', \xi'_0, 0) \; ; \; |x''| \leqslant r\} \cap WF(u) = \phi . \qquad (3.2)$$

For $\epsilon > 0$ put $V'_\epsilon = \{(x', \xi') \in \mathbf{R}^{n-d} \times (\mathbf{R}^{n-d} \backslash \{0\}) \; ; \; |x'| < \epsilon, |\xi'/|\xi'| - \xi'_0/|\xi'_0|| < \epsilon\}$. If $\epsilon > 0$ is small enough we have

$$\{(x, \xi', 0) \; ; (x', \xi') \in \overline{V}'_\epsilon, |x''| \leqslant r + \epsilon\} \cap WF(u) = \phi , \qquad (3.3)$$

and if $R > r$ is as in (3.1) above, we can also assume that

$$\{(x, \xi', 0) \; ; (x', \xi') \in \overline{V}'_\epsilon, |x''| \leqslant R\} \cap WF(w) = \phi . \qquad (3.4)$$

(Here we assume that $R \leqslant 1$ which is no restriction).

Now we write $P_0 u = w + (P_0 - P) u$ and we introduce

$$v(x, \xi') = T_\chi u(x, \xi') ,$$

where $\chi \in K^0$ is of the form (2.1). Applying Lemma 2.1 we get

$$P_0(x , \xi' , D_{x''}) v(x , \xi') + \sum_{|\alpha| \leqslant m} T_{\chi_\alpha}(D_{x''}^\alpha u)(x , \xi') = T_\chi w(x , \xi') +$$
$$+ T_\chi((P_0 - P) u)(x , \xi') ,$$

where $\chi_\alpha \in K^{-1/2}$. We rewrite this as

$$P_0(x , \xi' , D_{x''}) v(x , \xi') = T_\chi w(x , \xi') + A_{-1/2} u(x , \xi') . \quad (3.5)$$

If $f \in C^\infty(\mathbf{R}^n \times \mathbf{R}^{n-d})$ and $(x_0 , \xi_0') \in \mathbf{R}^n \times (\mathbf{R}^{n-d} \setminus \{0\})$, we put $F_f(x_0 , \xi_0') = \sup \{ s \in \mathbf{R} \ ; (1 + |\xi'|)^s f(x , \xi')$ is square integrable in some conic neighbourhood of $(x_0 , \xi_0')\}$. Then if $f = T_\chi g$ for some $g \in \mathcal{D}'(\mathbf{R}^n)$ with $WF(g) \subset \Sigma$, we have $F_f(x , \xi') = S_g(x , \xi' , 0)$ in view of Lemma 2.2. In our particular situation it follows from Lemma 2.2 and Lemma 1.3 that

$$S_{D_{x''}^\alpha u}(x , \xi' , 0) \geqslant \min_{|\beta| \leqslant m-1} F_{T_\chi(D_{x''}^\beta u)}(x , \xi')$$

for all α when $(x' , \xi') \in V_\epsilon'$ and $|x''| \leqslant R$. Then from Lemma 2.2 and Lemma 1.4 it follows that

$$F_{A_{-1/2} u}(x , \xi') \geqslant \min_{|\alpha| \leqslant m-1} F_{D_{x''}^\alpha v}(x , \xi') + 1/2 , \quad (3.6)$$

$$(x' , \xi') \in V_\epsilon' , \ |x''| \leqslant R .$$

Let $\psi_R(x'') \in C_0^\infty(\mathbf{R}^d)$ have support in $r < |x''| < R$ and be such that $\psi_R(x'') = 1$ near the domain $r + \epsilon \leqslant |x''| \leqslant (R + r)/2$. From (3.5) we obtain

$$P_0(x , \xi' , D_{x''})(\psi_R(x'') v(x , \xi')) = \psi_R(x'')(T_\chi w(x , \xi') +$$
$$+ A_{-1/2} u(x , \xi')) + [P_0(x , \xi' , D_{x''}) , \psi_R] v(x , \xi') . \quad (3.7)$$

Now we shall apply (3.1) to this equation, with $\tau = \nu \log(1 + |\xi'|)$, $\nu \in \mathbf{Z}^+$. Since by (3.3) $v(x , \xi')$ is rapidly decreasing in

$$\{(x , \xi') \ ; (x' , \xi') \in \overline{V}_\epsilon' , \ |x''| \leqslant r + \epsilon\}$$

and $T_\chi w(x , \xi')$ is rapidly decreasing in

$$\{(x , \xi') \ ; (x' , \xi') \in \overline{V}_\epsilon' , \ |x''| < R\} ,$$

we get for all real N and M :

$$\sum_{|\alpha|\leqslant m-1} \|(1+|\xi'|)^{\nu\varphi_R(x'')-M} D^{\alpha}_{x''} v(x,\xi')\|_{L^2(B_{(R+r)/2})} \leqslant$$

$$\leqslant C_{N,\nu}((1+|\xi'|)^{-N-M} + \|(1+|\xi'|)^{\nu\varphi_R(x'')-M} A_{-1/2}u(x,\xi')\|_{L^2(B_R)} +$$

$$+ \sum_{|\alpha|\leqslant m-1} \|(1+|\xi'|)^{\nu\varphi_R(x'')-M} D^{\alpha}_{x''} v(x,\xi')\|_{L^2(B_{(R+r)/2,R})}) \tag{3.8}$$

when $(x',\xi') \in V'_\epsilon$. Here we use the notations $B_T = \{x'' \in \mathbf{R}^k \,;\, |x''| \leqslant T\}$ and $B_{t,T} = \{x'' \in \mathbf{R}^k \,;\, t \leqslant |x''| \leqslant T\}$ for $0 < t < T$.

We can assume that $R > r$ is so small that $\varphi_R(x'') < 1/2$. Take M so large that for $|\alpha| \leqslant m-1$ we have $F_{D^{\alpha}_{x''}v}(x,\xi') > -M$ in $W = \{(x,\xi') \,;\, (x',\xi') \in V'_\epsilon,\ |x''| \leqslant R\}$. Then by (3.6) we have $F_{A_{-1/2}u}(x,\xi') > -M + \varphi_R(x'')$ in W and from (3.8) with $\nu = 1$ we get $F_{D^{\alpha}_{x''}v}(x,\xi') > -M + \varphi_R(x'')$ in W for $|\alpha| \leqslant m-1$. (For $|x''| \geqslant (R+r)/2$ this inequality is trivial, since $\varphi_R(x'') \leqslant 0$ then). Then $F_{A_{-1/2}}u(x,\xi') > -M + 2\varphi_R(x'')$ in W and applying (3.8) with $\nu = 2$, we get $F_{D^{\alpha}_{x''}v}(x,\xi') > -M + 2\varphi_R(x'')$ in W when $|\alpha| \leqslant m-1$. Repeating this argument, we get

$$F_{D^{\alpha}_{x''}v}(x,\xi') > -M + \nu\varphi_R(x'')$$

in W for all ν, so that $F_{D^{\alpha}_{x''}v}(x,\xi') = +\infty$ when $(x',\xi') \in V'_\epsilon$, $|x''| < (R+r)/2$. In particular (by Lemma 2.2) :

$$WF(u) \cap \{(0,x'',\xi'_0,0) \,;\, |x''| < (R+r)/2\} = \phi\,. \tag{3.9}$$

Recall that we started with the assumption (3.2).

Now take $x''_0 \in \mathbf{R}^d$, such that

$$\{x'' \in \mathbf{R}^d \,;\, |x''-x''_0| \leqslant r\} \subset \{x'' \in \mathbf{R}^d \,;\, |x''| < (R+r)/2\}$$

and

$$\{x'' \in \mathbf{R}^d \,;\, |x''-x''_0| \leqslant R\} \subset \{x'' \in \mathbf{R}^d \,;\, |x''| \leqslant 1\}\,,$$

where $r < R < R_0$ are as in (3.1). Then by the same argument, that gave (3.9) from (3.2), we get

$$\{(0,x'',\xi'_0,0) \,;\, |x''-x''_0| < (R+r)/2\} \cap WF(u) = \phi\,.$$

Repeating this procedure, we obtain

$$\mathrm{WF}(u) \cap \{(0\,, x''\,, \xi_0'\,, 0)\ ;\ |x''| < 1\} = \phi$$

and this completes the proof of Theorem 1.1.

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Johannes SJÖSTRAND,
Purdue University
Division of Mathematical Sciences
West Lafayette, Indiana 47907 (USA).