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A GENERAL DEFINITION OF CAPACITY
by Makoto OHTSUKA

Introduction.

During the past 20 years the notion of extremal length proved its usefulness in many branches of analysis. Given a family $\Gamma$ of locally rectifiable curves in the $(x, y)$-plane, the extremal length of $\Gamma$ is defined to be the reciprocal of the infimum of $\int \int \rho^2 \, dx \, dy$ for the family of Borel measurable functions $\rho \geq 0$ satisfying $\int_\gamma \rho \, ds \geq 1$ for every $\gamma \in \Gamma$.

There are also many definitions of capacity. One way to define the Newtonian capacity in $\mathbb{R}^n$ is to consider the class $M$ of non-negative measures $\mu$ with finite energy. It is known that $\text{grad } U^\mu$ exists a.e., where $U^\mu$ denotes the Newtonian potential of $\mu$. The Newtonian capacity of a compact set $K$ is defined to be the infimum of $\int_{\mathbb{R}^n} |\text{grad } U^\mu|^2 \, dx$ taken with respect to $\mu \in M$ satisfying $U^\mu \geq 1$ on $K$.

Recently, Meyers [2] defined $C_{k; \rho; \rho}(A)$ for $A \subseteq \mathbb{R}^n$ by $\inf \int \rho^\rho \, d\mu_\rho$ taken with respect to $\rho \geq 0$ satisfying

$$\int k(x, y) \rho(y) \, d\mu_\rho(y) \geq 1$$
on $A$, where $\mu_\rho$ is a non-negative measure and $k(x, y)$ is a positive lower semicontinuous function on $\mathbb{R}^n \times \mathbb{R}^n$. 

"Dédié à Monsieur M. Brelot à l'occasion de son 70e anniversaire."
In the present note we shall give a general definition of capacity which includes the above three quantities as special cases, and prove that this general capacity is continuous from the left.

1. General definition.

Let $\Omega$ be a space, and $F$ be a family of non-negative functions defined on $\Omega$ such that $cf_1 \in F$ and $f_1 + f_2 \in F$ whenever $0 \leq c < \infty$ and $f_1, f_2 \in F$; we set $0 \cdot \infty = 0$ if $0 \cdot \infty$ happens for $cf_1$. It follows that $f \equiv 0$ belongs to $F$. Let $\Phi \neq \infty$ be a non-negative functional defined on $F$. Assume that there exist $p, q > 0$ such that $\Phi(cf) \leq c^p \Phi(f)$ for any constant $c \geq 0$ and $f \in F$, and

$$(\Phi(f_1 + f_2))^q \leq (\Phi(f_1))^q + (\Phi(f_2))^q$$

if $f_1, f_2 \in F$. In addition, we assume that, if $f_1, f_2, \ldots \in F$ and $\Phi(f_{m+1} + \cdots + f_n) \to 0$ as $n, m \to \infty$, then

$$f = \sum_{k=1}^{\infty} f_k \in F$$

and $\Phi\left(\sum_{k=1}^{n} f_k\right) \to \Phi(f)$ as $n \to \infty$. It follows that

$$(\Phi(\sum_{k=1}^{n} f_k))^q \leq \sum(\Phi(f_k))^q$$

for such $\{f_n\}$.

Let $\Gamma_0$ be another space, and $G$ be a class of subsets of $\Gamma_0$ such that $\Gamma_1, \Gamma_2, \ldots \in G$ implies $\bigcup_{n} \Gamma_n \in G$ and that $\Gamma \in G$ and $\Gamma' \subset \Gamma$ imply $\Gamma' \in G$. We shall say that a property holds G-a.e. on $\Gamma \subset \Gamma_0$ if the exceptional set belongs to $G$. For each $f \in F$ suppose a non-negative function $T_f(\gamma)$ is defined G-a.e. on $\Gamma_0$, and assume that, for any $f_1, f_2 \in F$ and $c \geq 0$, $T_{cf_1} = cT_{f_1}$ and $T_{f_1 + f_2} = T_{f_1} + T_{f_2}$ hold and $f_1 \leq f_2$ implies $T_{f_1} \leq T_{f_2}$, where all relations as to $T_{f_1}$ and $T_{f_2}$ are supposed to hold wherever they are defined.

We shall say that $f$ is $G$-almost admissible (or simply $G$-alm. ad.) for $\Gamma \subset \Gamma_0$ when $f \in F$ and $T_f \geq 1$ G-a.e. on $\Gamma$. We set

$$G_G(\Gamma) = \inf_{G \text{-alm. ad. } f} \Phi(f)$$
if there is at least one G-alm. ad. $f$, and otherwise
\[ C_G(\Gamma) = \infty. \]
Evidently \( C_G(\Gamma) \leq C_G(\Gamma') \) if \( \Gamma \subseteq \Gamma' \). We observe that \( C_G(\Gamma) = 0 \) for every \( \Gamma \in G \) because \( f \equiv 0 \) is G-alm. ad. and \( \Phi(0) = 0 \).

We shall denote by \( L \) the family of functions \( f \in F \) with finite \( \Phi(f) \).

**Theorem 1.** — \( C_G(\Gamma) = 0 \) if and only if there exists \( f \in L \) such that \( T_f = \infty \) G-a.e. on \( \Gamma \).

**Proof.** — The if part follows from the definition of \( C_G \) and the properties of \( \Phi \) and \( T_f \). To prove the only-if part take \( f_n \in F \) and \( \Gamma_n \in G \) for each \( n \) so that \( T_{f_n} \geq 1 \) on \( \Gamma - \Gamma_n \) and \( (\Phi(f_n))^q \leq 2^{-n} \), and set \( f = \sum_n f_n \). Then \( f \in F \) and
\[ (\Phi(f))^q \leq \sum(\Phi(f_n))^q \leq 1. \]

We have
\[ T_f(\gamma) \geq \sum_{k=1}^m T_{f_k}(\gamma) \geq m \]
for every \( \gamma \in \Gamma - \bigcup_n \Gamma_n \) and \( m \) so that \( T_f(\gamma) = \infty \) for every \( \gamma \in \Gamma - \bigcup_n \Gamma_n \). Since \( \bigcup_n \Gamma_n \in G \), our theorem is proved.

**Lemma 1.** — \( \left( C_G \left( \bigcup_n \Gamma_n \right) \right)^q \leq \Sigma(C_G(\Gamma_n))^q \).

**Proof.** — We may assume that \( \Sigma(C_G(\Gamma_n))^q < \infty \). Given \( \varepsilon > 0 \), let \( f_n \) be G-alm. ad. for \( \Gamma_n \) such that
\[ (\Phi(f_n))^q \leq (C_G(\Gamma_n))^q + \varepsilon 2^{-n}. \]
By our assumption on \( \Phi \), \( f = \sum f_n \in F \) and
\[ (\Phi(f))^q \leq \Sigma(\Phi(f_n))^q. \]
Evidently \( f \) is G-alm. ad. for \( \bigcup_n \Gamma_n \) so that
\[ (C_G(\bigcup \Gamma_n))^q \leq (\Phi(f))^q \leq \Sigma(\Phi(f_n))^q \leq \Sigma(C_G(\Gamma_n))^q + \varepsilon. \]
This gives the required inequality.
LEMMA 2. — Suppose \( f \in F \) satisfies \( T_f \geq 1 \) on \( \Gamma - \Gamma' \), where \( C_G(\Gamma') = 0 \). Then \( C_G(\Gamma) \leq \Phi(f) \).

Proof. — By Theorem 1 there exist \( f' \in L \) and \( \Gamma'' \in G \) such that \( T_f = \infty \) on \( \Gamma' - \Gamma'' \). For any \( \varepsilon > 0 \) we have \( T_{f+\varepsilon f'} \geq 1 \) on \( \Gamma - \Gamma'' \), and hence

\[
C_G(\Gamma) \leq \Phi(f + \varepsilon f') \leq \{(\Phi(f))^q + (\varepsilon^p \Phi(f'))^q\}^{1/q} \to \Phi(f)
\]
as \( \varepsilon \to 0 \). Thus \( C_G(\Gamma) \leq \Phi(f) \).

THEOREM 2. — Denote \( \{\Gamma^* \subset \Gamma_0; C_G(\Gamma^*) = 0\} \) by \( G_0 \). Then

\[
C_{G_0}(\Gamma) = C_G(\Gamma)
\]
for any \( \Gamma \subset \Gamma_0 \).

Proof. — We observe that \( \Gamma_1, \Gamma_2, \ldots \in G_0 \) implies \( \bigcup \Gamma_n \in G_0 \) in virtue of Lemma 1 and that \( \Gamma \in G_0 \) and \( \Gamma' \subset \Gamma \) imply \( \Gamma' \in G_0 \). Since \( G \subset G_0 \), \( C_{G_0}(\Gamma) \leq C_G(\Gamma) \). Assume that \( C_{G_0}(\Gamma) < \infty \), and take \( f \in F \) such that \( T_f \geq 1 \) on \( \Gamma - \Gamma' \) where \( \Gamma' \in G_0 \). By Lemma 2 \( C_G(\Gamma) \leq \Phi(f) \).

Because of the arbitrariness of \( f \) we derive

\[
C_G(\Gamma) \leq C_{G_0}(\Gamma).
\]
The equality now follows.

THEOREM 3 (cf. [2], Theorem 4). — Each of the following statements implies the succeeding one.

(i) \( T_{f_n} \to T_f \) in \( C_G \), namely, for any \( a > 0 \),

\[
C_G(\{\gamma \in \Gamma_0 - \Gamma; |T_{f_n}(\gamma) - T_f(\gamma)| \geq a\}) \to 0 \text{ as } n \to \infty,
\]
where \( \Gamma \in G \) and all \( T_{f_n} \) and \( T_f \) are defined on \( \Gamma_0 - \Gamma \); \( -\infty - -\infty \) is set to be 0 if it happens for \( T_{f_n} - T_f \).

(ii) We can find \( \{f_{nk}\} \) with the property that, given \( \varepsilon > 0 \), there exists \( \Gamma' \subset \Gamma_0 \) with \( C_G(\Gamma') < \varepsilon \) such that \( T_{f_{nk}} - T_f \to 0 \) uniformly on \( \Gamma_0 - \Gamma' \).

(iii) For the sequence \( \{f_{nk}\} \) in (ii), \( T_{f_{nk}} \to T_f \) on \( \Gamma_0 - \Gamma'' \), where \( C_G(\Gamma'') = 0 \).
Proof. — (i) → (ii). There exist \( \{f_{n_k}\} \) and \( \{\Gamma_k\} \) in \( \Gamma_0 \) such that, for each \( k, \Gamma_k \supset \Gamma, \) \( (C_G(\Gamma_k))^q \leq 2^{-k} \) and

\[
|T_{f_{n_k}} - T_f| \leq \frac{1}{k} \quad \text{on} \quad \Gamma_0 - \Gamma_k.
\]

Given \( \varepsilon > 0, \) choose \( k_0 \) so that \( 2^{-k_0+1} < \varepsilon^q. \) We see that

\[
T_{f_{n_k}} - T_f \to 0 \quad \text{uniformly on} \quad \Gamma_0 - \bigcup_{k=k_0}^\infty \Gamma_k,
\]

and

\[
\left( C_G \left( \bigcup_{k=k_0}^\infty \Gamma_k \right) \right)^q \leq \sum_{k=k_0}^\infty (C_G(\Gamma_k))^q \leq \varepsilon^q
\]

by Lemma 1. This establishes (i) → (ii).

(ii) → (iii) is evident.

Now, let \( \Psi(f, g) \) be a functional on \( F \times F \) such that, for any \( f_1, f_2 \in F, \) there exists \( \tilde{f} \in F \) satisfying \( \Phi(\tilde{f}) \leq \Psi(f_1, f_2) \) and \( |T_{f_i} - T_{\tilde{f}}| \leq T_{\tilde{f}} \) \( G \)-a.e. on \( \Gamma_0. \) Then we have

**Theorem 4.** — For all \( f_1, f_2 \in F \) and \( 0 < a < \infty \) we have

\[
C_G(\{ \gamma \in \Gamma_0 - \Gamma; |T_{f_i}(\gamma) - T_{\tilde{f}}(\gamma)| \geq a\}) \leq a^{-p}\Psi(f_1, f_2),
\]

where \( \Gamma \in G \) is chosen so that both \( T_{f_i} \) and \( T_{\tilde{f}} \) are defined on \( \Gamma_0 - \Gamma. \)

Proof. — Denote \( \{ \gamma \in \Gamma_0 - \Gamma; |T_{f_i}(\gamma) - T_{\tilde{f}}(\gamma)| \geq a\} \) by \( \Gamma'. \) By our assumption there exists \( \tilde{f} \in F \) such that \( \Phi(\tilde{f}) \leq \Psi(f_1, f_2) \) and \( |T_{f_i} - T_{\tilde{f}}| \leq T_{\tilde{f}} \) \( G \)-a.e. on \( \Gamma_0. \) Evidently \( \tilde{f}/a \) is \( G \)-alm. ad. for \( \Gamma' \) so that

\[
C_G(\Gamma') \leq \Phi \left( \frac{\tilde{f}}{a} \right) \leq a^{-p}\Psi(f_1, f_2).
\]

Hereafter we assume that the relation

\[
\limsup_{n, m \to \infty} \left\{ \frac{\Phi(f_n) + \Phi(f_m)}{2} - \Phi \left( \frac{f_n + f_m}{2} \right) \right\} \leq 0
\]

for \( \{f_n\} \subset F \) implies the existence of \( f \in F \) such that \( \Psi(f_n, f) \to 0 \) and \( \Phi(f_n) \to \Phi(f). \)
Theorem 5. — If $C_G(\Gamma) < \infty$, then there exist $f \in F$ and $\Gamma'$ with $C_G(\Gamma') = 0$ such that $T_f \geq 1$ on $\Gamma - \Gamma'$ and $\Phi(f) = C_G(\Gamma)$.

Proof. — Choose $f_1, f_2, \ldots$ G-alm. ad. for $\Gamma$ so that $\Phi(f_n) \to C_G(\Gamma)$. Then $(f_n + f_m)/2$ is G-alm. ad. for $\Gamma$, and hence $\Phi((f_n + f_m)/2) \geq C_G(\Gamma)$. Therefore

$$\limsup_{n, m \to \infty} \left\{ \frac{\Phi(f_n) + \Phi(f_m)}{2} - \Phi\left(\frac{f_n + f_m}{2}\right) \right\} \leq 0.$$ 

By our assumption there exists $f \in F$ such that $\Phi(f_n, f) \to 0$ and $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$. Choose $\Gamma' \in G$ so that $T_{f_n} \geq 1$ on $\Gamma - \Gamma'$ for every $n$. In view of (i) $\to$ (iii) of Theorem 3 and Theorem 4 we find $\{f_n\}$ and $\Gamma'' \subset \Gamma$ with $C_G(\Gamma'') = 0$ such that

$$T_f = \lim_{k \to \infty} T_{f_{nk}} \geq 1 \text{ on } \Gamma - \Gamma''.$$ 

We have $\Phi(f) = \lim_{n \to \infty} \Phi(f_n) = C_G(\Gamma)$.

Theorem 6. — If $\Gamma_n \uparrow \Gamma$, then $C_G(\Gamma_n) \uparrow C_G(\Gamma)$.

Proof. — Denote $\lim C_G(\Gamma_n)$ by $C$. Clearly $C \leq C_G(\Gamma)$.

Hence it suffices to establish $C_G(\Gamma) \leq C$. We may assume that $C < \infty$. Choose $f_n$ G-alm. ad. for $\Gamma_n$ so that $\Phi(f_n) \to C$. If $m > n$, then $f_m$ and hence $(f_n + f_m)/2$ is G-alm. ad. for $\Gamma_n$. Therefore $\Phi((f_n + f_m)/2) \geq C_G(\Gamma_n)$. As in the proof of Theorem 5 we find $f \in F$ and $\Gamma'$ with $C_G(\Gamma') = 0$ so that $T_f(\gamma) \geq 1$ on $\Gamma - \Gamma'$ and $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$.

By Lemma 2 $C_G(\Gamma) \leq \Phi(f)$. Hence

$$C_G(\Gamma) \leq \Phi(f) = \lim_{n \to \infty} \Phi(f_n) = C.$$ 

From this theorem we derive immediately

Theorem 7. — If $C_G(\Gamma' \cup \Gamma'') \leq C_G(\Gamma') + C_G(\Gamma'')$ for any $\Gamma'$, $\Gamma''$, then

$$C_G\left(\bigcup_{n} \Gamma_n\right) \leq \sum_{n} C_G(\Gamma_n).$$
Remark. — If $\max (f_1, f_2)$ belongs to $F$ and
$$\Phi (\max (f_1, f_2)) \leq \Phi (f_1) + \Phi (f_2)$$
whenever $f_1, f_2 \in F$, then $C_G(\Gamma_1 \cup \Gamma_2) \leq C_G(\Gamma_1) + C_G(\Gamma_2)$. It suffices to show $C_G(\Gamma_1 \cup \Gamma_2) \leq \Phi (f_1) + \Phi (f_2)$ whenever $C_G(\Gamma_i)$ is finite and $f_i$ is G-alm. ad. for $\Gamma_i$, $i = 1, 2$. This is actually true because $f = \max (f_1, f_2)$ is G-alm. ad. for $\Gamma_1 \cup \Gamma_2$ so that
$$C_G(\Gamma_1 \cup \Gamma_2) \leq \Phi (f) \leq \Phi (f_1) + \Phi (f_2).$$

2. Examples.

Let $\Omega$ be a general space. Hereafter take as $\Gamma_0$ the class $M_0$ of all non-negative measures defined on a $\sigma$-field $E$ of sets in $\Omega$, and let $F$ be a family of non-negative $E$-measurable functions on $\Omega$ such that $c f \in F$ and $f_1 + f_2 \in F$ whenever $c \geq 0$ and $f_1, f_2 \in F$. For every $f \in F$ we define $T_f(\mu)$ to be $\int f \, d\mu$. We take $G = \emptyset$ and denote $C_G(\Gamma)$ by $C(\Gamma)$ for $\Gamma = M \subseteq M_0$.

Example 1. — Let $F$ consist of all non-negative $E$-measurable functions. With a fixed $m \in M_0$ set
$$\Phi (f) = \int f^p \, dm \quad \text{for} \quad f \in F$$
and $\Psi (f_1, f_2) = \int |f_1 - f_2|^p \, dm$ for $f_1, f_2 \in F$ if
$$\int f_1^p \, dm + \int f_2^p \, dm < \infty.$$
If
$$\int f_1^p \, dm + \int f_2^p \, dm = \infty,$$
then define $\Psi (f_1, f_2)$ to be $\infty$. Then $\Psi$ satisfies the conditions required in § 1. We call $C(\Gamma)$ the module of $\Gamma$ of order $p$. Its reciprocal is called the extremal length of $\Gamma$ of order $p$. In this case we have $\sup f_n \in F$ for $f_1, f_2, \ldots \in F$ and
$$\Phi \left( \sup f_n \right) \leq \Sigma \Phi (f_n).$$
We obtain the subadditivity from this immediately without appealing to Theorem 7.

Example 2. — Let $F$, $\Phi$ and $\Psi$ be as above. Let $k(x, e) \geq 0$ be an $E$-measurable function of $x$ on $\Omega$ for every fixed
e \in E, and a measure for every fixed \( x \in \Omega \). Set
\[ v_\mu(e) = \int k(x, e) \, d\mu(x) \quad \text{for} \quad \mu \in M_0, \]
and \( N_M = \{v_\mu; \mu \in M\} \). We may consider \( C(N_M) \). This gives a generalization of \( C_{K; \Phi^p}(A) \) referred to in the introduction when \( M = \{\delta_x; x \in A\} \) and
\[ k(x, e) = \int k(x, y) \, d\mu_0(y), \]
where \( A \) is a subset of \( \mathbb{R}^n \), \( \delta_x \) is the unit point measure at \( x \), \( k(x, y) \geq 0 \) is \( E \)-measurable for every fixed \( x \) and \( \mu_0 \) is a fixed measure in \( M_0 \). As in Example 1, (1) follows immediately.

**Example 3.** — Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( E \) be the Borel class of sets in \( \Omega \), \( m \) be the Lebesgue measure, \( F \) consist of (some) non-negative \( p \)-precise functions \( f \) in \( \Omega \), take \( \Phi(f) = \int |\text{grad} f|^p \, dm \), and define \( \Psi(f_1, f_2) \) by
\[ \int |\text{grad} (f_1 - f_2)|^p \, dm. \]
See [3] for \( p \)-precise functions and properties of these functions. In order to assure \( \sum f_n \in F \) for every \( \{f_n\} \subset F \) satisfying \( \int |\text{grad} (f_{n+1} + \cdots + f_m)|^p \, dm \to 0 \), we assume that there is a family, with positive module of order \( p \), of curves in \( \Omega \) such that every \( f \in F \) tends to 0 along \( p \)-a.e. curve of the family. Then all the conditions required in the beginning of § 1 are satisfied. If \( \max (f_1, f_2) \in F \) for any \( f_1, f_2 \in F \), then (1) holds because \( |\text{grad} (\max (f_1, f_2))| = |\text{grad} f_1| \) or \( |\text{grad} f_2| \) \( m \)-a.e. so that \( \Phi(\max (f_1, f_2)) = \Phi(f_1) + \Phi(f_2) \). Consider the case when the module of order \( p \) of the family \( A \) of curves terminating at \( \partial \Omega \) is positive, \( M = \{\delta_x; x \in A \subset \Omega\} \) and \( F \) consists of all non-negative \( p \)-precise functions in \( \Omega \) tending to 0 along \( p \)-a.e. curve of \( A \). Then the capacity is called the \( p \)-capacity of \( A \) (relative to \( \Omega \)).

**Example 4.** — Let \( \Omega \) be a topological space, and \( E \) be the Borel class of sets in \( \Omega \). Let \( F \) be as in the beginning of this section, and \( \Phi \) be as in § 1. Denote by \( S \) the family of lower semicontinuous functions in \( \Omega \), and define \( c(K) \)
for every compact set $K$ by $\inf \Phi(f)$ for $f \in F \cap S$ satisfying $f \geq 1$ on $K$. Let us see that $c(K_n) \downarrow c(K)$ if a sequence \{$K_n$\} of compact sets decreases to $K$. Suppose $f \in F \cap S$ satisfies $f(x) \geq 1$ on $K$. Since $f$ is lower semicontinuous, $f/(1 - 1/n) > 1$ for each $n \geq 2$ on an open set $\omega$ containing $K$. There exists $m_0$ such that $K_{m_0} \subset \omega$, and hence

$$c(K) \leq \lim_{m \to \infty} c(K_m) \leq \Phi(f/(1 - 1/n))$$

$$\leq (1 - 1/n)^{-\beta} \Phi(f) \to \Phi(f)$$

as $n \to \infty$. The arbitrariness of $f$ yields $c(K) = \lim_{m \to \infty} c(K_m)$.

This together with Theorem 6 shows that $C(M_A)$ with $M_A = \{\varepsilon_x; x \in A\}$ is a true capacity if it is shown that

$$c(K) = C(M_K)$$

for every compact set $K$; cf. [1; Part II, Chap. 1]. This is the case, for example, when $\Omega = R^n (n \geq 3)$, $F$ consists of all Newtonian potentials of non-negative measures with finite energy and $\Phi(f) = \int |\text{grad} f|^2 \, dm$; the 2-capacity is equal to the Newtonian outer capacity. Evidently all $f \in F$ are superharmonic.

Another example of such a case is found in [4]. See [3] too.

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