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Some remarks on the existence of a resolvent

SOME REMARKS ON THE EXISTENCE
OF A RESOLVENT

by Masanori KISHI

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1. Introduction.

In potential theory, one of the important classes of convo-
lution kernels on a locally compact abelian group G is the
class of those kernels with which resolvents are associated.
Such a kernel satisfies the domination principle, and as
shown in [6], it induces a Hunt convolution kernel on a
quotient space G/H modulo a compact subgroup H. It is
a Hunt convolution kernel if and only if it satisfies the unicity
principle.

In this note, we shall deal with convolution kernels satis-
fying the domination principle and give some remarks on
the existence of resolvents.

A convolution kernel $\varkappa$ on G is a positive (Radon) measure
on G. Given a positive measure μ, the convolution $\varkappa * \mu$
is defined by

$$\int \varphi d(\varkappa * \mu) = \iint \varphi(x + y) \, d\varkappa(x) \, d\mu(y),$$

when the integral converges for every $\varphi \in C_K(G)$, the linear
space of continuous functions with compact support. The
convolution $\varkappa * \mu$ is a positive measure. We denote by
$\mathscr{D}^+(\varkappa)$ the totality of positive measures μ such that the
convolution $\varkappa * \mu$ is defined.

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A family of positive measures $\{\varkappa_p\}$ $(p > 0)$ is called a resolvent, if each $\varkappa_p$ belongs to $\mathcal{D}^+(\varkappa)$ and

(1) $$\varkappa - \varkappa_p = p\varkappa * \varkappa_p.$$

When $\{\varkappa_p\}$ verifies moreover the condition

(2) $$\lim_{p \downarrow 0} \varkappa_p = \varkappa \quad \text{(vaguely)},$$

we call it a resolvent of $\varkappa$ or we say that $\{\varkappa_p\}$ is associated with $\varkappa$. It is known that a resolvent $\{\varkappa_p\}$ of $\varkappa$ exists at most one and

(3) $$\varkappa + \frac{1}{p}\,\varepsilon = \frac{1}{p} \sum_{n=0}^{\infty} (p\varkappa_p)^n$$

where ε denotes the Dirac measure at the origin and

$$(p\varkappa_p)^0 = \varepsilon, \quad (p\varkappa_p)^n = (p\varkappa_p)^{n-1} * (p\varkappa_p).$$

We denote by (R) the family of all convolution kernels on G with which resolvents are associated.

2. Relatively balayaged measure.

We say that a convolution kernel $\varkappa'$ satisfies the *domination principle relative to a convolution kernel* $\varkappa''$, if for any φ, $\psi \in C_K^+(G)$ $(=$ the totality of non-negative functions of $C_K(G))$, $\varkappa' * \varphi \leqslant \varkappa'' * \psi$ in G, whenever $\varkappa' * \varphi \leqslant \varkappa'' * \psi$ on supp (φ), the support of φ. In this case under suitable assumptions on $\varkappa'$, we can balayage $\mu \in \mathcal{D}^+(\varkappa')$ to a given relatively compact open set ω relative to $\varkappa'$. Namely, there exists $\mu' \in \mathcal{D}^+(\varkappa')$, supported by $\bar{\omega}$, such that

$$\varkappa' * \mu' = \varkappa'' * \mu \quad \text{in} \quad \omega$$
$$\varkappa' * \mu' \leqslant \varkappa'' * \mu \quad \text{in} \quad G.$$

As to the existence of μ', we refer to [3] and [4]. In case that

$$\varkappa' = \varkappa + c\varepsilon (c > 0) \quad \text{and} \quad \varkappa'' = \varkappa,$$

the existence of μ' is shown also by modifying Choquet-

Deny's method in [1]. In case that $x' = x'' = x$, we say simply that x satisfies the domination principle and call the above μ' the balayaged measure.

3. Dominated convergence theorem.

For a convolution kernel x in (R), we have the following dominated convergence theorem [6], [7]: Let $\{\mu_\alpha\}$ be a net of positive measures, converging vaguely to μ, and suppose that there exists $\lambda \in \mathscr{D}^+(x)$ such that $x * \lambda$ dominates every $x * \mu_\alpha$. Then $\{x * \mu_\alpha\}$ converges vaguely to $x * \mu$. This is proved by the equality (3) and Deny's convergence lemma ([2], Lemme 5.2).

We shall show the converse.

THEOREM 1. — *If a convolution kernel x satisfies the domination principle and has the following dominated convergence property (*), then a resolvent is associated with x* :

(*) *A net $\{x * \mu_\alpha\}$ converges vaguely to $x * \mu$, if $\{\mu_\alpha\}$ converges vaguely to μ and every $x * \mu_\alpha$ is dominated by x.*

Proof. — We shall use the method of M. Itô [3]. Let p be a fixed positive number and denote by $\mathscr{W}$ the directed set of relatively compact open neighborhoods of the origin such that $\bigcup \{\omega_\alpha ; \omega_\alpha \in \mathscr{W}\} = G$. Since $px + \varepsilon$ satisfies the domination principle relative to x, we have the following relatively balayaged measure μ_α, supported by $\overline{\omega}_\alpha$ for every $\omega_\alpha \in \mathscr{W}$:

$$(px + \varepsilon) * \mu_\alpha = x \quad \text{in} \quad \omega_\alpha,$$
$$(px + \varepsilon) * \mu_\alpha \leqslant x \quad \text{in} \quad G.$$

The net $\{\mu_\alpha\}$ being vaguely bounded, we may suppose that as $\omega_\alpha \uparrow G$, $\{\mu_\alpha\}$ converges vaguely to a positive measure x_p. Then by the dominated convergence property (*), $\{px * \mu_\alpha\}$ converges vaguely to $px * x_p$, and hence $(px + \varepsilon) * x_p = x$. Thus we have constructed a resolvent $\{x_p\}$ $(p > 0)$.

To show the vague convergence $\lim_{p \downarrow 0} x_p = x$, it is sufficient to remark that $\lim_{p \downarrow 0} px_p = 0$ (vaguely) and

$$x * (px_p) \leq x,$$

because $\varkappa$ has the dominated convergence property $(\ast)$. This completes the proof.

From this theorem it follows immediately that if $\varkappa$ satisfies the domination principle and if $\varkappa \in \mathscr{D}^+(\varkappa)$, then $\varkappa$ belongs to (R). It is also seen that if $\varkappa$ satifies the domination principle and if $\varkappa * \varphi(x)$, for any $\varphi \in C_K(G)$, tends to 0 as x tends to the point at infinity, then $\varkappa$ belongs to (R).

Necessary and sufficient conditions for the existence of a resolvent are also given in [3].

LEMMA — *Suppose that $\varkappa$ satisfies the domination principle and there exists $\varkappa'(\neq 0) \in (R)$ such that $\varkappa' \leqslant \varkappa, \varkappa - \varkappa' \in \mathscr{D}^+(\varkappa')$ and $\varkappa'$ satisfies the domination principle relative to $\varkappa$. Then there exists $\varkappa'' \in \mathscr{D}^+(\varkappa')$ such that $\varkappa = \varkappa' * \varkappa''$.*

Proof. — Without loss of generality we may assume that $\varkappa \neq \varkappa'$. For every ω_β of the net $\mathscr{W}$ used in the proof of Theorem 1, there exists a relatively balayaged measure ν_β, supported by $\overline{\omega}_\beta$, such that

$$\varkappa' * \nu_\beta = \varkappa \quad \text{in} \quad \omega_\beta,$$
$$\varkappa' * \nu_\beta \leqslant \varkappa \quad \text{in} \quad G.$$

Since $\{\nu_\beta\}$ is vaguely bounded, we have a vaguely adherent measure $\varkappa''$ of $\{\nu_\beta\}$, which is a required one. We show it assuming that $\{\nu_\beta\}$ converges vaguely to $\varkappa''$. What we have to show is the vague convergence of $\{\varkappa' * \nu_\beta\}$ to $\varkappa' * \varkappa''$. Let $\{\varkappa'_p\}$ be the resolvent of $\varkappa$, and let p be fixed. By virtue of $(p\varkappa'_p)^n \leqslant p\varkappa'$ $(n \geqslant 1)$ and $\varkappa' \in \mathscr{D}^+(\varkappa - \varkappa')$, $(p\varkappa'_p)^n$ belongs to $\mathscr{D}^+(\varkappa - \varkappa')$ and hence to $\mathscr{D}^+(\varkappa)$. Therefore by Deny's convergence lemma,

$$\lim_\beta (p\varkappa'_p)^n * \nu_\beta = (p\varkappa'_p)^n * \varkappa'' \quad \text{(vaguely)}.$$

On the other hand, we have

$$\sum_{m=n+1}^{\infty} (p\varkappa'_p)^m * \nu_\beta = (p\varkappa'_p)^n * p\varkappa' * \nu_\beta$$
$$\leqslant (p\varkappa'_p)^n * p\varkappa = \sum_{m=n+1}^{\infty} (p\varkappa'_p)^m + (p\varkappa'_p)^n * p(\varkappa - \varkappa').$$

Hence for a given positive number η and $\varphi \in C_K^+(G)$, we

have, taking f in $C_K^+(G)$ such that $\varphi \leqslant \check{\varkappa}' * f$,

$$\frac{1}{p} \sum_{m=n+1}^{\infty} (p\varkappa_p')^m(\varphi) < \eta,$$
$$(p\varkappa_p')^n * (\varkappa - \varkappa')(\varphi) \leqslant (p\varkappa_p')^n * (\varkappa - \varkappa') * \varkappa'(f) < \eta$$

for every sufficiently large n. The last inequality is due to the fact that $\lim_n (p\varkappa_p')^n * \varkappa' = 0$ (vaguely) and

$$\varkappa' \in \mathscr{D}'(\varkappa - \varkappa')$$

and hence

$$\lim_n (p\varkappa_p')^n * (\varkappa - \varkappa') * \varkappa'(f) = 0.$$

Consequently we have

$$\lim_\beta \sup \varkappa' * \nu_\beta(\varphi) \leqslant \lim_\beta \sup \left(\varkappa' + \frac{1}{p} \varepsilon \right) * \nu_\beta(\varphi)$$

$$\leqslant \lim \frac{1}{p} \sum_{m=0}^{n} (p\varkappa_p')^m * \nu_\beta(\varphi) + \lim_\beta \sup \frac{1}{p} \sum_{m=n+1}^{\infty} (p\varkappa_p')^m * \nu_\beta(\varphi)$$

$$< \frac{1}{p} \sum_{m=0}^{\infty} (p\varkappa_p')^m * \varkappa''(\varphi) + 2\eta = \left(\varkappa' + \frac{1}{p} \varepsilon \right) * \varkappa''(\varphi) + 2\eta,$$

$$\lim_\beta \sup \varkappa' * \nu_\beta(\varphi) \leqslant \left(\varkappa' + \frac{1}{p} \varepsilon \right) * \varkappa''(\varphi).$$

Thus we obtain, letting p tend to infinity,

$$\lim_\beta \sup \varkappa' * \nu_\beta(\varphi) \leqslant \varkappa' * \varkappa''(\varphi)$$

and hence $\lim_\beta \varkappa' * \nu_\beta(\varphi) = \varkappa' * \varkappa''(\varphi)$.

Using this lemma we shall remark the following

THEOREM 2. — *Let* $\{\varkappa_p\}$ $(p > 0)$ *be a resolvent such that*

$$\varkappa - \varkappa_p = p\varkappa * \varkappa_p.$$

If $\varkappa_0 = \lim_{p \downarrow 0} \varkappa_p$ *(vaguely) does not coincide with* $\varkappa$, *then*

$$\varkappa - \varkappa_0 \notin \mathscr{D}^+(\varkappa).$$

Proof. — Put $\sigma = \varkappa - \varkappa_0 (\geqslant 0)$. Then for every $p > 0$, we have $\sigma * p\varkappa_p = \sigma$ and we see that $\varkappa = \varkappa_0 + \sigma$ satisfies the domination principle. Assuming that $\sigma \in \mathscr{D}^+(\varkappa)$, we shall show that $\varkappa$ belongs absurdly to (R). We may assume that $\varkappa_0 \neq 0$. Then by the above lemma, there exists

$$\varkappa_0' \in \mathscr{D}^+(\varkappa_0)$$

such that $\varkappa = \varkappa_0 * \varkappa_0'$, because $\varkappa_0$ satisfies the domination principle relative to $\varkappa$.

Now let $\{\mu_\alpha\}$ be a net of positive measures converging vaguely to μ and assume that $\varkappa * \mu_\alpha \leqslant \varkappa$. We shall show the vague convergence of $\{\varkappa_0 * \mu_\alpha\}$ to $\varkappa_0 * \mu$. Then we have

$$\lim_\alpha \varkappa * \mu_\alpha = \varkappa * \mu \quad \text{(vaguely)},$$

because $\{\sigma * \mu_\alpha\}$ converges vaguely to $\sigma * \mu$ by the facts $\sigma * \mu_\alpha \leqslant \varkappa$ and $\sigma \in \mathscr{D}^+(\varkappa)$. Let φ be a function in $C_K^+(G)$, and take f in $C_K^+(G)$ such that $\varphi \leqslant \check{\varkappa}_0' * f$. For a fixed $p > 0$, we have $(p\varkappa_p)^m * \varkappa \leqslant \varkappa$ and

$$\lim_\alpha (p\varkappa_p)^m * \mu_\alpha = (p\varkappa_p)^m * \mu \quad \text{(vaguely)}.$$

On the other hand, we have

$$\frac{1}{p} \sum_{m=n+1}^\infty (p\varkappa_p)^m * \mu_\alpha(\varphi) = (p\varkappa_p)^n * \varkappa_0 * \mu_\alpha(\varphi)$$

$$\leqslant (p\varkappa_p)^n * \varkappa * \mu_\alpha(f) \leqslant (p\varkappa_p)^n * \varkappa(f) = \frac{1}{p} \sum_{m=n+1}^\infty (p\varkappa_p)^m * \varkappa_0'(f).$$

Since $\varkappa_0'$ belongs to $\mathscr{D}^+(\varkappa_0)$, we have, given a positive number η,

$$\frac{1}{p} \sum_{m=n+1}^\infty (p\varkappa_p)^m * \mu_\alpha(\varphi) < \eta$$

for any sufficiently large n. Therefore we have

$$\limsup_\alpha \left(\varkappa_0 + \frac{1}{p} \varepsilon\right) * \mu_\alpha(\varphi) \leqslant \frac{1}{p} \sum_{m=0}^n (p\varkappa_p)^m * \mu(\varphi) + \eta$$

$$\leqslant \frac{1}{p} \sum_{m=0}^\infty (p\varkappa_p)^m * \mu(\varphi) + \eta = \left(\varkappa_0 + \frac{1}{p} \varepsilon\right) * \mu(\varphi) + \eta.$$

Consequently $\lim_\alpha \varkappa_0 * \mu_\alpha = \varkappa_0 * \mu$ (vaguely). This completes the proof.

4. Square root of a convolution kernel.

Let $\varkappa$ be in (R). It is shown in [5] that the convolution kernel $\varkappa^\alpha$ $(0 < \alpha < 1)$ defined by the following integral is in (R):

$$\varkappa^\alpha(\varphi) = \frac{\sin \alpha\pi}{\pi} \int_0^\infty p^{-\alpha}\varkappa_p(\varphi) \, dp \quad (\varphi \in C_K(G)),$$

and that if $0 < \alpha, \beta < 1, \alpha + \beta < 1$, then

$$\varkappa^\alpha * \varkappa^\beta = \varkappa^{\alpha+\beta}.$$

As shown in [7] we have also

$$\varkappa^\alpha * \varkappa^{1-\alpha} = \varkappa \quad (0 < \alpha < 1).$$

Hence setting $\varkappa' = \varkappa^{1/2}$ we have $\varkappa' * \varkappa' = \varkappa$. Namely if $\varkappa$ is in (R), its square root exists in (R). (Needless to say, there exists $\varkappa \in$ (R), which has a square root not in (R).) We shall discuss the converse.

THEOREM 3. — *Let $\varkappa$ be a convolution kernel satisfying the domination principle, and suppose that there exists a square root $\varkappa'$ of $\varkappa$ satisfying the domination principle. Then $\varkappa$ belongs to* (R).

Proof. — Let $\{\mu_\alpha\}$ be a net of positive measures converging vaguely to μ and suppose that $\varkappa * \mu_\alpha \leqslant \varkappa$. We shall first show the vague convergence of $\{\varkappa' * \mu_\alpha\}$ to $\varkappa' * \mu$, assuming $\varkappa' \neq 0$. $\varkappa'$ being in $\mathscr{D}^+(\varkappa')$, it is in (R) and has the dominated convergence property. For every $V_\beta \in \mathscr{V}$, a net of compact neighborhoods of the origin such that $V_\beta \uparrow G$, we have a balayaged measure ν_β, supported by $\overline{CV_\beta}$, such that

$$\varkappa' * \nu_\beta = \varkappa' \quad \text{in} \quad CV_\beta$$
$$\varkappa' * \nu_\beta \leqslant \varkappa' \quad \text{in} \quad G.$$

Evidently $\{\nu_\beta\}$ converges vaguely to 0 as $V_\beta \uparrow G$, and hence $\{\varkappa' * \nu_\beta\}$ converges vaguely to 0, by $\varkappa' * \nu_\beta \leqslant \varkappa'$ and by the dominated convergence property. Then by

$$\varkappa' * (\varkappa' * \nu_\beta) \leqslant \varkappa' * \varkappa', \quad \{\varkappa * \nu_\beta\}$$

converges vaguely to 0. Therefore, for any positive number η and any $\varphi \in C_K^+(G)$, there exists $V_\beta \in \mathscr{V}$ such that

$$\varkappa * \nu_\beta(f) < \eta,$$

where f is a function in $C_K^+(G)$ such that $\varphi \leqslant \check{\varkappa}' * f$. Then

$$\lim_{\alpha} \sup \varkappa' * \mu_\alpha(\varphi) \leqslant \lim (\varkappa' - \varkappa' * \nu_\beta) * \mu_\alpha(\varphi)$$
$$+ \lim_{\alpha} \sup \varkappa' * \nu_\beta * \mu_\alpha(\varphi)$$
$$\leqslant (\varkappa' - \varkappa' * \nu_\beta) * \mu(\varphi) + \lim_{\alpha} \sup \varkappa * \nu_\beta * \mu_\alpha(f)$$
$$\leqslant \varkappa' * \mu(\varphi) + \eta.$$

This proves the vague convergence of $\{\varkappa' * \mu_\alpha\}$ to $\varkappa' * \mu$. We now apply again the dominated convergence property of $\varkappa'$ to conclude $\lim_{\alpha} \varkappa * \mu_\alpha = \varkappa * \mu$ (vaguely) and complete the proof.

Similarly as above we can show the following

THEOREM 3'. — *Let $\varkappa$ be a convolution kernel satisfying the domination principle and suppose that there exist convolution kernels $\varkappa', \varkappa''$ in (R) such that $\varkappa = \varkappa' * \varkappa''$. Then $\varkappa$ is in (R).*

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