

# ANNALES DE L'INSTITUT FOURIER

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## **Multiply superharmonic functions**

*Annales de l'institut Fourier*, tome 25, n° 3-4 (1975), p. 235-244

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## MULTIPLY SUPERHARMONIC FUNCTIONS

by Kohur GOWRISANKARAN <sup>(1)</sup>

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*Dédié à Monsieur M. Brelot à l'occasion  
de son 70<sup>e</sup> anniversaire.*

### 1. Introduction.

Let  $\Omega$  be a locally compact, connected and locally connected Hausdorff space. We shall say that  $\Omega$  is a harmonic space of Brelot if  $\Omega$  has a countable base for open sets and if there is a system of harmonic functions defined on open subsets of  $\Omega$  satisfying the axioms 1, 2 and 3 of Brelot [2].

Let  $\Omega_1$  and  $\Omega_2$  be harmonic spaces of Brelot. A lower semi-continuous extended real valued function  $\nu$  on the product  $\Omega_1 \times \Omega_2$  is said to be multiply superharmonic if (i)  $\nu \not\equiv +\infty$  and (ii)  $\nu$  is hyperharmonic in each variable separately. In this article we shall be concerned with multiply superharmonic functions: their boundary behaviour at the distinguished boundary, the reduced functions and the integral representation of positive functions. There are a number of questions which arise in connection with these topics. Concerning these questions we shall present here some results though not exhaustive, and describe some of the problems of interest to us.

### 2. Integral Representation and a Consequence.

Let  $\Omega_1$  and  $\Omega_2$  be harmonic spaces of Brelot, each one of them with a countable base for open sets consisting of

<sup>(1)</sup> This research was partially supported by the grant A-5289 of the National Research Council of Canada.

completely determining domains [2]. Let  $MS^+$  denote the class of all multiply superharmonic positive functions on the product  $\Omega_1 \times \Omega_2$ . Let  $MS$  denote the differences of elements in  $MS^+$  defined in the natural way. Then  $MS$ , provided with the semi-norms  $\nu_1 - \nu_2 \mapsto \left| \int \nu_1 d\rho_z^\omega - \int \nu_2 d\rho_z^\omega \right|$ , where  $z = (x, y) \in \omega = \omega_1 \times \omega_2$ ,  $\omega_1$  and  $\omega_2$  are regular domains in the respective spaces, is a locally convex Hausdorff topological vector space over the real numbers. A. Drinkwater [5] showed that with the above topology  $MS^+$  has a compact metrizable base  $\Delta$  and hence deduced that every positive multiply superharmonic function  $\nu$  is represented as an integral of a positive finite Radon measure on  $\Delta$  carried by the set of extreme elements of  $\Delta$ . She also showed that this set of extreme elements contains elements of the form  $s_1(x)s_2(y)$  where  $s_1$  and  $s_2$  are extreme generators of the cone of positive superharmonic functions on the respective spaces  $\Omega_1$  and  $\Omega_2$ . An explicit determination of all the extreme elements of  $\Delta$  has not yet been done. Also, it is not known whether the measures carried by the extreme elements representing the elements of  $MS^+$  are unique. We feel that the representing measure is not unique in general. The results of Ahern and Rudin [1] may throw some light on this problem.

Let  $C$  be the subclass of  $MS^+$  consisting of the elements  $\nu$  which verify the Cairoli condition; viz.,  $x \mapsto \nu_x$  behaves hyperharmonically on  $\Omega_1$  where for every  $x \in \Omega_1$ ,  $\nu_x$  is the canonical measure, representing  $\nu(x, \cdot)$ , on a convenient compact base of the positive superharmonic functions on  $\Omega_2$ , [4, 10, 11]. It was shown in [10] that  $C$  is a cone with a compact base and the extreme elements of this base are precisely of the form  $s_1 s_2$  considered above. Further  $C$  is a lattice in its own order. Hence, the elements of  $C$  are represented uniquely by finite Radon measures carried by the extreme elements of a convenient compact base of  $C$ . The following is an application of the above integral representation.

Let  $U$  be the open unit disc in the plane. For any positive integer  $n$ ,  $N(U^n)$  denotes the Nevanlinna class of the polydisc  $U^n$ , i.e., the class of holomorphic functions  $f$  on  $U^n$  such that  $\log |f|$  has a non-negative multiply harmonic majorant. The following result characterises the functions  $f$

in  $N(U^n)$  such that  $-\log |f| +$  multiply harmonic positive function belongs to  $C(U^n)$ .

**THEOREM 2.1.** — *Let  $f \in N(U^n)$ . Then, the following are equivalent.*

a)  $\exists$  a positive multiply harmonic function  $\omega$  on  $U^n$  such  $-\log f + \omega$  belongs to  $C(U^n)$ .

b)  $f = hg$  where  $h$  is free of zeros in  $N(U^n)$  and  $g = B_1, \dots, B_n$  where  $B_k$  is a Blaschke product for each  $k$ .

*Proof.* — We shall prove the result in the case  $n = 2$ . The general case follows easily by induction. Let us first show that b)  $\implies$  a). Let  $f \in N(U^2)$  such that  $f = B_1 B_2 h$  where  $h$  is in  $N(U^2)$  and free of zeros,  $B_1, B_2$  are Blaschke products on  $U$ . Then  $\log |h|$  is multiply harmonic and is the difference of two such non-negative functions. Further  $-\log |B_1|$  and  $-\log |B_2|$  are both non-negative multiply superharmonic and are constants in at least one of the variables. Hence we deduce that  $-\log |B_1|$  and  $-\log |B_2|$  are both elements of  $C$  [th. II.7, 10]. Now it is easily seen that

$$b) \implies a).$$

Conversely, suppose  $f$  is in  $N(U^2)$  and is such that there exists a positive 2-harmonic function  $\omega$  such that

$$\nu = -\log |f| + \omega$$

belongs to  $C(U^2)$ . Let  $\nu_z$ , for every  $z$  in  $U$ , be the measure on  $U$  corresponding to the potential part of  $\nu(z, \cdot)$  on the unit disc. Since  $f$  is a holomorphic function in the Nevanlinna class,  $\nu(z, \zeta)$  is a harmonic function in  $\zeta$  except possibly at a countable set  $\{\zeta_k\}$  satisfying the Blaschke condition. Hence,  $\nu_z$  charges precisely the points  $\zeta_k$ ; i.e.,  $\nu_z$  is a positive measure which is a countable sum of point masses at  $\zeta_k$ . But, since  $\nu$  is in  $C(U^2)$ , the sets of  $\nu_z$  measure zero are independent of  $z$  in  $U$ . By symmetry, we can find a sequence,  $\{z_j\}$  such that 1) this sequence satisfies the Blaschke condition and 2) for every  $\zeta$  in  $U$ ,  $\nu(z, \zeta)$  is harmonic in  $z$  except at the points  $z_k$ . It follows that  $\nu$  is 2-harmonic on  $UX(U - \{\zeta_k\}) \cup (U - \{z_j\})XU$  and further

that the zero set  $Z(f)$  of  $f$  is precisely  $UX\{\zeta_k\} \cup \{z_j\}XU$ .

Let  $z_0 \in U$  and  $\zeta_0 \in U$  be such that  $f(z_0, \zeta_0) \neq 0$ . Suppose that  $f(z_0, \zeta)$  has a zero of order  $l$  at  $\zeta_k$ . Let

$$g(\zeta) = [(\zeta_k - \zeta)|\bar{\zeta}_k/(1 - \bar{\zeta}_k\zeta)\zeta_k]^l.$$

Then,  $\nu - \log|g|$  is clearly a 2-superharmonic function on  $U^2$ . Since  $-\log|g|$  is harmonic in  $z$  or  $\equiv +\infty$  for every fixed  $\zeta$ , it can be shown as in [10, Theorem II.7, Cor.] that  $-\log|g|$  belongs to  $C(U^2)$ . Now,  $\nu(z_0, \zeta) - \log|g(\zeta)|$  is harmonic at  $\zeta_k$ , and by the same argument as above, we conclude that  $\nu(z, \zeta) - \log|g(\zeta)|$  is harmonic in a neighbourhood of  $\zeta_k$ , for every  $z$  in  $U$ . This shows that the order of the zero of  $f(z, \cdot)$  at  $\zeta_k$  is independent of  $z$  in  $U$ . Let now  $B_1$  (resp.  $B_2$ ) be the Blaschke product corresponding to zeros at  $\zeta_k$  (resp.  $z_j$ ) of orders determined by  $f(z_0, \zeta_k)$  (resp.  $f(z_j, \zeta_0)$ ). We now conclude that  $f/B_1B_2$  is a holomorphic function in  $N(U^2)$  free of zeros. Hence,

$$g = B_1B_2 \quad \text{and} \quad h = f/B_1B_2$$

satisfy the requirements of the theorem. The proof is complete.

The following corollary can be proved utilising the results on integral representation of positive multiply superharmonic functions [5].

**COROLLARY.** — *Let  $f$  be in the Nevanlinna class of the polydisc  $U^n$ . Then  $f$  can be written in the form  $gh$ , where  $g$  is a product of Blaschke products and  $h$  is in the Nevanlinna class and further the zero set of  $h$  cannot contain a set of the form  $E_1XE_2X \dots XE_n$  with  $E_k = U$  for some  $k$  between 1 and  $n$ .*

### 3. Behaviour at the Distinguished Boundary.

Let  $MS^+$  be the set of positive multiply superharmonic functions on the product of two Brelot spaces  $\Omega_1$  and  $\Omega_2$ . Let  $MH^+$  and  $P$  denote respectively the class of positive multiply harmonic functions and the class of those  $\nu$  in  $MS^+$  with 0 as the greatest multiply harmonic minorant. It is clear that  $MS^+$  is the direct sum of  $P$  and  $MH^+$  and

we know that the pointwise infimum of two elements in  $MS^+$  is also in  $MS^+$ . Hence [Theorem 5,8], corresponding to each extremal generator  $h_1 h_2$  of  $MH^+$  [Th. 1.1, 1.2, 6], there is a ‘ fine filter ’  $G_{h_1 h_2}$  defined by

$$G_{h_1 h_2} = \{ E \subset \Omega_1 \times \Omega_2 : \varphi \in P, \varphi \geq h_1 h_2 \text{ on } E \}$$

Let  $F_{h_1}$  (respectively  $F_{h_2}$ ) be the fine filter on  $\Omega_1$  (resp.  $\Omega_2$ ) corresponding to the minimal harmonic function  $h_1$  (resp.  $h_2$ ). The following lemma shows the relation between  $G_{h_1 h_2}$  and  $F_{h_1} \times F_{h_2}$ .

LEMMA 3.1. — *The filter  $G_{h_1 h_2}$  is finer than  $F_{h_1} \times F_{h_2}$ .*

*Proof.* — It is enough to show that every set of the form  $E_1 \times E_2$  in  $F_{h_1} \times F_{h_2}$  is also in  $G_{h_1 h_2}$ . Given a set of this form, there exist potentials  $p_1$  and  $p_2$  on  $\Omega_1$  and  $\Omega_2$  respectively such that  $p_i \geq h_i$  on  $\Omega_i \setminus E_i$  for  $i = 1, 2$ . Let  $\varphi = p_1 h_2 + h_1 p_2$ . Then  $\varphi \in P$  and clearly  $\varphi$  majorises  $h_1 h_2$  on  $\Omega_1 \times \Omega_2 \setminus (E_1 \times E_2)$ . The proof is complete.

It is known that for positive harmonic functions  $u_1$  on  $\Omega_1$ ,  $u_2$  on  $\Omega_2$ , the limits of  $[\varphi(x, y)/u_1(x)u_2(y)]$  as  $x$  and  $y$  tend to the fine boundaries of the spaces and independent of each other do not exist in general, even when  $\varphi > 0$  is a multiply harmonic function [9]. The following however shows that the limit following the product of fine filters exist for certain special classes.

THEOREM 3.2. — *Let  $u_i > 0$  be positive harmonic function on  $\Omega_i$ ,  $\mu_i$  the canonical Radon measure representing  $u_i$  carried by the set of extreme harmonic functions  $\Delta_1^i$  of a conveniently chosen compact base of positive superharmonic functions on  $\Omega_i$ , for  $i = 1, 2$ . Let  $\varphi > 0$  be a multiply harmonic function on  $\Omega_1 \times \Omega_2$  such that the measure on  $\Delta_1^1 \times \Delta_1^2$  representing  $\varphi$  has a Radon-Nikodym derivative  $f$  with respect to  $\mu_1 \otimes \mu_2$  such that  $f$  is a bounded uniformly continuous function on  $\Delta_1^1 \times \Delta_1^2$ . Then,  $\varphi/u_1 u_2$  has finite limit following  $F_{h_1} \times F_{h_2}$ , equal to  $f$ , for  $\mu_1 \otimes \mu_2$  almost every  $(h_1, h_2)$  in  $\Delta_1^1 \times \Delta_1^2$ .*

*Proof.* — The space  $C_u(\Delta_1^1)$  of bounded uniformly continuous functions on  $\Delta_1^1$  provided with the topology of uniform

convergence on compact subsets of  $\Omega_1$  is separable. Hence, there exists a countable dense subset  $\{f_n\}_{n \in \mathbb{N}}$  for this space. Hence, it is possible to find a Borel set  $E_1 \subset \Delta_1^1$  of  $\mu_1$ -measure zero such that, for all  $n \in \mathbb{N}$ ,  $\frac{1}{u_1(x)} \int h(x)f_n(h)\mu_1 (dh)$  has the fine limit  $f_n$ , except for elements in  $E_1$ . Now, given any  $f$  in  $C_u(\Delta_1^1)$  there exists a sequence  $f_{n_k}$  which converges uniformly on  $\Delta_1^1$  to  $f$ . This implies that  $\int f_{n_k}(h)h(x)\mu_1 (dh)$  converges uniformly on  $\Omega_1$  to  $\int f(h)h(x)\mu_1 (dh)$ . Hence, we deduce that except for elements in  $E_1$ ,  $\frac{1}{u_1(x)} \int f(h)h(x)\mu_1 (dh)$  has the fine limit equal to  $f$ . Similarly we can find a set  $E_2 \subset \Delta_1^2$  of  $\mu_2$ -measure zero such that for all  $g$  in  $C_u(\Delta_1^2)$ , the fine limit of  $\frac{1}{u_2(y)} \int h(y)g(h)\mu_2 (dh)$  exists and equals  $g$  for all elements of  $\Delta_1^2$  except possibly in  $E_2$ .

Now consider any function on  $\Delta_1^1 \times \Delta_1^2$  of the form  $f(h_1)g(h_2)$  where  $f \in C_u(\Delta_1^1)$  and  $g \in C_u(\Delta_1^2)$ . Since  $\mu_1$  and  $\mu_2$  are totally finite measures we deduce that the limit of  $[1/u_1(x)u_2(y)] \iint f(h_1)g(h_2)h_1(x)h_2(y)\mu_1 (dh_1)\mu_2 (dh_2)$  following

$$F_{h_1} \times F_{h_2}$$

exists for  $\mu_1 \otimes \mu_2$  almost every element of  $\Delta_1^1 \times \Delta_1^2$ ; in fact the exceptional set is contained in  $(\Delta_1^1 \times E_2 \cup E_1 \times \Delta_1^2)$ . Evidently the same conclusion is true for any finite linear combination of such functions. Now a straight-forward application of Stone-Weierstrass theorem gives the required result.

The following theorem can be proved using techniques similar to that found in [3].

**THEOREM 3.3.** — *Let  $U$  be the open unit disc in the plane and  $T$  the unit circle. Let  $\omega$  be a bounded multiply harmonic function on  $U^k$ . Then,  $\omega$  has a finite limit following the product of fine filters, for Lebesgue almost every element of  $T^k$ .*

The following questions arise naturally in the above considerations; partial answers to some of them are known.

1) Let  $u_1, u_2, \mu_1$  and  $\mu_2$  be as in the Theorem 3.2. Let  $f$  be a bounded  $\mu_1 \otimes \mu_2$  measurable function on  $\Delta_1^1 \times \Delta_1^2$ .

Let  $\varpi = \int h_1 h_2 f(h_1, h_2) (\mu_1 \otimes \mu_2) (dh_1 dh_2)$ . Does  $\varpi/u_1 u_2$  have finite limit following the product of fine filters for  $\mu_1 \otimes \mu_2$  almost every element of  $\Delta_1^1 \times \Delta_1^2$ ?

2) Let  $u > 0$  be a multiply harmonic function on  $\Omega_1 \times \Omega_2$  and let  $\nu$  be the canonical measure on  $\Delta_1^1 \times \Delta_1^2$  representing  $u$ . Let  $\varpi > 0$  be any multiply superharmonic function on  $\Omega_1 \times \Omega_2$ . Do the limits of  $\varpi/u$  following  $G_{h_1, h_2}$  exist  $\nu$ -almost every-where?

3) In the particular case of  $\Omega_1 = \Omega_2 = U$ , is there any relation between the limits following  $G_{h_1, h_2}$  and the restricted non-tangential limits [12]?

#### 4. Reduced function and capacity.

Let  $\nu > 0$  be a multiply superharmonic function on the product  $\Omega_1 \times \Omega_2$  of two Brelot spaces. For any

$$E \subset \Omega_1 \times \Omega_2,$$

we define the reduced function  $B(E, \nu) = \inf. \{ \varpi : \varpi \in MS^+ \text{ and } \varpi \geq \nu \text{ on } E \}$ . The reduced functions satisfy the usual subadditive and regularisation properties [7]. We have the following property.

**THEOREM 4.1.** — *Let  $\nu_i > 0$  be superharmonic functions on  $\Omega_i, i = 1, 2$ . Then  $B(\omega_1 \times \omega_2, \nu_1 \nu_2) = R_{\nu_1}^{\omega_1} R_{\nu_2}^{\omega_2}$ .*

*Proof.* — Clearly  $R_{\nu_1}^{\omega_1} R_{\nu_2}^{\omega_2}$  is a multiply superharmonic function  $> 0$  on  $\Omega_1 \times \Omega_2$  and equals  $\nu_1 \nu_2$  on  $\omega_1 \times \omega_2$ ; hence it majorises the reduced function on the left side. On the other hand, let  $\varpi > 0$  be multiply superharmonic on  $\Omega_1 \times \Omega_2$  and  $\geq \nu_1 \nu_2$  on  $\omega_1 \times \omega_2$ . Then, for each  $y \in \omega_2$ ,  $\varpi(x, y) \geq \nu_1(x) \nu_2(y)$  for all  $x \in \omega_1$ . Hence,

$$\varpi(x, y) \geq \nu_2(y) R_{\nu_1}^{\omega_1}(x)$$

for all  $x \in \Omega_1$  and each  $y \in \omega_2$ . Now, we get that for every  $x \in \Omega_1$  and all  $y \in \Omega_2$ ,  $\varpi(x, y) \geq R_{\nu_1}^{\omega_1}(x) R_{\nu_2}^{\omega_2}(y)$ . This completes the proof.

The reduced function is closely related to the notion of



capacity and convergence theorems, etc. The following result suggests a possible approach to the treatment of these questions.

**THEOREM 4.2.** — *Let  $X$  and  $Y$  be regular Hausdorff topological spaces. Let  $K$  (resp.  $K_0$ ) denote the class of all compact subsets (resp. finite unions of compact rectangles) of  $X \times Y$ . Let  $\varphi$  be a non-negative valued set function on  $K_0$  satisfying 1)  $\varphi$  is monotone non-decreasing, 2) strongly sub-additive and 3) outer regular i.e., for  $C \in K_0$ ,  $\varphi(C) < +\infty$  and any  $\varepsilon > 0$ , there exists an open set  $V \supset C$  such that for all  $D \in K_0$  and  $C \subset D \subset V$ ,  $\varphi(D) < \varphi(C) + \varepsilon$ .*

Then,  $\varphi$  can be extended to a capacity on  $K$  and this extension is unique.

*Proof.* — Let  $K$  be any compact set in  $X \times Y$ . Let us first of all show that  $K$  is the intersection of all sets in  $K_0$  containing  $K$ . Since  $K$  is a compact set in a Hausdorff space,  $K$  is the intersection of all open sets containing  $K$ . Hence, it is sufficient to show that given an open set  $V$  containing  $K$ , there exists a set  $C \in K_0$  such that  $K \subset C \subset V$ . Now, it is possible to choose a finite number points  $(x_1, y_1), \dots, (x_n, y_n)$  in  $K$ , and neighbourhoods  $V_{x_i} \times W_{y_i}$  of  $(x_i, y_i)$  such that (i)  $\bar{V}_{x_i} \times \bar{W}_{y_i} \subset V$  and (ii)  $\bigcup_{i=1}^n (V_{x_i} \times W_{y_i}) \supset K$ . Let  $K_i = K \cap (\bar{V}_{x_i} \times \bar{W}_{y_i})$  and  $A_i$  and  $B_i$  be respectively the projections of  $K_i$  to  $X$  and  $Y$ . Then,  $A_i \times B_i$  are compact rectangles and  $C = \bigcup_{i=1}^n A_i \times B_i$  is in  $K_0$  and clearly  $K \subset C \subset V$ .

Now let,  $K$  be any compact set of  $X \times Y$ . Let  $\{C_i\}_{i \in I}$  be the collection of all sets in  $K_0$  which contain  $K$ . Since  $K_0$  is closed for finite intersections,  $\{C_i\}$  is decreasing directed and we have just shown that  $K = \bigcap_{i \in I} C_i$ . It follows that the numbers  $\varphi(C_i)$  are decreasing directed and we set

$$\tilde{\varphi}(K) = \inf. \{\varphi(C_i) : i \in I\}.$$

It is clear that  $\tilde{\varphi} \geq 0$  and monotone non-decreasing on  $K$ . The strong subadditivity of  $\tilde{\varphi}$  on  $K$  is an immediate conse-

quence of the strong subadditivity of  $\varphi$  on  $K_0$ . Finally, to show the outer regularity of  $\tilde{\varphi}$ , suppose  $K \in K$  is that  $\tilde{\varphi}(K) < +\infty$  and  $\varepsilon > 0$ . There exists a set  $C \in K_0$  such that  $C \subset K$  and  $\varphi(C) < \tilde{\varphi}(K) + \varepsilon/2$ . Choose an open set  $V \supset C$  such that for all  $C'$  in  $K_0$  with

$$C \subset C' \subset V, \quad \varphi(C') < \varphi(C) + \varepsilon/2.$$

Let  $K'$  be any compact set such that  $K \subset K' \subset V$ . Then, there exists a  $D$  in  $K_0$  such that  $K' \subset D \subset V$ . Now,  $D \cup C$  is again a finite union of compact rectangles and hence belongs to  $K_0$  and  $D \cup C \subset V$ . Hence

$$\tilde{\varphi}(K') \leq \varphi(D \cup C) \leq \varphi(C) + \varepsilon/2 \leq \tilde{\varphi}(K) + \varepsilon.$$

This proves that  $\tilde{\varphi}$  is outer regular, showing that  $\tilde{\varphi}$  is a strong capacity.

It is obvious from the definition of  $\tilde{\varphi}$  that  $\tilde{\varphi}$  coincides with  $\varphi$  on  $K_0$ . Finally, suppose  $\psi$  any strong (or even weak) capacity on  $K$  such that  $\psi$  restricted to  $K_0$  is  $\varphi$ . Given any compact set of  $K$  and any  $\varepsilon > 0$ , there exists an open set  $V \supset K$  and a set  $C$  in  $K_0$  with

$$K \subset C \subset V$$

such that  $\psi(K) > \psi(C) - \varepsilon = \varphi(C) - \varepsilon \geq \tilde{\varphi}(K) - \varepsilon$ . Hence,  $\psi(K) \geq \tilde{\varphi}(K)$ . However,

$$\psi(K) \leq \inf \{ \varphi(C) : C \in K_0, C \supset K \} = \tilde{\varphi}(K).$$

The proof is complete.

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Manuscrit reçu le 7 mars 1975.

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