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ON B_r-COMPLETENESS (*)

by Manuel VALDIVIA

Let E be a separated locally convex space and let E'_σ be its topological dual provided with the topology $\sigma(E', E)$ of the uniform convergence on the finite sets of E. E is said to be B_r-complete if every dense subspace Q of E'_σ such that $Q \cap A$ is $\sigma(E', E)$-closed in A for each equicontinuous set A in E', coincides with E', [8]. In this paper we prove that if $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are two sequences of infinite-dimensional Banach spaces then $H = \left(\bigoplus_{n=1}^\infty E_n\right) \times \prod_{n=1}^\infty F_n$ is not B_r-complete and if F coincides with $\prod_{n=1}^\infty F_n$ we have that $F \times F'[\mu(F', F)]$ is not B_r-complete, $\mu(F', F)$ being the topology of Mackey on the topological dual F' of F. We prove that if $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are also reflexive spaces there is on H a separated locally convex topology $\mathfrak{T}$ coarser than the initial one, such that $H[\mathfrak{T}]$ is a bornological barrelled space which is not an inductive limit of Baire spaces. We give also another results on B_r-completeness and bornological spaces.

The vector spaces we use here are non-zero and they are defined over the field K of the real or complex numbers. By "space" we mean "separated locally convex space". If $\langle E, F\rangle$ is a dual pair we denote by $\sigma(E, F)$ and $\mu(E, F)$ the weak and the Mackey topologies on E, respectively. If a space E has the topology $\mathfrak{T}$ and M is a subset of E, then $M[\mathfrak{T}]$ is the set M, provided with the topology induced by $\mathfrak{T}$. If A is a bounded closed absolutely convex subset of a space E, we mean by E_A the normed space over the linear hull of A, being A the closed unit ball of E_A. The topological dual of E is denote by E'. If u is a continuous linear mapping from E into F, we denote by ${}^t u$ the mapping from F' into E', transposed of u.

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In [15] we have proved the following result : a) *Let* E *be a separable space. Let* $\{E_n\}_{n=1}^{\infty}$ *be an increasing sequence of subspaces of* E, *with* E *as union. If there exists a bounded set* A *in* E *such that* $A \not\subset E_n$, $n = 1, 2, \ldots$ *there exists a dense subspace* F *of* E, $F \neq E$, *such that* $F \cap E_n$ *is finite-dimensional for every positive integer n.*

LEMMA 1. – *Let* E *be an infinite-dimensional space such that in* $E'[\sigma(E', E)]$ *there is an equicontinuous total sequence. Let* F *be a space with a separable absolutely convex weakly compact total subset. If* F *is infinite-dimensional, there is a linear mapping u, continuous and injective, from* E *into* F, *such that* $u(E)$ *is separable, dense in* F *and* $u(E) \neq F$.

Proof. – Let $\{u_n\}_{n=1}^{\infty}$ be a total sequence in $E'[\sigma(E', E)]$, equicontinuous in E and linearly independent. By a method due to Klee, (see [7] p. 118), we can find a sequence $\{v_n\}_{n=1}^{\infty}$ in E', such that its linear hull coincides with the linear hull of $\{u_n\}_{n=1}^{\infty}$, and a sequence $\{x_n\}_{n=1}^{\infty}$ in E, such that $\langle v_n, x_n \rangle = 1$, $\langle v_n, x_m \rangle = 0$, if $n \neq m$, $n, m = 1, 2, \ldots$ If B is the closed absolutely convex hull of $\{u_n\}_{n=1}^{\infty}$, then B absorbs v_n and, therefore, we can take the sequence $\{v_n\}_{n=1}^{\infty}$ equicontinuous in E. Let A be a weakly compact separable absolutely convex subset of F which is total in F. We can take in A a linearly independent sequence $\{y_n\}_{n=1}^{\infty}$ which is total in F. Applying the method of Klee, ([7], p. 118), we can find a sequence $\{z_n\}_{n=1}^{\infty}$ in A, such that its linear hull coincides with the linear hull of $\{y_n\}_{n=1}^{\infty}$, and a sequence $\{w_n\}_{n=1}^{\infty}$ in F' such that $\langle w_n, z_n \rangle = 1$, $\langle w_m, z_m \rangle = 0$, $n \neq m$, $n, m = 1, 2, \ldots$ Let u be the mapping from E into F defined by

$$u(x) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x \rangle z_n , \quad \text{for } x \in E .$$

Let us see, first, that u is well defined. Since $\{v_n\}_{n=1}^{\infty}$ is equicontinuous in E there is a positive real number h, such that

$$|\langle v_n, x \rangle| \leqslant h, \; n = 1, 2, \ldots$$

Given a neighbourhood U of the origin in F, there is a positive number λ such that $\lambda A \subset U$. Since $\{(1/n) z_n\}_{n=1}^{\infty}$ converges to the origin in the Banach space F_A, there is a positive integer n_0 such that $(1/n) z_n \in (\lambda/h) A$, for every positive integer $n \geqslant n_0$, and since λA is convex and

$$1/2^n + 1/2^{n+1} + \cdots + 1/2^{n+p} < 1 , p \geqslant 0$$

we have that

$$\sum_{q=1}^{n+p} (1/q2^q) \langle v_q , x \rangle z_q \in \lambda A \subset U , n \geqslant n_0 , p \geqslant 0$$

and, therefore, the sequence

$$\left\{ \sum_{n=1}^{r} (1/n2^n) \langle v_n , x \rangle z_n \right\}_{r=1}^{\infty}$$

is Cauchy in F. Since $z_n \in A$, it follows that the members of this sequence are contained in the weakly compact set h A and, therefore,

$$\sum_{n=1}^{\infty} (1/n2^n) \langle v_n , x \rangle z_n$$

is convergent in F. Obviously u is linear. If $x, y \in E$, $x \neq y$, there exists a positive integer n_1 such that $\langle v_{n_1} , x - y \rangle \neq 0$, since $\{v_n\}_{n=1}^{\infty}$ is total in $E'[\sigma(E' , E)]$. Then

$$\langle w_{n_1} , u(x - y) \rangle = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n , x - y \rangle \langle z_n , w_{n_1} \rangle =$$
$$= (1/n_1 2^{n_1}) \langle v_{n_1} , x - y \rangle \neq 0 ,$$

and, therefore, u is injective. If V is a neighbourhood of the origin in F, we can find a positive number μ such that $\mu A \subset V$. If W is the set of E, polar of $\{v_1 , v_2 , \ldots , v_n , \ldots\}$ then μW is a neighbourhood of the origin in E and if $z \in \mu W$ we have that

$$u(z) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n , z \rangle z_n \in \mu A \subset V$$

and, therefore, u is continuous. Since

$$u(x_p) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n , x_p \rangle z_n = (1/p2^p) z_p$$

it follows that $u(E)$ is separable and dense in F. Finally, given any element $x \in E$ there is a positive number $\alpha > 0$ such that $\alpha x \in W$, hence $|\langle v_n , \alpha x \rangle| \leqslant 1$, $n = 1 , 2 , \ldots ,$ and

$$u(\alpha x) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n , \alpha x \rangle z_n$$

belongs to the closed absolutely convex bull $M \subset A$ of $\{(1/n) z_n\}_{n=1}^{\infty}$ and, therefore, $u(E)$ is contained in the linear hull of M. The set M is compact in the infinite-dimensional Banach space F_A and, therefore, applying the theorem of Riesz, (see [5], p. 155), it follows that M is not absorbing in F_A, hence $u(E) \neq F$.

q.e.d.

THEOREM 1. – *Let* $\{E_n\}_{n=1}^{\infty}$ *and* $\{F_n\}_{n=1}^{\infty}$ *be two sequence of infinite-dimensional spaces, such that, for every positive integer n, the following conditions hold :*

1) *There exists in* E_n *a separable weakly compact absolutely convex subsets which is total in* E_n.

2) *There exists in* $F'_n[\sigma(F'_n , F_n)]$ *an equicontinuous total sequence.*

Then there is in $L = \left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)$ *a dense subspace* G, *different from* L, *which intersects every bounded and closed set of* L *in a closed set of* L.

Proof. – Since for every E_n and F_n, the conditions of Lemma 1 hold there exists an injective linear continuous mapping u_n from F_n into E_n such that $u_n(F_n)$ is separable, dense in E_n and $u(F_n) \neq E_n$. We set

$$u = (u_1 , u_2 , \ldots , u_n , \ldots) \quad \text{and} \quad {}^t u = ({}^t u_1 , {}^t u_2 , \ldots , {}^t u_n , \ldots)$$

If $y = (y_1 , y_2 , \ldots , y_n , \ldots) \in \prod_{n=1}^{\infty} F_n$ and

$x' = (x'_1 , x'_2 , \ldots , x'_n , \ldots) \in \bigoplus_{n=1}^{\infty} E'_n$ we put

$$u(y) = (u_1(y_1) , u_2(y_2) , \ldots , u_n(y_n) , \ldots) \quad \text{and}$$

$${}^t u(x') = ({}^t u_1(x'_1) , {}^t u_2(x'_2) , \ldots , {}^t u_n(x'_n) , \ldots) .$$

If $x \in \bigoplus_{n=1}^{\infty} E_n$ we define the mapping f from L into $\prod_{n=1}^{\infty} E_n$ putting $f(x , y) = x + u(y)$. It is immediate that f is continuous and linear and, therefore, ${}^tf$ is weakly continuous from $\overset{\infty}{\underset{n=1}{\oplus}} E'_n$ in

$$\left(\prod_{n=1}^{\infty} E'_n\right) \times \left(\bigoplus_{n=1}^{\infty} F'_n\right).$$

If $y' \in \prod_{n=1}^{\infty} E'_n$ and $z' \in \bigoplus_{n=1}^{\infty} F'_n$ are elements such that ${}^tf(x') = (y', z')$, we have that

$$\langle y', x\rangle + \langle z', y\rangle = \langle (y', z'), (x, y)\rangle = \langle {}^tf(x'), (x, y)\rangle =$$

$$= \langle x', f(x, y)\rangle = \langle x', x + u(y)\rangle = \langle x', x\rangle + \langle x', u(y)\rangle =$$

$$= \langle x', x\rangle + \langle {}^tu(x'), y\rangle .$$

then $\langle y', x\rangle + \langle z', y\rangle = \langle x', x\rangle + \langle {}^tu(x'), y\rangle$. In the last relation if we take $y = 0$ it follows that $y' = x'$, and if we take $x = 0$ it results that $z' = {}^tu(x')$. Therefore, ${}^tf(x') = (x', {}^tu(x'))$.

Let $M = \{(x', {}^tu(x')) : x' \in \prod_{n=1}^{\infty} E'_n\}$. Since, for every positive integer n, ${}^tu_n$ is weakly continuous from E'_n into F'_n we have that M is weakly closed in $\left(\prod_{n=1}^{\infty} E'_n\right) \times \left(\prod_{n=1}^{\infty} F'_n\right)$ and, therefore,

$$N = M \cap \left[\left(\prod_{n=1}^{\infty} E'_n\right) \times \left(\bigoplus_{n=1}^{\infty} F'_n\right)\right]$$

is weakly closed in $\left(\prod_{n=1}^{\infty} E'_n\right) \times \left(\bigoplus_{n=1}^{\infty} F'_n\right)$. On the other hand, if $(x', {}^tu(x')) \in N$, then ${}^tu(x') \in \bigoplus_{n=1}^{\infty} F'_n$ and, therefore, ${}^tu_n(x'_n)$ is zero for all indices except a finite number of them. Since ${}^tu_n$ is injective it follows that x'_n is zero for all indices except a finite number of them, hence $x' \in \bigoplus_{n=1}^{\infty} E'_n$ and, therefore, ${}^tf\left(\bigoplus_{n=1}^{\infty} E'_n\right) = N$. Since ${}^tf\left(\bigoplus_{n=1}^{\infty} E'_n\right)$ is weakly closed in $\left(\prod_{n=1}^{\infty} E'_n\right) \times \left(\bigoplus_{n=1}^{\infty} F'_n\right)$ we have that f is a topological homomorphism from $L[\sigma(L, L')]$ onto

$$H = f(L)\left[\sigma\left(f(L), \bigoplus_{n=1}^{\infty} E'_n\right)\right] .$$

Let $L_p = \left(\overset{p}{\underset{n=1}{\oplus}} E_n\right) \times \left(\overset{\infty}{\underset{n=1}{\Pi}} F_n\right)$ and let $H_p = f(L_p)$. Since $u_n(F_n)$ is separable and dense in E_n we have that H_1 is separable and dense in H. If z_n is an element of E_n such that $z_n \notin u_n(F_n)$ let

$$z^{(p)} = (z_1, z_2, \ldots, z_p, 0, 0, \ldots, 0, \ldots) .$$

The set $A = \{z^{(1)}, z^{(2)}, \ldots, z^{(n)}, \ldots\}$ is bounded in H and $z^{(p+1)} \notin H_p$. According to result a), there exists a dense subspace D of H, $D \neq H$, such that $D \cap H_p$ is finite-dimensional, $p = 1, 2, \ldots$ If $G = f^{-1}(D)$, then $G \neq L$ and G is dense in L, since f is weakly open from L into H. Given in L a bounded closed set B such that $G \cap B$ is not closed, there is a point z in B, which is in the closure of $G \cap B$, with $z \notin G$. There exists a positive integer p_0 such that $B \subset L_{p_0}$. Since f is continuous, $f(z) \notin D$ and $f(z) \in \overline{f(G \cap B)} \subset \overline{D \cap f(B)}$. On the other hand, $D \cap f(B)$ is contained in the closed subspace $D \cap H_{p_0}$, hence $f(z)$, which belongs to D, is not in the closure of $D \cap f(B)$, which is a contradiction.

q.e.d.

THEOREM 2. – *If* $\{E_n\}_{n=1}^{\infty}$ *and* $\{F_n\}_{n=1}^{\infty}$ *are two sequences of arbitrary infinite-dimensional Banach spaces, then* $\left(\overset{\infty}{\underset{n=1}{\oplus}} E_n\right) \times \left(\overset{\infty}{\underset{n=1}{\Pi}} F_n\right)$ *is not* B_r*-complete.*

Proof. – Let G_n and H_n be separable closed subspaces of infinite dimension of E_n and F_n, respectively. Since every closed subspace of a B_r-complete space is B_r-complete, [8], and $\left(\overset{\infty}{\underset{n=1}{\oplus}} G_n\right) \times \left(\overset{\infty}{\underset{n=1}{\Pi}} H_n\right)$ is closed in $\left(\overset{\infty}{\underset{n=1}{\oplus}} E_n\right) \times \left(\overset{\infty}{\underset{n=1}{\Pi}} F_n\right)$ it is enough to carry out the proof, taking E_n and F_n to be separable spaces, which will be supposed. If $\{x_p\}_{p=1}^{\infty}$ is a dense sequence in E_n, we can find a sequence $\{\alpha_p\}_{p=1}^{\infty}$ of non-zero numbers such that $\{\alpha_p x_p\}_{p=1}^{\infty}$ converges to the origin in E_n. The sequence $\{\alpha_p x_p\}_{p=1}^{\infty}$ is total in E_n, and it is equicontinuous in $E'_n[\mu(E'_n, E_n)]$. If V_n is the closed unit ball in F_n and V_n^0 is the polar set of V_n in F'_n, then V_n^0 is a separable weakly compact absolutely convex set which is total in $F'_n[\mu(F'_n, F_n)]$. Since $\{F'_n[\mu(F'_n, F_n)]\}_{n=1}^{\infty}$ and $\{E'_n[\mu(E'_n, E_n)]\}_{n=1}^{\infty}$ satisfy conditions 1 and 2, respectively, of Theorem 1, there exists in

$$L = \left(\overset{\infty}{\underset{n=1}{\oplus}} F'_n[\mu(F'_n F_n)] \right) \times \left(\prod_{n=1}^{\infty} E'_n[\mu(E'_n , E_n)] \right)$$

a dense subspace G, $G \neq L$, such that G intersects every bounded closed subset of L in a closed subset of L and, therefore,

$$\left(\overset{\infty}{\underset{n=1}{\oplus}} E_n \right) \times \left(\prod_{n=1}^{\infty} F_n \right) \quad \text{is not } B_r\text{-complete.} \qquad \text{q.e.d.}$$

THEOREM 3. – *If* $\{E_n\}_{n=1}^{\infty}$ *and* $\{F_n\}_{n=1}^{\infty}$ *are two sequences of infinite-dimensional Banach spaces, then*

$$\left(\prod_{n=1}^{\infty} E_n \right) \times \left(\overset{\infty}{\underset{n=1}{\oplus}} F'_n[\mu(F'_n , F_n)] \right)$$

is not B_r*-complete.*

Proof. – We take in E_n a separable closed subspace G_n, of infinite dimension. If V_n is the closed unit ball of F_n, let V_n^0 be the polar set of V_n in F'_n. We take in V_n^0 an infinite countable set B linearly independent. If H_n is the closed linear hull of B in $F'_n[\sigma(F'_n , F_n)]$ and A is the $\sigma(F'_n , F_n)$-closed absolutely convex hull of B, then $H_n[\mu(H_n , H'_n)]$ has a separable weakly compact absolutely convex set A which is total. Reasoning in the same way than in Theorem 2 it is sufficient to carry out the proof when E_n is a separable space and $F'_n[\mu(F'_n , F_n)]$ is of the form $H_n[\mu(H_n , H'_n)]$. Then the sequences $\{E'_n[\mu(E'_n , E_n)]\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem 1, hence it follows that the space $\left(\prod_{n=1}^{\infty} E_n \right) \times \left(\overset{\infty}{\underset{n=1}{\oplus}} F'_n[\mu(F'_n , F_n)] \right)$ is not B_r-complete.

q.e.d.

COROLLARY 1.3. – *Let* E *be a product of countable infinitely many Banach spaces of infinite-dimension. Then* $E \times E'[\mu(E' , E)]$ *is not* B_r*-complete.*

By analogous methods used in Theorems 2 and 3, we can obtain Theorems 4 and 5.

THEOREM 4. – *Let* $\{E_n\}_{n=1}^{\infty}$ *and* $\{F_n\}_{n=1}^{\infty}$ *be two sequences of Banach spaces of infinite dimension. If, for every positive integer n,* E_n *is separable, then* $\left(\prod_{n=1}^{\infty} E'_n[\mu(E'_n , E_n)] \right) \times \left(\overset{\infty}{\underset{n=1}{\oplus}} F'_n[\mu(F'_n , F_n)] \right)$ is *not* B_r*-complete.*

THEOREM 5. – *Let* $\{E_n\}_{n=1}^{\infty}$ *and* $\{F_n\}_{n=1}^{\infty}$ *be two sequences of Banach spaces of infinite dimension. If, for every positive integer n,* F_n *is separable, then* $\left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F'_n[\mu(F'_n, F_n)]\right)$ *is not* B_r-*complete.*

Note 1. – It is easy to show that Theorems 2, 3, 4 and 5 are valid changing the condition "Banach space" by "Fréchet space", with the additional hypothesis : In Theorem 3, the topology of E_n will be defined by a family of norms ; in Theorems 2 and 4 the topology of F_n will be also defined by a family of norms. Let us suppose, now, given an infinite-dimensional nuclear Fréchet space F, its topology is defined by a family of norms. Since F is a Montel space then it is separable, [3], (see [5], p. 370). If we take $E_n = F_n = F$ and we apply the generalized Corollary 1.3 it results that $E = \prod_{n=1}^{\infty} E_n$ is a nuclear Fréchet space such that $E \times E'[\mu(E', E)]$ is not B_r-complete. If we apply the generalized Theorem 2 it results that $G = \left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)$ is a nuclear strict (LF)-space which is not B_r-complete and, finally, if we apply the generalized Theorem 4 it follows that $G'[\mu(G', G)]$ is a countable product of complete (DF)-spaces which is not B_r-complete.

In ([1], p. 35) N. Bourbaki notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have obtained a wide class of bornological barrelled spaces which are not ultrabornological. In [10] we give an example of a bornological barrelled space which is not the inductive limit of Baire spaces. This example is not a metrizable space. In Theorem 6 we shall obtain a class $\mathfrak{A}$ of bornological barrelled spaces which are not inductive limits of Baire spaces, such that $\mathfrak{A}$ contains metrizable spaces.

In [10] we have given the following result : b) *Let* E *be a bornological barrelled space which has a family of subspaces* $\{E_n\}_{n=1}^{\infty}$ *such that the following conditions hold :* 1) $\bigcup_{n=1}^{\infty} E_n = E$. 2) *For every positive integer n, there is a topology* $\mathcal{T}_n$ *on* E_n, *finer than the initial one, such that* $E_n[\mathcal{T}_n]$ *is a Fréchet space.* 3) *There is in* E *a bounded set* A *such that* $A \not\subset E_n$, $n = 1, 2, \ldots$ *Then there is a bornological* *barrelled space* F *which is not an inductive limit of Baire space, such that* E *is a dense hyperplane of* F.

THEOREM 6. – *If* $\{G_i : i \in I\}$ *is an infinite family of ultrabornological spaces, there is in* $G = \Pi\{G_i : i \in I\}$ *a dense subspace* E, *bornological and barrelled, which is not an inductive limit of Baire spaces, so that* E *contains an ultrabornological subspace* F, *of codimension one.*

Proof. – We take in I an infinite countable subset $\{i_1, i_2, \ldots, i_n, \ldots\}$. If G_{i_n} is of dimension one we put $G_{i_n} = K_n$. If G_{i_n} is not of dimension one we can take $G_{i_n} = K_n \oplus H_n$, being H_n a closed subspace of codimension one of G_{i_n}. The space G can be put in the form

$$\left(\prod_{n=1}^{\infty} K_n\right) \times \Pi\{L_j : j \in J\},$$

such that L_j is ultrabornological for every $j \in J$. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of infinite-dimensional separable Banach spaces and let $\{E_n\}_{n=1}^{\infty}$ be a sequence such that $E_n = \prod_{p=1}^{\infty} K_p$, $n = 1, 2, \ldots$ The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ satisfy conditions of Theorem 1 and, therefore using the same notations as in Theorem 1 we have that $\mu\left(H, \bigoplus_{n=1}^{\infty} E'_n\right)$ can be identified with the topology induced in H by $\prod_{n=1}^{\infty} E_n$, since the last space is metrizable. Hence, $L/f^{-1}(0)$ can be identified with $H\left[\mu\left(H, \bigoplus_{n=1}^{\infty} E'_n\right)\right]$ and, therefore, there is on H_n a topology $\mathcal{F}_n$, finer than the one induced by $\prod_{n=1}^{\infty} E_n$, such that $H_n[\mathcal{F}_n]$ is a Fréchet space isomorphic to $L_n/(f^{-1}(0) \cap L_n)$. On the other hand, A is a bounded set of $H\left[\mu\left(H, \bigoplus_{n=1}^{\infty} E'_n\right)\right]$ such that $A \not\subset H_n$, $n = 1, 2, \ldots$, whence it follows, applying result b), and since $\prod_{n=1}^{\infty} E_n$ is complete, that there is a point $x \in \prod_{n=1}^{\infty} E_n$, $x \notin H$, such that the linear hull S of $H \cup \{x\}$ is a dense subspace of $\prod_{n=1}^{\infty} E_n$, bornological and barrelled, which is not an inductive limit of Baire spaces and $H\left[\mu\left(H, \overset{\infty}{\underset{n=1}{\oplus}} E'_n\right)\right]$ is an ultrabornological subspace of S, of codimension one. Since $\prod_{n=1}^{\infty} K_n$ is topologically isomorphic to $\prod_{n=1}^{\infty} E_n$ there is in $\prod_{n=1}^{\infty} K_n$ a dense subspace D which is bornological and barrelled, such that it is not an inductive limit of Baire spaces, and it has an ultrabornological subspace T, of codimension one. We take in $\{L_j : j \in J\}$ the subspace U such that $x \in U$ if, and only if, all the components of x are zero except a most a countable infinite number of them. The space U is ultrabornological, (see the proofs of Theorem 1 and Theorem 2 in [11]). If $E = D \times U$ and $F = T \times U$, then E and F hold the conditions of the theorem.

q.e.d.

In [12] and [15] we have given, respectively, the two following results : c) *If* E *is a reflexive strict* (LF)*-space, then* $E'[\mu(E', E)]$ *is ultrabornological.* d) *Let* Ω *be a non-empty open set in the n-dimensional euclidean space* R^n. *Let* $\mathcal{D}'(\Omega)$ *the space of distributions, with the strong topology. Then there is a topology* $\mathfrak{F}$ *on* $\mathcal{D}'(\Omega)$ *coarser than the initial one, so that* $\mathcal{D}'(\Omega)$ $[\mathfrak{F}]$ *is a bornological barrelled space which is not ultrabornological.* In Theorem 7 we extend the result d).

THEOREM 7. – *Let* E *be a reflexive strict* (LF)*-space. If* $E'[\mu(E', E)]$ *is not* B_r*-complete, then there exists in* E' *a topology* $\mathfrak{F}$, *coarser than* $\mu(E', E)$, *so that* $E'[\mathfrak{F}]$ *is a bornological barrelled space which is not an inductive limit of Baire spaces.*

Proof. – Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of Fréchet subspaces of E, such that E is the inductive limit of this sequence. Let G be a dense subspace of E, $G \neq E$, which intersects to every weakly compact absolutely convex subset of E in a closed set. Let $\mathfrak{F} = \mu(E', G)$. Obviously every closed subset of $G[\sigma(G, E')]$ is compact and, therefore, $E'[\mathfrak{F}]$ is barrelled. Let us see, now, that $E'[\mathfrak{F}]$ is bornological. By a theorem of Köthe, ([5], p. 386), we shall see that $G[\mathfrak{F}_{c_0}]$ is complete, $\mathfrak{F}_{c_0}$ being the topology of the uniform convergence on every sequence of $E'[\mathfrak{F}]$ which converges to the origin in the Mackey sense. According to result c), we have that $E[\mu(E', E)_{c_0}]$ is complete. Since $E'[\mu(E', E)]$ is the Mackey dual of a (LF)-space, it follows that $E'[\mu(E', E)]$ is complete and, therefore, $\mu(E', E)_{c_0}$ is compatible with the dual pair $\langle E, E' \rangle$. Since $G \cap E_n$ is closed in E, we have that $(G \cap E_n)[\mu(E', E)_{c_0}]$ is closed in $E[\mu(E', E)_{c_0}]$, hence it results that $(G \cap E_n)[\mu(E', E)_{c_0}]$ is complete and hence, applying a theorem of Bourbaki, ([5], p. 210) one deduces that $(G \cap E_n)[\mathcal{T}_{c_0}]$ is complete. Let us suppose, now, that $G[\mathcal{T}_{c_0}]$ is not complete. We take in the completion $\hat{G}[\hat{\mathcal{T}}_{c_0}]$ of $G[\mathcal{T}_{c_0}]$ an element x_0 which is not in G. Since $(G \cap E_n)[\mathcal{T}_{c_0}]$ is complete, we have that $G \cap E_n$ is closed in $\hat{G}[\hat{\mathcal{T}}_{c_0}]$, and we can find a continuous linear form u_n on $\hat{G}[\hat{\mathcal{T}}_{c_0}]$, such that $\langle u_n, x_0 \rangle = 1$ and $\langle u_n, x \rangle = 0$, for every $x \in G \cap E_n$. Given any point $y_0 \in G$ there is a positive integer n_0, such that $y_0 \in G \cap E_{n_0}$, and, therefore, $\langle nu_n, y_0 \rangle = 0$, for $n \geqslant n_0$, hence it deduces that $\{nu_n\}_{n=1}^{\infty}$ converges to the origin in $E'[\mu(E', G)]$, from here $\{u_n\}_{n=1}^{\infty}$ converges to the origin in $E'[\mu(E', G)]$ in the sense of Mackey and, therefore, $\{u_n\}_{n=1}^{\infty}$ is equicontinuous in $\hat{G}[\hat{\mathcal{T}}_{c_0}]$. Since $\{\langle u_n, x \rangle\}_{n=1}^{\infty}$ converges to the origin for every $x \in G$, and G is dense in $\hat{G}[\hat{\mathcal{T}}_{c_0}]$ it follows that $\{\langle u_n, x \rangle\}_{n=1}^{\infty}$ converges to the origin, for every $x \in \hat{G}[\hat{\mathcal{T}}_{c_0}]$, which is a contradiction since $\langle u_n, x_0 \rangle = 1$, $n = 1, 2, \ldots$ Thus, $G[\mathcal{T}_{c_0}]$ is complete. Finally, if f is the identity mapping from $E'[\mu(E', E)]$ onto $E'[\mathcal{T}]$, then f is continuous and f^{-1} is not continuous. Applying the closed graph theorem in the form given by De Wilde, [2], we can derive that $E'[\mathcal{T}]$ is not an inductive limit of Baire spaces.

q.e.d.

THEOREM 8. – *If* $\{E_n\}_{n=1}^{\infty}$ *and* $\{F_n\}_{n=1}^{\infty}$ *are two sequences of infinite-dimensional reflexive Banach spaces there is on*

$$E = \left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)$$

a topology $\mathcal{T}$, *coarser than the initial one, so that* $E[\mathcal{T}]$ *is a bornological barrelled space which is not an inductive limit of Baire spaces.*

Proof. – It is immediate consequence from Theorem 2 and Theorem 7.

Note 2. – In part, the method followed in the proof of Theorem 7 suggest to us the following short proof of the well-known result that if E is the strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of complete spaces, then E is complete, [6], ([5], p. 224-225) : Suppose that E is not complete and let x_0 be a point of the completion $\hat{E}$ of E, $x_0 \notin E$. Since E_n is closed in $\hat{E}$ we can find $u_n \in (\hat{E})'$ such that $\langle u_n , x_0 \rangle = 1$, $\langle u_n , x \rangle = 0$ for every $x \in E_n$. The set of restrictions of $A = \{u_1 , u_2 , \ldots , u_n , \ldots\}$ to E_n is a finite set which is, therefore, equicontinuous in E_n. Hence A is equicontinuous in E and, therefore, A is equicontinuous in $\hat{E}$, hence A is relatively compact in $(\hat{E})'[\sigma((\hat{E})' , \hat{E})]$. If u is a cluster point of the sequence $\{u_n\}_{n=1}^{\infty}$ in $(\hat{E})'[\sigma((\hat{E})' , \hat{E})]$ then it is immediate that u is zero ou E and $\langle u , x_0 \rangle = 1$, which is a contradiction since E is dense in $\hat{E}$.

J. Dieudonné has proved in [4] the following theorem : e) *If* F *is a subspace of finite codimension of a bornological space* E, *then* F *is bornological.* We have proved in [13] the following result : f) *If* F *is a subspace of finite codimension of a quasi-barrelled space* E, *then* F *is quasi-barrelled.*

In e) and f) it is not possible to change "finite codimension" by "infinite countable codimension". In [10] we have given a example of a bornological space E which has a subspace F, of infinite countable codimension, which is not quasi-barrelled. In this example F is not dense in E. In Theorem 9, using in part the method followed in [10] we shall give a class $\mathfrak{A}$ of bornological spaces, such that if $E \in \mathfrak{A}$ there is a dense subspace F of E, of infinite countable codimension, which is not quasi-barrelled.

We shall need the following results given in [10] and [14], respectively : g) *Let* E *be the strict inductive limit of an increasing sequence of metrizable spaces. Let* F *be a sequentially dense subspace of* E. *If* E *is a barrelled then* F *is bornological.* h) *Let* E *be a barrelled space. If* $\{E_n\}_{n=1}^{\infty}$ *is an increasing sequence of subspaces of* E, *such that* $E = \bigcup_{n=1}^{\infty} E_n$, *then* E *is the strict inductive limit of* $\{E_n\}_{n=1}^{\infty}$.

THEOREM 9. – *If* $\{E_n\}_{n=1}^{\infty}$ *and* $\{F_n\}_{n=1}^{\infty}$ *are two sequences of infinite-dimensional separable Banach spaces, there is in*

$$L = \left(\bigoplus_{n=1}^{\infty} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)$$

a bornological dense subspace E, *such that* E *contains a dense subspace* F, *of infinite countable codimension, which is not quasi-barrelled.*

Proof. – The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem 1 and, therefore, there is in L a dense subspace F, $F \neq L$, which intersects to every bounded closed subset of L in a closed subset of L. Let A_n and B_n be respectively two countable dense subsets of E_n and F_n, considered as subspaces of L. Let H be the linear hull of $\bigcup_{n=1}^{\infty} (A_n \cup B_n)$. If E is the linear hull of $H \cup F$, with the topology induced by the topology of L, then E is bornological, according to result g), since $E \cap \left(\bigoplus_{n=1}^{p} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)$ is dense in

$$\left(\bigoplus_{n=1}^{p} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right).$$

Suppose that F is quasi-barrelled. Then F is barrelled, since F is quasi-complete and, therefore, by result h), F is the inductive limit of the sequence of complete spaces

$$\left\{F \cap \left(\bigoplus_{n=1}^{p} E_n\right) \times \left(\prod_{n=1}^{\infty} F_n\right)\right\}_{p=1}^{\infty}$$

and, therefore, F is complete, hence $F = L$, which is a contradiction. Thus, F is not quasi-barrelled. Since H has countable dimension, then F if of countable codimension in E and, by result f), F is of infinite countable codimension in E.

q.e.d.

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