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THREE SPECTRAL NOTIONS FOR REPRESENTATIONS OF COMMUTATIVE BANACH ALGEBRAS

by Yngve DOMAR and Lars-Åke LINDAHL

1. Introduction.

The results in Lyubich [11] and Lyubich, Matsaev and Fel'dman [12] on the spectrum of group representations have inspired us to reconsider our papers [7] and [9], where we discussed narrow spectral analysis, a concept originating from Beurling [2] and earlier studied in [5], pp. 55-66, and [6]. During our work we have found close connections with another independent line of research, represented by the papers Arens [1], Żelazko [19], Slodkowski [16] and Choi and Davis [3]. The main results of [3], [7], [9], [11], [12], [16] and [19] can in fact all be formulated in terms of three different spectral notions for representations of commutative Banach algebras. The aim of this paper is to present such a unified approach and to prove some new results.

Let T be a bounded representation of a commutative Banach algebra B on a normed linear space V, and denote by $\mathfrak{M}(B)$ the Gelfand space of B. With T we shall associate the following three spectral sets.

The first one is the hull $\Lambda_1(T)$ in $\mathfrak{M}(B)$ of Ker T. This is an old and very much studied notion.

The second spectral set $\Lambda_2(T)$ consists of those $\varphi \in \Lambda_1(T)$ which are bounded with respect to the seminorm $|\cdot|$ in B, defined by putting $|b| = \|T_b\|$. (T_b is here the image of b under the mapping T). This spectral definition turns out to be an extension of the old notion of narrow spectrum, studied in [7] and [9].

The third spectral set $\Lambda_3(T)$, finally, consists of all $\varphi \in \mathfrak{M}(B)$ such that, for every $\epsilon > 0$ and every finite $F \subset B$, there is a $v \in V$

satisfying $\|v\| = 1$ and $\|T_b v - b(\varphi) v\| < \epsilon$, for all $b \in F$. This definition can be considered as a generalization both of the essential spectrum in [11] and of the set of maximal ideals consisting of joint topological divisors of zero, studied in [16] and [19].

The paper is organized in the following way. In Section 2, $\Lambda_1(T)$ and $\Lambda_2(T)$ are introduced and some simple properties of them are derived. The set $\Lambda_3(T)$ is introduced in Section 3, and it is proved to be a closed subset of $\Lambda_2(T)$. Section 4 contains some elementary results on restrictions and reductions of representations. The regular representation R and its adjoint representation L are studied in Section 5, and we give a new proof of a theorem of Żelazko, which in our terminology states that $\Lambda_3(R)$ contains the Shilov boundary of B. In Section 6, this result is used to prove that $\Lambda_3(T)$ is nonempty if $\Lambda_2(T)$ is nonempty. Various conditions which suffice for equality of two, or all three, of the spectra are also given, and connections with results in [11] and [12] on group representations are pointed out. Section 7 contains a generalization of extension theorems in Lyubich [11] and Slodkowski [16]. Restrictions of L to invariant subspaces of B^* are studied in Section 8. Theorem 8.1 shows that, for these representations, Λ_1 and Λ_2 coincide with the two spectral sets studied in [7] and [9]. As an application we study the case when B is a Beurling algebra, and we obtain an extension of the theorem in Beurling [2]. Comparisons are also made with results of Dixmier [4] and Warner [18]. In Section 9 we give some examples which show the possibility of having $\Lambda_1 \neq \Lambda_2 \neq \Lambda_3$ for restrictions of L.

2. The spectrum and the narrow spectrum.

Let B be a commutative Banach algebra, and denote by $\mathfrak{M}(B)$ the Gelfand space of B, that is the (possibly empty) locally compact space of all nontrivial complex-valued homomorphisms of B. It deserves to be pointed out that, except for Section 8, our investigation will not depend on the choice of norm in B. In particular, we need not assume that $\|ab\| \leqslant \|a\| \, \|b\|$.

Let

$$T : B \to \mathrm{End}(V)\,, \quad b \curvearrowright T_b\,,$$

be a bounded representation of B on a normed linear space V, i.e. a homomorphism from B into the normed algebra End(V) of bounded operators on V such that $\|T\| = \sup \|T_b\| \, \|b\|^{-1} < \infty$. The kernel and the image of T will be denoted by Ker T and Im T, respectively.

The *spectrum* $\Lambda_1(T)$ *of* T is defined to be the hull in $\mathfrak{M}(B)$ of the closed ideal Ker T. It is immediate that $\Lambda_1(T)$ is a closed (possibly empty) subset of $\mathfrak{M}(B)$.

Put $B(T) = B/\operatorname{Ker} T$, and denote the cosets $b + \operatorname{Ker} T$ by b_T. We shall give B(T) two different norms. The first one is the usual quotient norm $\|\cdot\|_1$ under which B(T) becomes a commutative Banach algebra, which will be denoted by $B_1(T)$. The canonical homomorphism

$$\pi_1 : B \to B_1(T)\,,\ b \curvearrowright b_T\,,$$

induces a homeomorphism

$$\pi_1^* : \mathfrak{M}(B_1(T)) \to \Lambda_1(T)$$

between the Gelfand space of $B_1(T)$ and the spectrum of T. As usual we identify these two spaces. Then we have

$$\hat{b}_T(\varphi) = \hat{b}(\varphi) \qquad \forall\, \varphi \in \Lambda_1(T)\,,\ \forall\, b \in B\,.$$

Since B(T) is algebraically isomorphic to Im T, we obtain a second, submultiplicative norm $\|\cdot\|_2$ in B(T) by putting

$$\|b_T\|_2 = \|T_b\|\,.$$

Let $B_2(T)$ denote the completion of B(T) under this norm. Then $B_2(T)$ is a commutative Banach algebra. The injection $j : B_1(T) \hookrightarrow B_2(T)$ is continuous, because $\|b_T\|_2 \leqslant \|T\| \, \|b_T\|_1$. It follows that the dual map

$$j^* : \mathfrak{M}(B_2(T)) \to \Lambda_1(T) = \mathfrak{M}(B_1(T))$$

is an embedding. The closed subset

$$\Lambda_2(T) = j^*(\mathfrak{M}(B_2(T)))$$

of $\Lambda_1(T)$ will be called the *narrow spectrum of* T. It is convenient to identify $\mathfrak{M}(B_2(T))$ with $\Lambda_2(T)$.

Since the elements of $\Lambda_2(T)$ can be regarded as bounded complex-valued homomorphisms on Im T, and since the norm of these homomorphisms is at most one, the narrow spectrum has the following characterization, which does not mention $B_2(T)$ explicitly.

PROPOSITION 2.1. – *Let* $\varphi \in \mathfrak{M}(B)$.

i) *If* $\varphi \in \Lambda_2(T)$, *then* $|\hat{b}(\varphi)| \leqslant \|T_b\|$ *for all* $b \in B$.

ii) *If there is a constant* $C > 0$ *such that* $|\hat{b}(\varphi)| \leqslant C\,\|T_b\|$ *for all* $b \in B$, *then* $\varphi \in \Lambda_2(T)$.

A simple but useful consequence of Proposition 2.1 and of the definition of the spectrum is the following :

PROPOSITION 2.2. – *Let* S *and* T *be two representations of* B. *If* $\|S_b\| \leqslant \|T_b\|$ *for all* $b \in B$, *then* $\Lambda_i(S) \subset \Lambda_i(T)$, $i = 1, 2$.

We now give some examples which show that, in general, $\Lambda_2(T)$ is a proper subset of $\Lambda_1(T)$.

Example 2.3. – Let D denote the closed unit disc in the complex plane, and let A(D) be the sup normed algebra of all continuous functions on D that are analytic in the interior of D. The Gelfand space of A(D) can be identified with D, and under this identification $\hat{f}(z) = f(z)$ for all $f \in A(D)$ and all $z \in D$. Let E be a closed subset of D such that no nontrivial $f \in A(D)$ vanishes on E, and define a representation $T : A(D) \to \mathrm{End}\,(C(E))$ by

$$T_f g = f_{|E} \cdot g \qquad \forall f \in A(D)\ ,\ \forall g \in C(E)\ .$$

Then

$$\|T_f\| = \sup_{z \in E} |f(z)|\ ,$$

and hence $\Lambda_1(T) = D$ and $\Lambda_2(T) = \hat{E}$, where $\hat{E}$ denotes the polynomially convex hull of E.

Example 2.4. – Let $(\alpha_n)_0^\infty$ be a sequence of positive numbers with the following properties

i) $$\alpha_0 = 1$$

ii) $$\alpha_{m+n} \leqslant \alpha_m \cdot \alpha_n\ ,\ \forall m, n\ ,$$

iii) $$\lim_{n \to \infty} \alpha_n^{1/n} = 0\ .$$

Let B be the Banach space of all complex sequences $b = (b_n)_0^\infty$ with $b_0 = 0$ and $\|b\| = \Sigma_0^\infty |b_n| < \infty$. B is a Banach algebra under convolution and $\mathfrak{M}(\mathrm{B}) = \mathrm{D}\backslash\{0\}$. Let V be the Banach space of all complex sequences $v = (v_n)_0^\infty$ with $\|v\| = \Sigma_0^\infty |v_n| \alpha_n < \infty$, and let V_0 be the subspace of all $v \in \mathrm{V}$ with $v_0 = 0$. By ii), V_0 is a Banach algebra under convolution, and by iii), $\mathfrak{M}(\mathrm{V}_0) = \phi$. Convoluting elements in B with elements in V we obtain a bounded representation T of B on V, and it is easy to see that $\mathrm{B}_1(\mathrm{T}) = \mathrm{B}$ and $\mathrm{B}_2(\mathrm{T}) = \mathrm{V}_0$. This gives $\Lambda_1(\mathrm{T}) = \mathrm{D}\backslash\{0\}$ and $\Lambda_2(\mathrm{T}) = \phi$.

In order to assure that the spectrum of a representation T is nonempty one has to impose some condition on T. The following condition turns out to be sufficient, and it will be assumed to hold at some places.

Assumption I. – There exists an element $u \in \mathrm{B}$ *such that* $\mathrm{T}_u \neq 0$ *and* $\mathrm{T}_{ub} = \mathrm{T}_b$ *for all* $b \in \mathrm{B}$.

In particular, if B is an algebra with unit and $\mathrm{T} \neq 0$, then Assumption I is fulfilled.

PROPOSITION 2.5. – *Under Assumption I, both* $\Lambda_1(\mathrm{T})$ *and* $\Lambda_2(\mathrm{T})$ *are compact and nonempty.*

Proof. – If T fulfills Assumption I, then Ker T is a regular ideal in B with u acting as a unit modulo Ker T, and u_T is an identity element for both $\mathrm{B}_1(\mathrm{T})$ and $\mathrm{B}_2(\mathrm{T})$. So in this case $\Lambda_1(\mathrm{T})$ and $\Lambda_2(\mathrm{T})$ are Gelfand spaces of unitary Banach algebras, and this proves the proposition.

The study of the spectrum of a representation can be reduced to the case when the representation satisfies Assumption I, simply by adjoining a unit to B and extending the representation in a natural way. Let $\widetilde{\mathrm{B}}$ be the Banach algebra obtained from B by adjoining a unit 1. We shall regard B as a subalgebra of $\widetilde{\mathrm{B}}$. The Gelfand space $\mathfrak{M}(\widetilde{\mathrm{B}})$ of $\widetilde{\mathrm{B}}$ can be identified with $\mathfrak{M}(\mathrm{B}) \cup \{\varphi_\infty\}$, where φ_∞ denotes the zero homomorphism of B, if we define

$$(b + \lambda)\hat{}(\varphi) = \hat{b}(\varphi) + \lambda \ , \ \forall b \in \mathrm{B} \ , \ \forall \lambda \in \mathrm{C} \ , \ \forall \varphi \in \mathfrak{M}(\mathrm{B}) \cup \{\varphi_\infty\} .$$

Let I be the identity operator on V, and extend the representation T of B to a representation $\widetilde{\mathrm{T}}$ of $\widetilde{\mathrm{B}}$ on V by defining

$$\widetilde{\mathrm{T}}_{b+\lambda} = \mathrm{T}_b + \lambda \mathrm{I} .$$

Obviously, $\widetilde{\mathrm{T}}$ fulfills Assumption I. The following proposition relates the spectra and the narrow spectra of $\widetilde{\mathrm{T}}$ and T.

PROPOSITION 2.6. – $\Lambda_i(\mathrm{T}) = \Lambda_i(\widetilde{\mathrm{T}}) \cap \mathfrak{M}(\mathrm{B})$ *for* $i = 1, 2$.

Proof. – The proof is the same for $i = 1$ and $i = 2$. We start from the commutative diagram

$$\begin{tikzcd} \mathrm{B} \arrow[r, hook] \arrow[d, "\mathrm{T}"'] & \widetilde{\mathrm{B}} \arrow[d, "\widetilde{\mathrm{T}}"] \\ \mathrm{Im}\ \mathrm{T} \arrow[r, hook] & \mathrm{Im}\ \widetilde{\mathrm{T}} \end{tikzcd}$$

This induces in a natural way a commutative diagram

$$\begin{tikzcd} \mathrm{B} \arrow[r, hook] \arrow[d, "\pi_i"'] & \widetilde{\mathrm{B}} \arrow[d, "\widetilde{\pi}_i"] \\ \mathrm{B}_i(\mathrm{T}) \arrow[r, "\mu_i"'] & \widetilde{\mathrm{B}}_i(\widetilde{\mathrm{T}}) \end{tikzcd} \tag{2.1}$$

Since Im T is an ideal in Im $\widetilde{\mathrm{T}}$ of codimension 0 or 1, it follows that the codimension of $\mathrm{B}_i(\mathrm{T})$, considered as an ideal in $\widetilde{\mathrm{B}}_i(\widetilde{\mathrm{T}})$ via μ_i, is 0 or 1. (The codimension of $\mathrm{B}_1(\mathrm{T})$ is 0 if and only if Im T = Im $\widetilde{\mathrm{T}}$, and the codimension of $\mathrm{B}_2(\mathrm{T})$ is 0 if and only if Im T is dense in Im $\widetilde{\mathrm{T}}$). By [14, Theorem 3.1.18] the dual maps of the horizontal maps in (2.1) induce homeomorphisms

$$\mathfrak{M}(\widetilde{\mathrm{B}})\backslash\{\varphi_\infty\} \to \mathfrak{M}(\mathrm{B}) \quad \text{and} \quad \mathfrak{M}(\widetilde{\mathrm{B}}_i(\widetilde{\mathrm{T}}))\backslash\mathcal{H}_i \to \mathfrak{M}(\mathrm{B}_i(\mathrm{T})) .$$

Here, $\mathcal{H}_i$ denotes the hull of $\mathrm{B}_i(\mathrm{T})$ in $\mathfrak{M}(\widetilde{\mathrm{B}}_i(\widetilde{\mathrm{T}}))$. Thus, depending on the codimension of $\mathrm{B}_i(\mathrm{T})$, either $\mathcal{H}_i = \phi$ or $\mathcal{H}_i = \{\varphi^i_\infty\}$, where φ^i_∞ is the homomorphism that annihilates $\mathrm{B}_i(\mathrm{T})$, and in the latter case, $\widetilde{\pi}_i^*$ obviously maps φ^i_∞ on φ_∞. Thus, by duality, we obtain from (2.1) and from the definition of Λ_i either of the following two commutative diagrams in which all straight arrows are homeomorphisms.

$$\begin{tikzcd} \mathfrak{M}(\mathrm{B}) & \mathfrak{M}(\widetilde{\mathrm{B}})\backslash\{\varphi_\infty\} \arrow[l] \\ \Lambda_i(\mathrm{T}) \arrow[u, hook] & \Lambda_i(\widetilde{\mathrm{T}}) \arrow[u, hook] \\ \mathfrak{M}(\mathrm{B}_i(\mathrm{T})) \arrow[u] & \mathfrak{M}(\widetilde{\mathrm{B}}_i(\widetilde{\mathrm{T}})) \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{M}(\mathrm{B}) & \mathfrak{M}(\widetilde{\mathrm{B}})\backslash\{\varphi_\infty\} \arrow[l] \\ \Lambda_i(\mathrm{T}) \arrow[u, hook] & \Lambda_i(\widetilde{\mathrm{T}})\backslash\{\varphi_\infty\} \arrow[u, hook] \\ \mathfrak{M}(\mathrm{B}_i(\mathrm{T})) \arrow[u] & \mathfrak{M}(\widetilde{\mathrm{B}}_i(\widetilde{\mathrm{T}}))\backslash\{\varphi^i_\infty\} \arrow[l] \arrow[u] \end{tikzcd}$$

These diagrams show that either $\Lambda_i(\widetilde{\mathrm{T}}) = \Lambda_i(\mathrm{T})$ or

$$\Lambda_i(\widetilde{\mathrm{T}}) = \Lambda_i(\mathrm{T}) \cup \{\varphi_\infty\}.$$

In both cases, $\Lambda_i(\mathrm{T}) = \Lambda_i(\widetilde{\mathrm{T}}) \cap \mathfrak{M}(\mathrm{B})$.

3. The approximate point spectrum.

We are now going to introduce and study a third spectral concept which has its roots in some more special definitions of Lyubich [11] and Żelazko [19].(*)

Let T be a bounded representation of B on V. If $\mathrm{F} \subset \mathrm{B}$ and ω is a complex-valued function defined on F, we put

$$\nu_{\mathrm{T}}(\omega\,;\mathrm{F}) = \inf_{\|v\|=1} \sup_{b\in\mathrm{F}} \|\mathrm{T}_b v - \omega(b)\, v\| \,.$$

The set $\Lambda_3(\mathrm{T})$ of all $\varphi \in \mathfrak{M}(\mathrm{B})$ such that $\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}) = 0$, for all *finite* subsets F of B, will be called the *approximate point spectrum of* T.

PROPOSITION 3.1. – *$\Lambda_3(\mathrm{T})$ is a closed subset of $\Lambda_2(\mathrm{T})$.*

Proof. – For each finite set F, the map $\varphi \rightsquigarrow \nu_{\mathrm{T}}(\varphi\,;\mathrm{F})$ is continuous. Indeed, if $|\hat{b}(\varphi) - \hat{b}(\varphi_0)| < \epsilon$ for all $b \in \mathrm{F}$, then

$$|\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}) - \nu_{\mathrm{T}}(\varphi_0\,;\mathrm{F})| < \epsilon \,.$$

It follows that $\Lambda_3(\mathrm{T})$, being the intersection of the closed sets $\{\varphi \in \mathfrak{M}(\mathrm{B})\,;\,\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}) = 0\}$ (F finite), is a closed subset of $\mathfrak{M}(\mathrm{B})$. Since $\Lambda_2(\mathrm{T})$ is closed, it now suffices to prove that $\Lambda_3(\mathrm{T})$ is a subset of $\Lambda_2(\mathrm{T})$. Suppose that $\varphi \in \Lambda_3(\mathrm{T})$. Then

$$\begin{aligned} |\hat{b}(\varphi)| &= \inf_{\|v\|=1} \|\hat{b}(\varphi)\, v\| \leqslant \inf_{\|v\|=1} (\|\hat{b}(\varphi)\, v - \mathrm{T}_b v\| + \|\mathrm{T}_b v\|) \leqslant \\ &\leqslant \nu_{\mathrm{T}}(\varphi\,;\{b\}) + \|\mathrm{T}_b\| = 0 + \|\mathrm{T}_b\| = \|\mathrm{T}_b\| \end{aligned}$$

for every $b \in \mathrm{B}$. Hence $\varphi \in \Lambda_2(\mathrm{T})$, by Proposition 2.1.

(*) *Added in proof.* The approximate point spectrum has also been studied by Slodkowski and Żelazko in their recent paper "On joint spectra of commuting families of operators", Studia Math., 50 (1974), 127-148, which among other things contains equivalent versions of our Theorems 6.3 and 7.1.

In general, $\Lambda_3(\mathrm{T})$ is a proper subset of $\Lambda_2(\mathrm{T})$ (see e.g. Example 3.4). Of course, it is important to know whether $\Lambda_3(\mathrm{T})$ is nonempty, and one of our main results (Theorem 6.3) will be that so is the case, whenever $\Lambda_2(\mathrm{T})$ is nonempty. There is no corresponding relation between $\Lambda_1(\mathrm{T})$ and $\Lambda_2(\mathrm{T})$ as is shown by Example 2.4.

We now give some equivalent definitions of the approximate point spectrum. Of these, iii) is particularly useful for the determination of $\Lambda_3(\mathrm{T})$ in concrete cases.

PROPOSITION 3.2. – *The following five conditions are equivalent :*

i) $\varphi \in \Lambda_3(\mathrm{T})$.

ii) $\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}) = 0$ *for every finite subset* F *of some dense set in* B.

iii) $\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}) = 0$ *for every finite subset* F *of some set of generators of* B.

iv) $\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}) = 0$ *for every compact subset* F *of* B.

v) *There exists a net* $\{v_\alpha\}_{\alpha\in\mathfrak{a}}$ *in* V *such that* $\|v_\alpha\| = 1$, *for all* $\alpha \in \mathfrak{a}$, *and* $\lim_\alpha \|\mathrm{T}_b v_\alpha - \hat{b}(\varphi)\, v_\alpha\| = 0$, *for all* $b \in \mathrm{B}$.

Proof. – i) $\Rightarrow$ iv) follows by a standard compactness argument.

iv) $\Rightarrow$ iii) is trivial.

iii) $\Rightarrow$ ii) : It suffices to prove that if $\mathrm{F} = \{b_1, b_2, \ldots, b_n\}$ and if F' is a finite set of elements of the form $a = \mathrm{P}(b_1, b_2, \ldots, b_n)$, where P is a polynomial in n variables without constant term, then

$$\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}') \leqslant \mathrm{C}\,\nu_{\mathrm{T}}(\varphi\,;\mathrm{F}) \tag{3.1}$$

for some constant C (depending on F and F'). By Taylor's formula, there exist elements $d_j \in \widetilde{\mathrm{B}}$ such that

$$a - \hat{a}(\varphi) = \mathrm{P}(b_1, \ldots, b_n) - \mathrm{P}(\hat{b}_1(\varphi), \ldots, \hat{b}_n(\varphi)) = \\ = \sum_{j=1}^{n} d_j(b_j - \hat{b}_j(\varphi)) .$$

Hence

$$\|\mathrm{T}_a v - \hat{a}(\varphi)\, v\| = \|\sum_{j=1}^{n} \widetilde{\mathrm{T}}_{d_j}(\mathrm{T}_{b_j} v - \hat{b}_j(\varphi)\, v)\|$$

$$\leqslant \sum_{j=1}^{n} \|\widetilde{\mathrm{T}}_{d_j}\|\;\|\mathrm{T}_{b_j} v - \hat{b}_j(\varphi)\, v\| \leqslant \mathrm{C}(a) \cdot \sup_{b\in\mathrm{F}} \|\mathrm{T}_b v - \hat{b}(\varphi)\, v\| ,$$

with $C(a) = \sum_{j=1}^{n} \|\widetilde{T}_{d_j}\|$. It follows that (3.1) holds with $C = \max_{a \in F'} C(a)$.

ii) $\Rightarrow$ i) is obvious.

i) $\Leftrightarrow$ v) : If $\varphi \in \Lambda_3(T)$, then we can construct a net $\{v_\alpha\}_{\alpha \in \mathfrak{A}}$, with the set of all finite subsets of B as index set $\mathfrak{A}$, and such that v) holds. The converse is trivial.

Example 3.3. – Let $A_1, \ldots, A_n$ be a commuting family of bounded linear operators on a Banach space X, and let B be the Banach subalgebra of End (X) generated by $A_1, \ldots, A_n$. If T is the identity representation of B, then $\Lambda_3(T)$ coincides with the joint approximate point spectrum of $A_1, \ldots, A_n$ as defined in e.g. [3].

Example 3.4. – Let B and T be as in Example 2.3. Then $\Lambda_3(T) = E$. Thus, if for instance $E = \left\{ z \; ; \; |z| = \frac{1}{2} \right\}$, then

$$\Lambda_3(T) \neq \Lambda_2(T) \neq \Lambda_1(T) .$$

The following relation between $\Lambda_3(T)$ and $\Lambda_3(\widetilde{T})$ follows immediately from the definition of the approximate point spectrum.

PROPOSITION 3.5. – $\Lambda_3(T) = \Lambda_3(\widetilde{T}) \cap \mathfrak{M}(B)$.

4. Induced representations.

Let $T : B \to \mathrm{End}(V)$ be a bounded representation. If $I \subset \mathrm{Ker}\, T$ is a closed ideal in B, we obtain a bounded representation ${}^{I}T$ of the quotient Banach algebra B/I on V, by defining ${}^{I}T_{b+I} = T_b$. Obviously, $\|{}^{I}T_{b+I}\| = \|T_b\| \leqslant \|T\| \, \|b + I\|_{B/I}$. Identify $\mathfrak{M}(B/I)$ with the hull of I, which contains $\Lambda_1(T)$. We then have the following result, the proof of which is immediate from the relations ${}^{I}T_{b+I} = T_b$ and

$$\mathrm{Ker}\, {}^{I}T = (\mathrm{Ker}\, T)/I .$$

PROPOSITION 4.1. – $\Lambda_i({}^{I}T) = \Lambda_i(T)$ *for* $i = 1, 2, 3$.

Thus, in particular, by taking $I = \mathrm{Ker}\, T$ and passing to the quotient $B_1(T)$, if necessary, we may reduce the study of the spectrum of a representation to the case when Λ_1 coincides with the Gelfand space.

Let W be a (T-)invariant linear subspace of V, i.e. $T_b W \subset W$ for all $b \in B$. By restricting each T_b to W we obtain a representation T^W of B on W, which is called the *restriction of* T *to* W. Obviously, $\| T_b^W \| \leqslant \| T_b \|$, and hence $\Lambda_i(T^W) \subset \Lambda_i(T)$ for $i = 1, 2$, by Proposition 2.2. Clearly, this inclusion is valid for $i = 3$, too.

If W is closed and invariant, the *reduction* $T^{V/W}$ *of* T *to the quotient space* V/W is defined by

$$T_b^{V/W}(v + W) = T_b v + W .$$

Since $\| T_b^{V/W} \| \leqslant \| T_b \|$, we have $\Lambda_i(T^{V/W}) \subset \Lambda_i(T)$ for $i = 1, 2$. Simple counterexamples show that this inclusion fails when $i = 3$.

PROPOSITION 4.2. – *Let* W *be a closed invariant subspace of* V. *Then*

i) $\Lambda_1(T) = \Lambda_1(T^W) \cup \Lambda_1(T^{V/W})$

ii) $\Lambda_2(T) \supset \Lambda_2(T^W) \cup \Lambda_2(T^{V/W})$.

If T *is completely reduced by a pair of closed subspaces* W *and* X, *then*

iii) $\Lambda_3(T) = \Lambda_3(T^W) \cup \Lambda_3(T^X)$.

Proof. – All that remains to be proved is the inclusion $\subset$ in i) and iii). The first one follows from the easily verified inclusion

$$(\operatorname{Ker} T^W) \cdot (\operatorname{Ker} T^{V/W}) \subset \operatorname{Ker} T ,$$

where the left hand side denotes the ideal generated by all products ab with $a \in \operatorname{Ker} T^W$ and $b \in \operatorname{Ker} T^{V/W}$. The proof of iii) is left to the reader.

Remark. – There are examples where the inclusion in ii) is proper, even when T is completely reduced by a pair W, X. See e.g. Example 9.3.

PROPOSITION 4.3. – *Let* $b \in B$. *If* $W = T_b V$, *then*

$$\{\varphi \in \Lambda_i(T) \ ; \ \hat{b}(\varphi) \neq 0\} \subset \Lambda_i(T^W) \quad \textit{for} \quad i = 1, 2, 3 .$$

Proof. – Assume that $\varphi_0 \in \Lambda_i(T)$ and $\hat{b}(\varphi_0) \neq 0$. We consider the cases $i = 1, 2, 3$ separately.

$i = 1$: We have $\operatorname{Ker} T^W = \{a \,;\, ab \in \operatorname{Ker} T\}$. Therefore $a \in \operatorname{Ker} T^W$ implies $\hat{a}(\varphi_0)\,\hat{b}(\varphi_0) = 0$, that is $\hat{a}(\varphi_0) = 0$, and it follows that $\varphi_0 \in \Lambda_1(T^W)$.

$i = 2$: Firstly, $\hat{b}(\varphi_0) \neq 0$ implies $\| T_b \| \neq 0$. Next,

$$\| T_a^W \| = \sup_{\| T_b v \| \leqslant 1} \| T_a T_b v \| \geqslant \sup_{\| v \| \leqslant \| T_b \|^{-1}} \| T_{ab} v \| = \| T_{ab} \| \, \| T_b \|^{-1}$$

and hence, by Proposition 2.1 i),

$$| \hat{a}(\varphi_0) | = \frac{| (ab)\hat{}\,(\varphi_0) |}{| \hat{b}(\varphi_0) |} \leqslant \frac{\| T_{ab} \|}{| \hat{b}(\varphi_0) |} \leqslant \frac{\| T_b \|}{| \hat{b}(\varphi_0) |} \cdot \| T_a^W \|$$

for all $a \in B$. This shows that condition ii) of Proposition 2.1 is fulfilled with $C = \| T_b \| \cdot | \hat{b}(\varphi_0) |^{-1}$, and hence $\varphi_0 \in \Lambda_2(T^W)$.

$i = 3$: Let $\epsilon > 0$ and a finite subset $F \subset B$ be given. Put

$$C = \max_{a \in F} (1 \,,\, | \hat{a}(\varphi_0) |) \quad \text{and} \quad F' = \{ab \,;\, a \in F\} \cup \{b\} \,.$$

Since $\varphi_0 \in \Lambda_3(T)$, there exists a $v \in V$ such that $\| v \| = | \hat{b}(\varphi_0) |^{-1}$ and $\| T_c v - \hat{c}(\varphi_0)\, v \| < \epsilon / C$ for all $c \in F'$. Define $w = T_b v$. Then $w \in W$, $\| w \| \geqslant \| \hat{b}(\varphi_0)\, v \| - \epsilon/C \geqslant 1 - \epsilon$ and

$$\begin{aligned} \| T_a w - \hat{a}(\varphi_0)\, w \| &\leqslant \| T_{ab} v - \hat{a}(\varphi_0)\, \hat{b}(\varphi_0)\, v \| + \\ &\quad + | \hat{a}(\varphi_0) | \;\; \| \hat{b}(\varphi_0)\, v - T_b v \| \leqslant (1 + C)\, \epsilon / C \leqslant 2\epsilon \end{aligned}$$

for all $a \in F$. It follows that $\varphi_0 \in \Lambda_3(T^W)$.

We have considered representations T on arbitrary normed linear spaces V. However, as far as our three spectra are concerned, it is no restriction to assume that V is a Banach space, because if $\overline{V}$ is the completion of V and if $\overline{T}_b$ denotes the unique extension of T_b to $\overline{V}$, then $\overline{T}$ is a bounded representation on $\overline{V}$ with $\Lambda_i(\overline{T}) = \Lambda_i(T)$ for $i = 1, 2, 3$.

5. The regular representation and its adjoint.

Let R be the regular representation of B, defined by

$$R_b a = ba \,, \qquad \forall\, a \,,\, b \in B \,.$$

Denote the dual space of B by B^* and let L be the adjoint of R. Thus L is the representation of B on B^* defined by $L_b = R_b^*$ for all $b \in B$.

R and L are obviously bounded representations. In this section we shall study their spectra.

PROPOSITION 5.1. – $\Lambda_i(L) = \mathfrak{M}(B)$ *for* $i = 1, 2, 3$.

Proof. – It suffices to prove that $\Lambda_3(L) = \mathfrak{M}(B)$, and this is trivial, because $L_b \varphi = \hat{b}(\varphi)\varphi$ for every $\varphi \in \mathfrak{M}(B)$.

COROLLARY 5.2. – $\Lambda_i(R) = \mathfrak{M}(B)$ *for* $i = 1, 2$.

Proof. – Since $\| L_b \| = \| R_b \|$, we have $\Lambda_2(R) = \Lambda_2(L)$, by Proposition 2.2.

Remark. – Corollary 5.2 is equivalent to the inequality

$$\sup_{\varphi \in \mathfrak{M}(B)} |\hat{b}(\varphi)| \leqslant \sup_{\|a\| \leqslant 1} \|ab\| , \quad \forall\, b \in B ,$$

by Proposition 2.1. Of course, this inequality is trivial when B has a submultiplicative norm and a bounded approximate identity with bound 1. In the general case, it can also be deduced from the spectral radius formula.

COROLLARY 5.3. – *Let* I *be a closed ideal in* B. *Then*

$$\Lambda_1(R^{B/I}) = \Lambda_2(R^{B/I}) = \text{Hull}(I) .$$

Proof. – Since $I \subset \text{Ker}\, R^{B/I}$, the representation ${}^I(R^{B/I})$ of B/I on B/I is well-defined. Obviously, ${}^I(R^{B/I})$ is the regular representation of B/I. Hence, Corollary 5.3 follows from Proposition 4.1 and Corollary 5.2.

The approximate point spectrum of R coincides with a concept studied by Zelazko [19]. In [19], an ideal $I \subset B$ is said to consist of *joint topological divisors of zero* (abbreviated *j.t.d.z.*) if $\nu_R(\varphi_\infty ; F) = 0$ for all finite subsets F of I. (Recall that φ_∞ is the zero map on B). If B is a Banach algebra with unit, then $\varphi \in \Lambda_3(R)$ if and only if $\text{Ker}\,\varphi$ consists of j.t.d.z. (In the general case, we have $\varphi \in \Lambda_3(\widetilde{R})$ if and only if $\text{Ker}\,\varphi \subset \widetilde{B}$ consists of j.t.d.z). The following theorem is thus only a reformulation of the main theorem in [19].

THEOREM 5.4. – *The approximate point spectrum* $\Lambda_3(\mathrm{R})$ *of* R *contains the Shilov boundary* $\Gamma(\mathrm{B})$ *of* B.

Proof. – We shall give a proof that differs from Żelazko's proof. Let $\varphi_0 \in \Gamma(\mathrm{B})$, let $\epsilon > 0$, and let $\mathrm{F} = \{b_1, b_2, \dots, b_k\} \subset \widetilde{\mathrm{B}}$ be a finite subset of $\mathrm{Ker}\, \varphi_0$. (φ_0 is here considered as an element of $\mathfrak{M}(\widetilde{\mathrm{B}})$). We have to show that there is an element $x \in \mathrm{B}$ such that

$$\| bx \| < \epsilon \, \| x \| \quad \text{for all} \quad b \in \mathrm{F} . \tag{5.1}$$

Put

$$\mathrm{C} = \sup_{b \in \mathrm{F},\, \varphi \in \mathfrak{M}(\mathrm{B})} | \hat{b}(\varphi) |$$

and

$$\mathrm{U} = \{ \varphi \in \mathfrak{M}(\mathrm{B}) \; ; \, | \hat{b}(\varphi) | < \frac{\epsilon}{2} \, , \; \forall b \in \mathrm{F} \} .$$

Since $\varphi_0 \in \Gamma(\mathrm{B})$ and U is a neighbourhood of φ_0, there exists an element $c \in \mathrm{B}$ such that

$$\sup_{\mathfrak{M}(\mathrm{B})} | \hat{c}(\varphi) | = 2 \quad \text{and} \quad \sup_{\mathrm{C\,U}} | \hat{c}(\varphi) | < \epsilon / \mathrm{C} .$$

It follows from the spectral radius formula that

$$\lim_{n \to \infty} \| c^n \|^{1/n} = 2 ,$$

and that (see [7, Lemma 1])

$$\overline{\lim_{n \to \infty}} \sup_{n_1 + \dots + n_k = n} \| b_1^{n_1} b_2^{n_2} \cdots b_k^{n_k} c^n \|^{1/n} \leqslant \sup_{b \in \mathrm{F},\, \varphi \in \mathfrak{M}(\mathrm{B})} | \hat{b}(\varphi) \, \hat{c}(\varphi) | < \epsilon .$$

Now fix n such that

$$\| c^n \| \geqslant 1 , \tag{5.2}$$

and

$$\| b_1^{n_1} b_2^{n_2} \cdots b_k^{n_k} c^n \| < \epsilon^n \tag{5.3}$$

for every set of nonnegative integers $n_1, n_2, \dots, n_k$ with $\Sigma\, n_i = n$. It then follows from (5.2) and (5.3) that there exists an element $x \in \mathrm{B}$ of the form

$$x = b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k} c^n ,$$

where $m_i \geqslant 0$ for every i, and $\Sigma\, m_i \leqslant n - 1$, such that (5.1) holds. For details, see [5], p. 59, where the same argument has been used.

Remark. – There are examples of algebras B with identity such that $\Gamma(B) \neq \Lambda_3(R)$. See e.g. [1]. However, it is easy to see that $\Lambda_3(R) = \Gamma(B)$ when B is a sup norm algebra.

If $\mathfrak{M}(B)$ is nonempty, then the Shilov boundary is nonempty, too. Hence, Theorem 5.4 has the following corollary.

COROLLARY 5.5. – *If $\mathfrak{M}(B)$ is nonempty, then $\Lambda_3(R)$ is nonempty.*

6. Nonemptiness of the approximate point spectrum.

Our first aim is to generalize Corollary 5.5 to arbitrary representations T. To this end we shall need the following :

LEMMA 6.1. – *Let R_T denote the regular representation of $B_2(T)$. Then*

$$\Lambda_3(R_T) \subset \Lambda_3(T) .$$

Proof. – Let $\varphi \in \Lambda_3(R_T)$, $\epsilon > 0$, and a finite subset F of B be given. Since B(T) is dense in $B_2(T)$, there exists $x \in B$ such that

$$\| T_x \| = \| x_T \|_2 = 1$$

and

$$\| T_b T_x - \hat{b}(\varphi) T_x \| = \| b_T x_T - \hat{b}_T(\varphi) x_T \|_2 < \epsilon$$

for all $b \in F$. Next, choose $w \in V$ such that $\| w \| \leqslant 2$ and $\| T_x w \| = 1$. Then

$$\| T_b T_x w - \hat{b}(\varphi) T_x w \| < \epsilon \quad \| w \| \leqslant 2\epsilon$$

for all $b \in F$, and hence $\nu_T(\varphi ; F) \leqslant 2\epsilon$. Since $\epsilon > 0$ and F are arbitrary, this proves that $\varphi \in \Lambda_3(T)$.

Example 6.2. – This example shows that the inclusion in Lemma 6.1 may be proper. Let B be the disc algebra A(D), and let L be the adjoint of the regular representation. Then $B_2(L) = B$, so R_L is the regular representation of B. Hence

$$\Lambda_3(L) = D \quad \text{and} \quad \Lambda_3(R_L) = \partial D = \{z \,;\, |z| = 1\} .$$

THEOREM 6.3. – *If* $\Lambda_2(T) \neq \phi$, *then* $\Lambda_3(T) \neq \phi$. *In particular, if* T *satisfies Assumption 1, then* $\Lambda_3(T) \neq \phi$.

Proof. – Since $\Lambda_2(T) = \mathfrak{M}(B_2(T))$, the theorem follows immediately from Corollary 5.5, Lemma 6.1 and Proposition 2.5.

By Lemma 6.1 and Theorem 5.4, we have the following chain of inclusions :

$$\Gamma(B_2(T)) \subset \Lambda_3(R_T) \subset \Lambda_3(T) \subset \Lambda_2(T) \subset \Lambda_1(T) \subset \mathfrak{M}(B) .$$

In general, none of these inclusions can be replaced by equality (cf. Examples 3.4 and 6.2 and the remark preceding Corollary 5.5). However, if $B_2(T)$ is regular, then $\Gamma(B_2(T)) = \mathfrak{M}(B_2(T))$, so it follows that $\Lambda_3(T) = \Lambda_2(T)$. This relation holds, in particular, when B is regular, because the regularity is inherited by $B_2(T)$. If B is regular and semisimple, then in fact $\Lambda_1(T) = \Lambda_3(T)$. The proof of this uses the following lemma (cf. [5, Theorem 3.21] and [9]).

LEMMA 6.4. – *Let* $b \in B$ *and put* $W = T_b V$.

i) *If* b *is an idempotent modulo* Ker T, *then*

$$\Lambda_1(T^W) = \operatorname{supp} \hat{b}_T = \{\varphi \in \Lambda_1(T) \ ; \hat{b}(\varphi) = 1\} .$$

ii) *If* B *is regular and semisimple, then*

$$\Lambda_1(T^W) \subset \Lambda_1(T) \cap \operatorname{supp} \hat{b} .$$

Proof. – i) Assume that b is an idempotent modulo Ker T. If $\varphi_0 \in \Lambda_1(T)$ and $\hat{b}(\varphi_0) = 1$, then $\varphi_0 \in \Lambda_1(T^W)$, by Proposition 4.3. So assume that $\varphi_0 \in \Lambda_1(T)$ and $\hat{b}(\varphi_0) = 0$. We must show that $\varphi_0 \notin \Lambda_1(T^W)$. Choose $c \in B$ such that $\hat{c}(\varphi_0) \neq 0$ and put $a = c - cb$. Then $ab = c(b - b^2) \in \operatorname{Ker} T$ so that $a \in \operatorname{Ker} T^W$, but $\hat{a}(\varphi_0) = \hat{c}(\varphi_0) \neq 0$. Hence $\varphi_0 \notin \Lambda_1(T^W)$.

ii) Assume that B is regular and semisimple. We already know that $\Lambda_1(T^W) \subset \Lambda_1(T)$. So assume that $\varphi_0 \in \Lambda_1(T) \backslash \operatorname{supp} \hat{b}$. Then, by regularity, we can choose a such that $\hat{a}(\varphi_0) \neq 0$ and $\hat{a}(\varphi) = 0$ on $\operatorname{supp} \hat{b}$. Then $\hat{a}(\varphi)\hat{b}(\varphi) = 0$ on $\mathfrak{M}(B)$ so that $ab = 0$ by semisimplicity. It follows that $a \in \operatorname{Ker} T^W$ and, since $\hat{a}(\varphi_0) \neq 0$, we conclude that $\varphi_0 \notin \Lambda_1(T^W)$.

We can now prove the following theorem (cf. [9]).

THEOREM 6.5. – i) *If either* $\Lambda_2(T)$ *is totally disconnected or* B *is regular, then* $\Lambda_2(T) = \Lambda_3(T)$.

ii) *If either* $\Lambda_1(T)$ *is totally disconnected or* B *is regular and semisimple, then* $\Lambda_1(T) = \Lambda_3(T)$.

Remark. – We do not know whether $\Lambda_1(T) = \Lambda_3(T)$ for all regular algebras.

Proof. – The first statement has already been proved in the paragraph preceding Lemma 6.4, because the hypothesis on $\Lambda_2(T)$ (or on B) implies that $B_2(T)$ is regular. As for the second statement, Propositions 2.6 and 3.5 reduce the proof to the case when B has an identity. Then every non-zero representation of B has a nonempty approximate point spectrum by Theorem 6.3. Assume that $\varphi_0 \in \Lambda_1(T)$, and let U be an arbitrary neighbourhood of φ_0. We shall prove that $U \cap \Lambda_3(T)$ is nonempty, and, since $\Lambda_3(T)$ is closed, this implies that $\varphi_0 \in \Lambda_3(T)$. If B is regular and semisimple we use regularity to choose $b \in B$ so that $\hat{b}(\varphi_0) = 1$ and $\operatorname{supp} \hat{b} \subset U$. If instead $\Lambda_1(T)$ is totally disconnected, an application of Shilov's idempotent theorem to the algebra $B_1(T)$ gives an element $b \in B$ such that b_T is an idempotent in $B_1(T)$ with $\hat{b}(\varphi_0) = \hat{b}_T(\varphi_0) = 1$ and $\operatorname{supp} \hat{b}_T \subset U$. In both cases we set $W = T_b V$ and deduce from Lemma 6.4 that $\Lambda_1(T^W) \subset U$ and from Proposition 4.3 that $\varphi_0 \in \Lambda_1(T^W)$ so that T^W is non-zero. Hence $\Lambda_3(T^W) \neq \phi$ and, since $\Lambda_3(T^W) \subset \Lambda_1(T^W)$ and $\Lambda_3(T^W) \subset \Lambda_3(T)$, it follows that $U \cap \Lambda_3(T) \neq \phi$.

As an application of our results on the approximate point spectrum we shall now derive two extensions of theorems in [11] and [12]. Let $\tau : G \to \operatorname{Aut}(X)$ be a uniformly continuous representation of an abelian topological group G on a Banach space X, and let G^* denote the group of all continuous (unbounded) characters of G, i.e. continuous homomorphisms $G \to C\backslash\{0\}$. Following Lyubich [11], we define the *essential spectrum* $\sigma_e(\tau)$ *of* τ to be the set of all $\chi \in G^*$ for which there is a net $\{x_\alpha\}_{\alpha \in \mathfrak{A}}$ in X such that $\|x_\alpha\| = 1$, for all $\alpha \in \mathfrak{A}$, and $\lim_\alpha \|\tau_g x_\alpha - \chi(g) x_\alpha\| = 0$, for all $g \in G$. The *spectrum* $\sigma(\tau)$ *of* τ consists of those characters χ which satisfy the relation $|\Sigma_{j=1}^n \alpha_j \chi(g_j)| \leqslant \|\Sigma_{j=1}^n \alpha_j \tau_{g_j}\|$, for every choice of n ;

$$\alpha_1, \alpha_2, \ldots, \alpha_n \in C \quad \text{and} \quad g_1, g_2, \ldots, g_n \in G .$$

THEOREM 6.6 [11]. – $\sigma_e(\tau) \neq \phi$.

Proof. – The operators τ_g, $g \in G$, generate a Banach subalgebra B of End (X). For every $\varphi \in \mathfrak{M}(B)$ we obtain a character $\chi_\varphi \in G^*$ by defining $\chi_\varphi(g) = \hat{\tau}_g(\varphi)$. The map $\varphi \curvearrowright \chi_\varphi$ is a bijection of $\mathfrak{M}(B)$ onto $\sigma(\tau)$ which allows us to identify $\mathfrak{M}(B)$ with $\sigma(\tau)$. Consider the identity representation T of B on X, defined by $T_\omega = \omega$ for every $\omega \in B$. It is obvious that $\Lambda_3(T) = \sigma_e(\tau)$, and since B has an identity, it follows from Theorem 6.3 that $\sigma_e(\tau) \neq \phi$.

THEOREM 6.7 [12]. – *Assume that* $\lim\limits_{n\to\infty} \|\tau_{ng}\|^{1/n} = 1$ *for all* $g \in G$. *Then* $\sigma_e(\tau) = \sigma(\tau)$.

Proof. – Let B and T be as in the proof of Theorem 6.6. The condition on τ implies that $\mathfrak{M}(B)$ can be identified with a subset of some Cartesian product of unit circles. It follows that the Shilov boundary $\Gamma(B)$ equals $\mathfrak{M}(B)$. Since $B_2(T) = B$, Theorem 5.4 and Lemma 6.1 now yield $\mathfrak{M}(B) = \Gamma(B) \subset \Lambda_3(R_T) \subset \Lambda_3(T) = \sigma_e(\tau)$. This proves the theorem.

Remark. – Lyubich [11] proves Theorem 6.6 for separable groups only. His proof is entirely different and does not use any Banach algebra methods. However, a close look at his proof reveals that it does not use the full group structure of G but only the semi-group property. It follows that his proof can be extended to deal with representations of Banach algebras. In order to obtain extensions to the non-separable case, it seems however necessary to use Banach algebra techniques. [12] uses such techniques, but again only separable groups are considered.

7. An extension theorem.

The following extension theorem is useful for the study of the approximate point spectrum. It contains the extension lemma of Lyubich [11] and the results of Slodkowski [16] and Choi-Davis [3] as special cases. Another special case is Proposition 3.2 iii) $\Rightarrow$ i). Our proof is inspired by the proof of Lyubich [11] which uses an idea that goes back to F. Quigley [14, p. 25]. The new feature in our proof is the use of Theorem 6.3.

THEOREM 7.1. – *Let* T *be a representation of* B *satisfying Assumption I with* T_u *equal to the identity operator on* V. *Let* $E \subset B$, *let* ω *be a complex-valued function defined on* E, *and assume that* $\nu_T(\omega\,;F) = 0$ *for all finite subsets* F *of* E. *Then there exists a* $\varphi \in \Lambda_3(T)$ *such that* $\omega(b) = \hat{b}(\varphi)$ *for all* $b \in E$.

Proof. – By assumption, there exists a net $\boldsymbol{v}^0 = \{v_\alpha^0\}_{\alpha\in\mathfrak{A}}$ in V such that $\|v_\alpha^0\| = 1$ for all $\alpha \in \mathfrak{A}$, and $\lim_\alpha \|T_b v_\alpha^0 - \omega(b) v_\alpha^0\| = 0$ for all $b \in E$. Define $l^\infty(\mathfrak{A}\,;V)$ to be the normed linear space of all nets $\boldsymbol{v} = \{v_\alpha\}_{\alpha\in\mathfrak{A}}$ of elements in V with the norm

$$\|\boldsymbol{v}\| = \sup_\alpha \|v_\alpha\| < \infty \ ,$$

and let $c_0(\mathfrak{A}\,;V)$ be the closed linear subspace of all nets that converge to 0.

The representation T gives rise to a bounded representation T^∞ of B on $l^\infty(\mathfrak{A}\,;V)$, defined by $T_b^\infty(\{v_\alpha\}) = \{T_b v_\alpha\}$. Obviously, $c_0(\mathfrak{A}\,;V)$ is invariant under T^∞. Let S be the reduction of T^∞ to the quotient space $X = l^\infty(\mathfrak{A}\,;V)/c_0(\mathfrak{A}\,;V)$. We note that the norm in X is given by

$$\|\{v_\alpha\} + c_0(\mathfrak{A}\,;V)\| = \inf_{\{w_\alpha\}\in c_0} \sup_\alpha \|v_\alpha + w_\alpha\| = \limsup_\alpha \|v_\alpha\| \ ,$$

Put
$$Y = \{x \in X \ ;\ S_b x = \omega(b) x \ ,\ \forall b \in E\} \ .$$

Then Y is a non-zero linear subspace of X, because

$$x^0 = \boldsymbol{v}^0 + c_0(\mathfrak{A}\,;V) \in Y \quad \text{and} \quad \|x^0\| = 1 \ .$$

Since Y is invariant under S, the restriction S^Y is well-defined. Obviously, S_u^Y is the identity operator on Y, and hence $\Lambda_3(S^Y) \neq \phi$, by Theorem 6.3.

Let $\varphi \in \Lambda_3(S^Y)$. If $b \in E$, then

$$0 = \nu_{S^Y}(\varphi\,;\{b\}) = \inf_{x\in Y,\ \|x\|=1} \|S_b x - \hat{b}(\varphi)x\| =$$

$$= \inf_{x\in Y,\ \|x\|=1} \|\omega(b)x - \hat{b}(\varphi)x\| = |\omega(b) - \hat{b}(\varphi)| \ .$$

Hence $\omega(b) = \hat{b}(\varphi)$ for all $b \in E$. The theorem thus follows if we prove that $\varphi \in \Lambda_3(T)$. To this end, let $\epsilon > 0$ and a finite subset F of B be given. Then there exists an element $x = \{v_\alpha\} + c_0(\mathfrak{A}\,;V)$ in Y such that $\|x\| = 1$ and $\|S_b^Y x - \hat{b}(\varphi) x\| < \epsilon$ for all $b \in F$. By the definition of S^Y and of the norm in Y, this means that

$$\limsup_{\alpha} \|v_\alpha\| = 1 \quad \text{and} \quad \limsup_{\alpha} \|T_b v_\alpha - \hat{b}(\varphi) v_\alpha\| < \epsilon$$

for all $b \in F$, and it follows that $\varphi \in \Lambda_3(T)$.

Remark 1. – If we apply the theorem to the case when T is the regular representation of an algebra B with unit, E is an ideal in B, and ω is the zero function, we obtain a positive answer to a conjecture of Żelazko [19], namely that every ideal consisting of joint topological divisors of zero is contained in a maximal ideal consisting of joint topological divisors of zero. This result has also been obtained by Slodkowski [16] by a different method.

Remark 2. – Our proof of Theorem 7.1 uses Theorem 5.4. However, it is possible to prove the extension theorem without using Żelazko's theorem. Lyubich does this in a special case, and so does Slodkowski. Therefore it might be of interest to note that there is an easy way to deduce Żelazko's theorem from the extension theorem. This is shown in [10].

8. The adjoint of the regular representation.

In this section we assume that B is normed so that

$$\|ab\| \leqslant \|a\| \, \|b\| \, .$$

We denote the closed ball in B^* of radius ρ and centered at 0 by S_ρ.

Let L denote the adjoint of the regular representation of B, and let V be an L-invariant subspace of B^*. The aim of this section is to give an alternative characterization of the spectrum and the narrow spectrum of L^V. In order to get a simpler notation we shall write $\Lambda_i(V)$ instead of $\Lambda_i(L^V)$, and we shall also speak of $\Lambda_i(V)$ as the spectrum (narrow, approximate point spectrum) of V. The spectrum $\Lambda_i(\nu)$ of an element $\nu \in B^*$ is defined to be the corresponding spectrum of the subspace $\{L_b \nu \; ; b \in B\}$.

THEOREM 8.1. – i) $\Lambda_1(V) = \overline{V} \cap \mathfrak{M}(B)$.

ii) $\Lambda_2(V) = \overline{V \cap S_\rho} \cap \mathfrak{M}(B)$ *for all* $\rho \geqslant 1$.

(The closure bar refers to the weak-* topology).

Proof. – i) Denote the annihilator of Ker L^V in B^* by $(\text{Ker } L^V)^\perp$. If $\nu_0 \notin \overline{V}$, then there exists a $b \in B$ such that $\langle \nu , b \rangle = 0$ for all $\nu \in V$ whereas $\langle \nu_0 , b \rangle \neq 0$. It follows that $b \in \text{Ker } L^V$, because

$$\langle L_b \nu , a \rangle = \langle L_a \nu , b \rangle = 0$$

for all $a \in B$ and $\nu \in V$. Hence $\nu_0 \notin (\text{Ker } L^V)^\perp$, and this proves that $(\text{Ker } L^V)^\perp \subset \overline{V}$. In particular, $\Lambda_1(V) = (\text{Ker } L^V)^\perp \cap \mathfrak{M}(B) \subset \overline{V} \cap \mathfrak{M}(B)$.

To prove the converse inclusion, assume that $\nu_0 \in \overline{V}$ and $b \in \text{Ker } L^V$. Then $\langle \nu_0 , ab \rangle = 0$ for all $a \in B$, because $\langle \nu , ab \rangle = \langle L_b \nu , a \rangle = 0$ for all $a \in B$ and $\nu \in V$. If in addition $\nu_0 \in \mathfrak{M}(B)$, then $\hat{a}(\nu_0)\, \hat{b}(\nu_0) = 0$ for all $a \in B$, and it follows that $\hat{b}(\nu_0) = 0$. This shows that

$$\overline{V} \cap \mathfrak{M}(B) \subset (\text{Ker } L^V)^\perp \cap \mathfrak{M}(B) = \Lambda_1(V) \ ,$$

and the proof of i) is complete.

The proof above also shows that $\overline{V} = (\text{Ker } L^V)^\perp$, if b lies in the closure of $b \cdot B$ for each $b \in B$ (e.g. if B has an approximate identity), because then $\langle \nu_0 , ab \rangle = 0$ for all a implies that $\langle \nu_0 , b \rangle = 0$, without any assumptions on ν_0. Thus, in that case $B_1(L^V)^*$ is isomorphic to $\overline{V}$.

ii) $V \cap S_\rho$ is a convex, balanced set with compact closure, so it follows from the Hahn-Banach theorem that a functional $\nu_0 \in B^*$ belongs to $\overline{V \cap S_\rho}$ if and only if ν_0 and $V \cap S_\rho$ are not separated by any hyperplane. Since the continuous functionals on B^* (with the weak-* topology) are of the form $\nu \curvearrowright \langle \nu , b \rangle$ with $b \in B$, we conclude that

$$\nu_0 \in \overline{V \cap S_\rho} \Leftrightarrow \forall b \in B : |\langle \nu_0 , b \rangle| \leqslant \rho \cdot \sup_{\nu \in V \cap S_1} |\langle \nu , b \rangle| . \quad (8.1)$$

Since $L_a V \subset V$ and $\| L_a \| \leqslant \| a \|$, we obtain the following estimate for the right hand side of the inequality in (8.1) :

$$\sup_{\nu \in V \cap S_1} |\langle \nu , b \rangle| \geqslant \sup_{\nu \in V \cap S_1 , \|a\| \leqslant 1} |\langle L_a \nu , b \rangle| =$$

$$= \sup_{\nu \in V \cap S_1 , \|a\| \leqslant 1} |\langle L_b \nu , a \rangle| = \| L_b^V \| . \quad (8.2)$$

In the opposite direction we have trivially

$$\sup_{\nu \in V \cap S_1} |\langle \nu, ab \rangle| = \sup_{\nu \in V \cap S_1} |\langle L_b \nu, a \rangle| \leqslant \|a\| \; \|L_b^V\| . \qquad (8.3)$$

It follows immediately from (8.1), (8.2) and Proposition 2.1 i) that $\Lambda_2(V) \subset \overline{V \cap S_1} \cap \mathfrak{M}(B)$, so to complete the proof of ii) it suffices to prove that $\overline{V \cap S_\rho} \cap \mathfrak{M}(B) \subset \Lambda_2(V)$. To this end, assume that $\varphi \in \overline{V \cap S_\rho} \cap \mathfrak{M}(B)$ and choose $a \in B$ such that $\hat{a}(\varphi) = 1$. Then, by (8.1) and (8.3),

$$|\hat{b}(\varphi)| = |\langle \varphi, ab \rangle| \leqslant \rho \; \|a\| \; \|L_b^V\|$$

for all $b \in B$, and Proposition 2.1 ii) now shows that $\varphi \in \Lambda_2(V)$.

Remarks. – In [7] the narrow spectrum of V was defined as the set $\overline{V \cap S_1} \cap \mathfrak{M}(B)$, and the main problem was to show that $\overline{V \cap S_1} \cap \mathfrak{M}(B)$ is nonempty, whenever B is an algebra with identity and $V \neq \{0\}$. Only partial results in that direction were obtained in [6] and [8]. Theorem 8.1 together with Proposition 2.5 now solve the problem completely.

Part ii) of Theorem 8.1 implies that every homomorphism in $\mathfrak{M}(B)$ which is the weak-* limit of a bounded filter in V is the weak-* limit of a filter in V with bound one. This result is somewhat surprising in view of the results for the related problem of bounded spectral synthesis (cf. Varopoulos [17]).

Theorem 8.1 has connections with the work of Dixmier [4] on the characteristic of weak-* dense subspaces of dual Banach spaces. If W is a weak-* dense linear subspace of the dual B^* of an arbitrary Banach space B, then Dixmier defines the *characteristic* $ch(W)$ *of* W as

$$ch(W) = \inf_{b \in B, \|b\| = 1} \; \sup_{\nu \in W \cap S_1} |\langle \nu, b \rangle| ,$$

and he shows that $ch(W) > 0$ if and only if $\bigcup_{\rho > 0} \overline{W \cap S_\rho} = B^*$.

Let V be an L-invariant subspace of the dual of a Banach algebra B with a 1-bounded approximate identity. If we apply Dixmier's definition to V, considered as a dense subspace of $B_1(L^V)^*$, we obtain

$$ch(V) = \inf_{b_{L^V} \neq 0} \frac{\|L_b^V\|}{\|b_{L^V}\|_1} = \inf_{b_{L^V} \neq 0} \frac{\|b_{L^V}\|_2}{\|b_{L^V}\|_1} .$$

If $ch(V) > 0$, then the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $B(L^V)$ are equivalent, and it follows that $\Lambda_1(V) = \Lambda_2(V)$. Consequently, if $\Lambda_1(V) \neq \Lambda_2(V)$, then we have an example of a subspace V of characteristic 0. However, the condition $\Lambda_1(V) \neq \Lambda_2(V)$ is not necessary for V to be of characteristic 0. Warner [18] gives an example of a weak-* dense subspace H of $L^\infty(\mathbf{R})$ with $ch(H) = 0$. This subspace happens to be L-invariant so we can define the representation L^H of $L^1(\mathbf{R})$. It follows from Theorem 6.5 that $\Lambda_1(H) = \Lambda_2(H)$.

As an application of Theorem 8.1 we shall give a concrete interpretation of $\Lambda_i(V)$ for the Beurling algebras. We start by collecting some basic definitions and facts about these algebras.

Let $\mathcal{G}$ be a locally compact abelian group, and denote a Haar measure by m. Let $\mathcal{S} \subset \mathcal{G}$ be a closed semigroup containing the identity 0 of $\mathcal{G}$, and assume that $\mathcal{S}$ is the closure of its interior. Let p be a positive, m-measurable, submultiplicative function on $\mathcal{S}$, i.e. satisfying $p(x + y) \leqslant p(x)\, p(y)$ for all $x, y \in \mathcal{S}$, and assume that p and $1/p$ are bounded on every compact set. Assume further that $p(0) = 1$ and that p is continuous at 0. (This last assumption is not essential. It is used only in order to get the constants in Conditions (2) and (3) below equal to 1). The Banach space of all m-measurable functions f on $\mathcal{S}$ such that

$$\|f\| = \int_{\mathcal{S}} |f(x)|\, p(x)\, dm(x) < \infty$$

is denoted by $L_p^1(\mathcal{S})$. Let for each $y \in \mathcal{S}$ and $f \in L_p^1(\mathcal{S})$ the translate f_y be defined by

$$f_y(x) = \begin{cases} f(x-y) & \text{if} \quad x \in y + \mathcal{S} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_y \in L_p^1(\mathcal{S})$, $\|f_y\| \leqslant p(y)\, \|f\|$, and the map $\mathcal{S} \to L_p^1(\mathcal{S})$, $y \rightsquigarrow f_y$ is continuous. $L_p^1(\mathcal{S})$ is a Banach algebra with 1-bounded approximate identity under convolution *, defined by

$$f * g = \int_{\mathcal{S}} g(x)\, f_x\, dm(x)\ ,$$

and $\|f * g\| \leqslant \|f\|\, \|g\|$. The dual $L_p^1(\mathcal{S})^*$ is isometrically isomorphic to the Banach space $L_p^\infty(\mathcal{S})$ of all m-measurable functions F on $\mathcal{S}$ satisfying

$$\| F \| = \text{ess sup } \frac{|F(x)|}{p(x)} < \infty .$$

The duality between $L_p^\infty(\mathfrak{S})$ and $L_p^1(\mathfrak{S})$ is given by

$$\langle F , f \rangle = \int_{\mathfrak{S}} F(x) f(x) \, dm(x) .$$

For $F \in L_p^\infty(\mathfrak{S})$ and $y \in \mathfrak{S}$ we define the translate F_y by the duality relation $\langle F_y , f \rangle = \langle F , f_y \rangle$. Then

$$F_y(x) = F(y + x) \quad \text{and} \quad \| F_y \| \leqslant p(y) \| F \| .$$

The Gelfand space $\mathfrak{M}(L_p^1(\mathfrak{S}))$ consists of all multiplicative functions χ in $L_p^\infty(\mathfrak{S})$, i.e. all $\chi \in L_p^\infty(\mathfrak{S})$ satisfying $\chi(x + y) = \chi(x) \chi(y)$. Note that $\hat{f}_y(\chi) = \chi(y) \hat{f}(\chi)$.

The representation L of $L_p^1(\mathfrak{S})$ on $L_p^\infty(\mathfrak{S})$ is given by

$$(L_f F)(x) = \langle F , f_x \rangle , \quad f \in L_p^1(\mathfrak{S}) , \ F \in L_p^\infty(\mathfrak{S}) , \ x \in \mathfrak{S} .$$

Note that $L_f F$ is continuous, that $\| L_f F \| \leqslant \| F \| \, \| f \|$, and that $(L_f F)_y = L_f F_y = L_{f_y} F$.

Let V be an L-invariant subspace of $L_p^\infty(\mathfrak{S})$. For $\chi \in \mathfrak{M}(L_p^1(\mathfrak{S}))$ we introduce the following three conditions :

Condition (1). – For every $\epsilon > 0$ *and every nonnegative, m-measurable, locally bounded function* ω *on* $\mathfrak{S}$ *such that* $\lim_{x \to \infty} \omega(x) p(x) = 0$, *there exists a function* $F \in V$ *such that* $\text{ess sup}_{x \in \mathfrak{S}} |F(x) - \chi(x)| \, \omega(x) < \epsilon$.

Condition (2). – For every $\epsilon > 0$ *and every compact subset* K *of* $\mathfrak{S}$, *there exists a continuous function* $F \in V$ *such that* $\| F \| \leqslant 1$ *and* $\sup_{x \in K} |F(x) - \chi(x)| < \epsilon$.

Condition (3). – For every $\epsilon > 0$ *and every compact subset* K *of* $\mathfrak{S}$, *there exists a continuous function* $F \in V$ *such that* $\| F \| \leqslant 1$, $F(0) > 1 - \epsilon$ *and* $\sup_{x \in K} \| F_x - \chi(x) F \| < \epsilon$.

Obviously, Condition (3) implies Condition (2), and Condition (2) implies Condition (1). The different spectra of V are characterized in terms of Conditions (1) – (3) by the following theorem.

THEOREM 8.2. – $\chi \in \Lambda_i(\mathrm{V})$ *if and only if* χ *satisfies Condition (i),* $i = 1, 2, 3$.

Proof. – The proof of the sufficiency of Condition (i) is rather obvious in all three cases and is omitted. So let us prove the necessity.

$i = 1$: The result and the idea of the proof goes back to Beurling, and special cases have been used by several authors (e.g. [13, p. 25]). Since the proof does not seem to be available in the literature, we give it here. Let $\chi \in \Lambda_1(\mathrm{V})$ and let ω be given as in Condition (1). The space A_ω of all m-measurable functions f on $\mathfrak{S}$ such that

$$\|f\|_\omega = \operatorname*{ess\,sup}_{x \in \mathfrak{S}} |f(x)|\, \omega(x) < \infty \quad \text{and} \quad \lim_{x \to \infty} f(x)\, \omega(x) = 0$$

is a Banach space under the $\|\cdot\|_\omega$-norm. (Of course, functions f and g with $\|f - g\|_\omega = 0$ have been identified). Note that $\mathrm{L}_p^\infty(\mathfrak{S}) \subset \mathrm{A}_\omega$ and that the inclusion map is continuous.

In order to show that Condition (1) is fulfilled for χ we have to show that χ belongs to the $\|\cdot\|_\omega$-closure of V. Arguing by contradiction, we assume the contrary. Then, by the Hahn-Banach theorem, there is a $\mu_0 \in \mathrm{A}_\omega^*$ such that $\langle \mu_0 , \mathrm{F} \rangle = 0$ for all $\mathrm{F} \in \mathrm{V}$ whereas $\langle \mu_0 , \chi \rangle = 1$. Choose a continuous function $f \in \mathrm{L}_p^1(\mathfrak{S})$, supported by a compact subset of the interior of $\mathfrak{S}$, and such that $\hat{f}(\chi) = 1$. Since $\mathrm{F} \curvearrowright \langle \mu_0 , \mathrm{L}_f \mathrm{F} \rangle$ is a continuous linear functional on $\mathrm{L}_p^\infty(\mathfrak{S})$, there exists an element $\nu \in \mathrm{L}_p^\infty(\mathfrak{S})^*$ such that $\langle \nu , \mathrm{F} \rangle = \langle \mu_0 , \mathrm{L}_f \mathrm{F} \rangle$ for all $\mathrm{F} \in \mathrm{L}_p^\infty(\mathfrak{S})$. We shall prove that ν can be identified with an element g of $\mathrm{L}_p^1(\mathfrak{S})$, i.e. that

$$\langle \nu , \mathrm{F} \rangle = \langle \mathrm{F} , g \rangle = \int_{\mathfrak{S}} \mathrm{F}(x)\, g(x)\, dm(x) .$$

But then $\langle \mathrm{F} , g \rangle = \langle \mu_0 , \mathrm{L}_f \mathrm{F} \rangle = 0$ for all $\mathrm{F} \in \mathrm{V}$, and

$$\hat{g}(\chi) = \langle \mu_0 , \mathrm{L}_f \chi \rangle = \hat{f}(\chi) \langle \mu_0 , \chi \rangle = 1 ,$$

which contradicts the fact that $\chi \in \Lambda_1(\mathrm{V}) = \overline{\mathrm{V}} \cap \mathfrak{M}(\mathrm{L}_p^1(\mathfrak{S}))$.

The rest of the proof thus consists in showing that $\nu \in \mathrm{L}_p^1(\mathfrak{S})$. To this end, put $f^y(x) = f_x(y)$, $x, y \in \mathfrak{S}$. Each f^y is a continuous function on $\mathfrak{S}$ with compact support, and the map $\mathfrak{S} \to \mathrm{A}_\omega, y \curvearrowright f^y$ is continuous. It follows that the vector-valued integral $\int_{\mathfrak{S}} \mathrm{F}(y)\, f^y\, dm(y)$

exists and defines an element of A_ω for each $F \in L_p^\infty(\mathfrak{S})$ with compact support. An easy calculation shows that $\int_{\mathfrak{S}} F(y) f^y \, dm(y) = L_f F$. Put $g(y) = \langle \mu_0, f^y \rangle$. Then g is a continuous function on $\mathfrak{S}$, and

$$\langle \nu, F \rangle = \langle \mu_0, L_f F \rangle = \int_{\mathfrak{S}} F(y) \langle \mu_0, f^y \rangle \, dm(y) =$$

$$= \int_{\mathfrak{S}} F(y) \, g(y) \, dm(y)$$

for each $F \in L_p^\infty(\mathfrak{S})$ with compact support. It follows that

$$\left| \int_{\mathfrak{S}} F(y) \, g(y) \, dm(y) \right| \leqslant \| \nu \| \, \| F \|_{L_p^\infty} .$$

We conclude that $g \in L_p^1(\mathfrak{S})$ and that the linear functional $\tau = \nu - g$ on $L_p^\infty(\mathfrak{S})$ annihilates the subspace of functions with compact support. We shall finish the proof by showing that $\tau = 0$. Let $G \in L_p^\infty(\mathfrak{S})$ and $\epsilon > 0$ be given. Choose a compact subset K of $\mathfrak{S}$ such that $p(x) \, \omega(x) < \epsilon / (\| \mu_0 \| \, \| G \| \, \| f \|)$ outside K and

$$\int_{CK} | G(x) \, g(x) | \, dm(x) < \epsilon ,$$

and define $G_K(x) = \begin{cases} 0 & \text{if } x \in K \cup (K + \operatorname{supp} f) , \\ G(x) & \text{otherwise.} \end{cases}$

Then $$| \langle G_K, g \rangle | = \left| \int_{\mathfrak{S}} G_K(x) \, g(x) \, dm(x) \right| < \epsilon .$$

From $(L_f G_K)(x) = \langle G_K, f_x \rangle$, it follows that

$$| (L_f G_K)(x) | \leqslant \| G_K \| \, \| f_x \| \leqslant \| G \| \, \| f \| \, p(x) \quad \text{for all} \quad x \in \mathfrak{S}$$

and that

$$(L_f G_K)(x) = \int_{x + \operatorname{supp} f} G_K(y) \, f_x(y) \, dm(y) = 0 \quad \text{for all} \quad x \in K .$$

Hence

$$\| L_f G_K \|_\omega = \operatorname*{ess\,sup}_{x \notin K} | (L_f G_K)(x) | \, \omega(x) < \epsilon / \| \mu_0 \| ,$$

so that

$$| \langle \nu, G_K \rangle | = | \langle \mu_0, L_f G_K \rangle | \leqslant \| \mu_0 \| \, \| L_f G_K \|_\omega < \epsilon .$$

Since $G - G_K$ has compact support,

$$\langle \tau , G\rangle = \langle \tau , G_K\rangle = \langle \nu , G_K\rangle - \langle G_K , g\rangle .$$

Hence

$$|\langle \tau , G\rangle| \leqslant |\langle \nu , G_K\rangle| + |\langle G_K , g\rangle| < 2\epsilon ,$$

and, $\epsilon > 0$ being arbitrary, this shows that $\langle \tau , G\rangle = 0$, i.e. $\tau = 0$. The proof of the case $i = 1$ is complete.

$i = 2$: Let $\chi \in \Lambda_2(V)$, $\epsilon > 0$ and K, a compact subset of $\mathfrak{S}$, be given. Since $L_p^1(\mathfrak{S})$ has an approximate identity, bounded in norm by one, we can choose $g \in L_p^1(\mathfrak{S})$ such that $\hat{g}(\chi) = 1$ and $\|g\| \leqslant 1 + \epsilon$. It follows from Theorem 8.1 ii) and an easy covering argument that, if C is an arbitrary compact subset of $L_p^1(\mathfrak{S})$, then there exists a $G \in V$ such that $\|G\| \leqslant 1$ and $|\langle G , f\rangle - \hat{f}(\chi)| < \epsilon$, for all $f \in C$. In particular, since $\{g_x ; x \in K\}$ is a compact subset of $L_p^1(\mathfrak{S})$, there is a $G \in V$ with $\|G\| \leqslant 1$ satisfying

$$|\langle G , g_x\rangle - \hat{g}_x(\chi)| < \epsilon ,$$

for all $x \in K$. But $\langle G , g_x\rangle = (L_g G)(x)$ and $\hat{g}_x(\chi) = \hat{g}(\chi)\, \chi(x) = \chi(x)$. With $F = L_g G \in V$ we thus have

$$\|F\| \leqslant \|g\| \, \|G\| \leqslant 1 + \epsilon \quad \text{and} \quad \sup_{x \in K} |F(x) - \chi(x)| < \epsilon .$$

It follows that Condition (2) is satisfied by χ.

$i = 3$: Let $\chi \in \Lambda_3(V)$, $\epsilon > 0$ and K be given, and choose g such that $\hat{g}(\chi) = 1$ and $\|g\| \leqslant 1 + \epsilon$. By Proposition 3.2 iv), there exists $G \in V$ such that $\|G\| = 1$ and

$$\|L_{g_x} G - \chi(x)\, G\| = \|L_{g_x} G - \hat{g}_x(\chi)\, G\| < \epsilon , \quad \forall x \in K \cup \{0\} .$$

Since $\|G\| = 1$ and $\|L_g G - G\| < \epsilon$, we can choose $y \in \mathfrak{S}$ such that

$$|G(y)| > (1 - \epsilon)\, p(y) \quad \text{and} \quad |(L_g G)(y) - G(y)| < \epsilon\, p(y) .$$

Put $F = p(y)^{-1} (L_g G)_y$. Then :

a) $F \in V$, because $F = p(y)^{-1} L_{g_y} G$.

b) $\|F\| \leqslant \|L_g G\| \leqslant \|g\| \, \|G\| \leqslant 1 + \epsilon$.

c) $|F(0)| = p(y)^{-1} |(L_g G)(y)| \geqslant p(y)^{-1} |G(y)| - \epsilon \geqslant 1 - 2\epsilon$.

d) $F_x - \chi(x)\, F = p(y)^{-1}((L_{g_x} G - \chi(x)\, G) + \chi(x)\, (G - L_g G))_y$.

Hence

$$\|F_x - \chi(x)F\| \leqslant \|L_{g_x}G - \chi(x)G\| + |\chi(x)|\,\|G - L_g G\| <$$
$$< \epsilon(1 + p(x)) \ , \ \forall x \in K \ .$$

It follows from this that χ satisfies Condition (3), and the proof of Theorem 8.2 is complete.

If $\mathcal{G}$ is discrete, then $L_p^1(\mathcal{S})$ has an identity, so it follows that $\Lambda_3(V) \neq \phi$ for all nontrivial $V \subset L_p^\infty(\mathcal{S})$. If $\mathcal{G}$ is non-discrete, this need not be true any longer. Nyman [13] gives examples of weight functions p and of L-invariant subspaces V of $L_p^\infty(\mathbf{R})$ and $L_p^\infty(\mathbf{R}^+)$ with empty spectrum $\Lambda_1(V)$. However, if $\mathcal{S} = \mathcal{G}$ and the weight function p satisfies the following two conditions

$$p(x) \geqslant 1 \ , \ \forall x \in \mathcal{G} \ , \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\log p(nx)}{n^2} < \infty \ , \ \forall x \in \mathcal{G} \ ,$$

then, by a theorem of Domar [5], $L_p^1(\mathcal{G})$ is regular and semisimple, and every closed proper ideal of $L_p^1(\mathcal{G})$ is included in the kernel of some $\chi \in \mathfrak{M}(L_p^1(\mathcal{G}))$. Thus in this case, by Theorem 6.5 and the definition of the spectrum, $\Lambda_3(V) = \Lambda_1(V) \neq \phi$ for all L-invariant, nontrivial $V \subset L_p^\infty(\mathcal{G})$. In particular, there exists a character χ satisfying Condition (3). This was first proved in [5], and for $\mathcal{G} = \mathbf{R}$ and $p(x) \equiv 1$ by Beurling in [2], though the results of these papers were only stated in terms of Λ_2 and Condition (2).

9. Counterexamples.

In this final section we shall show that there exist Banach algebras B and L-invariant subspaces $V \subset B^*$ such that $\Lambda_1(V) \neq \Lambda_2(V) \neq \Lambda_3(V)$.

Let B be either the disc algebra A(D) (cf. Example 2.3) or the algebra $A^+(D)$ of all analytic functions $f(z) = \Sigma_0^\infty a_n z^n$ such that $\|f\| = \Sigma_0^\infty |a_n| < \infty$. (Of course, $A^+(D)$ is isomorphic to the algebra $L_p^1(\mathcal{S})$ with $\mathcal{S} = \mathbf{Z}^+$ and $p(x) \equiv 1$). In both cases, $\mathfrak{M}(B) = D$, and we shall write φ_z for the multiplicative functional that corresponds to $z \in D$.

Let $C = \{z_1, z_2, z_3, \ldots\}$ be a countable subset of ∂D and denote its closure by E. When $B = A^+(D)$ we furthermore assume that C is independent. Let $V = V(C)$ be the linear subspace of all $\mu \in B^*$ such that

$$\mu = \sum_{j=1}^{\infty} \alpha_j \varphi_{z_j} \quad \text{with} \quad \sum_{j=1}^{\infty} |\alpha_j| < \infty .$$

Since $L_f \mu = \Sigma_1^\infty \alpha_j f(z_j) \varphi_{z_j}$, V is invariant under L. The spectra of L^V are characterized in terms of E by the following theorem (cf. [9]).

THEOREM 9.1. – i) *If* E *is a* B-*zero set, then*

$$\Lambda_1(V) = \Lambda_2(V) = \Lambda_3(V) = E .$$

ii) *If* E *is not a* B-*zero set and* $E \neq \partial D$, *then* $\Lambda_1(V) = D$ *and* $\Lambda_2(V) = \Lambda_3(V) = E$.

iii) *If* $E = \partial D$, *then* $\Lambda_1(V) = \Lambda_2(V) = D$ *and* $\Lambda_3(V) = E$.

We recall that E is said to be a B-*zero set* if there is a nontrivial $f \in B$ that vanishes on E. It is well-known that E is an A(D)-zero set, if and only if the linear Lebesgue measure $m(E)$ is zero. A fortiori, if E is an $A^+(D)$-zero set, then $m(E) = 0$.

Proof. – We begin by proving that if $\mu = \Sigma_1^\infty \alpha_j \varphi_{z_j}$, then

$$\|\mu\| = \sum_{j=1}^{\infty} |\alpha_j| . \tag{9.1}$$

Let $H = \{z_1, z_2, \ldots, z_n\}$. When $B = A(D)$, we apply a wellknown theorem of Carleson and Rudin (see e.g. [15]) and conclude that, given $f \in C(H)$, there exists a $g \in B$ such that

$$g_{|H} = f \quad \text{and} \quad \|g\|_B = \|f\|_{C(H)} .$$

When $B = A^+(D)$, a similar conclusion follows from Kronecker's theorem, H being independent. In both cases, we obtain

$$\sum_{j=1}^{\infty} |\alpha_j| \geqslant \|\mu\| = \sup_{f \in B, \|f\| \leqslant 1} \left| \sum_{j=1}^{\infty} \alpha_j f(z_j) \right| \geqslant$$

$$\geqslant \sup_{f \in B, \|f\| \leqslant 1} \left| \sum_{j=1}^{n} \alpha_j f(z_j) \right| - \sum_{j=n+1}^{\infty} |\alpha_j| =$$

$$= \sup_{f\in C(H),\, \|f\|_{C(H)} \leq 1} \left| \sum_{j=1}^{n} \alpha_j f(z_j) \right| - \sum_{j=n+1}^{\infty} |\alpha_j| =$$

$$= \sum_{j=1}^{n} |\alpha_j| - \sum_{j=n+1}^{\infty} |\alpha_j| ,$$

and (9.1) follows by letting n tend to infinity.

From (9.1) we now obtain the following two equalities :

$$\| L_f^V \| = \sup_{\mu\in V, \|\mu\| \leq 1} \| L_f \mu \| = \sup_{\Sigma|\alpha_j| \leq 1} \sum_{j=1}^{\infty} |\alpha_j f(z_j)| = \sup_{z\in E} |f(z)| \tag{9.2}$$

$$\nu_{L^V}(\varphi_{z_0} ; \{z \curvearrowright z\}) = \inf_{\mu\in V, \|\mu\| = 1} \| L_z \mu - z_0 \mu \| =$$

$$= \inf_{\Sigma |\alpha_j| = 1} \sum_{j=1}^{\infty} |\alpha_j (z_j - z_0)| = \operatorname{dist}(z_0 , E) . \tag{9.3}$$

By (9.2) and Proposition 2.1, $\Lambda_2(V) = \hat{E}$, the polynomially convex hull of E. Since the function $z \curvearrowright z$ generates B,

$$\Lambda_3(V) = \{z ; \operatorname{dist}(z , E) = 0\} = E ,$$

by (9.3) and Proposition 3.2 iii). This proves the theorem for $\Lambda_2(V)$ and $\Lambda_3(V)$. To prove the theorem for $\Lambda_1(V)$, we first note that, since $\operatorname{Ker} L^V = \{f \in B ; f_{|E} = 0\}$, $\Lambda_1(V) = D$ if and only if E is not a B-zero set. So assume that $\Lambda_1(V) \neq D$. Then $\Lambda_1(V)$ is a B-zero set, and it follows that $\Lambda_1(V)$ must be totally disconnected. By Theorem 6.5 and by what we have already proved, we conclude that $\Lambda_1(V) = \Lambda_3(V) = E$.

Remarks. – If $E \neq \partial D$ is not a B-zero set, then, since $\Lambda_1(V) \neq \Lambda_2(V)$ and $\operatorname{Ker} L^V = \{0\}$, V is a weak-* dense subspace of B* of characteristic zero (cf. [4], [18]).

If $\mu = \Sigma_1^\infty \alpha_j \varphi_{z_j} \in V$ and $\alpha_j \neq 0$ for all j, then the norm closure of $\{L_f \mu ; f \in B\}$ equals V, so it follows that $\Lambda_i(\mu) = \Lambda_i(V)$. Hence Theorem 9.1 gives us examples of elements in B* with different spectrum and narrow spectrum and also of elements with different narrow and approximate point spectrum (cf. [9]).

Nyman [13] and Gurarii [8] give a characterization of $\Lambda_1(\nu)$, $\nu \in A^+(D)^*$, in terms of a certain analytic transform of ν, and from this characterization it is easy to deduce the result of Theorem 9.1 concerning $\Lambda_1(V)$.

The existence of Banach algebras B and L-invariant subspaces $V \subset B^*$ such that $\Lambda_1(V) \neq \Lambda_2(V) \neq \Lambda_3(V)$ now follows from Theorem 9.1 ii) and iii) and the following easily proved proposition.

PROPOSITION 9.2. – *Let* B *be the normed direct sum* $B_1 \oplus B_2$ *of two Banach algebras* B_1 *and* B_2, *and identify* $\mathfrak{M}(B)$ *with the disjoint union of* $\mathfrak{M}(B_1)$ *and* $\mathfrak{M}(B_2)$. *Let* $V_1 \subset B_1^*$ *and* $V_2 \subset B_2^*$ *be* L-*invariant subspaces and put* $V = V_1 \oplus V_2 \subset B^*$. *Then* V *is* L-*invariant, and* $\Lambda_i(V)$ *is the disjoint union of* $\Lambda_i(V_1)$ *and* $\Lambda_i(V_2)$ *for* $i = 1, 2, 3$.

We can also use Theorem 9.1 to obtain an example with $\Lambda_2(S \oplus T) \neq \Lambda_2(S) \cup \Lambda_2(T)$ (cf. Proposition 4.2 ii)).

Example 9.3. – Let E_1 and E_2 be two closed proper subsets of ∂D such that $E_1 \cup E_2 = \partial D$, let C_1 and C_2 be countable dense subsets of E_1 and E_2, respectively, and define the corresponding subspaces $V_1 = V(C_1)$ and $V_2 = V(C_2)$ of $A(D)^*$ as in Theorem 9.1. Put $S = L^{V_1}$ and $T = L^{V_2}$. Then $\Lambda_2(S) \cup \Lambda_2(T) = \partial D$ and, by Proposition 4.2 ii), $\Lambda_2(S \oplus T) \supset \partial D$. Since ∂D is the Shilov boundary of A(D), it follows from Proposition 2.1 that $\Lambda_2(S \oplus T) = D$. Thus

$$\Lambda_2(S \oplus T) \neq \Lambda_2(S) \cup \Lambda_2(T)$$

in this case.

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