ANNALES DE L'INSTITUT FOURIER

CHARLES FEFFERMAN

Convergence on almost every line for functions with gradient in $L^p(\mathbf{R}^n)$

Annales de l'institut Fourier, tome 24, nº 3 (1974), p. 159-164
<http://www.numdam.org/item?id=AIF_1974__24_3_159_0>

Ann. Inst. Fourier, Grenoble
24,3 (1974), 153-164

CONVERGENCE ON ALMOST EVERY LINE FOR FUNCTIONS WITH GRADIENT IN $L^p(R^n)$

by Charles FEFFERMAN

This note answers a question asked by L.D. Kudrjačev ([1], p. 264, problem 1). The question was suggested by the following result of Uspenskiĭ [2] : if $u(x)$ is a smooth function $\mathbf{R}^n \to \mathbf{R}$ and $\int_{\mathbf{R}^n} |\operatorname{grad} u|^p \, dx < \infty$ $(1 < p < n)$, there exists a constant c such that $\lim_{r\to\infty} u(rx') = c$ for almost every $x' \in S^{n-1}$. Professor Kudrjačev kindly informed me that V. Portnov answered his question independently by another method [3].

We use now the notation

$$x = (x_1, x'), \quad \text{where} \quad x_1 \in \mathbf{R} \quad \text{and} \quad x' \in \mathbf{R}^{n-1}.$$

THEOREM. – *Let $u(x_1, x')$ be a smooth function* : $R^n \to R$ *and suppose* $\int_{R^n} |\operatorname{grad} u|^p \, dx < \infty$ $(1 < p < n)$. *Then for a constant* c, $\lim_{x_1\to\infty} u(x_1, x') = c$ *for almost all* x'.

Prof. – Set $u_j = \dfrac{\partial u}{\partial x_j}$, let R_j denote j^{th} Riesz transform, and let I^1 denote fractional integration of first order in R^n.

We begin with the standard formula

$$\sum_i R_i R_j v_j = v_i + \sum_j R_j I^1 \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right),$$

valid for C^∞ functions of compact support. (To check this, just take Fourier transforms). Take a function φ_N on R^n equal to one for $|x| \leqslant 2^N$, supported in $|x| \leqslant 2^{N+1}$, and satisfying $|\nabla\varphi_N| \leqslant C2^{-N}$; and apply the above formula to the functions $v_j = \varphi_N u_j \in C_0^\infty$. As

$N \to \infty$, the left-hand side of the formula tends to $\sum_j R_i R_j u_j$ in L^p, since the Riesz transforms are bounded on L^p. On the other hand, $\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} = -\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \varphi_N + u_j \frac{\partial \varphi_N}{\partial x_i} - u \frac{\partial \varphi_N}{\partial x_j}$, and the first term is zero since (u_i) is a gradient. Thus,

$$\left\| \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \right\|_p \leqslant \|\text{grad } u\|_p \, \|\text{grad } \varphi_N\|_\infty \to 0 \quad \text{as} \quad N \to \infty .$$

Since $R_j I^1$ is bounded from L^p to L^s with $\frac{1}{s} = \frac{1}{p} - \frac{1}{n}$ $(1 < s < \infty$ since $1 < p < n)$, we see that as $N \to \infty$, the final term in our formula tends to zero in L^s. Thus, taking the limit in measure of both sides of our formula, we obtain $u_i = \sum_j R_i R_j u_j$.

In other words, setting $f = \sum_j R_j \frac{\partial u}{\partial x_j} \in L^p$, we have $u_i = R_i f$. Now for $f \in L^p$, we know that $\frac{\partial}{\partial x_i} (I^1 f) = R_i f$ in the sense of tempered distributions. (It's trivial for $f \in C_0^\infty$; in general we express f as the limit in L^p of smooth compactly supported f_N as $N \to \infty$. $R_i f_N \to R_i f$ in L^p and hence weakly as distributions, and $I^1 f_N \to I^1 f$ in L^s and hence weakly, so that $\frac{\partial}{\partial x_i} (I^1 f_N)$ also converges weakly to $\frac{\partial}{\partial x_i} (I^1 f)$.)

On the other hand, the Riesz transforms or 1^{rst} fractional integral of a smooth function in L^p are smooth functions, so that $\frac{\partial}{\partial x_i} (u - I^1 f) = R_i f - R_i f = 0$, not just as distributions, but as pointwise derivatives of smooth functions. Thus $u = \text{constant} + I^1 f$. So to prove the claim, it will be enough to show that

$$\lim_{x_1 \to \infty} I^1 f(x_1, x') = 0$$

for almost all x', whenever f is a smooth function in L^p.

Now

$$\mathrm{I}^1 f(x_1, x') = \int_{\mathbf{R}^n} \frac{f(x)\,dy}{|x-y|^{n-1}} = \int_{|x-y|<1} \frac{f(y)\,dy}{|x-y|^{n-1}} + $$
$$+ \int_{|x-y|\geqslant 1} \frac{f(y)\,dy}{|x-y|^{n-1}} = \mathrm{I} + \mathrm{II}.$$

Then II is easy ; it tends to zero as $x \to \infty$. (In fact, we just break up f into $f = g + b$ where $\|g\|_p \leqslant \|f\|_p$ and g lives on a bounded set, while $\|b\|_p < \epsilon$. The contribution of g to II clearly tends to zero, while b produces at most

$$\|b\|_p \cdot \left(\int_{|x-y|\geqslant 1} |x-y|^{-(n-1)q}\,dy\right) \left[\frac{1}{p} + \frac{1}{q} = 1\right] \leqslant$$
$$\leqslant \mathrm{C}\,\|b\|_p \leqslant \mathrm{C}\,\epsilon, \quad \text{since} \quad p < n.$$

So the main problem is term I.

The main step in dealing with term I is to prove the

LEMMA. – *Let* $f \in \mathrm{L}^p(\mathrm{Q})$ *where* Q *is a cube of side* 10, *and* $1 < p < n$. *Say* $\mathrm{Q} = [0, 10] \times \mathrm{Q}' \subseteq \mathrm{R}^1 \times \mathrm{R}^{n-1}$. *Then*

$$\mathrm{M}(x') = \sup_{0 \leqslant x_1 \leqslant 10} \left| \int_{\mathrm{Q}} \frac{f(y_1, y')\,dy_1\,dy'}{((y_1 - x_1)^2 + (y'-x')^2)^{\frac{n-1}{2}}} \right|$$

belongs to $\mathrm{L}^p(\mathrm{Q}')$, *and* $\|\mathrm{M}\|_p \leqslant \mathrm{C}\,\|f\|_p$.

Proof. – Say $f \geqslant 0$ and $f = 0$ outside Q. Let $f^{\downarrow}(\cdot\,, y')$ be the decreasing re-arrangement of $f(\cdot\,, y')$ on $[0, 10]$, for each fixed $y' \in \mathrm{Q}'$. Of course $\|f\|_p = \|f^{\downarrow}\|_p$. We claim that $\mathrm{M}f(x') \leqslant 2\mathrm{M}f^{\downarrow}(x')$ pointwise. In fact, fix x', x_1, y', and consider

$$\int_{y_1 > x_1} \frac{f(y_1, y')\,dy_1}{((y_1 - x_1)^2 + (y'-x')^2)^{\frac{n-1}{2}}}.$$

Since $\chi_{\{y_1 | y_1 > x_1\}}(y_1) \cdot ((y_1 - x_1)^2 + (y'-x')^2)^{-\frac{(n-1)}{2}}$ is monotone decreasing in the interval (x_1, ∞) in which it is supported, we know that

$$\int_{y_1 > x_1} \frac{f(y_1, y')\, dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \leqslant \int_0^{10} \frac{f^{\downarrow}(y_1, y')\, dy_1}{((y_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \cdot$$

Integrating over y' yields

$$\int_{\substack{y_1 > x_1 \\ y' \in Q'}} \frac{f(y_1, y')\, dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \leqslant \int_Q \frac{f^{\downarrow}(y_1, y')\, dy_1}{(y_1^2 + (y' - x')^2)^{\frac{n-1}{2}}} \cdot$$

Similarly,

$$\int_{\substack{y_1 < x_1 \\ y' \in Q'}} \frac{f(y_1, y')\, dy_1\, dy'}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \leqslant \int_Q \frac{f^{\downarrow}(y_1, y')\, dy_1\, dy'}{(y_1 + (y' - x')^2)^{\frac{n-1}{2}}} \cdot$$

Adding and taking the sup over x_1 gives us

$$\mathrm{M}f(x') \leqslant 2 \int_Q \frac{f^{\downarrow}(y_1, y')\, dy_1\, dy'}{(y_1^2 + (y' - x')^2)^{\frac{n-1}{2}}} \cdot \qquad (*)$$

which is stronger than the claim.

We shall use what we proved in full strength. For a fixed y_1, the function $\mathrm{M}_{y_1}(x') = \int_{y' \in Q'} \frac{f^{\downarrow}(y_1, y')\, dy'}{(y^2 + (y' - x')^2)^{\frac{n-1}{2}}}$ is just the convolution of $f^{\downarrow}(y_1, .)$ with $(y_1^2 + (y' - x')^2)^{-\frac{n-1}{2}}$ on the cube $Q' \subseteq R^{n-1}$. Thus

$$\|\mathrm{M}_{y_1}(\cdot)\|_p \leqslant \|\text{convolution kernel}_{y_1}\|_{L^1(Q')} \, \|f(y_1, .)\|_p \sim$$

$$\sim C\left(\log \frac{1}{|y_1|}\right) \|f^{\downarrow}(y_1, .)\| .$$

Therefore by estimate $(*)$, $\mathrm{M}f(x') \leqslant \int_0^{10} \mathrm{M}_{y_1}(x')\, dy'$, and

$$\|\mathrm{M}f\|_p \leqslant \int_0^{10} C\left(\log \frac{1}{|y_1|}\right) \|f^{\downarrow}(y_1, .)\|_p\, dy_1 \leqslant$$

$$\leqslant \left(\int_0^{10} \left(C \log \frac{1}{|y_1|}\right)^q dy_1\right)^{1/q} \left(\int_0^{10} \|f(y_1, .)\|_p\, dy_1\right)^{1/p}$$

(Note : $q < \infty$ since $p > 1$) $\leqslant C \|f\|_p = C \|f\|_p$.

Q.E.D.

Now we return to term I above. Divide R^{n-1} into a mesh of cubes of side 2, $R^{n-1} = \bigcup_j Q'_j$, and write $R^n = \bigcup_{i,j} Q_{ij}$ where

$$Q_{ij} = \{(x_1, x') \in R^n \mid x' \in Q_j,\ 2i \leqslant x_1 < 2(i+1)\} \ (-\infty < i < \infty).$$

Then let Q^* be the cube concentric with Q_{ij} but with side 10, and set $f_{ij} = f\chi_{Q^*_{ij}}$. For

$$g_{ij}(x') = \begin{cases} \displaystyle\sup_{2i<x_1<2(i+1)} \int_{|x-y|<1} \frac{|f(y_1, y')|\, dy_1\, dy'}{|x-y|^{n-1}}, \\ 0 \text{ otherwise} \end{cases}$$

where $x = (x_1, x')$ if $x' \in Q'_j$ we have $\|g_{ij}\|_{L^p(R^{n-1})} \leqslant C \|f_{ij}\|_{L^p(R^n)}$ by the lemma. For

$$x' \in Q_j,\ \sup_{x_1 \in R^1} \int_{\substack{|x-y|<1 \\ \text{where } x=(x_1,x')}} \frac{|f(y)|\, dy}{|x-y|^{\frac{n-1}{2}}} =$$

$$= \sup_i g_{ij}(x') \leqslant \Big(\sum_i g_{ij}^p(x')\Big)^{1/p}.$$

So certainly

$$\int_{R^{n-1}} \left[\sup_{x_1 \in R^1} \int_{|(x_1,x')-y|<1} \frac{|f(y)|\, dy}{|x-y|^{n-1}}\right]^p dx' =$$

$$= \sum_j \int_{x' \in Q'_j} [\sup \text{(etc. etc.)}]^p\, dx' \leqslant \sum_j \int_{R^{n-1}} \Big(\sum_i g_{ij}^p(x')\Big)^{p/p} dx' =$$

$$= \sum_{i,j} \int_{R^{n-1}} g_{ij}^p(x')\, dx' \leqslant C \sum_{i,j} \|f_{ij}\|_p^p \leqslant C\, \|f\|^p_{L^p(R^n)}.$$

In other words, if

$$N(x') = Nf(x') = \sup_{x_1 \in R^1} \int_{|(x_1,x')-y|<1} \frac{|f(y)|\, dy}{|x-y|^{n-1}},$$

then

$$\|Nf\|_{L^p(R^{n-1})} \leqslant C\, \|f\|_{L^p(R^n)}.$$

Now term I becomes easy. Again split up if $f = g + b$ where g is supported in a bounded set and $\|b\|_p < \epsilon$.

In evaluating the claim

$$\lim_{x_1 \to \infty} \int_{|(x_1, x') - y| < 1} \frac{|f(y)|\, dy}{|x - y|^{n-1}} = 0, \ ((x_1, x') = x)$$

we find that g makes no contribution at all to the last integral for x_1 large enough. On the other hand, for b we know that

$$\left| \limsup_{x_1 \to \infty} \int_{|(x_1, x') - y| < 1} \frac{|f(y)|\, dy}{|(x_1, x') - y|^{n-1}} \right|_{L^p(dx')} < C\epsilon \quad \text{by} \quad (**)$$

Since $\epsilon > 0$ is arbitrary, the proof is complete.

Q.E.D.

BIBLIOGRAPHIE

[1] L.D. Kudrjavcev, Svoǐctba graničnyh značeniǐ funkciǐ iz vesovyh prostranctv i ih priloženija k kraevym zadačam. Mehanika Splošnoǐ sredy i rodstvennye problemy analiza. Moskva 1972.

[2] S.V. Uspenskiǐ, O teoremah vloženija dlja vesovyh klassov, *Trudi Mat. Instta AN SSSR,* 60 (1961), 282-303.

[3] V. Portnov, *Doklady AN SSSR*, to appear.

Manuscrit reçu le 9 juillet 1973
accepté par J.P. Kahane.

Charles Fefferman,
Departement of Mathematics
University of Chicago
5734 University Avenue
Chicago, Ill. 60637.