Convergence on almost every line for functions with gradient in $L^p(\mathbb{R}^n)$


<http://www.numdam.org/item?id=AIF_1974__24_3_159_0>
This note answers a question asked by L.D. Kudrjačev ([1], p. 264, problem 1). The question was suggested by the following result of Uspenskii [2]: if $u(x)$ is a smooth function $\mathbb{R}^n \to \mathbb{R}$ and 
$$\int_{\mathbb{R}^n} |\text{grad } u|^p \, dx < \infty \quad (1 < p < n),$$
there exists a constant $c$ such that $\lim_{r \to \infty} u(rx') = c$ for almost every $x' \in S^{n-1}$. Professor Kudrjačev kindly informed me that V. Portnov answered his question independently by another method [3].

We use now the notation 
$$x = (x_1, x'), \quad \text{where } x_1 \in \mathbb{R} \quad \text{and} \quad x' \in \mathbb{R}^{n-1}.$$ 

**THEOREM.** Let $u(x_1, x')$ be a smooth function $\mathbb{R}^n \to \mathbb{R}$ and suppose 
$$\int_{\mathbb{R}^n} |\text{grad } u|^p \, dx < \infty \quad (1 < p < n).$$
Then for a constant $c$, 
$$\lim_{x_1 \to \infty} u(x_1, x') = c \text{ for almost all } x'.$$

*Prof. — Set $u_j = \frac{\partial u}{\partial x_j}$, let $R_j$ denote $j^{th}$ Riesz transform, and let $I^1$ denote fractional integration of first order in $\mathbb{R}^n$. We begin with the standard formula 
$$\sum_i R_i R_j v_j = v_i + \sum_j R_j I^1 \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right),$$
valid for $C^\infty$ functions of compact support. (To check this, just take Fourier transforms). Take a function $\varphi_N$ on $\mathbb{R}^n$ equal to one for $|x| \leq 2^N$, supported in $|x| \leq 2^{N+1}$, and satisfying $|\nabla \varphi_N| \leq C2^{-N}$; and apply the above formula to the functions $v_j = \varphi_N u_j \in C_0^\infty$. As
N → ∞, the left-hand side of the formula tends to \( \sum_j R_j R_j u_j \) in \( L^p \), since the Riesz transforms are bounded on \( L^p \). On the other hand,

\[
\frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_j} = - \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \varphi_N + u_j \frac{\partial \varphi_N}{\partial x_i} - u \frac{\partial \varphi_N}{\partial x_j},
\]

and the first term is zero since \((u_i)\) is a gradient. Thus,

\[
\left| \frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_j} \right| \leq \| \text{grad } u \|_p \| \text{grad } \varphi_N \|_\infty \to 0 \quad \text{as } N \to \infty.
\]

Since \( R_j l^1 \) is bounded from \( L^p \) to \( L^s \) with

\[
\frac{1}{s} = \frac{1}{p} - \frac{1}{n} \quad (1 < s < \infty)
\]

since \( 1 < p < n \), we see that as \( N \to \infty \), the final term in our formula tends to zero in \( L^s \). Thus, taking the limit in measure of both sides of our formula, we obtain

\[
u_i = \sum_j R_j R_j u_j.
\]

In other words, setting \( f = \sum_j R_j \frac{\partial u}{\partial x_j} \in L^p \), we have \( u_i = R_i f \).

Now for \( f \in L^p \), we know that \( \frac{\partial}{\partial x_i} (1^1 f) = R_i f \) in the sense of tempered distributions. (It's trivial for \( f \in C^0_\infty \); in general we express \( f \) as the limit in \( L^p \) of smooth compactly supported \( f_N \) as \( N \to \infty \). \( R_i f_N \to R_i f \) in \( L^p \) and hence weakly as distributions, and \( 1^1 f_N \to 1^1 f \) in \( L^s \) and hence weakly, so that \( \frac{\partial}{\partial x_i} (1^1 f_N) \) also converges weakly to \( \frac{\partial}{\partial x_i} (1^1 f) \).

On the other hand, the Riesz transforms or \( 1^1 \) fractional integral of a smooth function in \( L^p \) are smooth functions, so that

\[
\frac{\partial}{\partial x_i} (u - 1^1 f) = R_i f - R_i f = 0,
\]

not just as distributions, but as pointwise derivatives of smooth functions. Thus \( u = \text{constant} + 1^1 f \).

So to prove the claim, it will be enough to show that

\[
\lim_{x_1 \to \infty} 1^1 f(x_1, x') = 0
\]

for almost all \( x' \), whenever \( f \) is a smooth function in \( L^p \).
Now
\[ I^1 f(x_1, x') = \int_{\mathbb{R}^n} \frac{f(x) \, dy}{|x-y|^{n-1}} = \int_{|x-y| < 1} \frac{f(y) \, dy}{|x-y|^{n-1}} + \int_{|x-y| \geq 1} \frac{f(y) \, dy}{|x-y|^{n-1}} = I + II. \]

Then \(II\) is easy; it tends to zero as \(x \to \infty\). (In fact, we just breakup \(f \) into \(f = g + b\) where \(\|g\|_p \leq \|f\|_p\) and \(g\) lives on a bounded set, while \(\|b\|_p < \epsilon\). The contribution of \(g\) to \(II\) clearly tends to zero, while \(b\) produces at most
\[
\|b\|_p \cdot \left( \int_{|x-y| \geq 1} |x-y|^{-(n-1)q} \, dy \right) \left[ \frac{1}{p} + \frac{1}{q} = 1 \right] \leq C \|b\|_p \leq C \epsilon, \text{ since } p < n.
\]

So the main problem is term I.

The main step in dealing with term I is to prove the

**Lemma.** — Let \(f \in L^p(Q)\) where \(Q\) is a cube of side 10, and \(1 < p < n\). Say \(Q = [0, 10] \times Q' \subseteq \mathbb{R}^1 \times \mathbb{R}^{n-1}\). Then
\[
M(x') = \sup_{0 < x_1 < 10} \left| \int_{Q} \frac{f(y_1, y') \, dy_1 \, dy'}{((y_1 - x_1)^2 + (y' - x')^2)^{n-1/2}} \right|
\]
belongs to \(L^p(Q')\), and \(\|M\|_p \leq C \|f\|_p\).

**Proof.** — Say \(f \geq 0\) and \(f = 0\) outside \(Q\). Let \(f^i(., y')\) be the decreasing re-arrangement of \(f(., y')\) on \([0, 10]\), for each fixed \(y' \in Q'\). Of course \(\|f\|_p = \|f^i\|_p\). We claim that \(Mf(x') \leq 2Mf^i(x')\) pointwise. In fact, fix \(x', x_1, y', \) and consider
\[
\int_{y_1 > x_1} \frac{f(y_1, y') \, dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{n-1/2}}.
\]
Since \(\chi_{\{y_1 \mid y_1 > x_1\}} (y_1) \cdot ((y_1 - x_1)^2 + (y' - x')^2)^{-\frac{n-1}{2}}\) is monotone decreasing in the interval \((x_1, \infty)\) in which it is supported, we know that
\[
\int_{y_1 > x_1} \frac{f(y_1, y') \, dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{n-1}/2} \leq \int_0^{10} \frac{f^i(y_1, y') \, dy_1}{((y_1)^2 + (y' - x')^2)^{n-1}/2}.
\]

Integrating over \( y' \) yields
\[
\int_{y_1 > x_1} \frac{f(y_1, y') \, dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{n-1}/2} \leq \int_Q \frac{f^i(y_1, y') \, dy_1}{(y_1^2 + (y' - x')^2)^{n-1}/2}.
\]

Similarly,
\[
\int_{y_1 < x_1} \frac{f(y_1, y') \, dy_1 \, dy'}{((y_1 - x_1)^2 + (y' - x')^2)^{n-1}/2} \leq \int_Q \frac{f^i(y_1, y') \, dy_1 \, dy'}{(y_1 + (y' - x')^2)^{n-1}/2}.
\]

Adding and taking the sup over \( x_1 \) gives us
\[
Mf(x') \leq 2 \int_Q \frac{f^i(y_1, y') \, dy_1 \, dy'}{(y_1^2 + (y' - x')^2)^{n-1}/2}.
\]

which is stronger than the claim.

We shall use what we proved in full strength. For a fixed \( y_1 \),
the function \( M_{y_1}(x') = \int_{y' \in Q'} \frac{f^i(y_1, y') \, dy'}{(y_2 + (y' - x')^2)^{n-1}/2} \) is just the
convolution of \( f^i(y_1, \cdot) \) with \((y_1^2 + (y' - x')^2)^{n-1}/2\) on the cube
\( Q' \subseteq R^{n-1}. \) Thus
\[
\|M_{y_1}(\cdot)\|_p \leq \|\text{convolution kernel}_{y_1} \|_{L^1(Q')} \|f(y_1, \cdot)\|_p \sim \\
\sim C \left( \log \frac{1}{|y_1|} \right) \|f^i(y_1, \cdot)\|.
\]

Therefore by estimate (\(*\)), \( Mf(x') \leq \int_0^{10} M_{y_1}(x') \, dy', \) and
\[
\|Mf\|_p \leq \int_0^{10} C \left( \log \frac{1}{|y_1|} \right) \|f^i(y_1, \cdot)\|_p \, dy_1 \leq \\
\leq \left( \int_0^{10} C \left( \log \frac{1}{|y_1|} \right)^q \, dy_1 \right)^{1/q} \left( \int_0^{10} \|f(y_1, \cdot)\|_p \, dy_1 \right)^{1/p}.
\]

(Note : \( q < \infty \) since \( p > 1 \)) \( \leq C \|f\|_p = C \|f\|_p. \)

Q.E.D.
Now we return to term I above. Divide $\mathbb{R}^{n-1}$ into a mesh of cubes of side 2, $\mathbb{R}^{n-1} = \bigcup Q'_j$, and write $\mathbb{R}^n = \bigcup Q_{ij}$ where

$$Q_{ij} = \{(x_1, x') \in \mathbb{R}^n \mid x' \in Q_j, \ 2i \leq x_1 < 2(i + 1)\} \ (-\infty < i < \infty).$$

Then let $Q^*$ be the cube concentric with $Q_{ij}$ but with side 10, and set $f_{ij} = f \chi_{Q^*}$. For

$$g_{ij} (x') = \begin{cases} \sup_{2i < x_1 < 2(i + 1)} \int_{|x - y| < 1} \frac{|f(y_1, y')| \ dy_1 \ dy'}{|x - y|^{n-1}} , \\ 0 \text{ otherwise} \end{cases}$$

where $x = (x_1, x')$ if $x' \in Q_j$ we have $\|g_{ij}\|_{L^p(\mathbb{R}^{n-1})} \leq C \|f_{ij}\|_{L^p(\mathbb{R}^n)}$ by the lemma. For

$$x' \in Q_j, \ \sup_{x_1 \in \mathbb{R}^1} \int_{|x - y| < 1} \frac{|f(y)| \ dy}{|x - y|^{n-1}} = \sup_i g_{ij} (x') \leq \left( \sum_i g_{ij}^p (x') \right)^{1/p}.$$ 

So certainly

$$\int_{\mathbb{R}^{n-1}} \left[ \sup_{x_1 \in \mathbb{R}^1} \int_{|x_1, x' \in Q_j, y| < 1} \frac{|f(y)| \ dy}{|x - y|^{n-1}} \right]^p \ dx' =$$

$$= \sum_i \int_{x' \in Q_j} \sup (\text{etc. etc.})^p \ dx' \leq \sum_i \int_{\mathbb{R}^{n-1}} \left( \sum_i g_{ij}^p (x') \right)^{p/p} \ dx' =$$

$$= \sum_i \int_{\mathbb{R}^{n-1}} g_{ij}^p (x') \ dx' \leq C \sum_i \|f_{ij}\|_p^p \leq C \|f\|_{L^p(\mathbb{R}^n)}^p.$$ 

In other words, if

$$N(x') = Nf(x') = \sup_{x_1 \in \mathbb{R}^1} \int_{|(x_1, x') - y| < 1} \frac{|f(y)| \ dy}{|x - y|^{n-1}},$$

then

$$\|Nf\|_{L^p(\mathbb{R}^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$ 

Now term I becomes easy. Again split up if $f = g + b$ where $g$ is supported in a bounded set and $\|b\|_p < \epsilon$. 
In evaluating the claim
\[ \lim_{x_1 \to \infty} \int_{|x_1, x'| - y| < 1} \frac{|f(y)\,dy}{|x - y|^{n-1}} = 0, \quad ((x_1, x') = x) \]
we find that \( g \) makes no contribution at all to the last integral for \( x_1 \) large enough. On the other hand, for \( b \) we know that
\[
\limsup_{x_1 \to \infty} \int_{|x_1, x'| - y| < 1} \frac{|f(y)|\,dy}{|x_1, x'\, - y|^{n-1}} L^p(dx') < C \varepsilon \quad \text{by (**)}
\]
Since \( \varepsilon > 0 \) is arbitrary, the proof is complete.

Q.E.D.

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