A.S. Galbraith has communicated to us the following intriguing problem: Does the completeness of a manifold imply, or is it implied by, the emptiness of the class $H^2B$ of bounded nonharmonic biharmonic functions? Among all manifolds considered thus far in biharmonic classification theory (cf. Bibliography), those that are complete fail to carry $H^2B$-functions, and one might suspect that this is always the case. We shall show, however, that there do exist complete manifolds of any dimension that carry $H^2B$-functions. Moreover, there exist both complete and incomplete manifolds not permitting these functions, and, trivially, incomplete manifolds possessing them.

We attach a Bibliography of recent work in the field.

1. Let $C$ be the totality of complete Riemannian manifolds $M$, characterized by an infinite distance of any point of $M$ to the ideal boundary. Denote by $\mathcal{E}_{H^2B}^N$ and $\tilde{\mathcal{E}}_{H^2B}^N$ the classes of $N$-manifolds, $N \geq 2$, for which $H^2B = \emptyset$ or $H^2B \neq \emptyset$, respectively.

**Theorem 1.** - $C \cap \tilde{\mathcal{E}}_{H^2B}^N \neq \emptyset$ for every $N$.

**Proof.** - Take the $N$-cylinder

$$|x| < \infty, \quad |y_i| \leq 1, \quad i = 1, 2, \ldots, N - 1,$$

with each face $y_i = 1$ identified with $y_i = -1$, so as to obtain a covering space of the $N$-torus in the same manner as a conventional cylinder is a covering surface of the torus. Let $T$ be this $N$-cylinder with the Riemannian metric

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\[ ds^2 = \mu^{-2}(x) \, dx^2 + \mu^{4/(N-1)}(x) \sum_{i=1}^{N-1} dy_i^2 \]

where
\[ \mu(x) = (2 + x^2)^2 \log(2 + x^2). \]

To see that \( T \in C \), it suffices to show, in view of the symmetry, that \( \int_0^\infty \mu^{-1}(x) \, dx = \infty \). The verification is immediate:

\[
\int_0^\infty (2 + x^2)^{-\frac{1}{2}} \log^{-1}(2 + x^2) \, dx > \frac{1}{2} \int_0^\infty (2 + x)^{-1} \log^{-1}(2 + x) \, dx
\]

\[
= \frac{1}{2} \int_0^\infty \log(2 + x) = \infty.
\]

We introduce the function
\[ u(x) = \int_0^x \mu^{-3}(t) \int_0^t \mu(s) \int_0^s \mu^{-3}(r) \, dr \, ds \, dt. \]

The Laplace-Beltrami operator \( \Delta = d\delta + \delta d \) gives
\[
\Delta u = -g^{-\frac{1}{2}} (g^{xx} u')' = -\mu^{-1} (\mu \mu^2 (u')') = -\int_0^x \mu^{-3}(r) \, dr
\]

and
\[ \Delta^2 u = -\mu^{-1} (\mu \mu^2 (-\mu^{-3}))' = 0. \]

Thus \( u \) is nonharmonic biharmonic.

To see that \( u \) is bounded it suffices to show that it is so for \( x > 0 \). For all \( s > 0 \),

\[
\int_0^s \mu^{-3}(r) \, dr = \int_0^s (2 + r^2)^{-3/2} \log^{-3}(2 + r^2) \, dr = o(1),
\]

and for all \( t > 0 \),

\[
\int_0^t \mu(s) \int_0^s \mu^{-3}(r) \, dr \, ds < c \int_0^t (2 + s^2)^{\frac{1}{2}} \log(2 + s^2) \, ds
\]
\[
< 2c \int_0^t (2 + s) \log(2 + s) \, ds
\]
\[
= c \left[ (2 + t)^2 \log(2 + t) - \frac{1}{2} (2 + t)^2 + \text{const.} \right].
\]
Here and later \( c \) is a constant, not always the same. We let \( [ \ldots ] \) stand for the expression in brackets and obtain
\[
u(x) < c \int_0^x (2 + t^2)^{-3/2} \log^{-3}(2 + t^2) \ [ \ldots ] dt.
\]
The dominating term in the integrand is majorized by
\[
\frac{1}{2} t^{-3} \log^{-3}t \times (2 + t)^2 \log(2 + t).
\]
The integral from 1 to \( x > 1 \) is bounded, and consequently so is \( u \) for all \( x \).

This completes the proof of Theorem 1.

2. The following simple example, valid for \( N \geq 3 \), is perhaps also of interest. Let
\[
T : \ |x| < \infty, \ \ |y| \leq \pi, \ \ |z_i| \leq 1, \ \ i = 1, \ldots, N - 2,
\]
with the metric
\[
ds^2 = dx^2 + e^{-x} dy^2 + e^{(2e^x - x)/(N-2)} \sum_{i=1}^{N-2} dz_i^2,
\]
the opposite faces again identified by pairs. Clearly \( T \in C \).

The function
\[
u = \cos y
\]
belongs to \( H^2B \). In fact,
\[
\Delta u = -e^{-e^x + x} (e^{e^x - x} e^x) (-\cos y)
\]
\[
= e^x \cos y,
\]
and
\[
\Delta^2 u = -e^{-e^x + x} (e^{e^x - x} e^x)' \cos y + e^{e^x - x} e^x e^x (-\cos y) = 0.
\]
Thus \( T \in C \cap \tilde{\partial}_{H^2B} \).

3. The reason that we are only interested in nonharmonic biharmonic functions is, of course, that completeness is known not
to exclude bounded harmonic functions (Nakai-Sario [6]). For $N \geq 3$, we insert here a simple proof of this fact.

Take the $N$-cylinder

$$T: |x| < \infty, \quad |y| \leq 1, \quad |z_i| \leq 1, \quad i = 1, \ldots, N - 2,$$

with the metric

$$ds^2 = dx^2 + e^{2x^2} dy^2 + \sum_{i=1}^{N-2} dz_i^2.$$

Trivially $T \subseteq C$. The function

$$k(x) = \left[ x e^{-t^2} dt \right]$$

is harmonic,

$$\Delta h = -e^{-x^2} (e^{x^2} e^{-x^2})' = 0.$$

It also is bounded and, in fact, even Dirichlet finite:

$$D(h) = c \int_{-\infty}^{\infty} e^{-2x^2} e^{x^2} dx < \infty.$$

4. We return to nonharmonic biharmonic functions.

**Theorem 2.** $- C \cap H^2_B \neq \emptyset$ for every $N$.

**Proof.** — The Euclidean $N$-space $E^N \subseteq C$. Every biharmonic function $u$ has an expansion in spherical harmonics $S_{nm}$

$$u = \sum_{u=0}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{n+2} + b_{nm} r^n) S_{nm}.$$

If $u \in H^2_B$, then

$$\int_{|x|=r} u S_{nm} d\omega = c(a_{nm} r^{n+2} + b_{nm} r^n)$$

is bounded in $r$, hence $a_{nm} = b_{nm} = 0$ for all $n$, except for $b_{01}$. Therefore $u$ is constant.

5. In view of $u = r^2 \in H^2_B$ on the Euclidean $N$-ball, we have trivially $\widehat{C} \cap \widehat{H}^N_{H^2_B} \neq \emptyset$ for every $N$, with $\widehat{C}$ the totality of incomplete Riemannian manifolds. It remains to show:
THEOREM 3. - \( C \cap \Theta^N_{H^2_B} \neq \emptyset \) for every \( N \).

Proof. - Let \( E^N_\alpha \) be the \( N \)-space \( 0 < r < \infty \) with the metric
\[
ds = r^\alpha |dx|,
\]
\( \alpha \) a constant. It is known (Sario-Wang [19, 21]) that if \( N \geq 4 \),
\( E^N_\alpha \in \Theta^N_{H^2_B} \) for every \( \alpha \); \( E^2_\alpha \in \Theta^N_{H^2_B} \) if and only if \( \alpha \neq -1 + n/2 \),
\( n = 1, 2, \ldots \); \( E^3_\alpha \in \Theta^N_{H^2_B} \) if and only if \( \alpha \neq -1 + \left[ \frac{1}{2} n(n + 1) \right]^{1/2} \).
On the other hand, \( E^N_\alpha \in \overline{C} \) for every \( \alpha \), hence the theorem.

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