ANNALES DE L'INSTITUT FOURIER

P. K. JAIN
D. R. JAIN

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Annales de l'institut Fourier, tome 24, nº 1 (1974), p. 123-129
<http://www.numdam.org/item?id=AIF_1974__24_1_123_0>

Ann. Inst. Fourier, Grenoble
24,1 (1974), 123-129

ON THE LOWER ORDER (R) OF AN ENTIRE DIRICHLET SERIES

by P.K. JAIN [1] and D.R. JAIN

1. Introduction.

For an entire Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \ (s = \sigma + it, \ \lambda_1 \geqslant 0, \ \lambda_n \to \infty \text{ with } n) \quad (1.1)$$

the lower order (R) λ is defined as :

$$\lim_{\sigma \to \infty} \inf \frac{\log \log M(\sigma)}{\sigma} = \lambda, \ (0 \leqslant \lambda \leqslant \infty), \quad (1.2)$$

where $M(\sigma) = \sup \{ | f(\sigma + it) | : -\infty < t < \infty \}$.

Improving upon a result of Rahman [6], Juneja and Singh [4] have, very recently, proved the following theorem :

THEOREM A. – *Let* $f(s)$ *be an entire Dirichlet series given by* (1.1) *of lower order* (R) λ $(0 \leqslant \lambda \leqslant \infty)$. *Then*

$$\lim_{n \to \infty} \inf \frac{\lambda_n \log \lambda_{n-1}}{\log | a_n |^{-1}} \leqslant \lambda \ . \quad (1.3)$$

Further, if

$$\lim_{n \to \infty} \sup \frac{\log n}{\lambda_n} < \infty \ , \quad (1.4)$$

and

[1] The research work of this author has been supported partially by the University Grants Commission (India).

$$\varphi_n \equiv \frac{\log |a_{n-1}/a_n|}{\lambda_n - \lambda_{n-1}} \tag{1.5}$$

forms a non-decreasing function of n for $n > n_0$, then

$$\lim_{n\to\infty} \inf \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} = \lambda \, . \tag{1.6}$$

In this paper, we obtain the estimations for the lower order (R) λ in terms of the sequences $\{\lambda_n\}$ and $\{a_n\}$ which hold for every entire Dirichlet series, and one of our estimations includes (1.6) and the result of Rahman [6] as the special cases. In fact, we prove :

THEOREM. – *Let* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *be an entire Dirichlet series given by* (1.1) *of lower order* (R) λ ($0 \leqslant \lambda \leqslant \infty$) *such that* (1.4) *is satisfied. Then*

$$\lambda = \max_{\{\lambda_{n_p}\}} \lim_{p\to\infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_{p-1}}}{\log |a_{n_p}|^{-1}} \tag{1.7}$$

$$\lambda = \max_{\{\lambda_{n_p}\}} \lim_{p\to\infty} \inf \frac{(\lambda_{n_p} - \lambda_{n_{p-1}}) \log \lambda_{n_{p-1}}}{\log |a_{n_{p-1}}/a_{n_p}|} \tag{1.8}$$

2. Preliminary Discussions.

Let $\mu(\sigma)$ denote the maximum term of $f(s)$ for $\mathrm{Re}(s) = \sigma$ and $\lambda_{\nu(\sigma)} = \max\{\lambda_n : \mu(\sigma) = |a_n| e^{\sigma\lambda_n}\}$. Let $\{\rho_n\}$ be the sequence of jump points of $\lambda_{\nu(\sigma)}$ (points of discontinuity of $\lambda_{\nu(\sigma)}$), every jump point is listed with multiplicity equal to size of the jump, such that $\rho_1 \leqslant \rho_2 \leqslant \rho_3 \leqslant \cdots \rho_n \leqslant \cdots$. Since $\lambda_{\nu(\sigma)} \to \infty$ as $\sigma \to \infty$, $\rho_n \to \infty$ as $n \to \infty$. We denote by $\{\lambda_{n_k}\}$ the range of $\lambda_{\nu(\sigma)}$, so that $\lambda_{\nu(\sigma)} = \lambda_{n_k}$ for $\sigma = \rho_{n_k}$. Then

i) $0 < \rho_{n_k} < \rho_{n_k+1} = \rho_{n_k+2} = \cdots = \rho_{n_{k+1}}$, $k = 1, 2, 3, \ldots$

ii) $\lambda_{\nu(\sigma)} = \lambda_{n_k}$, when $\rho_{n_k} \leqslant \sigma < \rho_{n_{k+1}}$, $k = 1, 2, 3, \ldots$.

These arguments are analogue to those for entire power series, given and used by Gray and Shah in their works [1,2,3].

Since $|a_{n_{k-1}}| e^{\sigma \lambda_{n_{k-1}}}$ and $|a_{n_k}| e^{\sigma \lambda_{n_k}}$ are the two consecutive maximum terms, we have

$$\rho_{n_k} = \log |a_{n_{k-1}}/a_{n_k}|/(\lambda_{n_k} - \lambda_{n_{k-1}}) . \tag{2.1}$$

Also, we need the following :

LEMMA

$$\lim_{k \to \infty} \inf. \frac{\log \lambda_{n_k}}{\rho_{n_{k+1}}} = \lambda .$$

Proof. – Since ([5], Theorem B)

$$\lim_{\sigma \to \infty} \inf \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \lambda, \ (0 \leqslant \lambda \leqslant \infty) ,$$

there exists a sequence $\{x_i\}$, $x_i \to \infty$ with i such that

$$\lim_{i \to \infty} \frac{\log \lambda_{\nu(x_i)}}{x_i} = \lambda .$$

It is always possible to find a subsequence $\{\rho_{n_{k_i}}\}$ of $\{\rho_{n_k}\}$ which satisfies the inequalities :

$$\rho_{n_{k_i}} \leqslant x_i < \rho_{n_{k_i+1}}, \ i = 1, 2, 3, \ldots$$

In either of the cases, for $i \geqslant i_0 = i_0(\epsilon)$, $\epsilon > 0$, we have

$$\frac{\log \lambda_{n_{k_i}}}{\rho_{n_{k_i+1}}} \leqslant \frac{\log \lambda_{\nu(x_i)}}{x_i} \leqslant \lambda + \epsilon ,$$

which implies

$$\lim_{k \to \infty} \inf \frac{\log \lambda_{n_k}}{\rho_{n_{k+1}}} \leqslant \lambda .$$

The reverse inequality is obvious. Hence the lemma is proved.

3. Proof of the Theorem.

Using (2.1) in the lemma, we get

$$\lambda = \lim_{k \to \infty} \inf \frac{(\lambda_{n_k} - \lambda_{n_{k-1}}) \log \lambda_{n_{k-1}}}{\log | a_{n_{k-1}} / a_{n_k} |} . \tag{3.1}$$

We have proved (3.1) for a particular subsequence $\{\lambda_{n_k}\}$ which is the range of the rank function $\lambda_{\nu(\sigma)}$. Thus the theorem will be proved completely if we establish, for any arbitrary subsequence (say) $\{\lambda_{n_p}\}$ of $\{\lambda_n\}$, the following inequalities :

$$\lambda \geqslant \lim_{p \to \infty} \inf . \frac{\lambda_{n_p} \log \lambda_{n_{p-1}}}{\log | a_{n_p} |^{-1}} \geqslant \lim_{p \to \infty} \inf . \times \tag{3.2}$$

$$\frac{(\lambda_{n_p} - \lambda_{n_{p-1}}) \log \lambda_{n_{p-1}}}{\log | a_{n_{p-1}} / a_{n_p} |} .$$

Proof of the first inequality in (3.2) : Let

$$\lim_{p \to \infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_{p-1}}}{\log | a_{n_p} |^{-1}} = \alpha .$$

Assume that $\alpha > 0$, for otherwise the result is trivially true. Therefore, for any $\epsilon > 0$, $\exists\, a$ $N = N(\epsilon)$ such that

$$| a_{n_p} | > \lambda_{n_{p-1}}^{\frac{-\lambda_{n_p}}{\alpha - \epsilon}} , \ (p \geqslant N) .$$

Let $e^{\sigma_p} = 2 \lambda_{n_{p-1}}^{\frac{1}{\alpha - \epsilon}}$, $p = 1, 2, 3, \ldots$. So if

$$\sigma_p \leqslant \sigma \leqslant \sigma_{p+1} , \text{ we have}$$

$$\begin{aligned} \log M(\sigma) &\geqslant \log | a_{n_p} | + \sigma_p \lambda_{n_p} \\ &\geqslant \log | a_{n_p} | + \sigma_p \lambda_{n_p} \\ &\geqslant \lambda_{n_p} \log 2 \\ &= e^{(\alpha - \epsilon) \sigma_{p+1}} \log 2 / 2^{\alpha - \epsilon} , \end{aligned}$$

i.e.

$$\log\log M(\sigma) \geqslant (\alpha - \epsilon)\,\sigma_{p+1} + \log\log 2 - (\alpha - \epsilon)\log 2 \,.$$

which gives

$$\lambda = \lim_{\sigma \to \infty} \inf. \frac{\log\log M(\sigma)}{\sigma} \geqslant \alpha \,.$$

Proof of the Second Inequality in (3.2) : Let

$$\lim_{p \to \infty} \inf \frac{(\lambda_{n_p} - \lambda_{n_{p-1}}) \log \lambda_{n_{p-1}}}{\log |a_{n_{p-1}}/a_{n_p})} = \beta \,.$$

Again, without any loss of generality, assume $\beta > 0$, so that

$$|a_{n_{p-1}}/a_{n_p}| < \lambda_{n_{p-1}}^{\frac{\lambda_{n_p} - \lambda_{n_{p-1}}}{\beta - \epsilon}} \,,$$

for $p \geqslant p_0 = p_0(\epsilon)$, $\epsilon > 0$. This implies

$$\left|\frac{a_{n_{p_0}}}{a_{n_p}}\right| = \left|\frac{a_{n_{p_0}}}{a_{n_{p_0+1}}}\right| \cdot \left|\frac{a_{n_{p_0+1}}}{a_{n_{p_0+2}}}\right| \cdots \left|\frac{a_{n_{p-1}}}{a_{n_p}}\right|$$

$$< \prod_{m=p_0+1}^{p} \lambda_{n_{m-1}}^{\frac{\lambda_{n_m} - \lambda_{n_{m-1}}}{\beta - \epsilon}}$$

$$\Rightarrow \log |a_{n_p}|^{-1} < 0(1) + \frac{1}{\beta - \epsilon} \sum_{m=p_0+1}^{p} (\lambda_{n_m} - \lambda_{n_{m-1}}) \log \lambda_{n_{m-1}}$$

$$\Rightarrow \frac{\log |a_{n_p}|^{-1}}{\lambda_{n_p} \log \lambda_{n_{p-1}}} < o(1) + \frac{1}{\beta - \epsilon} - \frac{1}{\beta - \epsilon} \frac{\lambda_{n_{p_0}} \log \lambda_{n_{p_0}}}{\lambda_{n_p} \log \lambda_{n_{p-1}}}$$

$$- \frac{(\beta - \epsilon)^{-1} \sum_{m=p_0+1}^{p-1} \lambda_{n_m}(\log \lambda_{n_m} - \log \lambda_{n_{m-1}})}{\lambda_{n_p} \log \lambda_{n_{p-1}}}$$

$$\Rightarrow \lim_{p \to \infty} \sup \frac{\log |a_{n_p}|^{-1}}{\lambda_{n_p} \log \lambda_{n_{p-1}}} \leqslant \frac{1}{\beta} \,.$$

Hence

$$\lim_{p\to\infty} \inf. \frac{\lambda_{n_p} \log \lambda_{n_{p-1}}}{\log |a_{n_p}|^{-1}} \geqslant \lim_{p\to\infty} \inf. \frac{(\lambda_{n_p} - \lambda_{n_{p-1}}) \log \lambda_{n_{p-1}}}{\log |a_{n_{p-1}}/a_{n_p}|},$$

Remark. – If, in addition to the hypothesis of our theorem, (1.5) is satisfied, then our result (1.7) reduces to (1.6). Further, if $\log \lambda_{n+1} \sim \log \lambda_n$, as $n \to \infty$, the result of Rahman [6] is also obtained.

Justification. – Since (1.5) is satisfied, each term of $f(s)$ is a maximum term and so $\lambda_{n_k} = \lambda_k$, for $k = 1, 2, \ldots$. Therefore, the result (3.1) reduces to

$$\lambda = \lim_{k\to\infty} \inf. \frac{(\lambda_k - \lambda_{k-1}) \log \lambda_{k-1}}{\log |a_{k-1}/a_k|} \cdot \tag{3.3}$$

Further, as the result (3.2) is true for every subsequence $\{\lambda_{n_p}\}$ of $\{\lambda_n\}$, it is also true for the sequence $\{\lambda_n\}$, since $\{\lambda_n\}$ may be regarded as a subsequence of $\{\lambda_n\}$. Hence

$$\lambda \geqslant \lim_{n\to\infty} \inf \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} \geqslant \lim_{n\to\infty} \inf \frac{(\lambda_n - \lambda_{n-1}) \log \lambda_{n-1}}{\log |a_{n-1}/a_n|} \tag{3.4}$$

Thus, the result (1.6) follows from (3.3) and (3.4).

Furthermore, if $\log \lambda_{n+1} \sim \log \lambda_n$, as $n \to \infty$, then (1.6) implies that

$$\lambda = \lim_{n\to\infty} \inf. \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} = \lim_{n\to\infty} \inf \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}$$

which is a result of Rahman [6].

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Manuscrit reçu le 25 janvier 1973
accepté par M. Brelot.

D.R. JAIN,
Department of Mathematics,
A.R.S.D. College,
Delhi University, NEW-DELHI-23, India

P.K. JAIN,
Department of Mathematics,
Hans Raj College,
Delhi University, DELHI-7, India.