ROBERT KAUFMAN

Topics on Kronecker sets


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In the first part of this note we consider relations between classes of differentiable functions and linear Kronecker sets. The problem in each of three theorems is to find a set $E$ of some narrow type, and a differentiable map $\varphi$, so that $\varphi(E)$ is a Kronecker set. The first theorem focusses on functions $\varphi$ in a prescribed non-quasi-analytic class $C(M_{\varphi})$, as described in [2, V] and [6, Ch. 19]. The second deals with a qualitative study of $\varphi'$, and the third uses van der Corput’s inequality to impose a very strong condition on $E$.

The second part illustrates the use of our method for constructing Kronecker sets; the lesson is that many special phenomena of exceptional sets are present in each set of multiplicity. Here we have in mind the work of Korner [5], and apply our method to strengthen a theorem on the union of two Kronecker sets.

1.

Let $E$ be a compact subset of $[0, 1]$; the exact condition on $E$ is known, that there be a function $\varphi$ of class $C^1[0, 1]$ so that $\varphi' > 0$ and $\varphi(E)$ is a Kronecker set [3; 1, VII; 4]. There exist, however, sets $E$ of this type such that $\varphi(E)$ is an $M_0$-set whenever $\varphi' > 0$ and $\varphi \in C^2$. Thus there is some interest in sets $E$ for which $\varphi \in C^\omega$ can be chosen as before; in the next theorem, we prove what is perhaps an extremal result on the possible smoothness of $\varphi$. Let $(\lambda_n)$ be an increasing sequence of positive numbers and $M_n = \lambda_1 \ldots \lambda_n$. 
THEOREM 1. — Let $F$ be an $M_0$-set, and let $\Sigma \lambda_n^{-1} < \infty$. Then there is a $M_0$-set $E \subseteq F$, a function $\varphi$ of class $C^\infty(R)$ so that

(a) $\varphi' \geq 1$ everywhere, $\varphi^{(n)} = 0(M_n)$ uniformly $(n \geq 1)$.

(b) $\varphi(E)$ is a Kronecker set.

To explain the significance of the condition $\Sigma \lambda_n^{-1} < \infty$, let $\varphi$ be $C^\infty$ of compact support and $P_n = \|\varphi^{(n)}\|_2$ so that by the Plancherel theorem $(P_n)$ is log-convex [2] and by the Denjoy-Carleman theorem $\Sigma P_n P_n^{-1} < \infty$ [2, 6]. Thus the condition on $(\lambda_n)$ is essential for the technique of partitions of unity, and Theorem 1 may be best-possible among results valid for all $M_0$-sets $E$.

In the proof of Theorem 1 we take a sequence $(a_n)$ decreasing to 0 so that $\Sigma (a_n \lambda_n)^{-1} < \infty$. Then there is a function $\varphi$ of compact support, $0 \leq \varphi \leq 1$, $\varphi = 1$ on a neighbourhood of 0, and $|\varphi^{(n)}| = 0(a_1 \lambda_1 \ldots a_n \lambda_n)$ [2, 6]. Next we can expand the interval on which $\varphi = 1$ at will, preserving the inequality $0 \leq \varphi \leq 1$ and the inequalities on $\varphi^{(n)}$. Because $\lim a_n = 0$, each homothety of $\varphi$, say $\varphi(x) = \varphi(Ax + B)$ also satisfies the inequalities $|\varphi^{(n)}| \leq C_A M_n (0 \leq n < \infty)$, with $C_A$ a function of $A$ alone. In particular, if $r < r_1 < s_1 < s$, one of the homotheties equals 1 on $(r_1, s_1)$ and 0 outside $(r, s)$.

After a few reductions we can suppose that $F$ is a closed, totally disconnected subset of $[0, 1]$ and $\mu$ is a probability measure in $F$ with $\hat{\mu}$ in $C_0(R)$. Let $(f_m)^\infty_1$ be a dense sequence in the real Banach space $C[0, 1]$, and let $(h_m)^\infty_1$ be a sequence of $2\pi$-periodic $C^\infty$-functions of mean 1, such that $h_m \geq 0$ and $h_m(t) = 0$ when $m^{-1} \leq |t| \leq \pi - m^{-1}$. Thus for any function $g$ and real number $y$, $h_m(yg - f_m) = 0$ except on the set $(\exp iyg - \exp if_m) \leq m^{-1}$.

We shall construct a sequence of measures $\mu_m$, beginning with $\mu_0 = \mu$ and

$$
(1) \quad \mu_m = h_m(y_m g_m - f_m) \cdot \mu_{m-1}
$$

where $1 < y_1 < \ldots < y_m < \ldots$ and $g_m$ converges so rapidly to a limit $\varphi$ that $y_m |g_m - \varphi| \leq 2m^{-1}$. Also, $|\hat{\mu}_m - \hat{\mu}_{m-1}| < 3^{-m}$ so that the weak limit $\sigma$ of the sequence
(\nu_m) has norm in \((1/2, 3/2)\) and \(\sigma\) is in \(C_0(\mathbb{R})\). On the support of \(\sigma\), which is contained in the support of \(\nu_m\), we have 
\[|\exp iy_m\varphi - \exp if_m| \leq 2m^{-1},\]
and Theorem 1 will be proved by constructing the limit \(\varphi\) with the special properties listed.

Each \(h_m\) has an absolutely convergent Fourier expansion 
\[h_m(t) = \sum a_k \exp ikt, \quad a_0 = 1.\]

Thus 
\[\mu_m(u) - \mu_{m-1}(u) = \sum a_k \int \exp ik(y_mg_m - f_m) \cdot \exp - iut. d\mu_{m-1}.\]

Now each \(g_m\) shall be constructed so that \(g'_m = 0\) on a neighbourhood of \(F\); hence there are disjoint intervals \(I_j\), covering \(F\), on which \(g'_m = b_j\), say. We express the integral above as a sum of integrals over \(I_j\)'s. The \(j\)-th term in the \(k\)-th integral has the same modulus as 
\[\vartheta(u - ky_m b_j), \quad \vartheta = \nu(k, j) \text{ is absolutely continuous with respect to } \mu_0 \text{ and } \sum_j \|\nu(k, j)\| = \|\mu_{m-1}\| \text{ for all } k \neq 0.\]

Each \(\vartheta \in C_0(\mathbb{R})\), whence to each \(\varepsilon > 0\) there is a \(T\) so that for all \(u\)
\[|\mu_m(u) - \mu_{m-1}(u)| \leq \varepsilon + \sum |a_k| \cdot \|\nu(k, j)\|\]

where \(\sum\) means summation over the set of indices 
\[k \neq 0, \quad |u - ky_m b_j| < T.\]

Next, let us suppose that the values \(b_j\), of \(g_m\) on \(I_j\), are distinct, differing among each other by at least \(r > 0\). Each inequality \(|u - ky b_j| < T\) has at most one solution \(j\) for each \(k \neq 0\), as soon as \(yr > 2T\). When this condition is imposed, the sum \(\sum\) does not exceed \(\max \mu_{m-1}(I_j) \cdot \Sigma|a_k|\).

From this point is clear how to proceed. Let \(g_0 = x\) and suppose that \(g_{m-1}\) is a known function of class \(C^n\), with \(g_{m-1} = 0\) on a neighbourhood of \(F\). Let \((I_j)\) be a covering of \(F\) by disjoint intervals so that \(\nu_{m-1}(I_j) \Sigma|a_n| < 4^{-m}\), say. Then there is a function \(p > 0\) of compact support, so that \(p = c_j > 0\) on \(I_j\), where the numbers \(c_j\) are distinct; the indefinite integral \(q\) of \(p\), with \(q(-\infty) = 0\), has the property that \(|q| = 0(1)\) while
\[|q^{(a+1)}| = |p^{(a)}| = 0(M_n) = 0(M_{n+1}).\]

For all \(\delta > 0\) sufficiently small, the level sets of \((g_{m-1} + \delta q)\),
form a covering of $F$ finer than the covering $(I_j)$ and we choose $g_m = g_{m-1} + \delta g$. Because $\delta$ can be arbitrarily small it is clear that the sequence $(g_m)$ can be made to converge to a function $\varphi$ fulfilling the inequalities $|\varphi^{(n)}| = 0(M_n)$ for $n \geq 1$, and the inequality $|g_m - \varphi|y_m < m^{-1}$. This completes the proof of Theorem 1.

Theorem 2 is a variation on the idea of building jumps into the derivative, but using Lebesgue measure for the initial measure $\mu$, and differential calculus, we give a more precise conclusion. Let $\varphi$ be of class $C^1[0, 1]$, let $F$ be a closed subset of $[0, 1]$, $m(F) > 0$, and let

$$H(y) = m(x \in F, \varphi'(x) < y), \quad -\infty < y < \infty.$$ 

be the relative distribution function of $\varphi'$.

**Theorem 2.** — $F$ contains an $M_\alpha$-set $E$ such that $\varphi(E)$ is a Kronecker set, provided $H$ is continuous.

An equivalent statement is that such a set $E \subset F$ can be found, provided $H$ is not a pure saltus-function. Theorem 2 is proved by the same inductive process as before, beginning now with the Lebesgue measure restricted to $F$, i.e. $\mu_0 = \chi_F m$. In place of arguments on the sequence of functions $g_m$ we use

**Lemma.** — Let $H$, $\varphi$, and $F$ be as in Theorem 2. Then

$$\lim_{T \to +\infty} \sup_u \bigg| \int_T \exp - iut \exp T\varphi(t) \, dt \bigg| = 0$$

Proof. — Let $k$ be a positive integer, $\delta > 0$, and $u$ real, and let $S(k, \delta, u)$ be the union of all intervals

$$[pk^{-1}, (p + 1)k^{-1}], \quad (p = 0, \ldots, k - 1)$$

containing a point $x$ at which $|\varphi' - u| < \delta$. Using the uniform continuity of $\varphi'$ on $[0, 1]$ and of $H$ on $(-\infty, \infty)$, we see that to each $\varepsilon > 0$ there exist $k, \delta$ so that $m(F \cap S(k, \delta, u)) < \varepsilon$ for all real $u$. When $u$ and $T$ are specified let us denote by $G$ any intersection $F \cap (a, b)$ where $-u + T\varphi' \geq \delta T$ throughout $(a, b)$. We shall give a uniform method of estimating $\int_a \exp - iut \exp T\varphi(t) \, dt$ and this will prove the lemma.

For definiteness we suppose $-u + T\varphi'(t) > 0$ on $[a, b]$ and construct the sequence $a = a_0, a_1, \ldots$ such that
$-ut + T\varphi$ increases by exactly $2\pi T^{-1}$ between each pair $a_n, a_{n+1}$. Thus $a_{n+1} - a_n < 2\pi (\delta T)^{-1}$ and $(a, b)$ is covered by intervals $(a_n, a_{n+1})$ and a remainder $< 2\pi (\delta T)^{-1}$. The polygonal interpolation $\tilde{\varphi}$ of $\varphi$, with nodes $a_0, a_1, \ldots, b$ has the property $|\tilde{\varphi} - \varphi| = o(T^{-1})$ by the mean-value theorem, and the estimation

$$\int_c \exp - iut. \exp T\tilde{\varphi}(t).dt \to 0 \quad \text{as} \quad T \to +\infty$$

follows from the Lebesgue density theorem. This concludes the proof of the lemma.

**Theorem 3.** — Let $\varphi$ have an absolutely continuous derivative on $[0, 1]$, and $C_1 \leq \varphi'' \leq C_2$ almost everywhere ($0 < C_1 < C_2 < \infty$). Let $w(u)$ be positive on $[0, \infty)$, increasing to $+\infty$. Then there is a subset $E$ so that $\varphi(E)$ is a Kronecker set, and a measure $\mu \geq 0$ in $E$ such that

$$\hat{\mu}(u) = o \left( |u|^{-\frac{1}{2}} \right) w(|u|).$$

As in the two previous proofs, all depends on a suitable estimate of an exponential integral. The sequence $(f_m)$, dense in $C[0, 1]$, is now supposed to contain functions of class $C^2$; thus beginning with the Lebesgue measure $m$ on $[0, 1]$, all measures constructed by the inductive process have the form $\nu_m = p_m m$, with $p_m \in C^2$. Thus there is a constant $C_m$ so that

$$\left| \int f \, d\nu_m \right| \leq C_m \sup_x \left| \int_0^x f(t) \, dt \right| \quad (0 \leq t \leq 1).$$

Thus the following estimation enables us to complete the proof.

For all $y > y_0$, $k \geq 1$ or $k \leq -1$, and real $u$

$$\left| \int_0^x \exp - iut \cdot \exp ik(y\varphi - f_m) \cdot dt \right| \leq C_m |u|^{-\frac{1}{2}},$$

and moreover, the integrals are uniformly $o(1)$ as $y \to \infty$.

To prove this we use the inequality $\varphi'' \geq C_1 > 0$ and $f_m \in C^2$ to choose $y$ so large that $y\varphi'' - f''_m \geq \frac{1}{2} C_1 y$. By van der Corput’s inequality [7, p. 197] the integrals are uniformly $0 \left( |ky|^{-\frac{1}{2}} \right)$, so the second part is disposed of. More-
over, the first inequalities are valid on domains of the type $|ky| \geq \varepsilon |u|$, for any fixed $\varepsilon > 0$. For the complementary domain $|u|\varepsilon > |ky|$, $|k(y\varphi - f_m)'| \leq \varepsilon |u| \cdot \varphi' + y^{-1} \varepsilon |u| |f'_m|$. Thus for small $\varepsilon$ and large $y$, $-ut + k(y\varphi - f_m)$ has derivative $\geq \frac{1}{2} |u|$ or $\leq -\frac{1}{2} |u|$. We can then write $-ut + k(y\varphi - f_m) = up(t)$ where $\frac{1}{2} \leq |p'(t)|$ and $|p''(t)| \leq C_m^*$. The integral then takes the form $\int_a^b \exp -ius q(s) \, ds$ where $|q(s)| \leq 2$ and $q(s)$ has total variation $\leq 2C_m^*$. The integral then has modulus $\leq 8C_m^* |u|^{-1} \leq C_m^* |u|^{-\frac{1}{2}}$, because $|u| > |y| \to +\infty$. This proves the required estimation.

The last theorem about differentiable functions is a complement to the first and second; its proof involves a lemma on interpolation of differentiable functions. The set $E$ of Theorem 1 can be mapped by a diffeomorphism $\varphi$ of class $C(M_n)$ onto a Kronecker set, and in fact $\varphi'' = 0$ on $E$; by [1] $E$ can also be mapped by a $C^1$-diffeomorphism $\psi$ onto a Kronecker set, and here $\psi' = 1$ on $E$. Quite possibly $E$ could be constructed so that the diffeomorphism $\varphi$ is smooth and has derivative 1 on $E$, but the method of Theorem 1 plainly fails to accomplish this. The theorem to be proved shows that the existence of diffeomorphisms $\varphi$ and $\psi$ by no means implies that their characteristics can be attained simultaneously. A similar property of stability of $M_0$-sets is obtained in [4] by an entirely different technique.

**Theorem 2'.** — Let $F$ and $\varphi$ be as in Theorem 2, and let $\omega(u)$ be positive and increasing on $(0, \infty)$, $\omega(0+) = 0$. Then the set $E \subseteq F$ can be so chosen that $\varphi(E)$ is a Kronecker set, but $\psi(E)$ is an $M_0$-set whenever $\psi \in C^1[0, 1]$, $\psi' = 1$ on $E$, and $|\psi'(s) - \psi'(t)| \leq \omega(|s - t|)$ for $0 \leq s < t \leq 1$.

When $F$ is totally disconnected, $\varphi$ can be constructed in any non-quasi-analytic class $C(M_n)$ so that $\varphi'$ is strictly increasing and $\varphi'' = 0$ on $F$; the simple example $\varphi(x) = x^2$ illustrates that analyticity has no obvious consequences about $E$. The necessity of the lemma needs to be explained.
Beginning from the set $F$, the subset $E \subseteq F$ is to be defined, and thereby a certain subset of $C^1$, say $S(E)$. But $S_1(E)$ is then known only in principle and is obviously much larger than $S(F)$. Thus the construction seems to be circular, because it requires some knowledge of $S(E)$ to proceed. To circumvent this obstacle we consider all sets $S(E)$ simultaneously, attempting to replace each function $\psi_1$ in $S(E)$ by a function $\psi_1$ in $C^1[0, 1]$, such that $\psi = \psi_1$ in $E$ and $\psi'_1 = 1$ in $E \cup F$. This, however, is possible only if $[0, 1] \sim F$ meets each interval $(a, b) \subseteq (0, 1)$ in a subset whose measure is not much smaller than $(b - a) \cdot \omega(b - a)$, because $\psi'_1 = 1$ on $F$. To solve this problem of interpolation we must therefore replace $F_1$ by a subset whose complementary intervals are specially constructed.

**Lemma.** — Corresponding to the function $\omega$ there is a closed set $F_1$ with this property: whenever $E \subseteq [0, 1]$ and $\psi \in S_1(E)$, then $\psi$ coincides with a function $\psi_1 \in C^1[0, 1]$ whose derivative is 1 on $E \cup F_1$. Moreover all the derivatives $\psi'_1$ so constructed are equicontinuous on $[0, 1]$. Finally, $m(F \cap F_1) > 0$.

**Proof.** — Let $T_n$ be an increasing sequence of positive numbers and $R \sim F_1$ the set defined by $x \notin F_1$ if $|T_n - q| \leq n^{-2}$ for some integer $q$ and $n \geq 1$. When $(T_n)$ increases rapidly, $m(F \cap F_1) > 0$; we specify that $T_1 > 8 \omega(8T_1^{-1}) \leq (n + 1)^{-s}$. Thus, if $b - a > 8T_n^{-1}$, $(R \sim F_1) \cap (a, b)$ contains intervals of length $2n^{-2}T_n^{-1}$, whose total length exceeds $n^{-2}(b - a)$. Let now $E$ be a closed subset of $[0, 1]$ and $\psi$ a function in the class $S_1(E)$, and write $\psi_2(x) = x - \psi(x)$. Now $\psi_2$ is considered as a function defined only on $E$; then

$$|\psi_2(s) - \psi_2(t)| \leq |s - t| \omega(|s - t|)$$

by the mean-value theorem. To each interval $[a, b]$ meeting $E$ only in its end-points $a$ and $b$, there is a least integer $n$ with $b - a > 8T_n^{-1}$. Then $(a, b)$ meets $R \sim F_1$ in a certain set of intervals, of total length $> n^{-2}(b - a)$. The derivative of $\psi_2$ will have a triangular graph over these intervals, of height $h$, and will vanish elsewhere in $(a, b)$. 
The common height $h$ of these triangles fulfills an inequality $n^{-2}(b-a).|h| \leq 2|\psi_2(b) - \psi_2(a)| \leq 2(b-a)\omega(b-a)$. In case $1 > b-a > 8T_1^{-1}$, we obtain $|h| \leq 2\omega(1)$; when $8T_1^{-1} > b-a > 8T_1^{-1}$, we have $\omega(b-a) \leq n^{-3}$ and then $|h| \leq 2n^{-1}$. To complete the extension of $\psi_2$ onto $[0,1]$, we extend to be constant to the left and right of $[0,1]$. The equicontinuity of the aggregate $\{\psi_1\}$ follows from the triangular shape and the fact that $|\psi_1| \leq 2n^{-1}$ on the interval $(a,b)$ provided $b-a < 2n^{-1}$. Finally, set

$$\psi_1(x) = x - \psi_2(x)$$

and the lemma is complete.

To prove the main result, we can assume that $F_1 = F$, and construct $E$ as in Theorem 2 so that $\psi_1(E)$ is an $M_0$-set for each function $\psi_1$ constructed in the lemma. This can be accomplished with the aid of uniform estimates for integrals of the form

$$\int_{\mathbb{R}} \exp - iu\psi_1(t) \exp iy\varphi(t) dt.$$

To each $\delta > 0$ there is a neighbourhood $V_\delta \supseteq F$ on which $|\psi_1 - 1| < \delta$ for each function $\psi_1$, and from this point the argument of Theorem 2 is valid, so the integrals tend to 0 uniformly as $y \to \infty$.

Theorem 2' is valid for the weaker inequality $|\psi'(s) - \psi'(t)| = 0(\omega|t-s|)$, since there is a function $g$, locally constant on $E$, so that $g' + \psi'$ has modulus of continuity at most $\omega^{1/2}$.

2.

**Theorem 4.** — Let $\lambda$ be a continuous, finite measure on $\mathbb{R}$, and $F$ an $M_0$-set. Then there is a Kronecker set $E$ and a positive measure $\mu \neq 0$ in $E$ such that each set $\{|\hat{\mu}(u)| \geq \delta, |\hat{\lambda}(u)| \geq \delta\}$ is compact. Moreover to each $\delta > 0$ there is a $u_0$ so that the set $\{|\hat{\mu}(u + u_0)| \geq \delta, |\hat{\lambda}(u)| \geq \delta\}$ is empty.

Here we set $\varphi(x) = x$ so that the support of the limit measure $\mu$ is a Kronecker set. Of course we cannot obtain
uniform convergence of \( \hat{\mu}_m \), but only pointwise convergence, sufficient to ensure that \( \frac{3}{2} > \mu(E) > \frac{1}{2} \). However, we can obtain uniform convergence on each of the subsets \( R_\delta = \{ |\hat{\lambda}(u)| \geq \delta \} \). A classical theorem of Wiener shows that \(|\hat{\lambda}|^2\) has mean-value 0, and for each \( \delta \) there is an \( \eta > 0 \) such that \( R_\delta + (0, \eta) \subseteq R_{\frac{1}{2}\delta} \). Thus \( R_\delta + (0, \eta) \) meets \([-a, a]\) in a set of measure \( o(a) \). An elementary covering argument shows that this remains valid for \( R_\delta + I, I \) being a fixed, finite interval, and plainly that property is preserved by dilations and translations of \( R_\delta \). Thus we obtain the important property of \( R_\delta \): there is a sequence \( y_m \) such that \( \lim d(ky_m, R_\delta) = +\infty \) for each \( \delta > 0 \) and each integer \( k \neq 0 \). Examination of the formula for \( \hat{\mu}_m(u) - \hat{\mu}_{m-1}(u) \) shows that it is possible to force the sequence \( \hat{\mu}_m \) to converge uniformly on each \( R_\delta \). Because each \( \hat{\mu}_m \in C_0 \), the limit \( \hat{\mu} \in C_0(R_\delta) \) and this expresses the first property claimed for \( \hat{\mu} \).

To obtain the second we choose a sequence \((u_m)\) along with \((\mu_m)\). Now \( \hat{\mu}_m \in C_0(R) \) so there is a number \( -u_m \) so far from \( R_m - 1 \) that \( |\hat{\mu}_m(u - u_m)| < m^{-1} \) whenever \( |\hat{\lambda}(u)| \geq m^{-1} \); or \( |\hat{\mu}_m| < m^{-1} \) on \( R_m - 1 + u_m \). From here the argument is almost as before, except that \( \hat{\mu}_m - \hat{\mu}_{m-1} \) must be controlled on a set of the type finite \( + R_\delta \) and this is easily attained. In the limit we have, for example, \( |\hat{\mu}| < 2m^{-1} \) on \( R_m - 1 + u_m \) so that the inequalities \( |\hat{\lambda}| > m^{-1} \) and \( |\hat{\mu}(u - u_m)| > 2m^{-1} \) exclude each other.

Here is a simple consequence of Theorem 4. To each uncountable closed set \( E \) there are a Kronecker set \( E_1 \), disjoint from \( E \) and probability measures \( \mu \) in \( E_1 \), \( \lambda \) in \( E \), so that \( \lim \sup |\hat{\mu} + \hat{\lambda}| \leq 1 \), and a sequence of characters \( \chi_m \) so that \( \overline{|\chi_m\mu|} + |\hat{\lambda}| < +m^{-1} + 1 \). Thus \( E \cup E_1 \) is at most \( H_{\frac{1}{2}} \), in a sense somewhat stronger than in [5]; of course the most interesting case occurs when \( E \) is itself a Kronecker set.

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Accepté par J.-P. Kahane,

R. Kaufman,
Department of Mathematics,
University of Illinois
Urbana, Ill. 61801 (USA).