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TOPOLOGICAL CLASSIFICATION OF LINEAR ENDOMORPHISMS

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Endomorphisms $f\colon X \to X$ and $g : Y \to Y$ of topological spaces X and Y are called *homeomorphic* (notation : $f \overset{t}{\sim} g$) if there is a homeomorphism $h : X \to Y$ such that $g = hfh^{-1}$. When X and Y are smooth manifolds and h is a diffeomorphism, f and g are called *diffeomorphic* (notation : $f \overset{d}{\sim} g$). When X and Y are vector spaces and h is a linear isomorphism, f and g are called *linearly isomorphic* (notation : $f \overset{l}{\sim} g$). A general problem in the theory of dynamical systems is to classify large classes of endomorphisms up to the relation $\sim$ (where $\sim$ is say one of $\overset{t}{\sim}$, $\overset{d}{\sim}$, $\overset{l}{\sim}$).

One classifies linear endomorphisms of finite dimensional (real) vector spaces up to $\overset{l}{\sim}$ via the jordan normal form; $\overset{d}{\sim}$ gives nothing new : if f and g are linear, then $f \overset{d}{\sim} g \Longleftrightarrow f \overset{l}{\sim} g$. For $\overset{t}{\sim}$ the situation changes : for $X = Y = \mathbf{R}$, $f(x) = 2x$, $g(x) = 8x$ $(x \in \mathbf{R})$, we see that f and g are linearly distinct (different eigenvalues) but topologically the same. To check $f \overset{t}{\sim} g$ note that $g = hfh^{-1}$ where $h(x) = x^3$.

Our problem is to classify linear endomorphism up to homeomorphism : $\overset{t}{\sim}$. To state our theorem we need some notation. Any linear endomorphism f can be written as a direct sum of endomorphisms :

$$f = f_\infty \oplus f_+ \oplus f_0 \oplus f_-$$

where the eigenvalues λ of f_∞ (resp. f_+; f_0; f_-) satisfy

$\lambda = 0$ (resp. $0 < |\lambda| < 1$; $|\lambda| = |$; $1 < |\lambda|$). If f is an endomorphism denote by dim (f) the dimension of the domain (= target) of f; thus

$$\dim (f) = \dim (f_\infty) + \dim (f_+) + \dim (f_0) + \dim (f_-).$$

When f is an automorphism, or (f) denotes the sign of the determinant of f.

Conjecture. — Let f and g be linear endomorphisms of $\mathbf{R}^n$. Then $f \overset{t}{\sim} g$ if and only if $f_\infty \overset{l}{\sim} g_\infty$, $\dim (f_+) = \dim (g_+)$, or $(f_+) =$ or (g_+), $f_0 \overset{l}{\sim} g_0$,

$$\dim (f_-) = \dim (g_-) \quad \text{or } (f_-) = \text{or } (g_-).$$

As a special case consider the situation when f and g have finite order (some positive power is the identity). Then all eigenvalues are roots of unity so $f = f_0$ and $g = g_0$. Thus:

Special case. — If f and g are linear automorphisms of $\mathbf{R}^n$ of finite order, then $f \overset{t}{\sim} g$ if and only of $f \overset{l}{\sim} g$.

Fortunately, the special case is very difficult. It is easily seen to be equivalent to:

Special case reformulated. — If $f, g \in 0(n+1, \mathbf{R})$ have finite order, then $f|S^n \overset{t}{\sim} g|S^n$ if and only if $f \overset{l}{\sim} g$.

This last assertion is known to be true when the action of f on S^n is free (Attiyah-Bott-Lefschetz Theorem as modified by Wall). A theorem of the Rham asserts that for $f, g \in 0(n+1, \mathbf{R})$ one has $f|S^n \overset{d}{\sim} g|S^n$ if and only if $f \overset{l}{\sim} g$. The proof uses a kind of Whitehead-Reidemeister torsion; this is easily seen to be a differentiable invariant; if it is a topological invariant, our special case is true. Finally, Sullivan claims that our special case is true when the order is a prime power. These three cases suggest strongly that the special case always obtains.

We prove that the special case implies the conjecture and that the conjecture holds when no eigenvalue λ of the linear maps involved is a root of unity except possibly $\lambda = \pm 1$, $\pm i$ or $\sqrt[3]{-1}$.

To illustrate our theorem consider the two linear automorphisms of $\mathbf{R}^4$ given by:

$$f = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

These two automorphisms are clearly not linearly isomorphic. (They are in jordan normal form). Their sole eigenvalue is one. Thus according to our theorem, they are not homeomorphic. We invite the reader to prove this himself.

Our work will appear shortly in Inventiones Mathematicae.

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