

EQUIVARIANT ALGEBRAIC TOPOLOGY

by Sören ILLMAN

This talk covered the main parts of my thesis [5], see also [4] and [6]. Our main objective is to provide the equivariant analogue of ordinary singular homology and cohomology, and the equivariant analogue of the theory of CW complexes. For actions of **discrete** groups we have the work by Breton [1], [2], and more recently also the article by Bröcker [3].

1. Equivariant singular homology and cohomology.

Let G be a topological group. By a G -space X we mean a topological space X together with a left action of G on X . A G -pair (X, A) consists of a G -space X and a G -subspace A of X . The notions G -map, G -homotopy, G -homotopy equivalence, etc... have the usual meaning.

DEFINITION 1.1. — *An orbit type family \mathcal{F} for G is a family of subgroups of G which is closed under conjugation.*

Thus the family of all subgroups and the family of all finite subgroups of G are examples of orbit type families for G . A more special example is the following. Let $G = O(n)$ and let \mathcal{F} be the family of all subgroups conjugate to $O(r)$ (standard imbedding) for some r , where $0 \leq r \leq n$.

Let R be a ring with unit. By an R -module we mean a unitary left R -module.

DEFINITION 1.2. — *Let \mathcal{F} be an orbit type family for G . A covariant coefficient system k for \mathcal{F} , over the ring R ,*

is a covariant functor from the category of G -spaces of the form G/H , where $H \in \mathcal{F}$, and G -homotopy classes of G -maps, to the category of R -modules.

A contravariant coefficient system m is defined by the contravariant version of the above definition.

THEOREM 1.1. — *Let G be a topological group. Let \mathcal{F} be an orbit type family for G , and k a covariant coefficient system for \mathcal{F} , over the ring R . Then there exists an equivariant homology theory $H_*^G(\ ; k)$ defined on the category of all G -pairs and G -maps (and with values in the category of R -modules), which satisfies all seven equivariant Eilenberg-Steenrod axioms, and which has the given coefficient system k as coefficients.*

By the statement that $H_*^G(\ ; k)$ satisfies the equivariant dimension axiom and has k as coefficients we mean the following. If $H \in \mathcal{F}$ then

$$H_n^G(G/H; k) = 0 \quad \text{for} \quad n \neq 0,$$

and there exists a natural isomorphism

$$\gamma : H_0^G(G/H; k) \xrightarrow{\cong} k(G/H).$$

The meaning of the rest of Theorem 1.1. is clear.

THEOREM 1.2. — *Let G and \mathcal{F} be as above, and let m be a contravariant coefficient system for \mathcal{F} , over the ring R . Then there exists an equivariant cohomology theory $H_G^*(\ ; m)$ defined on the category of all G -pairs and G -maps (and with values in the category of R -modules), which satisfies all seven equivariant Eilenberg-Steenrod axioms, and which has the given coefficient system m as coefficients.*

For the details we refer to [5] and [6]. We call $H_*^G(\ ; k)$ for equivariant singular homology with coefficients in k , and $H_G^*(\ ; m)$ for equivariant singular cohomology with coefficients in m . Further properties, like functoriality in the transformation group G , transfer homomorphisms, Kronecker index, and a cup-product for equivariant singular cohomology are also given in [5] and [6].

2. Equivariant CW complexes.

In this section G denotes a compact Lie group.

DEFINITION 2.1. — Let X be a Hausdorff G -space and A a closed G -subset of X , and n a non-negative integer. We say that X is obtainable from A by adjoining equivariant n -cells if there exists a collection $\{c_j^n\}_{j \in J}$ of closed G -subsets of X such that.

1) $X = A \cup \left(\bigcup_{j \in J} c_j^n \right)$, and X has the topology coherent with $\{A, c_j^n\}_{j \in J}$.

2) Denote $\dot{c}_j^n = c_j^n \cap A$, then

$$(c_j^n - \dot{c}_j^n) \cap (c_i^n - \dot{c}_i^n) = \emptyset \quad \text{if } i \neq j.$$

3) For each $j \in J$ there exists a closed subgroup H_j of G and a G -map

$$f_j: (E^n \times G/H_j, S^{n-1} \times G/H_j) \rightarrow (c_j^n, \dot{c}_j^n)$$

such that $f_j(E^n \times G/H_j) = c_j^n$, and f_j maps $(E^n - S^{n-1}) \times G/H_j$ homeomorphically onto $c_j^n - \dot{c}_j^n$.

DEFINITION 2.2. — An equivariant relative CW complex (X, A) consists of a Hausdorff G -space X , a closed G -subset A of X , and an increasing filtration of X by closed G -subsets $(X, A)^k$ $k = 0, 1, \dots$, such that.

1) $(X, A)^0$ is obtainable from A by adjoining equivariant 0-cells, and for $k \geq 1$ $(X, A)^k$ is obtainable from $(X, A)^{k-1}$ by adjoining equivariant k -cells.

2) $X = \bigcup_{k=0} (X, A)^k$, and X has the topology coherent with $\{(X, A)^k\}_{k \geq 0}$.

The closed G -subset $(X, A)^k$ is called the k -skeleton of (X, A) . If $A = \emptyset$ we call X an equivariant CW complex and denote the k -skeleton by X^k .

It is not difficult to prove the equivariant analogues of the standard elementary properties of CW complexes, that is,

the equivariant homotopy extension property for an equivariant relative CW complex (X, A) , the equivariant skeletal approximation theorem, and an equivariant Whitehead theorem. These results are stated as Propositions 2.3-2.5 in [4], and will not be repeated here. I understand that both C. Vasekaran and S. Willson have also proved these three results.

Our main result about equivariant CW complexes is the following theorem, see [4] and [5].

THEOREM 2.1. — *Every differentiable G -manifold M is an equivariant CW complex.*

This result has also been proved by T. Matsumoto, see [7] Proposition 4.4. In fact a stronger result is true. We proved in [5] that every differentiable G -manifold can be equivariantly triangulated. The proof of this uses the theorem of C.T. Yang [9] that the orbit space G/M can be triangulated, the existence of slices, and the « covering homotopy theorem » of Palais [8] Theorem 2.4.1. It should be observed that Proposition 4.4. in Matsumoto [7] also proves this stronger result.

3. Equivariant singular homology and cohomology of finite dimensional equivariant CW complexes.

We are still assuming that G is a compact Lie group. An equivariant CW complex X is called finite dimensional if $X = X^m$ for some m . Let us for simplicity assume that the orbit type family \mathcal{F} under consideration is the family of all closed subgroups of G . We say that a covariant coefficient system k is finitely generated if $k(G/H)$ is a finitely generated R -module for every closed subgroup H of G .

THEOREM 3.1. — *Let X be a finite dimensional equivariant CW complex. Then the n -th homology of the chain complex*

$$\dots \xleftarrow{\partial} H_{n-1}^G(X^{n-1}, X^{n-2}; k) \xleftarrow{\partial} H_n^G(X^n, X^{n-1}; k) \xleftarrow{\partial} \dots$$

is isomorphic to $H_n^G(X; k)$.

Together with Theorem 2.1 this gives us.

COROLLARY 3.2. — *Let M^m be an m -dimensional differentiable G -manifold. Then*

$$H_r^G(M^m; k) = 0 \quad \text{for} \quad r > m.$$

If M^m moreover is compact, and k is a finitely generated coefficient system over a noetherian ring R , then $H_r^G(M^m; k)$ is a finitely generated R -module for every r .

The corresponding results for cohomology are true.

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