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ROGER C. MCCANN

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## ON ABSOLUTE STABILITY

by Roger C. McCANN

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It is well known that absolute stability of a compact subset  $M$  of a locally compact metric space can be characterized by the presence of a fundamental system of absolutely stable neighbourhoods, and also by the existence of a continuous Liapunov function  $V$  defined on some neighbourhood of  $M = V^{-1}(0)$ , [1]. Here we characterize the absolute stability of  $M$  in terms of the cardinality of the set of positively invariant neighbourhoods of  $M$ .

Throughout this paper  $\mathbb{R}$  and  $\mathbb{R}^+$  will denote the reals and non-negative reals respectively. A rational number  $r$  is called dyadic if and only if there are integers  $n$  and  $j$  such that  $n \geq 0$ ,  $1 \leq j \leq 2^n$ , and  $r = \frac{j}{2^n}$ .

A dynamical system on a topological space  $X$  is a mapping  $\pi$  of  $X \times \mathbb{R}$  into  $X$  satisfying the following axioms (where  $x\pi t = \pi(x, t)$ ):

- (1)  $x\pi 0 = x$  for  $x \in X$ .
- (2)  $(x\pi t)\pi s = x\pi(t + s)$  for  $x \in X$  and  $t, s \in \mathbb{R}$ .
- (3)  $\pi$  is continuous in the product topology.

If  $M \subset X$  and  $N \subset \mathbb{R}$ , then  $M\pi N$  will denote the set  $\{x\pi t : x \in M, t \in N\}$ . A subset  $M$  of  $X$  is called positively invariant if and only if  $M\pi\mathbb{R}^+ = M$ . A point  $x \in X$  is called a critical point if and only if  $x\pi\mathbb{R} = \{x\}$ . A subset  $M$  of  $X$  is called stable if and only if every neighbourhood of  $M$  contains a positively invariant neighbourhood of  $M$ .

A Liapunov function for a positively invariant compact subset  $M$  of  $X$  is a continuous mapping  $V$  of a neighbour-

hood  $W$  of  $M$  into  $R^+$  such that  $V^{-1}(0) = M$  and  $V(x\pi t) \leq V(x)$  for  $x \in W$  and  $t \in R^+$ .

Absolute stability is defined in terms of a prolongation and is characterized by the following theorem, [1].

**THEOREM.** — *Let  $M$  be a compact subset of a locally compact metric space. Then the following are equivalent:*

- (i) *There is a Liapunov function  $V$  for  $M$ .*
- (ii)  *$M$  possesses a fundamental system of absolutely stable neighbourhoods.*
- (iii)  *$M$  is absolutely stable.*

**LEMMA 1.** — *Let  $A \subset R$  be uncountable. Then there exists an  $x \in A$  such that every neighbourhood of  $x$  contains uncountably many elements of  $A$ .*

*Proof.* — [4, 6,23, III].

The following is a consequence of Lemma 1.

**LEMMA 2.** — *Let  $A \subset R$  be uncountable. Then there exists an  $x \in A$  such that the sets  $\{y \in A : y < x\}$  and  $\{y \in A : x < y\}$  are uncountable.*

**LEMMA 3.** — *Let  $S$  and  $T$  be relatively compact sets of a locally compact connected metric space  $X$  and  $\mathcal{D}$  a family of open sets of  $X$  such that*

- (i) *for every  $U \in \mathcal{D}$ ,  $\bar{S} \subset U \subset \bar{U} \subset T$ ,*
  - (ii) *if  $U, V \in \mathcal{D}$ , then either  $\bar{U} \subset V$  or  $\bar{V} \subset U$ .*
- Then there is a  $W \in \mathcal{D}$  such that the sets  $\{U \in \mathcal{D} : U \subset W\}$  and  $\{U \in \mathcal{D} : W \subset U\}$  are uncountable.*

*Proof.* — Since  $X$  is connected, the boundary  $\partial U$  of  $U \in \mathcal{D}$  is nonempty. If  $U \in \mathcal{D}$ , then  $\partial U$  is compact since  $T$  is relatively compact. Let  $d$  be a metric on  $X$  and define  $f: \mathcal{D} \rightarrow R^+$  by  $f(U) = d(\bar{S}, \partial U)$ . If  $U, V \in \mathcal{D}$  with  $\bar{U} \subset V$ , then  $f(U) < f(V)$ . Let  $A$  be the image of  $\mathcal{D}$  under  $f$ .

Then  $f$  is a one-to-one order preserving mapping of  $\mathcal{D}$  onto  $A$ .  $A$  is uncountable since  $\mathcal{D}$  is such. By Lemma 2 there is an  $x \in A$  such that the sets  $\{y \in A : x < y\}$  and

$\{y \in A : y < x\}$  are uncountable. Set  $W = f^{-1}(x)$ . It is easily verified that

$$\begin{aligned}\{U \in \mathcal{D} : U \subset W\} &= \{f^{-1}(y) : y < x\}, \\ \{U \in \mathcal{D} : W \subset U\} &= \{f^{-1}(y) : x < y\},\end{aligned}$$

and that both sets are uncountable.

**THEOREM 4.** — *A nontrivial compact subset  $M$  of a locally compact connected metric space is absolutely stable if and only if  $M$  possesses a fundamental system  $\mathcal{F}$  of open positively invariant neighbourhoods such that*

(i) for each  $U \in \mathcal{F}$ , the set  $\{V \in \mathcal{F} : V \subset U\}$  is uncountable,

(ii) if  $U, V \in \mathcal{F}$ , then either  $\bar{U} \subset V$  or  $\bar{V} \subset U$ .

*Proof.* — Since  $X$  is connected, no nontrivial subset of  $X$  is both open and closed. If  $M$  is absolutely stable, then there is a continuous Liapunov function  $V$  for  $M$ . Set  $\mathcal{F} = \{V^{-1}([0, r]) : r \text{ in the range of } V\}$ . It is easily verified that  $\mathcal{F}$  possesses the desired properties. Now assume that  $\mathcal{F}$  is a fundamental system of open positively invariant neighbourhoods of  $M$  with properties (i) and (ii). For each dyadic rational we will construct a set  $U(r) \in \mathcal{F}$  such that  $U(r) \subset U(s)$  whenever  $r < s$ . We first obtain from  $\mathcal{F}$  a fundamental system of neighbourhoods  $\left\{U\left(\frac{1}{2^n}\right) : n \text{ a non-negative integer}\right\}$  such that  $U\left(\frac{1}{2^{n+1}}\right) \subset U\left(\frac{1}{2^n}\right)$  and the set  $\left\{A \in \mathcal{F} : U\left(\frac{1}{2^{n+1}}\right) \subset A \subset U\left(\frac{1}{2^n}\right)\right\}$  is uncountable. This is done by induction in the following manner. Let  $N_i$  be a countable fundamental system of neighbourhoods of  $M$ . Let  $U(1) \subset N_1$  be an element of  $\mathcal{F}$  which is relatively compact. Suppose that  $U\left(\frac{1}{2^n}\right)$  has been defined. By Lemma 3 and property (ii), there is a  $B \in \left\{W \in \mathcal{F} : W \subset U\left(\frac{1}{2^n}\right)\right\}$  such that  $B \subset N_{n+1}$  and both  $\{W \in \mathcal{F} : W \subset B\}$  and

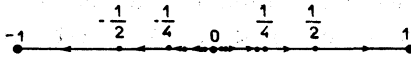
$$\left\{W \in \mathcal{F} : B \subset W \subset U\left(\frac{1}{2^n}\right)\right\}$$

are uncountable. Set  $U\left(\frac{1}{2^{n+1}}\right) = B$ . Now extend this system to one with the desired properties. For example, we chose  $U\left(\frac{3}{4}\right)$  to be any element  $C$  of  $\mathcal{F}$  such that the sets  $\left\{W \in \mathcal{F} : U\left(\frac{1}{2}\right) \subset V \subset C\right\}$  and  $\{W \in \mathcal{F} : C \subset V \subset U(1)\}$  are uncountable. This is possible by the properties of the sets  $U\left(\frac{1}{2^n}\right)$  and Lemma 3. Now define  $V : U(1) \rightarrow \mathbb{R}^+$  by  $V(x) = \inf \{r : x \in U(r)\}$ . Evidently  $V(x) = 0$  if and only if  $x \in M$ . If  $x \in U(r)$  and  $t \in \mathbb{R}^+$ , then  $x\pi t \in U(r)$  since  $U(r)$  is positively invariant. Therefore,

$$V(x) = \inf \{r : x \in U(r)\} \geq \inf \{r : x\pi t \in U(r)\} = V(x\pi t).$$

The continuity of  $V$  is proved as in the proof of Urysohn's lemma. Thus we have constructed a Liapunov function for  $M$ .  $M$  is absolutely stable.

*Example.* — Let  $X = [-1, 1]$ ,  $M = \{0\}$ , and  $\pi$  be the dynamical system indicated by the following diagram where the points  $\pm 2^{-n}$ ,  $n$  a non-negative integer, are critical points.



Clearly  $M$  is stable. The only open positively invariant neighbourhoods of  $M$  are  $X$  and intervals of the form  $(-2^{-m}, 2^{-n})$  where  $m$  and  $n$  are non-integers. There are only countably many such neighbourhoods. Hence,  $M$  is not absolutely stable.

**PROPOSITION 5.** — *Let  $X$  be the plane and  $p$  an isolated critical point. If each neighbourhood of  $p$  contains uncountably many periodic trajectories (cycles), then  $p$  is absolutely stable.*

*Proof.* — Let  $W$  be a disc neighbourhood of  $p$  which contains no critical points other than  $p$ . A cycle  $C$  is a Jordan curve and, hence, decomposes the plane into two components, one bounded (denoted by  $\text{int } C$ ) and the other unbounded. If  $C$  is a cycle, then  $\text{int } C$  contains a critical point, [3, VII,

4.8]. Hence, if  $C$  is a cycle in  $W$ , then  $C$  is the boundary of a neighbourhood (necessarily invariant) of  $p$ . It can be shown (the proof is almost identical with that of Proposition 1.10 of [6]) that if  $C_1$  and  $C_2$  are distinct cycles in  $W$ , then either  $\overline{\text{int } C_1} \subset \text{int } C_2$  or  $\overline{\text{int } C_2} \subset \text{int } C_1$ . Theorem 4 may now be applied to obtain the desired result.

Another characterization of absolute stability of compact sets is found in [5]. Non-compact absolutely stable sets are characterized in [3].

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Roger C. McCANN,  
Department of Mathematics,  
Case Western Reserve University,  
Cleveland, Ohio 44 106 (USA).

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