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LAYERING METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER*

by Avron DOUGLIS

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Introduction.

Global solutions of nonlinear first order partial differential equations have been an object of inquiry since 1950-51, when for the particular equation

$$u_t + uu_x = 0 \tag{1}$$

E. Hopf [16] and J. Cole [2] first succeeded in constructing "weak solutions" –solutions in a distribution sense– in the half-plane $t \geqslant 0$ with bounded, measurable initial values. Their methods, which were similar but independently arrived at, after some years were accomodated to other equations than (1) and additional approaches developed. At present, several ways are known to construct global weak solutions with prescribed initial values of fairly general types of scalar first order equations, including equations of Hamilton-Jacobi form

$$u_t + f(x, t, \text{grad } u) = 0 \tag{2}$$

and quasilinear "conservation laws"

$$u_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathrm{A}_i(x, t, u) = \mathrm{C}(x, t, u). \tag{3}$$

Here, $x = (x_1, x_2, \ldots, x_n) \in \mathrm{E}^n$, $u(x, t)$ denotes a scalar function, grad $u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$, $u_{x_i} = \partial u/\partial x_i$, $u_t = \partial u/\partial t$. A beginning has been made with scalar equations in two independant variables of the form $(f(u))_t + (g(u))_x = 0$ (D.P. Ballou [1]), and weak solutions of certain hyperbolic systems of quasilinear equations also have been obtained (see P.D. Lax [28], J. Glimm [14], Glimm and Lax [15], J.A. Smoller [32], J.L. Johnson and J.A. Smoller [19], ([1]) and the bibliographies in these papers). Nevertheless, global theory today is incomplete, and often it is complex and delicate. Hence, it seems worthwhile to continue to explore the scalar case, to which this paper is devoted.

([1]) See also J.A. Smoller and C.C. Conley [1-6] and the included references.

Our methods are based on the idea of approximating weak solutions by functions that are built up out of strict C^1 solutions in appropriately narrow domains. This kind of construction seems to have been first suggested in principle by B.L. Rozhdestvenskii [31] (1961) and was first carried out by N.N. Kuznetsov [27] (1967). I hit upon it later and outlined its application to equations (2) and (3) in an A.M.S. lecture([1]) in 1968, not knowing that Kuznetsov had already introduced it. In this paper, the original methods are greatly broadened and developed.

The principal estimates needed in this approach are, as Kuznetsov also stresses, suitable a priori inequalities for strict solutions —solutions of class C^1— of the given partial differential equations. Besides these inequalities, a smoothing transformation S of $C^1(E^n)$ into itself is required under which the inequalities are preserved. Supposing we have S, we can describe the idea as follows. Let $u_0(x)$ denote the prescribed initial data. The first step is to find, in a suitably thin layer $0 \leqslant t \leqslant h$, a strict solution $u_1(x, t)$ satisfying the initial condition $u_1(x, 0) = Su_0(x)$. The second step is to find in the layer $h \leqslant t \leqslant 2h$ a strict solution $u_2(x, t)$ such that $u_2(x, h) = Su_1(x, h)$. The third step is to find in the layer $2h \leqslant t \leqslant 3h$ a strict solution $u_3(x, t)$ such that $u_3(x, 2h) = Su_2(x, 2h)$, and so forth. The sectionally continuous, sectionally smooth function

$$\begin{aligned} u(x, t) &= u_1(x, t) \quad \text{for} \quad 0 \leqslant t < h, \\ &= u_2(x, t) \quad \text{for} \quad h \leqslant t < 2h, \\ &= u_3(x, t) \quad \text{for} \quad 2h \leqslant t < 3h, \end{aligned}$$

we call a *layered*, or *stratified, solution.* It obeys inequalities depending on u_0 that derive from the estimates originally established for strict solutions. It is of course not a weak solution of the original problem, but under appropriate conditions a sequence of layered solutions will approximate a weak solution if the smoothing performed is made finer and finer with h accordingly tending toward zero.

([1]) *The existence of weak solutions of first order partial differential equations,* an invited address before the Eastern Sectional Meeting of the American Mathematical Society at Johns Hopkins University, October 26, 1968.

A large class of smoothing operators, including arithmetic and other types of averaging as well as quite different kinds of transformations, exists with which layering works comparatively simply for equations of forms (2) and (3).

CHAPTER 2

GENERALIZED SOLUTIONS OF QUASILINEAR CONSERVATION LAWS WITH BOUNDED, MEASURABLE INITIAL DATA

Global weak solutions of multi-dimensional quasilinear conservation laws were first constructed by E.D. Conway and J.A. Smoller [4], who confined their efforts to equations of the form

$$u_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(u) = 0.$$

For equations

$$u_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(x, t, u) = C(x, t, u) \tag{$(A)_0$}$$

that depend explicitly upon x and t, similar results were then obtained by K. Kojima [20], extending Conway's and Smoller's finite difference methods, and by N.N. Kuznetsov [27], who used a layering procedure. Finite difference schemes in relation to the two equations $u_t + \nabla \cdot F(u) = 0$ and $u_t + \nabla \cdot F(u) = \nu \Delta u$ have been further elucidated by D.B. Kotlow [21].

These writers required the initial data to be of bounded variation in a sense of Tonelli and Cesari, but recently new estimates in conjunction with the "viscosity" method have made it possible for S.N. Kruzhkov [25, 26] to dispense with all demands upon initial data other than boundedness and measurability. In the same papers, Kruzhkov also has presented a new definition of "generalized solution" under which generalized solutions of $(A)_0$ are weak solutions but, almost regardless of the nature of the A_i, depend uniquely and continuously (in L^1) upon their initial data(1). In the one-dimensional case, Kruzhkov's generalized solutions satisfy the "entropy" or analogous conditions that hitherto have served as the basis of uniqueness theorems.

Ideas closely related to Kruzhkov's were presented by E. Hopf [18]. For sectionally continuous solutions, B.K. Quinn [1-5] has provided an elegant alternative to Kruzhkov's methods.

(1) P.A. Andreyanov [1-2] recently has extended Kruzhkov's methods and ideas to problems with just locally bounded initial data.

Viscosity methods like Kruzhkov's are of interest in themselves, as Kruyhkov stresses, but procedures not involving parabolic equations doubtless also are worthwhile. Such a procedure, which leads relatively easily to Kruzhkov's existence theorem, is to approximate the given initial data by functions of locally bounded variation (in the sense of Tonelli and Cesari) and then to pass from solutions that have these data to the solution with the data desired by means of a limit process based on Kruzhkov's theorem on continuous dependence alluded to earlier. (Kuznetsov [27] used a method somewhat like this for equations of the form $u_t + \sum_i (A_i(u))_{x_i} = 0$). In this approach, incidentally, second instead of third order differentiability of the A_i and C suffices.

This chapter is devoted to a direct approach, by means of layering, to the problem of bounded, measurable initial data and also provides a pattern for our discussion of the case of initial data of bounded variation (Chapter 4) and our treatment of Hamilton-Jacobi equations (Chapter 5). The lack of a needed estimate (an appropriate generalization of Theorem 3) restricts present considerations to conservation laws of the form

$$u_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(t, u) = C(t, u) \tag{A}$$

without explicit dependence upon x. Initial conditions

$$u(x, 0) = u_0(x) \tag{B}$$

with bounded, measurable u_0 are imposed.

1. Notation, assumptions, aims.

For an arbitrary point $x = (x_1, \ldots, x_n)$ of real Euclidean space E^n, we write

$$|x| = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}.$$

If v is a bounded function of x on E^n, we write

$$M_v = \sup_{x \in E^n} |v(x)| ;$$

if v is a function of x, t for $x \in E^n$ and $t \geqslant 0$, and $v(\bullet\,, t)$ is bounded for fixed t, we write

$$M_v(t) = \sup_{x \in E^n} |v(x\,, t)|.$$

Similar notation is used for vector functions.

Let $C(E^n)$ denote the class of functions that are bounded and continuous on E^n, and for $v \in C(E^n)$ write

$$|v|_0 = \sup_{x \in E^n} |v(x)|.$$

For a vector $f = (f_1, \ldots, f_n)$ with $f_i \in C(E^n)$ $(i = 1, \ldots, n)$, we write

$$|f|_0 = \left| \left[\sum_{i=1}^{n} f_i^2 \right]^{1/2} \right|_0 .$$

For $T > 0$, $M > 0$, $a < b$, $h > 0$, and positive integers m, define

$$\begin{aligned}
Z(T) &= \{(x\,, t) : x \in E^n\,, 0 \leqslant t < T\},\\
Z(a\,, b) &= \{(x\,, t) : x \in E^n\,, a \leqslant t \leqslant b\},\\
Z_m &= Z_{m,h} = Z((m-1)\,h\,, mh),\\
Z'_m &= \{(x\,, t) : x \in E^n\,, (m-1)\,h \leqslant t < mh\}\,;\\
Z(T\,; M) &= \{(x\,, t\,, u) : x \in E^n\,, 0 \leqslant t < T\,, |u| \leqslant M\},\\
Z(a\,, b\,; M) &= \{(x\,, t\,, u) : x \in E^n\,, a \leqslant t \leqslant b\,, |u| \leqslant M\},\\
\widetilde{Z}(T\,; M) &= \{(x\,, t) : 0 \leqslant t \leqslant T\,, |u| \leqslant M\}.
\end{aligned}$$

If Z represents any domain, by $C^k(Z)$ we shall mean the class of functions that are bounded and continuous and have bounded, continuous partial derivatives on Z of orders up to k(1). By $C_0^k(Z)$ we mean the subclass of $C^k(Z)$ consisting of the functions that have compact support in Z. Here, k denotes any nonnegative integer. If

(1) The conditions of boundedness will be dropped in Section 3, Chapter 3, and thereafter.

$f \in C^k(Z)$, we write $|f|_{C^k(Z)}$ to mean, as is usual, the sum of the least upper bounds in Z of the absolute values of f and of its partial derivatives of orders up to k. By $L^1(Z)$ we mean the class of Lebesgue summable functions on Z, and by $|f|_{L^1(Z)}$ the usual norm of f in that space.

For all positive constants M, T, we assume

$$C \in C^1(\widetilde{Z}(T\,;M))\,,\ A_i \in C^2(\widetilde{Z}(T\,;M))\,,\ i = 1, 2, \ldots, n.$$

We also assume u_0 to be bounded and measurable in E^n. Following Kruzhkov [26], we shall say that a bounded, measurable function u on the set Z(T) is a *generalized solution* of $(A)_0$, (B) if for all real constants k and all nonnegative functions $f \in C_0^1(Z(T))$ with $f(x\,,0) = 0$, we have

$$\iint_{Z(T)} \Big\{ |u - k|\, f_t + \operatorname{sign}(u-k) \bullet \sum_{i=1}^{n} [A_i(x\,,t\,,u) - A_i(x\,,t\,,k)]$$

$$+ \operatorname{sign}(u-k) \bullet \Big[C(x\,,t\,,u) - \sum_{i=1}^{n} A_{i,x_i}(x\,,t\,,k)\Big] f \Big\}\, dxdt$$

$$\geqslant 0\,; \tag{3a}$$

here, $f_t = \partial f/\partial t$, $f_{x_i} = \partial f/\partial x_i$. In addition, it is required that a subset $\mathscr{E}$ of $[0\,,T]$ of measure zero exist such that for every ball

$$S(r) = \{x \in E^n : |x| \leqslant r\}$$

$$\lim_{t\to 0} \int_{S(r)} |u(t\,,x) - u_0(x)|\, dx = 0, \tag{3b}$$

where during the limit process $0 < t < T$ and $t \notin \mathscr{E}$. As Kruzhkov proves, generalized solutions are weak solutions. Furthermore, all generalized solutions of an equation for which the A_i and C are of class C^1, and the first derivatives of the A_i with respect to x and t are Lipschitz-continuous, for $(x\,,\,t\,,u) \in Z(T\,,M)$ with arbitrary M, depend uniquely and continuously (in L^1) upon their initial data. (See also B.K. Quinn [1-5]).

For our purposes, we add to the previous assumptions the following growth condition upon C : for $x \in E^n$ and $t \geqslant 0$

$$\sup_{|\nu| \leqslant w} |C(t, \nu)| \leqslant E(w),$$

where E is a nonnegative, nondecreasing function defined for $w > 0$ such that

$$\int_a^\infty \frac{dw}{E(w)} = \infty \text{ for } a > 0.$$

Our principal aim in this chapter is to prove the following result :

THEOREM A. – *Under the assumptions stated, a generalized solution of* (A) *and* (B) *exists.*

To employ stratified solutions in constructing the solution demanded, we must first establish certain properties of strict solutions and also suitable smoothing operators. The latter will be discussed in detail in the next chapter, from which we shall borrow as required. We now turn to the former.

2. Strict solutions of equation (A_0).

By a strict solution of a partial differential equation we mean a continuously differentiable solution in the ordinary sense. In this section, we shall discuss strict solutions of first order conservation laws of the form

$$u_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(x, t, u) = C(x, t, u). \qquad (A)_0$$

where $C \in C^1(Z(T ; M))$, $A_i \in C^2(Z(T ; M))$ for $i = 1, \ldots, n$. We suppose a generalized form of the previous growth condition to hold : for $x \in E^n$ and $t \geqslant 0$,

$$\sup_{|\nu| \leqslant w} \left| \sum_{i=1}^{n} A_{i,x_i}(x, t, \nu) - C(x, t, \nu) \right| \leqslant E(w),$$

where $A_{i,x_i} = \dfrac{\partial A_i}{\partial x_i}$ and E, as before, is a nonnegative, nondecreasing function defined for $w > 0$ such that $\int_a^\infty \dfrac{dw}{E(w)} = \infty$ for $a > 0$. An initial condition of the form

$$u(x\,,\,t_0) = w(x) \tag{1}$$

is assumed, where w is an arbitrary bounded, continuously differentiable function with bounded gradient :

$$|w|_0 \leqslant w_0\,,\ |\mathrm{grad}\ w|_0 \leqslant w_1\,,$$

w_0 and w_1 being constants. Standard theory will provide positive constants A and a independent of w_1 such that u exists in the layer

$$Z(t_0\,,\,t_0 + 1/(Aw_1 + a))$$

and will furnish an upper bound pertaining to this layer for $M_u(t)$. Well known methods will also enable us to show that u –a strict solution of $(A)_0$ and (1)– is a generalized solution in an appropriate sense. Finally, we shall prove u to be mean continuous (in the L_1 sense) with respect to x, uniformly with respect to w_1. This information is the main content of the following three theorems.

THEOREM 1. *A strict solution u of* $(A)_0$, (1) *exists in the layer* $Z(t_0\,,t_0 + h)$, $h = 1/(Aw_1 + a)$, *where the constants* A *and* a *may depend upon* w_0, *the bounds assumed for the first derivatives of* C, *and those for the second derivatives of the* A_i. *This solution u is subject to a bound described as follows. Define the function*

$$\phi(t) \equiv \phi(t\ ;\,t_0\,,\,w_0)$$

by the condition

$$\int_{w_0}^{\phi(t)} \frac{dv}{E(v)} = t - t_0\,.$$

(Under our growth assumption, ϕ exists and is an increasing, absolutely continuous function for $t \geqslant t_0$). Then

$$M_u(t) \leqslant \phi(t\,;t_0\,,w_0) \ \text{ for } \ t_0 \leqslant t \leqslant t_0 + h. \tag{2}$$

THEOREM 2. – *Let u be a strict solution of* $(A)_0$ *in* $Z(t_0, t_1)$, *where* $t_0 < t_1$, *and suppose* $f \in C^1(Z(t_0, t_1))$ *to vanish identically for* $|x| > r$, *r being any positive constant. Then for any constant k*

$$\int_{Z(t_0,t_1)} \Big\{ |u-k| f_t + \operatorname{sign}(u-k) \bullet \sum_{i=1}^{n} [A_i(x,t,u) - A_i(x,t,k)] f_{x_i} + \operatorname{sign}(u-k) \bullet [C(x,t,u) - \sum_{i=1}^{n} A_{i,x_i}(x,t,k)] f \Big\} dxdt = \int_{E^n} |u-k| f dx \Big|_{t=t_0}^{t_1} .$$

THEOREM 3([1]). – *Let u be a strict solution of* (A) *in a layer* $Z(t_0, t_1)$ $(t_0 < t_1)$, *in which* $|u(x,t)| \leqslant M$. *Let*

$$N = \sup_{Z(t_0,t_1;M)} \left[\sum_{r=1}^{n} (\partial A_r/\partial u)^2 \right]^{1/2}.$$

With $\xi \in E^n$ and $T \geqslant t_1$, denote by $D(t;\xi,T)$ the horizontal disk

$$D(t;\xi,T) = \{x : |x-\xi| < N(T-t)\}$$

cut out of the plane with ordinate t by the back horizontal cone with vertex (ξ, T) and "slope" N. Then for any $z \in E^n$ and for $t_0 \leqslant t \leqslant t_1$, we have :

$$\int_{D(t;\xi,T)} |u(x+z,t) - u(x,t)|\, dx \leqslant \exp[C_1(t-t_0)] \bullet \int_{D(t_0;\xi,T)} |u(x+z,t_0) - u(x,t_0)| dx.$$

Here, $C_1 = \sup_{Z(t_0,t_1;M)} |C_u|$.

Proof of Theorem 1. – For strict solutions, equation $(A)_0$ is equivalent to

([1]) Kuznetsov [27] gives such a result in the case in which C = 0.

$$u_t + \sum_{i=1}^{n} a_i(x, t, u)\, u_{x_i} = c(x, t, u), \tag{3}$$

where $a_i = \partial A_i/\partial u$ and $c = C - \sum_{i=1}^{n} (\partial A_i/\partial x_i)$. The coefficients in (3) satisfy the following conditions :

i) For all positive constants M, T,

$$c, a_i \in C^1(Z(T ; M)), \; i = 1, \ldots, n ;$$

ii) in terms of the function E previously defined, c satisfies the condition

$$\sup_{|v| \leqslant w} |c(x, t, v)| \leqslant E(w)$$

for $x \in E^n$ and $t \geqslant 0$.

We use the method of characteristics as follows. For an arbitrary point ξ of the initial plane $t_0 = 0$, let the $n + 1$ functions

$$U(t ; \xi), \; x_j(t ; \xi), \; j = 1, 2, \ldots, n, \tag{4}$$

denote solutions of the differential equations

$$dU/dt = c(x, t, U), \; dx_j/dt = a_j(x, t, U), \quad j = 1, 2, \ldots, n, \tag{5}$$

satisfying the initial conditions

$$U(t_0 ; \xi) = w(\xi), \; x_j(t_0 ; \xi) = \xi_j, \quad j = 1, 2, \ldots, n. \tag{6}$$

We shall find an a priori bound for U in any interval

$$I : t_0 \leqslant t \leqslant t_1 \qquad (t_0 < t_1)$$

within which, for fixed ξ, the solution (4) exists and is of class C^1. By (5),

$$U(t ; \xi) = w(\xi) + \int_0^t c(x(s ; \xi), s, U(s ; \xi))\, dx \quad \text{for } t \in I.$$

Assumption (ii) implies that

$$U(t\,;\xi) \leqslant |w|_0 + \int_0^t E(|U(s\,;\xi)|)\,ds$$

$$\leqslant w_0 + \int_0^t E(|U(s\,;\xi)|)\,ds,$$

where, as required, w_0 is a constant such that

$$|w|_0 \leqslant w_0.$$

Since ϕ satisfies the condition $\phi'(t) = E(\phi(t))$ almost everywhere and $\phi(t_0) = w_0$, ϕ satisfies the condition

$$\phi(t) = w_0 + \int_{t_0}^t E(\phi(s))\,ds,$$

and reasoning such as is used for Gronwall's inequality shows that

$$|U(t\,;\xi)| \leqslant \phi(t) \quad \text{for} \quad t_0 \leqslant t \leqslant t_1. \tag{7}$$

Therefore, in particular, $|U(t_1\,, \xi)| \leqslant \phi(t_1)$, and standard theorems show that the values τ such that the solution of (5), (6) exists for $t_0 \leqslant t \leqslant \tau$ have no finite least upper bound. Hence, this solution (4) will exist for $t \geqslant t_0$. By well known theorems, this solution also is of class C^1 with respect to $t\,, \xi$ for $\xi \in E^n$, $t \geqslant t_0$.

For fixed t, consider the transformation

$$x = x(t\,;\xi). \tag{8}$$

We can invert (8), say as

$$\xi = \xi(x\,, t) \tag{9}$$

in any layer

$$x \in E^n, \qquad t_0 \leqslant t \leqslant t_0 + \delta \tag{10}$$

in which the functional determinant

$$J(t) \equiv J(t\,;\xi) = \partial(x)/\partial(\xi)$$

exceeds a positive constant. In such a layer, Cauchy's theory says that the function

$$u(x\,, t) \equiv U(t\,;\xi(x\,, t))$$

in a solution of (3) and the only solution that satisfies the initial conditions (1). Furthermore, the functions

$$p_j(t\,;\xi) \equiv u_{x_j}(x(t\,;\xi)\,,t)\,, \qquad j = 1, 2, \ldots, n,$$

which are the derivatives of u regarded as functions of t , ξ, satisfy the differential equations

$$dp_j/dt = -\sum_{i=1}^{n} (a_{i,x_j}p_i + a_{i,u}\,p_ip_j) + c_{x_j} + c_up_j\,. \tag{11}$$

These show that positive constants α and β depending on the first derivatives of the a_i and c exist such that

$$|dp_j/dt| \leqslant \alpha(p(t) + \beta)^2,$$

where

$$p(t) = p(t\,;\,\xi) = \max_i\,|p_i(t\,;\,\xi)|.$$

By reasoning similar to that which led to (7) we have

$$p(t\,;\,\xi) \leqslant q(t),$$

where $q(t)$ is the function that satisfies the differential equation $dq/dt = \alpha(q + \beta)^2$ and the initial condition $q(t_0) = w_1 \geqslant p(t_0)$. Consequently,

$$p(t\,;\xi) \leqslant \frac{w_1 + \alpha\beta(w_1 + \beta)(t - t_0)}{1 - \alpha(w_1 + \beta)(t - t_0)},$$

the right hand side in fact being $q(t)$. This shows, in particular, that if the transformation $x = x(t\,;\xi)$ can be inverted as $\xi = \xi(x\,,t)$ in a layer

$$\xi \in \mathrm{E}^n, \qquad t_0 \leqslant t \leqslant t_0 + \delta \tag{12}$$

with

$$\delta \leqslant 1/[2\alpha(w_1 + \beta)] \equiv h.$$

then in that layer

$$p(t) \leqslant 2w_1 + \beta. \tag{13}$$

We shall now exhibit a positive lower bound for $\mathrm{J}(t) \equiv \mathrm{J}(t\,;\xi)$ in this layer. It is well known –and easily proved from (5) and (11)– that

$$dJ/dt = \sum_{i=1}^{n} (a_{i,x_i} + a_{i,u}\, p_i)\, J. \tag{14}$$

It follows from this that

$$|dJ/dt| \leqslant (\alpha p + \gamma)J \quad \text{for} \quad t_0 \leqslant t \leqslant t_0 + \delta \tag{15}$$

with the same α as before and γ a bound for $\left|\sum_i a_{i,x_i}\right|$. In view of (6), we have $J(t_0) = 1$, and the homogeneous differential equation (14) implies that $J > 0$. By (15) and (13),

$$\begin{aligned}\log J &\geqslant -[\alpha(2w_1 + \beta) + \gamma]\, h \\ &= -\frac{\alpha(2w_1 + \beta) + \gamma}{2\alpha(w_1 + \beta)} \\ &= -1 - \frac{\gamma - \alpha\beta}{2\alpha w_1 + 2\alpha\beta} \\ &\geqslant -j_0,\end{aligned}$$

where

$$j_0 = \begin{cases} 1 & \text{if} \quad \gamma \leqslant \alpha\beta \\ \dfrac{\gamma + \alpha\beta}{2\alpha\beta} & \text{if} \quad \gamma > \alpha\beta . \end{cases}$$

Thus,

$$J(t) \geqslant e^{-j_0} \quad \text{for} \quad t_0 \leqslant t \leqslant t_0 + \delta. \tag{16}$$

But δ is any positive number not exceeding h such that the equations $x = x(t\,;\xi)$ can be solved for ξ in the layer (12). Hence, in view of this estimate for the Jacobian of the transformation, the least upper bound of such δ must be h. This implies that the inversion (9) can be performed in the layer $Z(t_0\,, t_0 + h)$, and Cauchy's theory now guarantees that $U(t\,;\xi(x\,,t))$ is a solution of (1), (3) in this layer. In view of (7), the absolute value of this solution is not greater than $\phi(t\,;t_0\,, w_0)$, as the theorem states. Thus, this theorem is wholly proved.

Proof of Theorem 2. – Since k is a constant, we have

$$(u-k)_t + \sum_{i=1}^{n} (A_i(x,t,u) - A_i(x,t,k))_{x_i}$$

$$= C(x,t,u) - \sum_{i=1}^{n} A_{i,x_i}(x,t,k) \equiv C^*. \qquad (17)$$

Let S denote any maximal subdomain of the cylinder

$$Z' = \{(x,t) : |x| < r,\ t_0 < t < t_1\}$$

within which $u \neq k$; by continuity, $u - k$ has constant sign in S. Letting S' denote the part of the boundary of S on which $t \neq t_0$, $t \neq t_1$, and $|x| < r$, we have by maximality

$$u - k = 0 \quad \text{on} \quad S'.$$

We intend to show that

$$\int_S \{(u-k) f_t + \sum_i (A_i(x,t,u) - A_i(x,t,k)) f_{x_i}$$

$$+ C^* f\}\, dxdt = \int_{E^n} f(u-k)\, dx \Big|_{S_0''}^{S_1''}, \qquad (*)$$

S_0'' denoting the part of the boundary of S on which $t = t_0$ and S_1'' the part on which $t = t_1$. (Either or both of these parts may be empty). Since sign $(u - k)$ is constant in S, when (*) is justified it will imply

$$\int_S \{f_t\, |u-k| + \text{sign}\,(u-k) \cdot [f_{x_i} (A_i(x,t,u) - A_i(x,t,k))$$

$$+ C^* f]\}\, dxdt = \int_{E^n} f\, |u-k|\, dx \Big|_{S_0''}^{S_1''}.$$

Then summing the relations of this form for all domains S as described will give

$$\int_{Z(t_0,t_1)} \{f_t\, |u-k| + \text{sign}\,(u-k) \cdot [(A_i(x,t,u) - A_i(x,t,k)) f_{x_i}$$

$$+ C^* f]\}\, dxdt = \int_{E^n} f\, |u-k|\, dx \Big|_{t=t_0}^{t_1},$$

as claimed, since f is zero for $|x| \geqslant r$.

We have still to justify (*). Let S_ν be an open subset of S with piecewise smooth boundary $\dot{S}_\nu$, $\nu = 1, 2, \ldots$, such that $S_\nu \cup \dot{S}_\nu \subset S_{\nu+1}$ and $\bigcup_{\nu=1}^{\infty} S_\nu = S$. We require in addition for any point P of $\dot{S}_\nu$

$$\mathrm{dist}\,(P, S') \leqslant \alpha d_\nu\,,\ d_\nu = \inf_{Q\in \dot{S}_\nu} \mathrm{dist}\,(Q, S'),$$

where α is a constant greater than 1, and for an arbitrary set V

$$\mathrm{dist}\,(P, V) = \inf_{R\in V} |P - R|.$$

(S_ν might be, for instance, a union of open cubes of edge $2^{-\nu-1}n^{-1/2}$, covering the part of S that is removed from S' by at least the distance $2^{-\nu}$, with $\alpha = 2$). Set

$$S^* = Z'\setminus S\,,\ S_\nu^* = Z'\setminus S_\nu\,,$$

and define the functions

$$\begin{aligned} \widetilde{u} &= u \quad \text{in} \quad S, & \widetilde{u}_\nu &= u \quad \text{in} \quad S_\nu\,,\\ &= k \quad \text{in} \quad S^*, & &= k \quad \text{in} \quad S_\nu^*. \end{aligned}$$

By our previous remarks, $\widetilde{u}$ is Lipschitz continuous in Z', say with Lipschitz constant γ.

For $\nu = 1, 2, \ldots$, let $\zeta_\nu(x, t)$ be a function in $C_0^1(E^{n+1})$ such that $\zeta_\nu \geqslant 0$, $\iint \zeta_\nu\, dxdt = 1$,

$$\zeta_\nu(x, t) = 0 \quad \text{for} \quad |x|^2 + t^2 > (d_\nu/2)^2,$$

and

$$|D\zeta_\nu(x, t)| \leqslant \beta/d_\nu\,,$$

where β is a suitable constant, D represents differentiation with respect to t or a component of x, and the integration with respect to x and t is over E^{n+1}. The continuously differentiable functions

$$u_\nu = \zeta_\nu * \widetilde{u}_\nu = \iint \zeta_\nu(x - y, t - s)\, \widetilde{u}_\nu(y, s)\, dyds$$

then have the following properties :

a) $u_\nu = k$ at all points nearer S' than the distance $d_\nu/2$;

b) $\lim_{\nu\to\infty} u_\nu = u$ and $\lim_{\nu\to\infty} Du_\nu = Du$ in S ;

c) the derivatives Du_ν are bounded uniformly with respect to ν.

Properties (a) and (b) are clear. To justify (c), we write

$$u_\nu = \zeta_\nu * \widetilde{u} + \zeta_\nu * (\widetilde{u}_\nu - \widetilde{u}) \equiv Y + Z.$$

The first derivatives of Y are uniformly bounded, because $\widetilde{u}$ is Lipschitz-continuous. Since $\widetilde{u}_\nu = \widetilde{u}$ in $S^* \cup S_\nu$, we have

$$DZ = \int_{S \setminus S_\nu} D\zeta_\nu (x - y, t - s)\,[k - u(y, s)]\,dyds,$$

and by the assumed estimates

$$|DZ| \leqslant (\beta/d_\nu)\,\gamma\alpha d_\nu \int_{S \setminus S_\nu} dyds = \alpha\beta\gamma\,\mu\,(S \setminus S_\nu),$$

μ referring to Lebesgue measure in $m + 1$ dimensions. Consequently, $DZ \to 0$ as $\nu \to \infty$, and we conclude, in particular, that the derivatives Du_ν are uniformly bounded, as asserted.

For this reason, if Lu_ν is the expression obtained by replacing u by u_ν in the first member of (17), then Lu_ν is bounded uniformly with respect to ν. In addition, by property (b), $\lim\limits_{\nu \to 0} Lu_\nu = C^*$ in S. Therefore, Lebesgue's theorem gives :

$$\lim_{\nu \to \infty} \int_S f L u_\nu\,dxdt = \int_S f C^* dxdt.$$

On the other hand, partial integrations in the usual way result in the relation

$$\int_S f L u_\nu\,dxdt = -\int_S \Big\{ (u_\nu - k)\,f_t + \sum_i f_{x_i}\,[A_i(x, t, u_\nu) - A_i(x, t, k)] \Big\}\,dxdt + \int f(u_\nu - k)\,dx \Big|_{S_0''}^{S_1''},$$

other boundary terms disappearing because of (a). Letting $\nu \to \infty$ in this condition results in (*), completing our proof of Theorem 2.

A different proof is given by Kruzhkov [25].

Proof of Theorem 3. – Our aim is to prove that each strict solution of (A) is mean continuous in x, uniformly with respect to t. Let $f \in C^1(Z(t_0, t_1))$, and suppose f to vanish for $|x| > r$, where r

is some constant. We multiply both sides of equation (A) by f, integrate over the layer $Z(t_0, t_1)$, and then integrate by parts to obtain :

$$\int_{Z(t_0,t_1)} (f_t u + \sum_i f_{x_i} A_i + fC)dxdt = \int_{E^n} fudx \Big|_{t=t_0}^{t_1} .$$

An analogous relation will also hold in which f is replaced by $g = T_{-s}f$, T_s denoting the translation defined by the condition $T_s f(x, t) = f(x + s, t)$ for $s \in E^n$. Since, for instance,

$$\int_{E^n} (T_{-s}f)udx = \int_{E^n} f(T_s u)\, dx ,$$

we can write this second relation as

$$\int_{Z(t_0,t_1)} (f_t T_s u + \sum_i f_{x_i} T_s A_i + fT_s C)dxdt = \int_{E^n} fT_s u\, dx \Big|_{t=t_0}^{t_1} .$$

Subtracting it from the first relation, we obtain :

$$\int_{Z(t_0,t_1)} \left\{ f_t(T_s u - u) + \sum_i f_{x_i}(T_s A_i - A_i) + f(T_s C - C)\right\} dxdt \tag{18}$$
$$= \int_{E^n} f(T_s u - u)\, dx \Big|_{t=t_0}^{t_1}$$

The rest of the argument depends upon a proper choice of f. Let

$$w(x) = \begin{cases} \operatorname{sign}\,(u(x+s, t_1) - u(x, t_1)) & \text{for } |x - \xi| \leqslant N(T - t_1) \\ 0 \quad \text{for} \quad |x - \xi| > N(T - t_1) \end{cases}$$

and let w_m, $m = 1, 2, \ldots$, be continuously differentiable functions on E^n such that

$$|w_m| \leqslant 1,\ w_m = 0 \ \text{ for } \ |x - \xi| > N(T - t_1) + 1/m ,$$

and $\lim_{m\to\infty} w_m = w$ almost everywhere. (For instance,

$$w_m(x) = m^n \int_{E^n} k(m(x - x'))\, w(x')\, dx'$$

with continuously differentiable kernel k such that $k \geqslant 0$, $k(x) = 0$ for $|x| \geqslant 1$, $\int k(x)\,dx = 1$). Soon we shall show that we can choose f so that

$$f(x\,,\,t_1) = w_m(x)\,,\,|f(x\,,\,t_0)| \leqslant e^{b(t_1 - t_0)},$$

$$f(x\,,\,t_0) = 0 \ \text{ for } \ |x - \xi| > \mathrm{N}(\mathrm{T} - t_0) + 1/m,$$

and the left member of (18) vanishes, b denoting a constant. Then it will be clear from (18) that

$$\int_{t=t_1} w_m(\mathrm{T}_s u - u)\,dx = \int_{t=t_0} f(\mathrm{T}_s u - u)\,dx.$$

Since the right side of this equality is not greater in absolute value than

$$e^{b(t_1 - t_0)} \int_{\mathrm{D}(t_0\,;\,\xi\,,\,\mathrm{U}_m)} |\mathrm{T}_s u - u|\,dx\,\Big|^{t=t_0},$$

where $\mathrm{U}_m = \mathrm{T} + 1/m$, letting $m \to \infty$ gives the required result.

To justify our choice of f, note first that

$$\begin{aligned}\mathrm{T}_s\mathrm{C}(t,\,u) - \mathrm{C}(t,\,u) &= \mathrm{C}(t\,,u(x + s\,,\,t)) - \mathrm{C}(t\,,\,u(x\,,\,t))\\ &= (u(x + s\,,\,t) - u(x\,,\,t))\,\mathrm{C}'(x\,,\,t\,,\,s),\end{aligned}$$

where

$$\mathrm{C}'(x\,,\,t\,,\,s) = \int_0^1 \mathrm{C}_u(t\,,\,u(x\,,\,t) + \theta\,(u(x + s\,,\,t) - u(x\,,\,t))\,d\theta.$$

More briefly,

$$\mathrm{T}_s\mathrm{C} - \mathrm{C} = (\mathrm{T}_s u - u)\,\mathrm{C}',$$

and, analogously,

$$\mathrm{T}_s\,\mathrm{A}_i - \mathrm{A}_i = (\mathrm{T}_s u - u)\,\mathrm{A}'_i, \quad i = 1, \ldots, n,$$

C' and A'_i all being continuously differentiable in the domain considered. Schwarz's inequality applied to the integral expressions for the A'_i shows that

$$\sum_{i=1}^{n} (\mathrm{A}'_i)^2 \leqslant \mathrm{N}^2. \tag{19}$$

Substituting from the previous results, we can write the expression in curly brackets in (18) as

$$(\mathrm{T}_s u - u)\, \mathrm{L}f,$$

where

$$\mathrm{L}f = f_t + \sum_i \mathrm{A}_i' f_{x_i} + \mathrm{C}'f,$$

and we determine f by the condition $\mathrm{L}f = 0$,

$$f(x, t_1) = w_m(x).$$

Condition (19) and the nature of the support of w_m show that $f(x, t_0)$ vanishes as required. The inequality stated for $f(x, t_0)$, with b a bound for $|\mathrm{C}_u|$ in the layer considered, follows from the assumption that $|w_m| \leqslant 1$. Thus, all the conditions demanded of f are met, and Theorem 3 is proved.

This theorem can also be handled by such means as Kruzhkov [25] employed to show that generalized solutions depend uniquely and continuously upon the initial data. But those methods are as involved for strict as for generalized solutions, simplifications apparently not occuring in the case with which we are concerned.

3. Stratified solutions of initial value problems for equation (A).

In constructing stratified solutions of this problem, we shall smooth by means of averaging operations. Let $k(x)$ be a function of class C^1 in the cube

$$\mathrm{C} : |x_i| \leqslant 1, \quad i = 1, \ldots, n.$$

Assume that $k \geqslant 0$, that $\int_{\mathrm{C}} k dx = 1$, and that k is an even function of each individual coordinate x_i of $x = (x_1, \ldots, x_n)$ when the other coordinates are held fixed. Then if, say, $f \in \mathrm{C}(\mathrm{E}^n)$ –recall that $\mathrm{C}(\mathrm{E}^n)$ consists of the bounded, continuous functions on E^n –define for $\varepsilon > 0$

$$\mathrm{K}_\epsilon f(x) = (\mathrm{K}_\epsilon f)(x) = \int_{\mathrm{C}} k(\xi)\, f(x + \varepsilon \xi)\, d\xi.$$

This operator has the following properties :

$$|K_\epsilon f|_0 \leqslant |f|_0 , \tag{1}$$

$$\int_{|x|\leqslant r} |K_\epsilon f| \, dx \leqslant \int_{|x|\leqslant r+\epsilon\sqrt{n}} |f| \, dx \tag{2}$$

$$|\text{grad } K_\epsilon f| \leqslant k_1 \, \varepsilon^{-1} \, |f|_0 , \tag{3}$$

where k_1 is a constant depending on the function $k(\cdot)$. Properties (1) and (2) are immediate, and property (3) follows from the obvious generalization to n dimensions of the following computation. Suppose x to be a scalar quantity (i.e., $n = 1$). Then

$$K_\epsilon f(x) = \int_{-1}^{1} k(\xi) \, f(x + \varepsilon \xi) \, d\xi = \varepsilon^{-1} \int_{x-\epsilon}^{x+\epsilon} k((y - x)/\varepsilon) \, f(y) dy,$$

and

$$(d/dx) \, K_\epsilon f(x) = \varepsilon^{-1} \, [k(1) f(x + \varepsilon) - k(-1) f(x - \varepsilon)] - \varepsilon^{-2} \int_{x-\epsilon}^{x+\epsilon} k'((y - x)/\varepsilon) f(y) dy,$$

while the integral on the right is equal to

$$\varepsilon^{-1} \int_{-1}^{1} k'(\xi) \, f(x + \varepsilon \xi) \, d\xi.$$

Therefore,

$$|(d/dx) \, K_\epsilon f|_0 \leqslant [\,|k(1)| + |k(-1)| + \int_{-1}^{1} |k'(\xi)| \, d\xi] \, \varepsilon^{-1} |f|_0$$

justifying (3) in this case.

Four more properties of K_ϵ will be used. In the notation

$$\bar{f}(x) = |f(x)|,$$

the fourth property is that

$$|K_\epsilon f(x)| \leqslant K_\epsilon \bar{f}(x) . \tag{4}$$

The fifth, referring to the translation operator T_z defined for $z \in E^n$ by the rule $T_z f(x) = f(x + z)$, is :

$$K_\epsilon T_z = T_z K_\epsilon . \tag{5}$$

The sixth property is that for any constant k,

$$K_\epsilon f + k = K_\epsilon (f + k). \tag{6}$$

These are all trivial. The seventh property pertains to the order of weak approximation of f by $K_\epsilon f$: for all $g \in C_0^2(E^n)$,

$$\left| \int_{E^n} g(K_\epsilon f - f)\, dx \right| \leqslant C_2 \; |g|_{C^2(E^n)} \; |f|_0 \; a(\varepsilon), \tag{7}$$

where C_2 and the function $a(\varepsilon)$ depend only upon n, and

$$\lim_{\epsilon \to 0} \varepsilon^{-1} a(\varepsilon) = 0.$$

This property will be justified in the next chapter.

Kuznetsov used averaging operators such as K_ϵ, but required the kernel k to belong to $C_0^1(E^n)$.

Let $T > 0$ and $0 < h \leqslant 1/2a$ with T/h an integer. (The constant a is that of Theorem 1, Section 2). We shall construct a stratified solution of (A), (B) in $Z(T)$ with layer thickness h. Smoothing will be performed with K_ϵ, where

$$\varepsilon = 2\, A k_1\, \phi(T\, ; 0\, , |u_0|_0)\, h.$$

The resulting stratified solution v will be subject to the estimate

$$|v|_{C^0(Z(0,T))} \leqslant \phi(T\, ; 0\, , |u_0|_0). \tag{8}$$

For convenience, set $\phi(t) = \phi(t\, ; 0\, , |u_0|_0)$. Since

$$|\mathrm{grad}\; K_\epsilon u_0|_0 \leqslant k_1\, \varepsilon^{-1}\, |u_0|_0 < k_1\, \varepsilon^{-1}\, \phi(T),$$

Theorem 1 of Section 2 shows that a strict solution u_1 of (A) satisfying the initial condition

$$u_1(x\, , 0) = K_\epsilon u_0(x)$$

exists in a layer of thickness

$$\frac{1}{A k_1\, \varepsilon^{-1} \,.\, \phi(T) + a} = \frac{2h}{1 + 2ah}\, ;$$

since $h \leqslant 1/2a$, this thickness is $\geqslant h$, and u exists, in particular, in Z_1.

Furthermore,

$$|u_1|_{C^0(Z_1)} \leqslant \phi(h).$$

To proceed inductively, let m be an arbitrary positive integer $< T/h$, and suppose u_m to belong to $C^1(Z_m)$ and to satisfy the condition

$$|u_m|_{C^0(Z_m)} \leqslant \phi(mh). \tag{8$_m$}$$

Since $\phi(mh) < \phi(T)$, considerations like those in the case $m = 1$ show that a strict solution u_{m+1} of (A) exists in Z_{m+1} and conforms to the restriction

$$|u_{m+1}|_{C^0(Z_{m+1})} \leqslant \phi((m+1)h\,; mh\,, w_m),$$

where $w_m = \phi(mh)$. To complete the induction, it suffices to prove that

$$\phi((m+1)h\,; mh\,, w_m) = \phi((m+1)h).$$

For this purpose, define the sequence

$$F_0 = |u_0|_0\,,\ F_m = \phi(mh\,; (m-1)h\,, F_{m-1})\,,\ m = 1, 2, \ldots$$

By definition of ϕ,

$$\int_{F_{i-1}}^{F_i} dv/E(v) = h \quad \text{for} \quad i = 1, \ldots, m.$$

Summing gives us

$$\int_{F_0}^{F_m} dv/E = mh$$

while, again by definition,

$$\int_{F_0}^{\phi(mh)} dv/E = mh.$$

It follows that $F_m = \phi(mh)$, as contended. Thus, the induction is complete and inequality $(8)_m$, in particular, proved.

We now define a stratified function v in $Z(T)$ as

$$v = u_m \quad \text{in} \quad Z'_m \quad \text{for} \quad m = 1, 2, \ldots, T/h,$$

inequality $(8)_m$ implying inequality (8). We shall also denote v by v_h.

4. Precompactness of stratified solutions.

Let $S(r)$, $r > 0$, denote the sphere,

$$S(r) = \{x : x \in E^n , |x| < r\}$$

in E^n of radius r. In this section, we shall prove the following result :

PRECOMPACTNESS THEOREM. *A bounded, measurable function u in $Z(T)$ and a null sequence $\{h_k\}$ exist such that, for $r > 0$, for $t = 0$, and for almost all t in the interval* $(0, T)$,

$$\lim_{k\to\infty} \int_{S(r)} |v_{h_k}(x, t) - u(x, t)| \, dx = 0.$$

Our demonstration is in several steps. Let $v = v_h$ denote a stratified solution, as constructed in Section 3, with arbitrary h. Let $f \in C_0^2(E^n)$, and for $0 \leqslant t < T$ set

$$V(t) = V_h(t) = \int_{E^n} v_h(x, t) \, f(x) \, dx.$$

We intend to show that the variation

$$\operatorname{var} V$$

of V in the interval $0 \leqslant t < T$ is finite with a bound that is independent of h. Since v_h is stratified, V is continuous except for possible jumps at $h, 2h, \ldots, T - h$, and its continuous variation in $(0, T)$ is given by

$$\sum_{m=1}^{T/h} \int_{(m-1)h}^{mh} |V'(t)| \, dt.$$

We can obtain an upper bound on $|V'|$. The stratified solution $v = v_h$ being continuously differentiable for $(m - 1)h < t < mh$ and satisfying equation (A), we have

$$\begin{aligned} V'(t) &= \int (\partial v/\partial t) \, f dx \\ &= \int (C - \sum_i \partial A_i/\partial x_i) \, f dx \\ &= \int \Big(Cf + \sum_i f_{x_i} A_i\Big) \, dx. \end{aligned}$$

Therefore,

$$|V'| \leqslant c' \, |f|_{C^1(E^n)} ,$$

where c' is a constant depending on $\phi(T\,;0, |u_0|_0)$ and the support of f, and the continuous variation of V in $(0\,,T)$ does not exceed

$$c'T\,|f|_{C^1(E^n)} .$$

We estimate the discontinuous variation of V from the formula

$$V(mh + 0) - V(mh - 0) = \int_{E^n} (K_\epsilon u_m(x\,,mh) - u_m(x\,,mh))\, f(x)dx,$$

in which u_m is as defined in the previous section. By property (7) of Section 3,

$$|V(mh + 0) - V(mh - 0)| \leqslant C_2 \, |f|_{C^2(E^n)} \, \phi(T\,;0\,, |u_0|_0) \, a(\varepsilon).$$

Since ε is proportional to h, and $a(\varepsilon)$ is of higher than first order in ε, summing these inequalities for $m = 1, 2, \ldots, T/h$ gives us as upper bound for the discontinuous variation of V in the interval considered

$$\text{const.}\ |f|_{C^2(E^n)} \, \phi(T\,;0\,, |u_0|_0).$$

Combining this with our previous estimate of the continuous variation of V now gives a bound for var V that is uniform with respect to h, as desired.

It follows that a null sequence $\{h_k\}$ exists such that the sequence $\{V_{h_k}(t)\}$ has a limit as $k \to \infty$ for every t in the interval $(0\,,T)$ (see Natanson [29]).

Furthermore, if $f_j \in C_0^2(E^n)$, $j = 1, 2, \ldots$, by a diagonal process we can find a null sequence, which we again denote by $\{h_k\}$, such that the limits as $k \to \infty$ of the functions

$$V_{h_k,j}(t) = \int_{E^n} v_{h_k}(x\,,t)\, f_j(x)\, dx$$

exist for all $j = 1, 2, \ldots$, and all t in the interval $(0\,,T)$. As f_j we can choose a sequence dense in $L^1(E^n)$. Since the v_h are uniformly bounded (see (8), Section 3), we thereby arrive at the following fact :

PROPOSITION 1. – *If f is any bounded, measurable function on* E^n *with compact support, then*

$$\int_{E^n} v_{h_k}(x, t) f(x)\, dx$$

converges as $k \to \infty$ for $0 \leqslant t < T$.

The next step is to prove :

PROPOSITION 2. – *Let t be a fixed value in* $(0, T)$, *and consider any subsequence $\{h_{k'}\}$ of $\{h_k\}$. A subsequence $\{h_{k''}\}$ of $\{h_{k'}\}$ and a bounded, measurable function w on* E^n *exist such that*

$$\lim_{k'' \to \infty} |v_{h_{k''}}(\cdot, t) - w|_{L^1(S(r))} = 0$$

for every $r > 0$.

Proof. – The main step in justifying this statement is to show that for arbitrary r and t as indicated, the integrals

$$\int_{S(r)} |v_h(x + z, t) - v_h(x, t)|\, dx$$

converge to zero uniformly with respect to h as $|z| \to 0, z \in E^n$. A result of M. Riesz [30] then will imply that a function $w \in L^1(S(r))$ and a subsequence $\{\tilde{h}_{k''}\}$ of the sequence $\{h_{k'}\}$ exist such that

$$\lim_{k'' \to \infty} |v_{\tilde{h}_{k''}}(\cdot, t) - w|_{L^1(S(r))} = 0.$$

A diagonal procedure will produce a further subsequence $\{h_{k''}\}$ of $\{h_{k'}\}$ and a function w bounded and measurable in E^n that satisfy the conclusion of the proposition.

Let

$$S_{t,\delta} = \{x : x \in E^n, |x| \leqslant N(T - t) + \delta + R\},$$

and set

$$\begin{aligned} I_\delta(t) = I_\delta(t, z; v) &= \int_{S_{t,\delta}} |v(x + z, t) - v(x, t)|\, dx \\ &= \int_{S_{t,\delta}} |T_z v - v|\, dx \end{aligned}$$

where $R > 0$, $\delta \geqslant 0$, $z \in E^n$, $0 \leqslant t < T$, and $\nu = \nu_h$ as above. It suffices for the previous argument to prove that for arbitrary R and t

$$\lim_{|z| \to 0} I_0(t, z; \nu_h) = 0 \text{ uniformly with respect to } h. \tag{1}$$

Theorem 3, Section 2, shows that

$$I_0(t) \leqslant e^{C_1[t-(m-1)h]}\, I_0((m-1)h) \tag{2}$$

$$\text{for } (m-1)h \leqslant t < mh, \qquad m = 1, 2, \ldots, T/h.$$

(T is arbitrary in Theorem 3 ; we replace it at present by T + R/N).

In addition, we can estimate $I_\delta(mh)$, $m = 1, \ldots, T/h$, as follows. On the plane $t = mh$, by definition of ν,

$$\nu = K_\epsilon u_{m-1} ;$$

therefore,

$$T_z \nu = T_z K_\epsilon u_{m-1} = K_\epsilon T_z u_{m-1}$$

on this plane, and we have for $\delta \geqslant 0$

$$I_\delta(mh) = \int_{S_{mh,\delta}} |T_z K_\epsilon u_{m-1} - K_\epsilon u_{m-1}|dx \Big|^{t=mh}$$

$$= \int_{S_{mh,\delta}} |K_\epsilon (T_z u_{m-1} - u_{m-1})|dx \Big|^{t=mh}$$

$$\leqslant \int_{S_{mh,\delta+\epsilon'}} |T_z u_{m-1} - u_{m-1}|\, dx \Big|^{t=mh}$$

$$= \lim_{t \to mh-0} I_{\delta+\epsilon'}(t),$$

where $\varepsilon' = \varepsilon\sqrt{n}$. This and inequality (2) imply that

$$I_\delta(mh) \leqslant e^{C_1 h}\, I_{\delta+\epsilon'}((m-1)h) \quad \text{for} \quad m = 1, \ldots, T/h.$$

It follows that if $qh \leqslant t < (q+1)h$, then

$$I_0(t) \leqslant e^{C_1(t-qh)}\, I_0(qh) \leqslant e^{C_1 t}\, I_{q\epsilon'}(0) \leqslant e^{C_1 t}\, I_{(\epsilon'/h)t}(0),$$

or, in other words,

$$I_0(t, z; \nu_h) \leqslant I_{(\epsilon'/h)t}(0, z; K_\epsilon u_0)\, e^{C_1 t} \quad \text{for} \quad 0 \leqslant t < T.$$

This estimate justifies (1) and thus completes the proof of Proposition 2.

Propositions 1 and 2 enable us to prove :

PROPOSITION 3. – *Suppose that for all* $r > 0$

$$\int_{S(r)} |v_{h_{k'}}(x, t) - w(x)|\, dx \to 0 \quad as \quad k' \to \infty,$$

where w *is a bounded, measurable function on* E^n, t *is a fixed value in* $(0, T)$, *and* $\{h_{k'}\}$ *a subsequence of* $\{h_k\}$. *Then if* f *is any bounded, measurable function on* E^n *with compact support*

$$\lim_{k\to\infty} \int_{E^n} f(x)\,(v_{h_k}(x, t) - w(x))\, dx = 0. \tag{3}$$

Proof. – Taking r so large that $S(r)$ contains the support of f, we have immediately

$$\lim_{k'\to\infty} \int f(x)\, v_{h_{k'}}(x, t)\, dx = \int f(x)\, w(x)\, dx,$$

the domain of integration in each case being E^n. By Proposition 1, however,

$$\lim_{k\to\infty} \int f(x)\, v_{h_k}(x, t)\, dx$$

exists, and the conclusion stated is obvious.

We can now finish proving the theorem. At most one function w can satisfy (3) for all admitted f. Hence, the function w in Proposition 2 –a different function for each t– is unique and independent of the subsequence $\{v_{h_{k'}}\}$. Thus, every subsequence of $v_{h_k}(\cdot, t)$ contains a subsequence of itself that converges to w in $L^1(S(r))$ for $r \to 0$. This means that every subsequence of the sequence of real numbers

$$|v_{h_k}(\cdot, t) - w|_{L^1(S(r))} \quad k = 1, 2, \ldots,$$

contains a null subsequence of itself. It follows that the sequence itself has the limit 0, and this implies that for $r > 0$ Cauchy's condition

$$\lim_{\substack{k\to\infty\\ \ell\to\infty}} |v_{h_k}(\cdot\,,t) - v_{h_\ell}(\cdot\,,t)|_{L^1(S(r))} = 0$$

is satisfied with any t in $(0\,,T)$. Integrating gives us for the cylinder

$$C(r\,,T) = \{(x\,,t) : |x| < r\,, 0 \leqslant t < T\}$$

the relation

$$\lim_{k\to\infty} |v_{h_k} - v_{h_\ell}|_{L^1(C(r,T))}$$

$$= \lim_{k\to\infty} \int_0^T |v_{h_k}(\cdot\,,t) - v_{h_\ell}(\cdot\,,t)|_{L^1(S(r))}\,dt$$

$$= 0.$$

By completeness, a function $W \in L^1(C(r\,,T))$ exists such that

$$\lim_{k\to\infty} |v_{h_k} - W|_{L^1(C(r,T))} = 0.$$

Hence, for a suitable subsequence $\{h_{k^*}\}$ of $\{h_k\}$

$$\lim_{k^*\to\infty} |v_{h_{k^*}}(\cdot\,,t) - W(\cdot\,,t)|_{L^1(S(r))} = 0$$

for almost all t in $(0\,,T)$. On the other hand, for each such t, a function $w(\cdot\,,t)$ —that previously denoted by w— exists such that

$$\lim_{k\to\infty} |v_{h_k}(\cdot\,,t) - w(\cdot\,,t)_{L^1(S(r))} = 0.$$

It follows that for almost all t in $(0\,,T)$, the functions $w(\cdot\,,t)$ and $W(\cdot\,,t)$ are equivalent members of $L^1(S(r))$ and thus may be identified. Therefore, for almost all t in $(0\,,T)$

$$\lim_{k\to\infty} |v_{h_k}(\cdot\,,t) - W(\cdot\,,t)|_{L^1(S(r))} = 0.$$

This being true for arbitrary r, W can be regarded as the restriction to $C(r\,,T)$ of a bounded, measurable function u on $Z(T)$, and this u obviously satisfies the demands of the theorem.

5. The existence of a generalized solution of (A), (B).

We shall now conclude the proof of Theorem A, Section 1. In the last section, we obtained in an arbitrary layer Z(T) a bounded measurable function u such that for all $r > 0$

$$\lim_{k\to\infty} |v_{h_k} - u|_{L^1(C(r,T))} = 0$$

$\{h_k\}$ being a suitable null sequence and the v_{h_k} stratified solutions as described. Our aim is to prove u to be a generalized solution of (A), (B) under the definition in Section 1. Choose arbitrarily a constant k and a function $f \in C_0^1(Z(T))$, and let r be so large that the cylinder $C(r, T)$ contains the support of f. Theorem 2, Section 2, says that for each $h = h_j$

$$\int_{Z_i} \Big\{ f_t |v_h - k| + \operatorname{sign}(v_h - k) \cdot \Big[\sum_i f_{x_i} (A_i(t, v_h) - A_i(t, k)) + fC \Big] dxdt = \int_{E^n} f|v_h - k|\, dx \Big|_{t=(i-1)h}^{t=ih}$$

Summing over i gives us

$$\int_{Z(T)} \Big\{ f_t |v_h - k| + \operatorname{sign}(v_h - k) \cdot \Big[\sum_i f_{x_i} (A_i(t, v_h) - A_i(t, k)) + fC \Big] dxdt + \int_{E^n} f(x, 0)\, |v_h(x, 0) - k|\, dx = R_h, \tag{1}$$

where

$$R_h = \sum_{i=1}^{T/h} \int_{E^n} f(x, ih) \Big[|v_h(x, ih - 0) - k| - |v_h(x, ih + 0) - k| \Big] dx.$$

If $k \to \infty$, the left member of (1) tends toward the left member of inequality (3), Section 1. Hence, u will certainly be a generalized solution of (A), (B) if $\liminf_{h\to 0} R_h \geqslant 0$. To discuss this, set

$$f(x) = f(x, ih), \quad v(x) = v_h(x, ih - 0).$$

Then

$$v_h(x,\ ih + 0) = \mathrm{K}_\epsilon\, v(x),$$

and we have

$$|v_h(x\,,\, ih + 0) - k\,| = |\mathrm{K}_\epsilon v - k| = |\mathrm{K}_\epsilon(v - k)| \leqslant \mathrm{K}_\epsilon(|\,v - k\,|)\,.$$

For the $i - th$ term in R_h,

$$\int f(|\,v - k\,| - |\mathrm{K}_\epsilon v - k|)\,dx \geqslant \int f(|\,v - k| - \mathrm{K}_\epsilon(|\,v - k|))\,dx$$

$$\geqslant - \,|f|_{\mathrm{C}^2(\mathrm{Z}(\mathrm{T}))}\ \mathrm{C}_2\ (\phi(\mathrm{T}\ ;\, 0\ ,\, |u_0|_0) + |\,k\,|)\ a(\varepsilon)$$

by property (7), Section 3 ; the bound (8) in that section also has been used. Multiplying the last expression by T/h gives a lower bound for R_h. Since h is proportional to ϵ and $\epsilon^{-1} a(\epsilon) \to 0$, it follows that the lower limit of R_h as $h \to 0$ is nonnegative, and condition (3a), Section 2, follows.

To justify the second condition (relation (3b), Section 1) for generalized solutions, consider again stratified solutions v_{h_k} with layer thickness h_k such that $\lim\limits_{k \to 0} v_{h_k} = u$ almost everywhere. Denoting by $\mathcal{E}$ the set of t in $[0\,,\,\mathrm{T}]$ for which this limit relation fails to hold for almost all x, we have $\mu\,\mathcal{E} = 0$. We first prove u_0 to be the weak limit of the $v_h(\cdot\,,\,t)$ as t approaches 0 in the complement of $\mathcal{E}$. This is meant in the following sense :

Proposition 4. – *If $f(x)$ is bounded, measurable, and has compact support in* E^n, *then*

$$\lim_{t \to 0}{}' \int_{\mathrm{E}^n} f(x)\, u\,(x\,,\, t)\, dx = \int_{\mathrm{E}^n} f(x)\ u_0\,(x)\, dx,$$

the prime indicating that t while approaching zero is positive and avoids the null set $\mathcal{E}$.

Proof. – Since u_0 and u are bounded, it suffices to prove the lemma for $f \in \mathrm{C}_0^2(\mathrm{E}^n)$, which henceforth we shall assume. If $v(x\,,\,t)$ is a strict solution of (A) in a layer $\mathrm{Z}\,(t_0\ ,\, t_1)$, integration within this layer of the condition $f\left(v_t + \sum\limits_i \left(\dfrac{\partial \mathrm{A}_i}{\partial x_i}\right) - \mathrm{C}\right) = 0$ and integration by parts give us

$$\int_{E^n} f(x)\, v(x\,,t)\, dx \Big|_{t=t_0}^{t_1} = \int_{Z(t_0,t_1)} \sum_i f_{x_i} A_i + fC)\, dxdt.$$

We apply this result to the layers of a stratified solution v_{h_k}. Summing over the layers that are contained within $Z(0\,,t)$, where t is an arbitrary ordinate such that $0 < t < T$, we obtain :

$$\int_{E^n} f(x)\, (v_{h_k}(x\,,t) - S_{\epsilon_k} u_0(x))\, dx =$$

$$\int_{Z(0,t)} \Big(\sum_i f_{x_i} A_i + fC\Big)\, dxdt + R_k(t),$$

where ε_k is proportional to h_k (Section 2), and as noted previously

$$|R_k(t)| \leqslant |f|_{C^2(E^n)}\, b_k \quad \text{with} \quad \lim_{k\to 0} b_k = 0.$$

The integral on the right side can be estimated by $c\,|f|_{C^1(E^n)} t$, c depending on upper bounds for the $|A_i|$ and $|C|$ and on the support of f. Denoting the support of f by S_f, we have also

$$\left|\int_{E^n} f(u_0 - S_{\epsilon_k} u_0)\, dx\right| \leqslant |f|_{C^0(E^n)}\, c_k,$$

where $c_k = \int_{S_f} |u_0 - S_{\epsilon_k} u_0|\, dx \to 0$ as $k \to \infty$. Hence in sum,

$$\left|\int_{E^n} f(x)\, (v_{h_k}(x\,,t) - u_0(x))\, dx\right| \leqslant |f|_{C^2(E^n)}\, (ct + b_k + c_k).$$

For $t \notin \mathcal{E}$, letting $k \to \infty$ gives

$$\left|\int_{E^n} f(x)\, (u(x\,,t) - u_0(x))\, dx\right| \leqslant |f|_{C^2(E^n)}\, ct,$$

from which Proposition 4 follows in the case in which $f \in C_0^2$. As observed, the entire proposition follows from this case.

Now consider any null sequence $\{t_m\}$ of positive values not in $\mathcal{E}$. Theorem 3, Section 2, insures that a subsequence $\{t_{m'}\}$ and a bounded measurable function $u^*(x)$ exist such that

$$\lim_{m'\to\infty} |u(\cdot\,,t_{m'}) - u^*|_{L^1(S(r))} = 0 \quad \text{for} \quad r > 0.$$

This and Proposition 4 imply that $\int_{E^n} f(x)\, (u^*(x) - u_0(x))\, dx = 0$

for arbitrary bounded, measurable f with compact support, and we conclude that $u^* = u_0$ almost everywhere. Therefore,

$$\lim_{m' \to 0} |u(\cdot\,, t_{m'}) - u_0|_{L^1(S(r))} = 0 \quad \text{for} \quad r > 0.$$

Since the original sequence $\{t_m\}$ was arbitrary, we conclude that

$$\lim_{t \to 0}{}' \, |u(\cdot\,, t) - u_0|_{L^1(S(r))} = 0 \quad \text{for} \quad r > 0,$$

which is (3b), the final condition to be justified.

CHAPTER 3

METHODS OF SMOOTHING

In this chapter, we first derive formulas for the modifications in an arbitrary function produced by averaging (Sections 1 and 2). These lead us immediately to the estimate applied in Chapter 1 (inequality (7), Section 3) ; Kuznetsov [27] had used a similar inequality, which he proved differently. In Section 3, we go to other smoothing methods of interest in connection with solutions of finite variation, which will be discussed in Chapters 4 and 5.

1. Symmetric averaging operators.

Let

$$C_\epsilon = \{x : x = (x_1, \ldots, x_n), |x_i| \leqslant \varepsilon, i = 1, \ldots, n\}$$

denote the cube in E^n with center at the origin and edges of length 2ε ($\varepsilon > 0$) parallel to the coordinate axes. Suppose k to be in $L^1(C_\epsilon)$ and to be an even function with respect to each argument x_i, $i = 1, \ldots, n$. For convenience, we also assume :

$$k \geqslant 0, \int_{C_\epsilon} k(\xi)\, d\xi = 1.$$

If u denotes an arbitrary bounded, measurable function on E^n, we define the average of u with kernel k to be the integral

$$Ku(x) = (Ku)(x) = \int_{C_\epsilon} k(\xi)\, u(x + \xi)\, d\xi.$$

For the deviation of u from its average we shall prove that

$$Ku - u = \sum_{i=1}^{n} D_i^2 J_i, \tag{1}$$

where $J_1, \ldots, J_n$ denote bounded, measurable functions defined almost everywhere on E^n such that

$$|J_i| \leqslant \varepsilon^2 \, |u|_0 \, , \, i = 1, \ldots, n, \tag{2}$$

and $D_i = \partial/\partial x_i$ in the distribution sense. If $u \in C^1(E^n)$, each J_i admits the differentiations indicated, and formula (1) holds literally.

Through integration by parts, this formula implies inequality (7), Section 3, Chapter 1, with $a(\varepsilon) = \varepsilon^2$.

Our proof of (1) is based on the one dimensional case, in which case $x = x_1$ and we define

$$J_1(x) = J(x) = \int_{-\epsilon}^{\epsilon} M(|s|)\, u(x+s)\, ds$$

where for $0 \leqslant r \leqslant \varepsilon$,

$$M(r) = \int_r^{\epsilon} dr' \int_{r'}^{\epsilon} k(r'')\, dr''.$$

The formula in this case asserts that $Ku - u = d^2J/dx^2$. It suffices to establish it for $u \in C^1(E^n)$. Writing

$$J(x) = \int_{-\epsilon}^{0} M(-s)\, u(x+s)\, ds + \int_0^{\epsilon} M(s)\, u(x+s)\, ds,$$

we obtain

$$DJ = dJ/dx = \int_{-\epsilon}^{0} M(-s)\, u'(x+s)\, ds + \int_0^{\epsilon} M(s)\, u'(x+s)\, ds$$

$$= \int_{-\epsilon}^{0} M'(-s)\, u(x+s)\, ds - \int_0^{\epsilon} M'(s)\, u(x+s)\, ds$$

and by similar manipulations

$$D^2J = d^2J/dx^2 = 2M'(0)\, u(x) + \int_{-\epsilon}^{0} M''(-s)\, u(x+s)\, ds$$

$$+ \int_0^{\epsilon} M''(s)\, u(x+s)\, ds = -u(x) + Ku(x),$$

since $M'' = k$, which is even. Thus, (1) is justified in the one dimensional case ; inequality (2) is immediate.

A change in notation will help in extending this result to n dimensions. Still with $x = x_1$ and $D = d/dx_1$, define

$$Ik(x) = -M'(|x|) = \int_{|x|}^{\epsilon} k(y)\, dy,$$

and let δ denote Dirac's point distribution. The formula in one dimension we have just proved can be written :

$$\begin{aligned} D^2 \int_{-\epsilon}^{\epsilon} I^2 k(y) \cdot u(x+y)\, dy &= \int_{-\epsilon}^{\epsilon} k(y)\, u(x+y)\, dy \\ &\quad - \int_{-\epsilon}^{\epsilon} \delta(y)\, u(x+y)\, dy \\ &= \int_{-\epsilon}^{\epsilon} (k(y) - \delta(y))\, u(x+y)\, dy. \end{aligned}$$

If the integral

$$Ak = \int_{-\epsilon}^{\epsilon} k(y)\, dy$$

is not necessarily normalized to be 1, as we previously required, by considering k/Ak in place of k we have, more generally,

$$D^2 \int_{-\epsilon}^{\epsilon} I^2 k(y) \cdot u(x+y)\, dy = \int_{-\epsilon}^{\epsilon} (k(y) - \delta(y)\,(Ak))\, u(x+y)\, dy. \tag{3}$$

To prove (1) in the general case, let

$$I_i k(x) = \int_{|x_i|}^{\epsilon} k(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)\, ds,$$

$$A_i k(x) = \int_{-\epsilon}^{\epsilon} k(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)\, ds, \quad i = 1, \ldots, n,$$

and

$$k_0 = k\,,\ k_j = \left(\prod_{i=1}^{j} A_i\right) k\,,\ j = 1, \ldots, n.$$

Then $A_i k$ is independent of the i-th coordinate, k_j is independant of the first j coordinates, and $k_n = 1$. Formula (3) shows that if ψ_i is an arbitrary function –or distribution– on C_ϵ, but independent of the i-th coordinate, we have

$$\begin{aligned} & D_i^2 \int_{C_\epsilon} I_i^2 k(y) \cdot \psi_i(y)\, u(x+y)\, dy \\ &\qquad = \int_{C_\epsilon} [k(y) - \delta(y_i)\, A_i k(y)]\, \psi_i(y)\, u(x+y)\, dy. \end{aligned}$$

Replacing k by k_{i-1} and $\psi_i(y)$ by $\prod_{k=1}^{i-1} \delta(y_k)$ gives us the following formulas we shall need :

$$D_i^2 \int_{C_\epsilon} I_i^2 k_{i-1}(y) \prod_{q=1}^{i-1} \delta(y_q)\, u(x+y)\, dy$$

$$= \int_{C_\epsilon} [k_{i-1}(y) - \delta(y_i) k_i(y)] \prod_{q=1}^{i-1} \delta(y_q) u(x+y)\, dy,\ i = 1, \ldots, n \quad (4)$$

Now to prove (1), we represent the left side as a telescoping sum :

$$\begin{aligned} Ku(x) - u(x) &= \int k(y)\, u(x+y)\, dy \\ &\quad - \int k_n(y) \prod_{q=1}^{n} \delta(y_q)\, u(x+y)\, dy \\ &= \int [k(y) - k_1(y)\, \delta(y_1)]\, u(x+y) \\ &\quad + \int [k_1(y) - k_2(y)\, \delta(y_2)]\, \delta(y_1)\, u(x+y) dy \\ &\quad + \int [k_2(y) - k_3(y)\, \delta(y_3)]\, \delta(y_1)\, \delta(y_2)\, u(x+y)\, dy \\ &\quad + \ldots \\ &\quad + \int [k_{n-1}(y) - k_n(y)\, \delta(y_n)] \prod_{q=1}^{n-1} \delta(y_q)\, u(x+y)\, dy, \end{aligned}$$

all integrations being over C_ϵ. Formula (1), i.e.,

$$Ku - u = \sum_i D_i^2 J_i,$$

with

$$J_i(x) = \int_{C_\epsilon} I_i^2 k_{i-1}(y) \prod_{q=1}^{i-1} \delta(y_q)\, u(x+y)\, dy, \quad (5)$$

results, in view of (4). Since $I_i k_{i-1} \leqslant \frac{1}{2} A_i k_{i-1} = k_i/2$ and k_i is independent of the first i coordinates, we have $I_i^2 k_{i-1} \leqslant \frac{1}{2} I_i k_i \leqslant \frac{\varepsilon}{2} k_i$ and

$$|J_i| \leqslant \varepsilon^2\, |u|_0 \int k_i\, dy_{i+1} \ldots dy_n = \varepsilon^2\, |u|_0,$$

the last integration being over the domain $|y_p| \leqslant \varepsilon, p = i + 1, \ldots, n$. Thus inequality (2) is proved.

Later in this chapter, we shall need an estimate we now derive concerning $D_i J_i$. For fixed $i = 1, \ldots, n$, consider the rectangular portion of hyperplane

$$H_i = \{(x_1, \ldots, x_n) : x_i = c_i \, , a_k \leqslant x_k \leqslant b_k$$

for $k = 1, \ldots, n$, but $k \neq i\}$,

where c_i and the a_k and b_k are constants with $a_k < b_k$. Obviously, H_i is contained in the rectangular parallelepiped

$$Q_i^\epsilon = \{(x_1, \ldots, x_n) : |x_i - c_i| \leqslant \varepsilon \, , a_k - \varepsilon \leqslant x_k \leqslant b_k + \varepsilon$$

for $k = 1, \ldots, n$, but $k \neq i\}$.

The estimate we shall require is that for $u \in C^1(Q_i^\epsilon)$,

$$\int_{H_i} |D_i J_i| \, dx_i' \leqslant (\varepsilon/2) \int_{Q_i^\epsilon} |D_i u| \, dx, \tag{6}$$

where $dx_i' = dx_1 \ldots dx_{i-1} \; dx_{i+1} \ldots dx_n$ is the element of volume in H_i. To obtain this, we first differentiate J_i with respect to x_i under the sign of integration and use again the inequality

$$I_i^2 k_{i-1} \leqslant \varepsilon k_i/2$$

to obtain

$$|D_i J_i(x)| \leqslant (\varepsilon/2) \int_{C_\epsilon} k_i(y) \prod_{q=1}^{i-1} \delta(y_q) |D_i u(x + y)| \, dy$$

$$= (\varepsilon/2) \int k_i(y) \Big(\int_{-\epsilon}^{\epsilon} |D_i u(x + y)| \, dy_i \Big) \prod_{q=1}^{i-1} \delta(y_q) dy_i',$$

all y_k with $k \neq i$ being integrated over the domain $|y_k| \leqslant \varepsilon$. Integrating with respect to x and y gives

$$\int_{H_i} |D_i J_i| \, dx_i'$$

$$\leqslant \frac{1}{2} \varepsilon \int k_i(y) \prod_{q=1}^{i-1} \delta(y_q) \left[\int_{H_i} \Big(\int_{-\epsilon}^{\epsilon} |D_i u(x + y)| \, dy_i \Big) dx_i' \right] dy_i',$$

while the integral in square brackets is $\leqslant \int_{Q_i^\epsilon} |D_i u|\, dx$. Inequality (6) follows at once.

2. Radially symmetric averaging operators.

An averaging kernel k is radially symmetric if $k(x)$ is a function of $|x|$, say

$$k(x) = R(|x|).$$

In that case, we also call the operator K radially symmetric. For radially symmetric averages, a more elegant deviation formula than the previous (equation (1), Section 1) can be given, as follows.

Let

$$S_\epsilon = \{x : x \in E^n \ , \ |x| \leqslant \varepsilon\}$$

denote the ball in E^n with center at the origin and radius $\varepsilon > 0$. Suppose k to be in $L^1(S_\epsilon)$ and to be radially symmetric. We also assume :

$$k \geqslant 0 \ , \int_{S_\epsilon} k(x)\, dx = 1.$$

If u denotes an arbitrary bounded, measurable function on E^n, we define the average of u with this kernel k to be the integral

$$Ku(x) = (Ku)(x) = \int_{S_\epsilon} k(y)\, u(x+y)\, dy.$$

For the deviation $Ku - u$ we shall prove in this case that

$$Ku(x) - u(x) = \Delta \int_{S_\epsilon} M(|y|)\, u(x+y)\, dy, \tag{1}$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ in the distribution sense, and for $r > 0$

$$M(r) = \int_r^\epsilon \left[\frac{1}{w_n \rho^{n-1}} - \int_0^\rho \left(\frac{\sigma}{\rho}\right)^{n-1} R(\sigma)\, d\sigma\right] d\rho,$$

w_n denoting the surface area of the unit sphere $|x| = 1$ in E^n. If $u \in C^1(E^n)$, the differentiations indicated can actually be carried out, and formula (1) holds in the strict sense.

The function M is the solution of the second order singular differential equation

$$(M'(r)\, r^{n-1})' = r^{n-1}\, R(r) \tag{2}$$

satisfying the initial conditions

$$M(\varepsilon) = 0 \tag{3}$$

and

$$M'(\varepsilon) = 0. \tag{4}$$

Verifying these statements is straightforward, the condition

$$\int_{S_\epsilon} k(x)\, dx = w_n \int_0^\epsilon R(r)\, r^{n-1}\, dr = 1$$

being used in connection with (4). Two other important properties of M are :

$$\lim_{r\to 0} r^{n-1}\, M(r) = 0\ ,\ w_n \lim_{r\to 0} r^{n-1}\, M'(r) = -\, 1. \tag{5}$$

It suffices to prove (1) for $u \in C^2(E^n)$. For such u, define

$$J(x) = \int_{S_\epsilon} M(|y|)\, u(x+y)\, dy.$$

In spherical coordinates, we have

$$J(x) = w_n \int_0^\epsilon M(r)\, r^{n-1}\, u(x\,,\, r)\, dr,$$

$u(x\,,\,r)$ denoting the mean value of u on the sphere with center x and radius r :

$$u(x\,,\,r) = \frac{1}{w_n} \int_{|\eta|=1} u(x + r\eta)\, dS_\eta\ ,\ r > 0.$$

By dS_η is meant the element of area of the sphere, $|\eta| = 1$ over which the integration takes place. Since u is continuous, we also define

$$u(x\,,\,0) = u(x)\ .$$

It is known (see Courant-Hilbert [5] Vol. II, p. 699) that

$$r^{n-1}\, \Delta u(x\,,\,r) = (r^{n-1}\, u_r(x\,,\,r))_r\ ,$$

Δ being the Laplacean operator with respect to x, x and r being regarded as independent variables (in all, $n + 1$ variables), and the subscript r indicating partial differentiation with respect to r. Using this formula, we have

$$\Delta J(x) = w_n \int_0^\epsilon M(r)\, r^{n-1}\, \Delta u(x, r)\, dr = w_n \int_0^\epsilon M(r)\, (r^{n-1} u_r(x, r))_r dr.$$

Integrating by parts gives us, in view of (3) and (5),

$$\Delta J(x) = - w_n \int_0^\epsilon M'(r)\, r^{n-1}\, u_r(x, r)\, dr.$$

Again integrating by parts and using (4) and (5) we obtain

$$\begin{aligned}\Delta J(x) &= - u(x, 0) + w_n \int_0^\epsilon (r^{n-1} M'(r))'\, u(x, r)\, dr \\ &= - u(x) + w_n \int_0^\epsilon R(r)\, r^{n-1}\, u(x, r)\, dr \\ &= - u(x) + \int_{S_\epsilon} R(|y|)\, u(x + y)\, dy.\end{aligned}$$

Thus, formula (1) is proved.

3. Other methods of smoothing.

Averaging is only one kind of process meeting the rather delicate requirements –such as inequality (7), Section 3, Chapter 1– of stratified solutions. Additional ways of smoothing and otherwise modifying functions while satisfying the requirements will be described below ; it is believed that some will be of interest in other problems.

These more general processes pertain to functions of locally finite variation in the sense of Tonelli and Cesari, although in this section we shall keep to the class $C^1(E^n)$. (From now on, we revert to the usual meaning of $C^k(E^n)$, dropping our previous requirement that the derivatives of orders up to k of a member of this class be bounded). If $f \in C^1(E^n)$ and D is a domain of E^n, we define the *TC-variation* of f on D to be

$$V(f\,; D) = \sum_{i=1}^{n} \int_D |\partial f/\partial x_i|\, dx.$$

Our criterion pertains to a system of grid points in E^n that must be selected in advance. Changing our previous notation, here let

$$x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$$

denote an arbitrary point in E^n. For $i = 1, \ldots, n$, divide the $x^{(i)}$-axis by partition points

$$x_{j_i}^{(i)}, j_i = 0, \pm 1, \pm 2, \ldots,$$

where

$$\ldots < x_{-2}^{(i)} < x_{-1}^{(i)} < x_0^{(i)} < x_1^{(i)} < x_2^{(i)} < \ldots$$

and

$$x_{-k}^{(i)} \to -\infty, x_k^{(i)} \to \infty \quad \text{as} \quad k \to \infty.$$

The grid points

$$x_j = (x_{j_1}^{(1)}, x_{j_2}^{(2)}, \ldots, x_{j_n}^{(n)}),$$

in which j stands for the multi-index $(j_1, j_2, \ldots, j_n)$, are the vertices of the rectangular parallelepipeds

$$C_j = \{x : x_{j_i}^{(i)} \leqslant x^{(i)} \leqslant x_{j_i+1}^{(i)} \quad \text{for} \quad i = 1, \ldots, n\}.$$

For any multi-index $j = (j_1, \ldots, j_n)$ and for $k = 1, \ldots, n$, define the translation

$$T_k j = (j_1, \ldots, j_{k-1}, j_k + 1, j_{k+1}, \ldots, j_n).$$

For any quantity a_j indexed by j, define

$$T_k a_j = a_{T_k j},$$

for a function f on the grid points x_j define

$$T_k f(x_j) = f(T_k x_j),$$

and so forth.

Let Q be a rectangular parallelepiped in E^n that is the union of certain C_j, say

$$Q = \{x : x_{j'_i}^{(i)} \leqslant x^{(i)} \leqslant x_{j''_i}^{(i)}, i = 1, \ldots, n\},$$

where $j''_i \geqslant j'_i + 2$. Let

$$Q_* = \{x : x^{(i)}_{j'_i+1} \leqslant x^{(i)} \leqslant x^{(i)}_{j''_i-1} \quad \text{for} \quad i = 1, \dots, n\},$$

$$Q_\epsilon = \{x : x^{(i)}_{j'_i} - \varepsilon \leqslant x^{(i)} \leqslant x^{(i)}_{j''_i} + \varepsilon \quad \text{for} \quad i = 1, \dots ; n\},$$

and

$$|\Delta|_Q = \max\,(x^{(i)}_{j_i+1} - x^{(i)}_{j_i})\ ,\ (\Delta)_Q = \min\,(x^{(i)}_{j_i+1} - x^{(i)}_{j_i}),$$

the maximum and the minimum being with respect to i and j_i such that $j'_i \leqslant j_i < j''_i$, $i = 1, \dots, n$.

Using some of this notation, we now state the basic result of this section. It takes the form of a condition that a function v approximate zero in a certain sense.

THEOREM OF APPROXIMATION. – *Let $v \in C^1(E^n)$, and let Q denote a parallelepiped as previously defined. Suppose that for each C_j in Q a set of n real numbers $B_{1j}, B_{2j}, \dots, B_{nj}$ exists such that*

$$\int_{C_j} v dx = \sum_{k=1}^{n} (T_k - I)\, B_{kj}, \tag{1}$$

where I denotes the identity. Suppose also that

$$\sum_{k=1}^{n} \sum_{j}^{Q} |B_{kj}| \leqslant c V(v\ ;\ Q_\epsilon)\ |\Delta|_Q, \tag{2}$$

where ε and c are constants, and $\sum\limits_{j}^{Q}$ indicates summation over such j that $C_j \subset Q$. Then

$$\int_{Q_*} |v|\, dx \leqslant (1 + 2c)\ V(v\ ;\ Q_\epsilon)\ |\Delta|_Q \tag{3}$$

and for $f \in C^1(E^n)$ with $f = 0$ in the complement of Q_*

$$\left|\int_Q f v dx\right| \leqslant (1 + 3c)\ V(v\ ;\ Q_\epsilon)\ |f|_{C^1(E^n)}\ |\Delta|^2_Q. \tag{4}$$

(It would suffice to assume $v \in C^1(Q_\epsilon)$).

To prove this, we introduce the step function

$$v_\Delta(x) = \frac{1}{|C_j|}\int_{C_j} v(y)\, dy = v_j \quad \text{for} \quad x \in C_j^0,$$

where C_j^0 denotes the interior of C_j and $|C_j|$ its measure. We shall first show that

$$\int_Q |v_\Delta - v|\, dx \leqslant V(v\ ;\ Q)\, |\Delta|_Q. \tag{5}$$

Indeed, we have

$$\int_Q |v_\Delta - v|\, dx = \sum_j^Q \int_{C_j} |v_j - v(x)|\, dx,$$

while

$$\begin{aligned}\int_{C_j} |v(x) - v_j|\, dx &= \int_{C_j} \left|\frac{1}{|C_j|}\int_{C_j} (v(x) - v(y))\, dy\right| dx \\ &\leqslant \frac{1}{|C_j|}\int_{C_j}\int_{C_j} |v(x) - v(y)|\, dxdy.\end{aligned}$$

Since

$$\begin{aligned}|f(x) - f(y)| \leqslant\ & |f(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) - f(y^{(1)}, x^{(2)}, \ldots, x^{(n)})| \\ & + |f(y^{(1)}, x^{(2)}, \ldots, x^{(n)}) - f(y^{(1)}, y^{(2)}, \ldots, x^{(n)})| \\ & + \cdots \\ & + |f(y^{(1)}, \ldots, y^{(n-1)}, x^{(n)}) - f(y^{(1)}, \ldots, y^{(n)})|,\end{aligned}$$

we easily see that

$$\int_{C_j} |v(x) - v_j|\, dx \leqslant |\Delta|_Q\, V(v\ ;\ C_j),$$

and (5) follows by summing over j.

Inequality (3) now results from the decomposition

$$\int_{Q^*} |v|\, dx \leqslant \int_{Q^*} |v - v_\Delta|\, dx + \int_{Q^*} |v_\Delta|\, dx = G_1 + G_2,$$

since G_1 is given by (5) and G_2 by the calculation :

$$G_2 = \sum_j^{Q_*} \int_{C_j} |v_\Delta|\, dx = \sum_j^{Q_*} \left| \int_{C_j} v_\Delta\, dx \right| = \sum_j^{Q_*} \left| \int_{C_j} v dx \right|$$

$$= \sum_j^{Q_*} \left| \sum_{k=1}^{n} (T_k - I)\, B_{kj} \right| \leqslant 2 \sum_j^{Q} \sum_{k=1}^{n} |B_{kj}|$$

$$\leqslant 2cV(v\ ;\ Q_\epsilon)\ |\Delta|_Q .$$

To prove (4), we write

$$\int_Q fv dx = \int_{Q^*} f_\Delta v dx + \int_{Q^*} (f - f_\Delta)\, v dx = J_1 + J_2 .$$

Since $|f - f_\Delta| \leqslant |f|_{C^1}\, |\Delta|_Q$ in Q, we have from (3)

$$|J_2| \leqslant |f|_{C^1}\, |\Delta|_Q \int_{Q^*} |v|\, dx \leqslant (1 + 2c)\ |f|_{C^1}\, V(v\ ;\ Q_\epsilon)\ |\Delta|_Q^2 ,$$

C^1 abbreviating $C^1(E^n)$. Since $\int_{Q^*} f_\Delta (v - v_\Delta)\, dx = 0$, we also obtain

$$J_1 = \int_{Q^*} f_\Delta v_\Delta\, dx = \sum_j^{Q_*} f_j \sum_{k=1}^{n} (T_k - I)\, B_{kj} ,$$

where f_j denotes the value of f_Δ in C_j. Thus,

$$J_1 = \sum_j^{Q_*} \sum_{k=1}^{n} f_j (T_k - I)\, B_{kj}$$

$$= - \sum_j^{Q} \sum_{k=1}^{n} (I - T_k^{-1})\, f_j \, .\, B_{kj} ,$$

f_j being zero outside Q_*. Therefore,

$$|J_1| \leqslant |\Delta|_Q\ |f|_{C^1} \sum_{k=1}^{n} \sum_j^{Q} |B_{kj}|$$

$$\leqslant c\ |f|_{C^1}\ V(v\ ;\ Q_\epsilon)\ |\Delta|_Q^2 ,$$

inequality (4) following from this and the previous estimate of J_2.

Notice that if $f \in C^1(Q)$ with $f = 0$ in the complement of Q_* and $v \in C^1(Q)$, then

$$\left| \int_Q f(v - v_\Delta)\, dx \right| \leqslant |f|_{C^1(Q)} V(v\,;\,Q)\, |\Delta|_Q^2. \tag{6}$$

In fact, the left member of (6) is equal to

$$\left| \int_Q (f - f_\Delta)(v - v_\Delta)\, dx \right| \leqslant |\Delta|_Q\, |f|_{C^1(Q)} \int_Q |v - v_\Delta|\, dx,$$

and (5) applies.

In using the Approximation Theorem just proved, we replace v by the deviation $u - Su$ of an arbitrary function u from a presumed approximation Su. So applied, this theorem covers, in particular, the averaging operators K of Sections 1 and 2. To verify this, we derive from the representation (1) of Section 1 for an arbitrary function $u \in C^1(E^n)$ the relation

$$\int_{C_j} (Ku - u)\, dx = \sum_{k=1}^{n} \int_{C_j} D_k^2 J_k\, dx$$

$$= \sum_k \left(\int_{C_{T_k j,k}} D_k J_k\, dx_k' - \int_{C_{j,k}} D_k J_k\, dx_k' \right),$$

where

$$C_{j,k} = \{x : x^{(k)} = x_{j_k}^{(i)},\ x_{j_i}^{(i)} \leqslant x^{(i)} \leqslant x_{j_i+1}^{(i)} \quad \text{for} \quad i \neq k\}.$$

This is of the form (1), with $Ku - u = v$ and

$$B_{kj} = \int_{C_{j,k}} D_k J_k\, dx_k'\,.$$

Furthermore,

$$\sum_k \sum_j^{Q} |B_{kj}| \leqslant \sum_k \sum_j^{Q} \int_{C_{j,k}} |D_k J_k|\, dx_k'$$

$$= \sum_k \sum_{j_k = j_k'}^{j_k''} \int_{H_{k,j_k}} |D_k J_k|\, dx_k'.$$

where

$$H_{k,j_k} = \{x : x^{(k)} = x^{(k)}_{j_k},\ x^{(i)}_{j'_i} \leqslant x^{(i)} \leqslant x^{(i)}_{j''_i} \quad \text{for} \quad i \neq k\}.$$

Hence, by inequality (6), Section 1,

$$\sum_k \sum_j^{Q} |B_{kj}| \leqslant \frac{n}{2}\, \varepsilon \sum_k \int_{Q_\epsilon} |D_k u|\, dx = \frac{n}{2}\, \varepsilon V(u\,;\,Q_\epsilon)$$

provided

$$\varepsilon \leqslant 1/2 \min_k (x^{(k)}_{j''_k} - x^{(k)}_{j'_k}).$$

Thus, condition (2) holds with $c = \dfrac{n}{4}\cdot$

We now give an example of a smoothing process that is not averaging in the sense we have considered, but falls under the previous Approximation Theorem. The effect of this process is to approximate an arbitrary C' function —more generally, a function of locally bounded variation in the sense of Tonelli and Cesari— by what we shall call a "weathered" step function, i.e., a smooth function coinciding with a step function except near the discontinuities of the latter. Let $\{x_j\}$ be a system of mesh points such as we have described, and relative to this system define the operator B_Δ on the space of bounded, measurable functions ν on E^n by

$$B_\Delta \nu = \nu_\Delta .$$

Let Q denote a parallelepiped of the previously indicated type, and choose ε such that

$$0 < \varepsilon < 1/2\,(\Delta)_Q .$$

Let K_ϵ denote an averaging operator, of the kind discussed in Sections 1 or 2, over a cube or ball of respective edge or diameter 2ε. The smoothing operator we wish to mention here is defined by :

$$S_\epsilon = K_\epsilon B_\Delta .$$

We apply this operator to functions $u \in C^1(E^n)$. Since

$$\int_{C_j} B_\Delta u dx = \int_{C_j} u dx,$$

we have

$$\int_{C_j} (S_\epsilon u - u)\, dx = \int_{C_j} (K_\epsilon B_\Delta u - B_\Delta u)\, dx,$$

while the last integral can be expressed in the form

$$\sum_{k=1}^{n} (T_k - I)\, B_{kj}\,,$$

where

$$\sum_k \sum_j^{Q} |B_{kj}| \leqslant \binom{n}{2} \varepsilon V(u\,;Q^*)\,, \tag{7}$$

Q^* being the parallelepiped

$$Q^* = \{x : x^{(i)}_{j_i'-1} \leqslant x^{(i)} \leqslant x^{(i)}_{j_i''+1} \quad \text{for} \quad i = 1, \ldots, n\}.$$

This is a consequence of the fact that averaging operators are subject to the Approximation Theorem, which however we have proved only if $u \in C^1$. We extend that theorem as follows. Requiring $0 < \eta < (\Delta)_Q/2$, define the approximations $w \equiv w_\eta = K_\eta u_\Delta$. Rather tedious calculations like the previous enable us to show that

$$V(w\,;Q_\epsilon) \leqslant V(u\,;Q^*).$$

On the other hand, w being in C^1 satisfies a condition of the form

$$\int_{C_j} (K_\epsilon w - w)\, dx = \sum_k (T_k - I)\, B_{kj}\,,$$

where

$$\sum_k \sum_j^{Q} |B_{kj}| \leqslant \binom{n}{2} \varepsilon V(w\,;Q_\epsilon) \leqslant \binom{n}{2} \varepsilon V(u\,;Q^*).$$

As $\eta \to 0$, by means of expression (5), Section 1, we can verify that the B_{kj} pertaining to w have limits. These limits, again called B_{kj}, must satisfy (7) and the condition

$$\int_{C_j} (K_\epsilon u_\Delta - u_\Delta)\, dx = \sum_k (T_k - I)\, B_{kj}.$$

This outcome justifies our use of the Approximation Theorem, and we may conclude that

$$\int_{C_j} (S_\epsilon u - u)\, dx = \sum_k (T_k - I)\, B_{kj},$$

as asserted.

CHAPTER 4

CONSERVATION LAWS WITH INITIAL DATA OF BOUNDED VARIATION

The condition of bounded variation early became prominent in the one-dimensional theory of nonlinear conservation laws and was first used in several dimensions, in the sense of Tonelli and Cesari, by Conway and Smoller [4]. K. Kojima [20] extended their methods and results to conservation laws that might depend upon x ; we excluded such x-dependence from equation (A), Chapter 2. N.N. Kuznetsov [27] used the bounded variation property in his stratifying approach. In this chapter, we wish to justify a variety of stratifying methods based on the smoothing procedures of Chapter 3 when the initial data are of locally bounded variation in the sense of Tonelli and Cesari.

1. Functions of bounded variation in the sense of Tonelli and Cesari.

Let $x = (x_1, \ldots, x_n)$ be an arbitrary point of n-dimensional real Euclidean space E^n, let x'_k denote the point in E^{n-1} that is obtained by suppressing the k-th coordinate of x : for instance,

$$x'_1 = (x_2, \ldots, x_n)\,, x'_2 = (x_1\,, x_3, \ldots, x_n).$$

For $Z \subset E^n$ and for any function f on E^n, define

$$V_1(f\,; Z\,; x'_1) = \sup \sum_{i=1}^{p} |f(s_i\,, x'_1) - f(s_{i-1}\,, x'_1)|\,,$$

taking the supremum with respect to all finite increasing sequences of real numbers s_i, $i = 0, 1, \ldots, p$, such that

$$(s_i\,, x'_1) \notin Z \quad \text{for} \quad i = 0, 1, \ldots, p$$

and also with respect to all positive integers p. We call $V_1(f\,; Z\,; x'_1)$ "the variation of f with respect to x_1, for fixed x'_1, outside Z". For

$k = 1, \ldots, n$, we define "the variation of f with respect to x_k, for fixed x'_k, outside Z" analogously, and denote it by $V_k(f; Z; x'_k)$. A function f defined in E^n is said *to have bounded TC-variation –bounded variation in the sense of Tonelli and Cesari–* if a set $Z \subset E^n$ of measure zero exists such that

$$V_k(f; Z; x'_k) \in L_1(E^{n-1}) \quad \text{for} \quad k = 1, 2, \ldots, n.$$

For such a function, we set

$$V(f) = \inf \sum_{k=1}^{n} \int V_k(f; Z; x'_k)\, dx'_k,$$

the infimum being over all subsets Z of E^n of measure zero, and the integration being over E^{n-1}. We call $V(f)$ the *TC-variation* of f on E^n.

A function f defined in E^n has *locally bounded* TC-variation if

$$V(f\chi_Q) < \infty \quad \text{for any cube} \quad Q \subset E^n,$$

where for any set $E \subset E^n$ χ_E denotes the characteristic function of E :

$$\begin{aligned} \chi_E(x) &= 1 \quad \text{if} \quad x \in E, \\ &= 0 \quad \text{if} \quad x \notin E. \end{aligned}$$

If $f \in C^1(E^n)$ with $|f| \leqslant M$, and Q is a cube in E^n with edges of length s, then

$$V(f\chi_Q) \leqslant 2ns^{n-1}M + V(f; Q),$$

where $V(f; Q)$ denotes the TC-variation of f on Q as defined in Section 3, Chapter 3.

To H. Federer [8], E. de Giorgi [12, 13], W.H. Fleming [11], and K. Krickeberg [22] (see also H. Federer [9], Section 4.5)) is due the following criterion of precompactness.

PRECOMPACTNESS THEOREM. – *Let* G *denote an infinite family of locally integrable functions defined on* E^n. *Assume that any cube* $Q \subset E^n$ *determines constants* M *and* N *such that, for any* $f \in G$,

$$|f(x)| \leqslant M \quad \textit{for} \quad x \in Q$$

and

$$V(f\chi_Q) \leqslant N.$$

Then a locally integrable function f_0 *and a sequence* $f_n \in G$, $n = 1, 2, \ldots$, *exist such that*

$$\lim_{n\to\infty} \int_Q |f_n - f_0| \, dx = 0$$

for every cube $Q \subset E^n$. *The limit* f_0 *has locally bounded TC-variation.*

2. Suitable smoothing operators.

Let B denote the class of functions defined on E^n that are bounded and have bounded, continuous first partial derivatives. For $f \in B$, let

$$|f|_0 = \sup_{x\in E^n} |f(x)| \, ;$$

if f is a member of B with continuous derivatives of orders up to k, let $|f|_k$ analogously denote the supremum of the moduli of the derivatives of f of order k at points of E^n.

We permit here families of smoothing operators S_ϵ, $\varepsilon > 0$, that map B into B and satisfy seven conditions, pertaining to an arbitrary $f \in B$, as follows :

i) Pointwise bounds are preserved : $|S_\epsilon f|_0 \leqslant |f|_0$.

ii) Sharp gradients are blunted : $|\mathrm{grad}\, S_\epsilon f|_0 \leqslant C_1 |f|_0/\varepsilon$, where C_1 is a constant independent of f, x, and ε.

iii) If $\bar{f}$ denotes the function defined by $\bar{f}(x) = |f(x)|$, then

$$|S_\epsilon f(x)| \leqslant S_\epsilon \bar{f}(x) \quad \text{for} \quad x \in E^n.$$

iv) The smoothed function approximates the original weakly to better than first order in ε. More precisely, for all $\phi \in C_0^2$(1),

$$\int_{E^n} \phi(S_\epsilon f - f) \, dx \leqslant C_2 |\phi|_2 \, a(\varepsilon),$$

(1) C_0^2 here denotes the class of functions that are of class C^2 in E^n and have compact support.

where $a(\cdot)$ is a function independant of ϕ and f such that

$$\varepsilon^{-1} a(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

and C_2 is a constant independent of ϕ and ε, but possibly depending on $|f|_0$ and $V(f\,; \text{supp}\ \phi)$, where supp ϕ denotes the support of ϕ. The last three conditions pertain to an arbitrary cube

$$Q = \{x : a_i \leqslant x_i \leqslant b_i, \quad i = 1, \ldots, n\}$$

with edges parallel to the axes. Let

$$Q_\epsilon = \{x : a_i - \varepsilon \leqslant x_i \leqslant b_i + \varepsilon, \quad i = 1, \ldots, n\}$$

denote the ε-neighborhood of Q. We require for any such cube Q and $f \in B$:

v) $\int_Q |S_\epsilon f|\, dx \leqslant \int_Q |f|\, dx + c_1 \varepsilon\, |f|_0$, c_1 being a constant independent of f and ε, but depending upon Q ;

vi) $\int_Q |S_\epsilon f - f|\, dx \leqslant c_2 \varepsilon$, where the constant c_2 may depend upon Q, $|f|_0$, and $V(f\,; Q_\epsilon)$, but otherwise is independent of f and ε ;

vii) $V(S_\epsilon f\,; Q) \leqslant V(f\,; Q_\epsilon)$.

These hypotheses are satisfied by the averaging operators of Sections 1-2, Chapter 3, and also by additional smoothing operators such as are described in Section 3 of that chapter. Smoothing operators that lead to "weathered step functions", in particular, satisfy these hypotheses.

3. The TC-variation of strict solutions of first order quasi-linear equations.

Conway and Smoller first showed that the TC-variation of an arbitrary weak solution of a first order conservation law behaves regularly, and their result was apparently new even in the case of a solution that is strict. Kuznetsov first proved the analogous thing for strict solutions of first order quasi-linear equations, applying his result to weak solutions of conservation laws by a method of stratifying. In this section, we give our approach to these problems treated by Kuznetsov. Our calculations are essentially like his, but show to advantage, for instance, in the following special case. For convenience limiting ourselves to three independent variables, consider the partial differential equation

$$u_t + f(u)\,u_x + g(u)\,u_y = 0, \tag{1}$$

where f and g are of class C^1 for all real values of their argument u. With $t_0 \geqslant 0$, we take an initial condition of the form

$$u(x\,,y\,,t_0) = w(x\,,y)\,, \tag{2}$$

assuming w to be of class C^1 and to have compact support. We shall prove that $V(u) = V(w)$ on every plane t = constant in the layer within which u is a strict solution of (1). More precisely, our result is as follows :

THEOREM 1. – *Under the previous assumptions, a strict solution* $u(x\,,y\,,t)$ *of* (1), (2) *exists in the layer*

$$t_0 \leqslant t \leqslant t_0 + h,$$

where

$$h = 1/[2\,|\nabla w|_0 \max_{|s|\leqslant |w|_0} \sqrt{f'(s)^2 + g'(s)^2}\,],$$

$$|\nabla w|_0 = \sup_{x\,,y} \sqrt{w_x(x\,,y)^2 + w_y(x\,,y)^2}.$$

The solution has compact support in this layer and satisfies the conditions

$$|u(\cdot\,,\cdot\,,t)|_0 \leqslant |w|_0 \tag{3}$$

and

$$\begin{aligned} \int |u_x(x\,,y\,,t)|\,dxdy &= \int |w_x(x\,,y)|\,dxdy,\\ \int |u_y(x\,,y\,,t)|\,dxdy &= \int |w_y(x\,,y)|\,dxdy, \end{aligned} \tag{4}$$

the integrations being over the supports of the functions involved.

Proof. – The method of characteristics gives us a formula for the solution as follows. For an arbitrary point (ξ, η) of the initial plane $t = t_0$, let

$$x(t) \equiv x(t\,;\xi,\eta)\,,\, y(t) \equiv y(t\,;\xi,\eta)\,,\, \mathrm{U}(t) \equiv \mathrm{U}(t\,;\xi,\eta)$$

denote solutions of the characteristic differential equations

$$\frac{dx}{dt} = f(\mathrm{U})\,,\, \frac{dy}{dt} = g(\mathrm{U})\,,\, \frac{d\mathrm{U}}{dt} = 0$$

satisfying the initial conditions

$$x(t_0) = \xi\,,\, y(t_0) = \eta\,,\, \mathrm{U}(t_0) = w(\xi, \eta).$$

We have immediately

$$\mathrm{U}(t) \equiv \mathrm{U}(t\,;\xi,\eta) = w(\xi,\eta).$$

Therefore,

$$\begin{aligned} x(t) &\equiv x(t\,;\xi,\eta) = \xi + f(w(\xi,\eta))\,(t - t_0), \\ y(t) &\equiv y(t\,;\xi,\eta) = \eta + g(w(\xi,\eta))\,(t - t_0). \end{aligned} \tag{5}$$

For each fixed t, consider the transformation

$$x = x(t\,;\xi,\eta)\,,\, y = y(t\,;\xi,\eta). \tag{6}$$

For the Jacobian determinant

$$\mathrm{J} = \frac{\partial(x, y)}{\partial(\xi, \eta)} = x_\xi y_\eta - x_\eta y_\xi$$

of this transformation, an explicit calculation shows :

$$\mathrm{J} = 1 + (f'(w)\,w_\xi + g'(w)\,w_\eta)\,(t - t_0).$$

Hence, with h defined as above,

$$\mathrm{J} \geqslant \frac{1}{2} \quad \text{if} \quad 0 \leqslant t - t_0 \leqslant h.$$

This implies that the transformation (6), in which t is to be regarded as a parameter, is invertible in the layer $0 \leqslant t - t_0 \leqslant h$. We denote the inverse transformation by

$$\xi = \xi(x, y, t), \eta = \eta(x, y, t).$$

Cauchy's theory says that the function

$$u(x, y, t) = w(\xi(x, y, t), \eta(x, y, t))$$

is a solution of (1), (2) in the layer $0 \leqslant t - t_0 \leqslant h$ and also that this solution is unique.

It is clear that u has compact support and satisfies condition (3). Conditions (4) will result from the relations

$$u_x = \mathrm{J}^{-1} w_\xi, u_y = \mathrm{J}^{-1} w_\eta, \tag{8}$$

which we now justify. To prove the first of these relations, we write

$$u_x = w_\xi \xi_x + w_\eta \eta_x$$

and make the substitutions

$$\xi_x = \mathrm{J}^{-1} y_\eta = \mathrm{J}^{-1}(1 + g'(w) w_\eta (t - t_0)),$$
$$\eta_x = -\mathrm{J}^{-1} y_\xi = -\mathrm{J}^{-1} g'(w) w_\xi (t - t_0).$$

The result is the first formula of (8), and the second is obtained in a similar way.

By the first formula of (8), we have

$$\int |u_x(x, y, t)|\, dxdy = \int |w_\xi(\xi, \eta)|\, \mathrm{J}^{-1}\, dxdy = \int |w_\xi(\xi, \eta)|\, d\xi d\eta,$$

giving the first formula of (4). The second formula of (4) is obtained from the second formula of (8). Thus the theorem is proved.

The idea just employed in the special case of an equation of the form (1) can be extended to the more general equations

$$u_t + \sum_{i=1}^{n'} a_i(x, t, u) u_{x_i} = c(x, t, u) \tag{9}$$

under assumptions (i) and (ii) formulated in the proof of Theorem 1, Section 2, Chapter 2. Concerning the initial condition

$$u(x, t_0) = w(x), \tag{10}$$

we assume, as in the earlier section, that w and grad w are bounded and continuous in E^n. Thus, in particular, (iii) w has locally bounded TC-variation.

Repeating the proof of the theorem referred to, we find that a strict solution of (9), (10) exists in a layer $Z(t_0, t_0 + h)$, where h is an arbitrary positive number not greater than $1/(A\,|\mathrm{grad}\, w|_0 + a)$; a and A are constants depending on the coefficients in (9) and on $|w|_0$, but not otherwise on w. Furthermore, the solution is subject to the bound

$$|u(x, t)| \leqslant \phi(t, t_0, |w|_0) < \phi(t_0 + h, t_0, |w|_0)$$

for $$(x, t) \in Z(t_0, t_0 + h). \tag{11}$$

Let

$$N' = \sup_{Z(t_0, t_0 + h; \phi(t_0 + h\,;\, t_0, |w|_0))} \left[\sum_i a_i^2\right]^{1/2}.$$

With $t_1 > t_0 + h$, $x_1 \in E^n$, and $N \geqslant N'$, define the disk

$$D_t = \{x : |x - x_1| \leqslant N(t_1 - t)\}.$$

Since $N \geqslant N'$, as is well known –and follows from the proof of the theorem in Chapter 2 to which we have been referring– the values of $u(x, t)$ for $x \in D_t$ and $t_0 \leqslant t \leqslant t_0 + h$ are determined by the values of w on D_{t_0}. Our aim is the following estimate, essentially first given by Kuznetsov :

THEOREM 2. – *Under the indicated hypotheses, the solution of* (9) *and* (10) *satisfies a condition of the form*

$$V(u\,; D_t) \leqslant e^{B(t - t_0)}\, V(w\,; D_{t_0}) + b(e^{B(t - t_0)} - 1),$$

where $b = (\omega_n/n)\, N^n (t_1 - t_0)^n$, *and* B *depends upon bounds for the first derivatives of the coefficients in (9). Here,* ω_n *represents the area of the unit sphere in* E^n.

Proof. – We continue to use the terminology and results of Section 2, Chapter 2. Let

$$P(t) \equiv V(u\,; D_t) = \int_{D_t} \sum_i |u_{x_i}(x, t)|\, dx.$$

Changing from x to ξ, we have in terms of the quantities $P_i = p_i J$,

$$P(t) = \int_{D_{t_0}} \sum_i |p_i(t\,;\xi)|\, J(t\,;\xi)\, d\xi = \int_{D_{t_0}} \sum_i |P_i(t\,;\xi)|\, d\xi$$

for $t_0 \leqslant t \leqslant t_0 + h$, J being positive in that layer.

Our aim is to estimate the last expression with the help of the expansion

$$P_i(t\,;\xi) = P_i(t_0\,;\xi) + \int_{t_0}^{t} \dot{P}_i(s\,;\xi)\, ds, \tag{12}$$

where $\dot{P}_i = dP_i/dt$. We have $\dot{P}_i = \dot{p}_i J + p_i \dot{J}$, and by substitution from equations (11) and (14) of Section 2, Chapter 2,

$$\dot{P}_i = \sum_k [a_{k,x_k} P_i - a_{k,x_i} P_k] + c_u P_i + c_{x_i} J,$$

the terms quadratic in the p_k all dropping out. Therefore,

$$\sum_i |\dot{P}_i| \leqslant B \sum_i P_i + BJ,$$

where B is a constant depending upon upper bounds for the absolute values of the first derivatives of the a_i and c. Hence, and because $P_i(t_0\,;\xi) = w_{x_i}(\xi)$, we have from (12)

$$\sum_i |P_i(t\,;\xi)| \leqslant \sum_i |w_{x_i}(\xi)| + B \int_{t_0}^{t} \left[\sum_i |P_i(s\,;\xi)| + J(s\,;\xi) \right] ds.$$

Integrating with respect to ξ over D_{t_0} and changing the order of the integrations with respect to s and ξ gives us :

$$P(t) \leqslant V(w\,; D_{t_0}) + B \int_{t_0}^{t} P(s)\, ds + B \int_{t_0}^{t} \left(\int_{D_{t_0}} J(s\,;\xi)\, d\xi \right) ds. \tag{13}$$

The last term is easily calculated, and we have :

$$\int_{t_0}^{t} \left(\int_{D_{t_0}} J(s\,;\xi)\, d\xi \right) ds = \int_{t_0}^{t} \left(\int_{D_s} dx \right) ds$$

$$= (\omega_n/n) \int_{t_0}^{t} [N(t_1 - s)]^n\, ds \leqslant b(t - t_0),$$

b being the constant previously defined. If we replace the last term on the right side of (13) by this estimate, we arrive by using Gronwall's reasoning at the inequality asserted in the theorem.

4. Stratified solutions and weak solutions.

Given $T > 0$, determine constants A and a as in the previous section. Fix h as a positive number not greater than $1/(2a)$ and such that, for convenience, T/h is an integer. The procedures of Section 3, Chapter 2, enable us to construct a stratified solution v with layer thickness h if we perform our smoothing with S_ϵ, where

$$\varepsilon = 2C_1 A\phi(T\,;0\,,|u_0|_0)h.$$

We will have

$$|v|_{C^0(Z(0,T))} \leqslant \phi(T\,;0\,,|u_0|_0) \tag{1}$$

and, in every stratum, will be able to use the determination

$$N = \sup_{Z(0,T\,;\phi(T\,;0\,,|u_0|_0))} \left[\sum_i a_i^2\right]^{1/2}$$

when applying Theorem 2 of the previous section.

To describe the TC-variation of v, we introduce suitable "stepped" cones, as follows. With arbitrary $t_1 > T$ and $x_1 \in E^n$, let

$$S_k = \{(x\,,t) : |x - x_1| \leqslant N(t_1 - t) + (Th^{-1} - k)\varepsilon\,, kh \leqslant t < (k+1)h\}$$

be a frustum of a "cone of determinacy" in Z_{k+1}. For

$$kh \leqslant \tau < (k+1)h, \quad \text{let} \quad E(\tau) = \{(x\,,\tau) : (x\,,\tau) \in S_k\}$$

be a "horizontal" section of S_k at the ordinate τ. We denote the base of S_k by

$$E_k = E(kh) = \{(x\,,kh) : |x - x_1| \leqslant N(t_1 - kh) + (Th^{-1} - k)\varepsilon\},$$

and we denote the top of S_{k-1} by

$$F_k = \{(x\,,kh) : |x - x_1| \leqslant N(t_1 - kh) + (Th^{-1} - k + 1)\varepsilon\}.$$

The union of the S_k for $k = 0, 1, \ldots, T/h$ is a figure we might describe as a portion of a stepped cone. We shall prove that for any of its horizontal sections $E(t)$, we have :

$$V(v\,;E(t)) \leqslant e^{Bt}\,V(u_0\,;E(0)) + b(e^{Bt} - 1). \tag{2}$$

We have first

$$V(\nu\,;E_k) = V(\nu\,;E(kh)) = V(S_\epsilon\,\nu\,;E(kh-0)),$$

the right hand member representing $\lim_{\sigma\downarrow 0} V(S_\epsilon\nu\,;E(kh-\sigma)$; this is by definition of $\nu(x\,;kh)$ in a stratified solution. Property (vii) of smoothing operators (Section 2) shows that

$$V(S_\epsilon\,\nu\,;E(kh-0)) \leqslant V(\nu\,;F_k),$$

while by Theorem 2, Section 3,

$$V(\nu\,;F_k) \leqslant e^{Bh}\,V(\nu\,;E_{k-1}) + b(e^{Bh}-1).$$

In sum,

$$V(\nu\,;E_k) \leqslant e^{Bh}\,V(\nu\,;E_{k-1}) + b(e^{Bh}-1) \quad \text{for} \quad k = 1, 2, \ldots, T/h.$$

From this, by induction, we have

$$V(\nu\,;E_k) \leqslant e^{Bkh}\,V(\nu\,;E_0) + b(e^{Bkh}-1) \quad \text{for} \quad k = 1, 2, \ldots T/h,$$

which is of the desired form (2) for $t = kh$. For $kh < t < (k+1)h$, Theorem 2, Section 3, tells us that

$$V(\nu\,;E(t)) \leqslant e^{B(t-kh)}\,V(\nu\,;E_k) + b(e^{B(t-kh)}-1),$$

and substitution for $V(\nu\,;E_k)$ from the previous result leads to (2) in the present case as well. In this way, inequality (2) is completely verified.

With inequalities (1) and (2) established, the precompactness theorem of Section 1 can be applied and then the limit function shown to be a weak solution, for instance by the methods of Chapter 2. We omit further details.

CHAPTER 5

EQUATIONS OF HAMILTON-JACOBI TYPE

The global theory of Hamilton-Jacobi equations is very similar to that of quasilinear conservation laws. That this should be so is obvious in the one-dimensional case, the Hamilton-Jacobi equation $u_t + f(u_x) = 0$, for instance, leading to the conservation law

$$v_t + f(v))_x = 0$$

with $v = u_x$.

Multi-dimensional equations of Hamilton-Jacobi type have been treated by several authors : S.N. Kruzhkov [23] employed artificial viscosity, E.D. Conway and E. Hopf [3] used a formulation within the calculus of variations, E. Hopf [17] adapted the method of envelopes ([1]), A. Douglis [7] set up approximating problems in which x is discrete, and W.H. Fleming [11,1-4] applied the stochastic calculus of variations and also control theory and the theory of differential games. Uniqueness theorems have been given by A. Douglis [7] and S.N. Kruzhkov [24].

1. Aims and assumptions.

Equations of the Hamilton-Jacobi type

$$u_t + f(x, t, u, \text{grad } u) = 0 \tag{F}$$

permit the method of stratified solutions to be applied more simply than do conservation laws. For given $T > 0$, suppose f to be of class C^2 in the $(2n + 2)$-dimensional region

$$W(T) = \{(x, t, u, p) : x \in E^n, 0 \leqslant t \leqslant T, u \in E^1, p \in E^n\},$$

and suppose that for any constant V the derivatives

$$f_{p_i}(x, t, u, p), i = 1, \ldots, n$$

([1]) Applied to boundary problems by S. Aigawa and N. Kikuchi [1-1] and by S.H. Benton, Jr. [1-3].

are bounded on the subregion

$$\{(x, t, u, p) : x \in E^n, 0 \leqslant t \leqslant T, |u| \leqslant V, |p| \leqslant V\}.$$

We have used the notation $f_{p_i} = \partial f/\partial p_i$ and shall also abbreviate $\partial f/\partial t$, $\partial f/\partial x_i$, $\partial f/\partial u$ by f_t, f_{x_i}, f_u, respectively, with analogous notation for second derivatives.

Prescribing an initial condition of the form

$$u(x, 0) = u_0(x), \tag{G}$$

where

$$|u_0(x)| \leqslant U, |u_0(x) - u_0(x')| \leqslant U |x - x'| \quad \text{for} \quad x, x' \in E^n \tag{1}$$

with constant U, we shall ask for a Lipschitz continuous function u satisfying (G) in E^n and (F) almost everywhere in

$$Z(T) = \{(x, t) : x \in E^n, 0 \leqslant t \leqslant T\}.$$

To obtain such solutions, we make further assumptions concerning f.

The first additional assumption is the following convexity condition : a positive constant μ shall exist such that for any vector $\xi = (\xi_1, \ldots, \xi_n) \in E^n$ we have

$$\sum_{i,j} f_{p_i p_j}(x, t, u, p)\, \xi_i \xi_j \geqslant \mu |\xi|^2 \quad \text{for} \quad (x, t, u, p) \in W(T), \tag{2}$$

where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$. This and the previous suppositions already imply that at most one Lipschitz continuous function u can satisfy (G) in E^n, (F) almost everywhere in Z(T), and a "semi-concavity" requirement of the form

$$u(x + \xi, t) + u(x - \xi, t) - 2u(x, t) \leqslant K(t) |\xi|^2 \quad \text{for} \quad x, \xi \in E^n, \tag{3}$$

where $K(t) < \infty$ for $t > 0$. A Lipschitz continuous function u that fulfills all these conditions is called a *generalized solution* of (F) and (G) in Z(T).

Now we make assymptions concerning the growth of f and its first and second derivatives. We presume a positive nondecreasing function $E(\rho)$ to exist for $\rho \geqslant 0$ such that

$$\int_a^\infty d\rho/E(\rho) = \infty \quad \text{for any} \quad a > 0 \tag{4}$$

and that if Ψ represents any of the $n + 1$ expressions

$$- f + \sum_j p_j f_{p_j} \, , f_{x_i} + p_i f_u \, , \, i = 1, \ldots, n,$$

then

$$|\Psi(x\, , t\, , u\, , p)| \leqslant E(\rho) \tag{5}$$

for

$$x \in E^n \, , 0 \leqslant t \leqslant T \, , |u| \leqslant \rho \, , \, |p| \leqslant \rho . \tag{6}$$

This assumption will be responsible for an a priori estimate of the Lipschitz constant for a solution of (F), (G). The next, our final supposition, guarantees that strict solutions will have domains of suitable widths. It is that if Φ represents f or any one of its first or second derivatives, then

$$\sup \, |\Phi(x\, , t\, , u\, , p)| \leqslant B(\rho), \tag{7}$$

the supremum being taken for the values (6), and $B(\rho)$ being a non-decreasing function finite for $\rho \geqslant 0$.

Our principal aim is to prove :

THEOREM C. *Under the assumptions stated, a generalized solution of* (F), (G) *exists in* Z(T).

2. Strict solutions of equation (F).

By a strict solution of (F) we mean a continuously differentiable function that satisfies (F) at all points of its domain. We are concerned here with strict solutions u of (F) defined in layers of the type

$$Z(t_0\, , t_0 + h) = \{(x\, , t) : x \in E^n \, , \, t_0 \leqslant t \leqslant t_0 + h\}$$

and satisfying initial conditions of the form

$$u(x\, , t_0) = w(x), \tag{1}$$

where w is an arbitrary bounded function of class C^2 with bounded first and second derivatives in E^n. Let W and M be constants such that

$$|w| \leqslant \mathrm{W}\ ,\ |\partial w/\partial x_i| \leqslant \mathrm{W}\ ,\ |\partial^2 w/\partial x_i\, \partial x_j| \leqslant \mathrm{M} \text{ in } \mathrm{E}^n \qquad (2)'$$

for $i , j = 1, \ldots , n$.

With fixed t_0 in the interval $0 \leqslant t_0 < \mathrm{T}$, let $y(t\,; t_0)$ be the function that satisfies the differential equation $dy/dt = \mathrm{E}(y)$ and the initial condition $y(t_0) = \mathrm{W}$. Since this function is characterized by the relation $\int_{\mathrm{W}}^{y(t;t_0)} dz/\mathrm{E}(z) = t - t_0$, in view of condition (4), Section 1, it exists for $t \geqslant t_0$; it is positive and increasing. Let A be a constant such that

$$|f_{p_i}(x\,,t\,,u\,,p)| \leqslant \frac{\mathrm{A}}{\sqrt{n}} \quad \text{for} \quad x \in \mathrm{E}^n\ , t_0 \leqslant t \leqslant \mathrm{T},$$

$$|u| \leqslant y(\mathrm{T}\,; t_0)\ ,\ |p| \leqslant y(\mathrm{T}\,; t_0)\ , i = 1, \ldots ,n. \quad (2)''$$

Any solid conical layer of the form

$$\mathrm{L} \equiv \mathrm{L}(h) \equiv \mathrm{L}(x_1\,, t_1\,; t_0\,, h)$$
$$= \{(x\,, t) : |x - x_1| \leqslant \mathrm{A}(t_1 - t)\ ,\ t_0 \leqslant t \leqslant t_0 + h\},$$

where $x_1 \in \mathrm{E}^n$ and $t_0 < t_0 + h < \mathrm{T} < t_1$, will be called a (conical) *layer of determinacy* of u. As will be seen, the values of u in L will be determined by those of w prescribed on the base

$$\mathrm{B} \equiv \mathrm{B}(x_1\,, t_1\,; t_0) = \{(x\,, t_0) : |x - x_1| \leqslant \mathrm{A}(t_1 - t_0)\}.$$

We can now formulate our main results concerning strict solutions of (F). In the following theorem, C and C′ denote appropriate constants depending on W and on bounds for f and its partial derivatives of first and second order ; C and C′ do not depend upon t_0, M, or μ.

THEOREM. – *Under the previous hypotheses, the initial value problem* (F), (G) *has a strict solution* u *in* $\mathrm{Z}(t_0\,, t_0 + h)$, *where*

i) $h = 1/[\mathrm{C}(1 + \mathrm{M})]$.

In $\mathrm{Z}(t_0\,, t_0 + h)$,

ii) $|u(x\,, t)| \leqslant y(t\,; t_0)\ ,\ |\partial u(x\,, t)/\partial x_i| \leqslant y(t\,; t_0)$

for $i = 1, \ldots ,n$,

and

iii) $|\partial^2 u/\partial x_i\, \partial x_j| \leqslant 2n\mathrm{M} + 1$ *for* $i\,,j = 1,\ldots,n.$

With an arbitrary unit vector $z = (z_1,\ldots,z_n)$, $|z| = 1$, *define*

$$\mathrm{P}^*(t\,;z) \equiv \mathrm{P}^*(t\,;z\,|\,u) = \sup_{\mathrm{E}^n} \sum_{i,j} \frac{\partial^2 u}{\partial x_i\, \partial x_j} z_i z_j\,,$$

$$\mathrm{P}(t) \equiv \mathrm{P}(t\,;z) = \mathrm{P}(t\,;z\,|\,u) = \max\,(\mathrm{C}'/\mu\,, \mathrm{P}^*(t\,;z\,|u).$$

Then we have

iv) $$\frac{\mu\mathrm{P}(t) - \mathrm{C}'}{\mu\mathrm{P}(t) + \mathrm{C}'} \leqslant e^{-\mathrm{C}'(t-t_0)}\,\frac{\mu\,\mathrm{P}(t_0) - \mathrm{C}'}{\mu\,\mathrm{P}(t_0) + \mathrm{C}'} \quad \text{for} \quad t_0 \leqslant t \leqslant t_0 + h.$$

Proof. – The characteristic equations corresponding to (F) are the differential equations

$$\begin{aligned} \dot{x}_i &= f_{p_i}\,, \dot{p}_i = -f_{x_i} - f_u p_i \equiv f_i\,, \\ \dot{v} &= \sum_i p_i f_{p_i} - f \equiv f_0,\ \ i = 1,\ldots,n, \end{aligned} \tag{3}$$

in which $\cdot = d/dt$, and f and its derivatives have $(x\,,t\,,v\,,p)$ as argument. Selecting an arbitrary point $\xi = (\xi_1,\ldots,\xi_n)$, we impose the initial conditions :

$$x_i = \xi_i\,, v = w(\xi)\,, p_i = \partial w(\xi)/\partial \xi_i \quad \text{for} \quad t = t_0, \quad i = 1,\ldots,n. \tag{4}$$

We shall see that the problem (3), (4) has a unique solution for $t_0 \leqslant t \leqslant \mathrm{T}$.

For us to be assured of a solution

$$x_i(t\,;\xi)\,, v(t\,;\xi)\,, p_i(t\,;\xi), \quad i = 1,\ldots,n\,, \tag{5}$$

of the initial value problem (3), (4) in the interval $t_0 \leqslant t \leqslant \mathrm{T}$, it will suffice to be able to find a priori bounds for $|v|$ and $|p|$ in any interval $t_0 \leqslant t \leqslant t'$ in which the solution (5) exists. For convenience, set $p_0 = v$ and $\mathrm{P} = (p_0\,, p_1,\ldots,p_n)$, then writing the last $n + 1$ characteristic equations in the form

$$\dot{p}_a = f_a(x\,,t\,,\mathrm{P}), \quad a = 0,\ 1,\ldots,n.$$

Integrating and using (4) gives us the integral relations

$$p_a(t\,;\xi) = w_a(\xi) + \int_{t_0}^{t} f_a(x(s\,;\xi)\,, s\,, \mathrm{P}(s\,;\xi))\,ds, \quad a = 0, 1, \ldots, n,$$

supposedly valid for $t \leqslant t \leqslant t'$; here,

$$w_0 = w \quad \text{and} \quad w_i = w_{x_i}\,, \quad i = 1, \ldots, n.$$

Defining

$$\rho(t) \equiv \rho(t\,;\xi) = \max_{a=0,1,\ldots,n} \sup_{t_0<s<t} |p_a(s\,;\xi)|,$$

in view of (2) and of assumption (5), Section 1, we have

$$|p_a(t\,;\xi)| \leqslant \mathrm{W} + \int_{t_0}^{t} \mathrm{E}(\rho(s))\,ds.$$

This implies that

$$\rho(t) \leqslant \mathrm{W} + \int_{t_0}^{t} \mathrm{E}(\rho(s))\,ds.$$

We conclude by Gronwall's reasoning that $\rho(t) \leqslant y(t\,;t_0)$ in the interval $t_0 \leqslant t \leqslant t'$. Thus, all the dependent variables (5) have a priori bounds in this interval, and the bounds do not depend upon t'. From well known results on the maximum extent of solutions of ordinary differential equations we can conclude that the problem (3), (4) has a unique solution in the interval $t_0 \leqslant t \leqslant \mathrm{T}$ and the estimates

$$|v(t\,;\xi)| \leqslant y(t\,;t_0)\,,\ |p_i(t\,;\xi)| \leqslant y(t\,;t_0), \quad i = 1, \ldots, n, \tag{6}$$

hold there.

Eventually, v is identified with the solution u of (F), (G) and p_i with $\partial u/\partial x_i$ in a suitable layer $\mathrm{Z}(t_0\,, t_0 + h)$. To estimate the height h of this layer, we are led to consider new quantities p_{ij}, $i\,,j = 1, \ldots, n$, which will ultimately be identified with the second derivatives $\partial^2 u/\partial x_i\,\partial x_j$. These p_{ij} satisfy ordinary differential equations obtained by applying $\partial^2/\partial x_i\,\partial x_j$ formally to equation (F) and the substituting p_{ij} for $\partial^2 u/\partial x_i \partial x_j$, p_i for $\partial u/\partial x_i$, and $\dot{p}_{ij} \equiv dp_{ij}/dt$ for $\sum_k (\partial p_{ij}/\partial x_k)\, f_{p_k} + \partial p_{ij}/\partial t$. These equations are

$$\dot{p}_{ij} = C_{ij} + \sum_k D_{ik}\, p_{kj} + \sum_k D_{jk}\, p_{ki} - f_u\, p_{ij}$$

$$- \sum_{k,\ell} f_{p_k p_\ell}\, p_{ik}\, p_{j\ell}\,, \quad i\,, j = 1, \ldots, n, \tag{7}$$

where

$$-C_{ij} = f_{x_i x_j} + f_{ux_i} p_j + f_{ux_j} p_i + f_{uu}\, p_i p_j\,,$$

$$-D_{ik} = f_{x_i p_k} + p_i f_{p_k u}\,.$$

In these equations, p_i stands for $p_i(t\,;\xi)$, and the derivatives of f have the argument $(x(t\,;\xi)\,, t\,, v(t\,;\xi)\,, p(t\,;\xi))$. We understand the p_{ij} to be functions of $(t\,;\xi)$ determined by (7) and the initial conditions

$$p_{ij}(t_0\,;\xi) = w_{x_i x_j}(\xi). \tag{8}$$

In view of (6) and assumption (7), Section 1, a constant C independent of M and μ exists such that the right side of (7) is not greater in absolute value than $C(1 + \sum_{k,\ell} |p_{k\ell}|)^2$. Hence, any solution of (7) defined, say, for $t_0 \leqslant t \leqslant t'$ will satisfy the inequalities

$$|\dot{p}_{ij}| \leqslant C\left(1 + \sum_{k,\ell} |p_{k\ell}|\right)^2$$

and by (8), on the other hand,

$$|p_{ij}(t_0\,;\xi)| \leqslant M.$$

Summing over i and j and reasoning as in the proof of Gronwall's inequality shows that $\sum_{i,j} |p_{ij}| \leqslant q$, where $q \equiv q(t)$ satisfies the integral relation

$$q(t) = nM + nC \int_{t_0}^{t} (1 + q(s))^2\, ds$$

and, equivalently, the differential equation $dq/dt = nC(1 + q)^2$ and the initial condition $q(t_0) = nM$. The latter imply that

$$q(t) = \frac{nM + nC(1 + nM)\,(t - t_0)}{1 - nC(1 + nM)\,(t - t_0)}$$

if t is such that the denominator is positive. Setting

$$h = 1/[2n\mathrm{C}(1 + n\mathrm{M})], \tag{9}$$

we conclude, in particular, that

$$|p_{ij}(t, \xi)| \leqslant 2n\mathrm{M} + 1 \quad \text{if} \quad t_0 \leqslant t \leqslant t_0 + h, \quad i, j = 1, \ldots, n, \tag{10}$$

the right side of this inequality being the maximum of q for the indicated domain of t.

In view of the a priori estimate (10) just achieved, standard methods now show that the initial value problem (7), (8) has a unique solution in the interval $t_0 \leqslant t \leqslant t_0 + h$; it is subject to (10).

The next step is to show that the mapping $x = x(t\,;\xi)$ can be inverted to give ξ as a function of x, t. To do so, choose $x_1 \in \mathrm{E}^n$ and $t_1 \geqslant \mathrm{T}$ arbitrarily, and require that $(\xi, t_0) \in \mathrm{B} \equiv \mathrm{B}(x_1, t_1\,; t_0)$. Let

$$\mathrm{Y} = (t\,;\xi) = (t\,;\xi_1, \ldots, \xi_n),\ \mathrm{X} = (x, t) = (x_1, \ldots, x_n, t),$$
$$\mathrm{X}(t\,;\xi) = (x_1(t\,;\xi), \ldots, x_n(t\,;\xi), t),$$

and define the set

$$\mathrm{K}(h) = \{\mathrm{Y} = (t\,;\xi) : (\xi, t_0) \in \mathrm{B},\ t_0 \leqslant t \leqslant t_0 + h\}.$$

In view of $(2)''$, the transformation

$$\mathrm{X} = \mathrm{X}(\mathrm{Y}) \tag{11}$$

maps $\mathrm{K}(h)$ into $\mathrm{L}(h)$. Below we shall prove that the functional determinant of the transformation has a positive lower bound α :

$$\frac{\partial(\mathrm{X})}{\partial(\mathrm{Y})} \geqslant \alpha \quad \text{for} \quad \mathrm{Y} \in \mathrm{K}(h)\,; \tag{12}$$

as will be seen, α can be selected to be independent of M. It will follow from (12) that the mapping (11) is locally one-to-one. Furthermore, it is easily seen that the range of the mapping is $\mathrm{L}(h)$. Indeed, a characteristic curve issuing from an arbitrary point of $\mathrm{L}(h)$ has spatial coordinates satisfying the conditions $\dot{x}_i = f_{p_i}$, $i = 1, \ldots, n$, and, owing to the definition of A, therefore will not escape from $\mathrm{L}(h)$ through the conical sides as the parameter t decreases to the value t_0. Hence, the characteristics that emanate from the points of B cover L(h), which means that (11) maps K(h) onto L(h), a simply connected region. Hence, this locally one-to-one mapping is globally one-to-one from K(h) onto L(h). Both the mapping and its inverse are continuous and have continuous first and second partial derivatives. This mapping (11) embraces the transformation

$$x = x(t\,;\xi), \tag{13}$$

which, for fixed t in the interval $t_0 \leqslant t \leqslant t_0 + h$, maps a subset S$(t)$ of

$$\mathrm{S} = \{\xi : |\xi - x_1| \leqslant \mathrm{A}(t_1 - t_0)\}$$

onto

$$\mathrm{S}_t = \{x : |x - x_1| \leqslant \mathrm{A}(t_1 - t)\}.$$

The inverse transformation to (11) embraces the inverse transformation to (13), which we write as

$$\xi = \xi(x\,, t), \tag{14}$$

mapping S_t onto S(t). The previous remarks concerning (11) and its inverse imply that both (13) and (14) are continuous and have continuous first and second partial derivatives. Consequently, by traditional theory, the function

$$u(x\,, t) = v(t\,, \xi(x\,, t))$$

is a solution of (F) in L(h), where also

$$u_{x_i}(x\,, t) = p_i(t\,, \xi(x\,, t))\,, u_{x_i x_j}(x\,, t) = p_{ij}(t\,, \xi(x\,, t))$$

for $i\,, j = 1, \ldots, n$. Since x_1 and t_1 are arbitrary, the same is true in $\mathrm{Z}(t_0\,, t_0 + h)$. Finally, u satisfies the initial condition (G).

We have still to prove (12). The functional determinant on the left equals

$$\mathrm{J} \equiv \mathrm{J}(t) \equiv \mathrm{J}(\xi\,, t) = \frac{\partial(x(t\,;\xi))}{\partial(\xi)},$$

the functional determinant for (13). By (4), $\mathrm{J}(t_0) = 1$, and by continuity a positive number h' exists such that $\mathrm{J}(\xi\,, t) > 0$ in K(h'). Denoting by h_1 the supremum of all h' such that $\mathrm{J}(\xi\,, t) > 0$ in K(h') and $h' \leqslant h$, we shall prove that $h_1 = h$. For any h' as described, the argument of the previous paragraph shows that the mapping (13) has an inverse (14) for which $u(x, t) = v(t\,; \xi(x, t)$ solves (F), (G), in $L(h')$ and $u_{x_i} = p_i$, $u_{x_i x_j} = p_{ij}$ there. Hence, in $L(h')$ we have the differentiation formulas

$$v_{\xi_j} \equiv \partial v/\partial \xi_j = \sum_k \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial \xi_j} = \sum_k p_k x_{k,\xi_j},$$

$$p_{i,\xi_j} \equiv \partial p_i/\partial \xi_j = \sum_k \frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial x_k}{\partial \xi_j} = \sum_k p_{ik}\, x_{k,\xi_j},$$

where $x_{k,\xi_j} = \partial x_k/\partial \xi_j$. Using these when differentiating with respect to ξ_j in the first n characteristic equations (3) gives us

$$\dot{x}_{i,\xi_j} = \sum_k f_{p_i x_k}\, x_{k,\xi_j} + f_{p_i u} \sum_k p_k x_{k,\xi_j} + \sum_k f_{p_i p_k} \sum_{\ell} p_{k\ell} x_{\ell,\xi_j}.$$

Hence, differentiating J by rows and cancelling as is appropriate, we obtain

$$\dot{J} = \sum_i (f_{p_i x_i} + f_{p_i u}\, p_i + \sum_k p_{ik}\, f_{p_i p_k})\ J.$$

Previous estimations show that the quantities in parentheses are bounded in absolute value by an expression of the form $(a + bM)/n$, where a and b are constants independant of M and μ. Therefore,

$$\dot{J} \geqslant -(a + bM)\, J,$$

and since $J(\xi, t_0) = 1$, we can deduce that

$$J(\xi, t) \geqslant e^{-(a+bM)(t-t_0)} \quad \text{in} \quad K(h).$$

In $K(h)$, we have $0 \leqslant t - t_0 \leqslant h$, while by (9) $h = 1/[2nC(1 + nM)]$. We conclude that a positive constant α independent of M and μ exists such that $J \geqslant \alpha$ in $K(h)$ and therefore such that (12) holds, as required. We have thus established statements (i), (ii), (iii) of the theorem being considered.

To prove (iv), let $z = (z_1, \dots, z_n)$ be an arbitrary unit vector : $|z| = 1$. Multiplying both sides of equation (7) by $z_i z_j$ and summing gives

$$\dot{P}^* = \sum_{i,j} \dot{p}_{ij} z_i z_j = \sum_{i,j} C_{ij} z_i z_j + 2 \sum_k \sum_i z_i D_{ik} \sum_j p_{kj} z_j$$
$$- f_u P^* - \sum_{k,\ell} f_{p_k p_\ell} \Big(\sum_i p_{ik}\, z_i\Big)\Big(\sum_j p_{j\ell} z_j\Big), \quad (15)$$

where $P^* = \sum_{i,j} p_{ij} z_i z_j$. Constants c' and c'' independent of M and μ exist such that $\sum_{i,j} C_{ij}^2 \leqslant (c')^2$, $f_u^2 \leqslant c''$, and $\sum_{i,k} D_{ik}^2 \leqslant c''$, the last condition implying

$$2 \left|\sum_k \sum_i z_i D_{ik} \sum_j p_{kj} z_j\right| \leqslant 4c''/\mu + (\mu/4) \sum_k \Big(\sum_j p_{kj} z_j\Big)^2.$$

In view of assumption (2), Section 1, we also have

$$\sum_{k,\ell} f_{p_k p_\ell} \Big(\sum_i p_{ik} z_i\Big) \Big(\sum_j p_{j\ell} z_j\Big) \geqslant \mu \sum_k \Big(\sum_j p_{kj} z_j\Big)^2.$$

Hence, (15) leads to the inequality

$$\dot{P}^* \leqslant c/\mu + (\mu/4) P^{*2} - (3\mu/4) \sum_k \Big(\sum_j p_{kj} z_j\Big)^2$$

where c is a bound, for instance, for $c'\mu + 5c''$ and is taken to be independent of μ for μ less than an agreed upper bound. Moreover,

$$\sum_k \Big(\sum_i p_{ik} z_i\Big)^2 = \Big[\sum_k z_k^2\Big] \Big[\sum_k \Big(\sum_i p_{ik} z_i\Big)^2\Big]$$

$$\geqslant \Big[\sum_k z_k \sum_i p_{ik} z_i\Big]^2 = P^{*2}.$$

It follows that

$$P^* \leqslant c/\mu - (\mu/2) P^{*2}. \tag{16}$$

In order to discuss this, set

$$s = \sqrt{c/2\, t}\,,\ s_0 = \sqrt{c/2}\, t_0\,,\ q = (\mu/\sqrt{2c})\, P^*\,,\ q_0 = q(s_0)\,,$$

inequality (16) becoming

$$dq/ds \leqslant 1 - q^2. \tag{17}$$

We shall prove from it that if

$$Q = \max(1\,, q)\,,\ Q_0 = \max(1\,, q_0),$$

then

$$\frac{Q-1}{Q+1} \leqslant \frac{Q_0 - 1}{Q_0 + 1} e^{-2(s-s_0)}. \tag{18}$$

Four cases arise.

Case 1. $|q_0| < 1$. In this case, (17) leads to $dq/(1 - q^2) \leqslant ds$ and, by integration, to the inequality

$$\frac{1+q}{1-q} \leqslant \frac{1+q_0}{1-q_0} \zeta,$$

where $\zeta = e^{2(s-s_0)}$. This implies that

$$q \leqslant \frac{\zeta - 1 + (\zeta + 1)q_0}{\zeta + 1 + (\zeta - 1)q_0}$$

and thus that $q \leqslant 1$ in this case.

Case 2. $q_0 < -1$. As s increases, q achieves no value exceeding q_0, in view of the sign of dq/ds. Hence, $q < -1$ in this case.

Case 3. $q_0 > 1$. Here, we have $dq/(q^2 - 1) \leqslant - ds$, and therefore

$$\frac{q-1}{q+1} \leqslant \frac{q_0 - 1}{q_0 + 1} \zeta^{-1}$$

or, equivalently,

$$q \leqslant \frac{(\zeta + 1) q_0 + \zeta - 1}{(\zeta - 1) q_0 + \zeta + 1}. \tag{19}$$

Case 4. $|q_0| = 1$. Either $q \leqslant 1$ or else q assumes a value $q_1 > 1$. Letting s_1 be the first value above s_0 for which this occurs, we have by Case 3 that for $s \geqslant s_1$

$$q \leqslant \frac{(\zeta_1 + 1) q_1 + \zeta_1 - 1}{(\zeta_1 - 1) q_1 + \zeta_1 + 1},$$

where $\zeta_1 = \exp[2(s - s_1)]$. Since we may select q_1 arbitrarily close to 1, we conclude that in fact $q \leqslant 1$ for $s \geqslant s_0$.

Except in Case 3, we have just seen that $q \leqslant 1$. Hence, in all four cases,

$$q \leqslant \frac{(\zeta + 1)\,Q_0 + \zeta - 1}{(\zeta - 1)\,Q_0 + \zeta + 1},$$

the right side of this inequality exceeding the right side of (19) and also exceeding 1. For the last reason,

$$Q \leqslant \frac{(\zeta + 1)\,Q_0 + \zeta - 1}{(\zeta - 1)\,Q_0 + \zeta + 1}.$$

Equivalently,

$$\frac{Q - 1}{Q + 1} \leqslant \frac{Q_0 - 1}{Q_0 + 1}\,\zeta^{-1},$$

which implies (iv) with $C' = \sqrt{2c}$. The theorem is now completely proved.

3. Stratified solutions of (F), (G).

To construct stratified solutions of this problem we employ smoothing operators S_ϵ with the properties that for an arbitrary function $f \in C^2(E^n)$:

$$|S_\epsilon f|_0 \leqslant |f|_0, \tag{1}$$

$$|(S_\epsilon f)_{x_i x_j}|_0 \leqslant k_1\, \varepsilon^{-1}\, |f|_1 \quad \text{for} \quad i, j = 1, \dots, n, \tag{2}$$

$$P^*(t\,; z\,|S_\epsilon f) \leqslant P^*(t\,; z|\,f) \quad \text{for all unit vectors } z \in E^n, \tag{3}$$

$$\left| \int (S_\epsilon f - f)\, \phi dx \right| \leqslant C_\phi\, |f|_0\, a(\varepsilon) \quad \text{for any} \quad \phi \in C_0^2(E^n). \tag{4}$$

Here, as on previous occasions, $C^2(E^n)$ denotes the class of bounded functions with bounded, continuous derivatives of first and second orders in E^n ; $|f|_0 = \sup_{x \in E^n} |f(x)|$; $|f|_1 = \max_{i=1,\dots,n} \sup_{E^n} |f_{x_i}|$; k_1 is an absolute constant ; C_ϕ is a constant depending on ϕ only ; P^* is the functional defined in the theorem of Section 2 ; $\lim_{\epsilon \to 0} a(\varepsilon)/\varepsilon = 0$.

Averaging operators, for instance, will satisfy all these conditions, see Chapter 3.

Let us fix once and for all the thickness T of the layer $Z(T)$ within which we are to solve the given problem. Thereby the constants C and C′ in the theorem of Section 2 are determined. We shall construct a stratified solution of (F), (G) with stratum thickness h such that $Ch \leqslant 1/2$ and that, for convenience, T/h is an integer. We determine ε by the condition

$$h = 1/\{C\,[1 + k_1\, y(T)/\varepsilon]\},$$

and thus as

$$\varepsilon = \frac{k_1\, y(T)\, Ch}{1 - Ch},$$

where $y(t)$ is the increasing nonnegative function defined by the relation $\int_U^{y(t)} d\rho/E(\rho) = t$. In view of (2) and part (i) in the theorem of the previous section, an inductive procedure analogous to that of Section 3, Chapter 2, shows that, for $m = 1, 2, \ldots, T/h$, a strict solution u_m of (F) in Z_m [1] exists that satisfies the initial condition

$$u_m(x\,,(m-1)\,h) = S_\epsilon\, u_{m-1}(x\,,(m-1)\,h)$$

and obeys the conditions

$$|u_m|_{C^0(Z_m)} \leqslant y(mh)\,,\ |u_{m,x_i}|_{C^0(Z_m)} \leqslant y(mh) \quad \text{for} \quad i = 1,\ldots,n,$$

$u_0(x\,,0)$ standing for $u_0(x)$. Thus, the stratified solution

$$v = u_m \quad \text{in} \quad Z'_m \quad (^1) \ \text{for}\ m = 1,\ldots,T/h$$

satisfies the inequalities

$$|v|_{C^0(Z(T))} \leqslant y(T)\,,\ |v_{x_i}|_{C^0(Z(T))} \leqslant y(T). \tag{5}$$

Condition (iv) of the theorem referred to tells us that for

$$(m-1)\,h \leqslant t \leqslant mh$$

$$\frac{\mu P(t) - C'}{\mu P(t) + C'} \leqslant e^{-C'(t-(m-1)h)}\, \frac{\mu P_{m-1} - C'}{\mu P_{m-1} + C'} \tag{6}$$

where $P(t) = P(t\,; z\,|\,u_m)$, $P_k = P(kh\,; z\,|\,u_k)$, and also gives us the relations

[1] The notation is that of Chapter 2, Section 1 (p. 44).

$$\frac{\mu P_{k+1} - C'}{\mu P_{k+1} + C'} = \frac{\mu P((k+1)h\,; z\,|u_{k+1}) - C'}{\mu P((k+1)h\,; z\,|u_{k+1}) + C'}$$

$$\leqslant e^{-C'h}\frac{\mu P(kh\,; z\,|u_{k+1}) - C'}{\mu P(kh\,; z\,|u_{k+1}) + C'}$$

for $k = 0, 1, \ldots, T/h - 1$.

But inequality (3) implies that $P(kh\,; z\,|u_{k+1}) \leqslant P(kh\,; z\,|u_k) = P_k$. Hence, the last member of the previous inequality is

$$\leqslant e^{-C'h}[\mu P_k - C']/[\mu P_k + C'],$$

and in sum we have the recursion

$$\frac{\mu P_{k+1} - C'}{\mu P_{k+1} + C'} \leqslant e^{-C'h}\frac{\mu P_k - C'}{\mu P_k + C'} \quad \text{for} \quad k = 1, 1, \ldots, T/h - 1.$$

Multiplying these for $k = 0, 1, \ldots, m-2$ and then also multiplying by (6) shows that

$$\frac{\mu P(t) - C'}{\mu P(t) + C'} \leqslant e^{-C't}\frac{\mu P_0 - C'}{\mu P_0 + C'} \leqslant e^{-C't}.$$

from which we immediately deduce that

$$P(t) \leqslant (C'/\mu)\coth(C't/2).$$

It follows that for an arbitrary unit vector $z \in E^n$, $|z| = 1$, we have

$$\sum_{i,j} v_{x_i x_j} z_i z_j \leqslant (C'/\mu)\coth(C't/2) \quad \text{in} \quad Z(T)$$

and thus

$$v(x + \rho z, t) + v(x - \rho z, t) + 2v(x, t)$$

$$= \int_0^1 \left[\int_{-1}^1 \rho^2 z_i z_j v_{x_i x_j}(x + \theta\theta'\rho z)\,d\theta'\right]\theta\,d\theta$$

$$\leqslant (C'/\mu)\rho^2 \coth(C't/2) \text{ for } (x, t) \in Z(T), z \in E^n, |z| = 1. \quad (7)$$

4. Precompactness of stratified solutions.

We shall demonstrate the following result, similar in its nature and its proof to the precompactness theorem of Section 4, Chapter 2. Again v_h denotes the stratified solution with layer thickness h.

PRECOMPACTNESS THEOREM. *A bounded, Lipschitz continuous function u in $Z(T)$ and a null sequence $\{h_k\}$ exist such that for any t in the interval* $0 \leqslant t < T$

$$\lim_{k \to \infty} v_{h_k}(\cdot\,, t) = u(\cdot\,, t)$$

in E^n, *uniformly in any bounded region of* E^n.

Again the proof is in several steps. First, we have by methods of the earlier section :

PROPOSITION 1. – *A null sequence $\{h_k\}$ exists such that for any bounded, measurable function ϕ with compact support in* E^n,

$$\lim_{k \to \infty} \int_{E^n} v_{h_k}(x\,, t)\, \phi(x)\, dx$$

exists for $0 \leqslant t < T$.

Secondly, we prove :

PROPOSITION 2. – *Let t be a fixed value in* $(0\,, T)$, *and let $\{h_{k'}\}$ be an arbitrary subsequence of the sequence $\{h_k\}$ in Proposition 1. Correspondingly, a function $u'(\cdot\,, t)$ continuous in* E^n *and a sub-subsequence $\{h_{k''}\}$ –a subsequence of $\{h_{k'}\}$ –exist such that*

$$\lim_{k'' \to \infty} v_{h_{k''}}(\cdot\,, t) = u'(\cdot\,, t)$$

in E^n, *uniformly in every bounded region of* E^n.

The proof is immediate from the equiboundedness and equicontinuity of the $v_h(\cdot\,, t)$ (inequalities (5), Section 3).

Propositions 1 and 2 imply that for any bounded measurable ϕ with compact support in E^n,

$$\lim_{k\to\infty} \int_{E^n} v_{h_k}(x\,,\,t)\,\phi(x)\,dx$$

$$\lim_{k''\to\infty} \int_{E^n} v_{h_{k''}}(x\,,\,t)\,\phi(x)\,dx = \int_{E^n} u'(x\,,\,t)\,\phi(x)\,dx$$

which in turn implies that $u'(\cdot\,,\,t)$ is independent of the subsequence $\{h_{k'}\}$. This and Proposition 2 give us :

PROPOSITION 3. – *For each fixed value of t in $(0\,,\,T)$, a function $u(\cdot\,,\,t)$ continuous in E^n exists such that*

$$\lim_{k\to\infty} v_{h_k}(\cdot\,,\,t) = u(\cdot\,,\,t)$$

in E^n, uniformly in any bounded region of E^n.

Next, we shall prove that u is Lipschitz continuous in $Z(T)$ and begin by discussing its continuity with respect to t. For this purpose, we return to the functions

$$V(t) \equiv V(t\,;v_h) = \int v_h(x\,,\,t)\,\phi(x)\,dx$$

with $\phi \in C_0^2(E^n)$ and $\phi \geqslant 0$. The domain of integration is E^n. Since the v_h are stratified solutions, the one-sided limits $V(kh \pm 0)$ exist, and for $(k-1)\,h < t < kh$ the derivative

$$V'(t) = \int v_{h,t}(x\,,\,t)\,\phi(x)\,dx$$

exists, with $k = 1, 2, \dots, T/h$. If $0 \leqslant t_1 \leqslant t_2 < T$ and, more specifically,

$$m_1 h \leqslant t_1 < (m_1+1)\,h \leqslant m_2 h \leqslant t_2 < (m_2+1)\,h,$$

we thus have

$$V(t_2) - V(t_1) = \left\{ \int_{t_1}^{(m_1+1)h} + \sum_{k=m_1+1}^{m_2-1} \int_{kh}^{(k+1)h} + \int_{m_2 h}^{t_2} \right\} V'(t)\,dt$$

$$+ \sum_{k=m_1+1}^{m_2} [V(kh-0) - V(kh+0)].$$

Property (4), Section 3, shows the second summation on the right, in absolute value, to be $\leqslant C_\phi(t_2 - t_1)\,h^{-1}\,y(T)\,a(\varepsilon)$ and thus to tend towards zero with h. The remaining part of the right hand side, in absolute value, is $\leqslant F_0(t_2 - t_1) \int \phi(x)\,dx$, where F_0 is an upper bound for $|f(x, t, u, p)|$ for $x \in E^n$, $0 \leqslant t \leqslant T$, $|u| \leqslant y(T)$, $|p| \leqslant y(T)$. Consequently, we have

$$\lim_{k\to\infty} |V(t_2 ; \nu_{h_k}) - V(t_1 ; \nu_{h_k})| = \left| \int (u(x, t_2) - u(x, t_1))\, \phi(x)\,dx \right|$$
$$\leqslant F_0(t_2 - t_1) \int \phi(x)\,dx.$$

This implies that

$$\int \phi(x) \{F_0(t_2 - t_1) \pm (u(x, t_2) - u(x, t_1))\}\,dx \geqslant 0,$$

and since u is continuous with respect to x and ϕ is arbitrary,

$$F_0(t_2 - t_1) \pm (u(x, t_2) - u(x, t_1)) \geqslant 0.$$

This says that u is Lipschitz continuous with respect to t with Lipschitz constant F_0. It follows that u is Lipschitz continuous with respect to (x, t).

Properties (5) and (7) of stratified solutions, Section 3, carry over to u, and we have

$$|u(x, t)| \leqslant y(T),\ |u(x + z, t) - u(x, t)| \leqslant y(T)\ |z|,$$

$$u(x + z, t) + u(x - z, t) - 2u(x, t) \leqslant (C'/\mu)\ |z|^2 \quad \coth\ (C't/2) \quad (2)$$

for $(x, t) \in Z(0, T)$ and $z \in E^n$.

5. Generalized solutions of (F), (G).

The Lipschitz continuous function u obtained in the previous section obviously fulfills the initial condition (G). Now we shall show that u satisfies the differential equation (F) almost everywhere in $Z(T)$.

By the previous section,

$$\lim_{m\to\infty} u^{(m)} = u \quad \text{in} \quad Z(T),$$

where $u^{(m)} \equiv u_{h_m}$, $m = 1, 2, \ldots,$ is a suitable sequence of stratified solutions as described. Since $u^{(m)}$ is a stratified solution, we have for $0 < t < T$ and, say, $kh_m \leqslant t < (k+1)h_m$,

$$u^{(m)}(x, t) - u^{(m)}(x, kh_m + 0)$$
$$= \int_{kh_m}^{t} f(x, s, u^{(m)}(x, s), \operatorname{grad} u^{(m)}(x, s))\, ds \;;$$

also,

$$u^{(m)}(x, jh_m - 0) - u^{(m)}(x, (j-1)h_m + 0)$$
$$= \int_{(j-1)h_m}^{jh_m} f(x, s, u^{(m)}(x, s), \operatorname{grad} u^{(m)}(x, s))\, ds$$

for $j = 1, \ldots, k$.

Adding these $k + 1$ relations, multiplying by an arbitrary test function $\phi \in C_0^2(Z(T))$, and integrating over $Z(T)$ gives us

$$\int_{Z(T)} \phi(x, t) \left\{ u^{(m)}(x, t) - u^{(m)}(x, 0) \right.$$
$$\left. - \int_0^t f(x, s, u^{(m)}(x, s), \operatorname{grad} u^{(m)}(x, s)\, ds \right\} dxdt \tag{1}$$
$$= \sum_{j=1}^{k} \int \phi(x, jh_m)\, [u^{(m)}(x, jh_m + 0) - u^{(m)}(x, jh_m - 0)]\, dx.$$

We shall wish to let $m \to \infty$ in this formula and therefore now consider the convergence of the first derivatives

$$p_i^{(m)} \equiv \partial u^{(m)} / \partial x_i$$

of the stratified solutions of the sequence to the respective first derivatives

$$p_i \equiv \partial u / \partial x_i$$

of their limit.

By Lipschitz continuity, the p_i exist on a set $G \subset Z(T)$ such that $Z(T) \setminus G$ has measure zero. We shall now argue that the p_i are the respective limits almost everywhere of the $p_i^{(m)}$ as $m \to \infty$.

Fix the index i. By Fubini's theorem, G includes almost all points of almost all lines in $Z(T)$ that are parallel to the x_i-axis.

On such a line L, almost all points of which belong to G, we regard x_i as the only variable in the functions

$$q^{(m)}(x_i) \equiv p_i^{(m)}(x, t), \; q(x_i) \equiv p_i(x, t),$$

and consider the other x_j and t to be fixed parameters. For $t > 0$, it follows from the last inequality (2), Section 4, that q satisfies the one-sided Lipschitz condition

$$\frac{q(x_i) - q(x_i')}{x_i - x_i'} \leqslant \kappa \quad \text{for} \quad x_i \neq x_i' \tag{2}$$

with $\kappa \equiv \kappa(t) = (C'/\mu) \coth(C't/2)$; hence, q is continuous at almost all points on L. (See a remark on "semi-monotonic" functions in [6], page 78). Inequality (7), Section 3, implies that the $q^{(m)}$ for $m = 1, 2, \ldots$ also satisfy a one-sided Lipschitz condition of the form (2) and all with the same constant κ independent of m. Consequently, from any subsequence $\{m'\}$ of indices, a sub-subsequence $\{m''\}$ can be selected such that $q^{(m'')}$ converges almost everywhere on L. (See [6]). The limit function q' again is subject to the one-sided Lipschitz condition (2) and therefore is continuous at almost all points of L. Furthermore, $q^{(m'')} \to q'$ at every point at which q' is continuous.

The next step is to identify q' with q at the points at which q is continuous. Let L$'$ consist of the points of L at which q is continuous and L$''$ consist of the i – th coordinates of these points. Setting

$$U^{(m)}(x_i) \equiv u^{(m)}(x), \; U(x_i) \equiv u(x) \quad \text{along L},$$

we have for $x_i, a_i \in L''$

$$U^{(m)}(x_i) = U^{(m)}(a_i) + \int_{a_i}^{x_i} q^{(m)}(\xi)\, d\xi.$$

Letting $m = m'' \to \infty$ gives

$$U(x_i) = U(a_i) + \int_{a_i}^{x_i} q'(\xi)\, d\xi,$$

while the same relation holds with q in place of q'. Therefore, $q' = q$ almost everywhere on L and, in particular, at every point of continuity of the two functions. This means that $\lim_{m'' \to \infty} q^{(m'')} = q$ at every point of continuity of q on L, while the limit q is independent of the arbitrarily selected subsequence $\{m'\}$. With respect to the original sequence, we conclude, therefore, that

$$\lim_{m\to\infty} q^{(m)} = q$$

at every point at which q is continuous on L. It follows that

$$\lim_{m\to\infty} p_i^{(m)} = p_i$$

almost everywhere in Z(T). Let p_i^* and p_{i*} denote the upper and lower limits of the $p_i^{(m)}$, respectively, as $m \to \infty$. The set of points S at which $p_i^* - p_{i*} > 0$ is measurable and is included in the set of discontinuities of p_i with respect to x_i. Hence, for almost every line L parallel to the x_i-axis the intersection $S \cap L$ has one-dimensional measure 0. Therefore, the $(n + 1)$-dimensional measure of S is 0, i.e., $\lim_{m\to\infty} p_i^{(m)}$ exists almost everywhere in Z(T). Similar reasoning, in which $p^* - p_i$ is considered, will show that $\lim_{m\to\infty} p_i^{(m)} = p_i$ almost everywhere.

It is now possible to let $m \to \infty$ in (1). The limit of the right hand side is 0 in view of property (4), Section 3, and we obtain :

$$\int_{Z(T)} \phi(x\,,\,t) \left\{ u(x\,,\,t) - u_0(x) - \int_0^t f(x\,,\,s\,,\,u(x\,,\,s)\,,\,\text{grad}\,u(x\,,\,s))\,ds \right\} dxdt = 0.$$

Since ϕ is arbitrary, we can infer that the quantity in curly brackets is 0 almost everywhere in Z(T) and therefore, in particular, at almost all points of almost all line segments $x = \text{const.}$, $0 \leqslant t < T$, parallel to the t-axis. If M is such a segment, by differentiation with respect to t we have $u_t = f$ almost everywhere on M. Since u_t and f are measurable, this relation, which is (F), holds almost everywhere in Z(T), as was to be proved.

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