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CERTAIN REAL SURFACES IN $\mathbb{C}^2$ WITH ISOLATED SINGULARITIES

by Sushil GORAI (*)

ABSTRACT. — The surfaces in $\mathbb{C}^2$ with an isolated CR singularity at the origin and with cubic lowest degree homogeneous term in its graph near the origin, under certain geometric condition, can be reduced — up to biholomorphism of $\mathbb{C}^2$ — to a one-parameter family of the form

$$M_t := \left\{ (z, w) \in \mathbb{C}^2 : w = z^2\bar{z} + tz\bar{z}^2 + \frac{t^2}{3}\bar{z}^3 + o(|z|^3) \right\}, \quad t \in (0, \infty),$$

near the origin. We prove that M_t is not locally polynomially convex if $t < 1$. The local hull contains a ball centred at the origin if $t < \sqrt{3}/2$. We also prove that M_t is locally polynomially convex for $t \geq \sqrt{3}/2$. We show that, for $\sqrt{3}/2 \leq t < 1$, the local hull of M_t contains a one-parameter family of analytic discs passing through the origin. We also prove that, if we remove the higher order terms from the graphing function of M_t , it is locally polynomially convex for $t \geq \frac{\sqrt{15}-\sqrt{33}}{2\sqrt{2}}$. Some new results about the local polynomial convexity of the union of three totally-real planes are also reported.

Keywords: polynomially convex, totally real, CR singularity, polynomial hull, analytic disc.

2020 Mathematics Subject Classification: 32E20, 32D10, 30E10.

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RÉSUMÉ. — Les surfaces dans $\mathbb{C}^2$ avec une singularité CR isolée à l'origine et avec un terme homogène de plus bas degré cubique dans son graphe près de l'origine, sous certaines conditions géométriques, peuvent être réduites — à un biholomorphisme près de $\mathbb{C}^2$ — à une famille à un paramètre de la forme

$$M_t := \left\{ (z, w) \in \mathbb{C}^2 : w = z^2\bar{z} + tz\bar{z}^2 + \frac{t^2}{3}\bar{z}^3 + o(|z|^3) \right\}, \quad t \in (0, \infty),$$

près de l'origine. Nous prouvons que M_t n'est pas localement polynomialement convexe si $t < 1$. L'enveloppe locale contient une boule centrée à l'origine si $t < \sqrt{3}/2$. Nous prouvons également que M_t est localement polynomialement convexe pour $t \geq \sqrt{3}/2$. Nous montrons que, pour $\sqrt{3}/2 \leq t < 1$, l'enveloppe locale de M_t contient une famille à un paramètre de disques analytiques passant par l'origine. Nous prouvons également que, si nous supprimons les termes d'ordre supérieur de la fonction graphique de M_t , elle est localement polynomialement convexe pour $t \geq \frac{\sqrt{15}-\sqrt{33}}{2\sqrt{2}}$. De nouveaux résultats sur la convexité polynomiale locale de l'union de trois plans totalement réels sont également rapportés.

1. Introduction and statement of results

Let K be a compact subset of $\mathbb{C}^n$. The polynomially convex hull of K is defined by $\hat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p|, p \in \mathbb{C}[z_1, \dots, z_n]\}$. The set K is said to be polynomially convex if $\hat{K} = K$. A closed subset E of $\mathbb{C}^n$ is said to be locally polynomially convex at $p \in E$ if $E \cap \overline{B(p; r)}$ is polynomially convex for some $r > 0$, where $B(p; r)$ denotes the open ball in $\mathbb{C}^n$ with centre p and radius r . In $\mathbb{C}$, polynomial convexity of a compact set K is equivalent to $\mathbb{C} \setminus K$ is connected which is purely a topological condition on the compact set. In higher dimensions, there is no such criterion. Polynomial convexity is one of the fundamental and classical concepts in several complex variables; it came up, mainly, due to its deep interconnections with polynomial approximations. We will state a couple of such results for motivations. The first result is a generalization of Runge's approximation theorem.

THEOREM 1.1 (Oka–Weil). — *Let $K \subset \mathbb{C}^n$ be a compact polynomially convex set. Then any function that is holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials in $z_1, \dots, z_n$.*

A few notations: for a compact $K \subset \mathbb{C}^n$, $\mathcal{C}(K)$ denotes the algebra of all continuous functions on K , and $\mathcal{P}(K)$ denotes the uniformly closed subalgebra of $\mathcal{C}(K)$ generated by polynomials in $z_1, \dots, z_n$. In this paper, planes will mean 2-dimensional real subspaces of $\mathbb{C}^2$ and surfaces will be 2-dimensional real submanifolds of $\mathbb{C}^2$.

A real submanifold M of $\mathbb{C}^n$ is said to be totally real at $p \in M$ if $T_p M \cap iT_p M = \{0\}$, where $T_p M$ denotes the tangent space of M at p .

viewed as a subspace of $\mathbb{C}^n$. The manifold M is said to be totally real if it is totally real at every point $p \in M$. A point p in a real surface M in $\mathbb{C}^2$ is said to be a CR singularity if M is not totally real at p , i.e., $T_p M$ is a complex subspace of $\mathbb{C}^2$. We now state another approximation result, which is a generalization of Stone–Weierstrass theorem.

THEOREM 1.2 (O’Farrell–Preskenis–Walsh). — *Let $K \subset \mathbb{C}^n$ be a compact polynomially convex subset of $\mathbb{C}^n$ and $E \subset K$ be such that $K \setminus E$ is locally contained in totally-real submanifolds of $\mathbb{C}^n$. Then*

$$\mathcal{P}(K) = \{f \in \mathcal{C}(K) : f|_E \in \mathcal{P}(E)\}.$$

If E is an arc or finitely many points in K , $\widehat{K} = K$ implies $\mathcal{P}(K) = \mathcal{C}(K)$. The subsets that we will consider in this paper will be totally real except at one point.

For a compact subset $K \subset \mathbb{C}^n$, the main obstruction to $\mathcal{P}(K) = \mathcal{C}(K)$ is having analytic structure in $\widehat{K}$. One of the ways to get analytic structure is by attaching analytic discs to K , that is, existence of a continuous map $\phi: \overline{\mathbb{D}} \rightarrow \mathbb{C}^n$ that is holomorphic on $\mathbb{D}$ and $\phi(\partial\mathbb{D}) \subset K$. Constructions of such families of analytic discs attached to a submanifold are crucial for the hull of holomorphy and extension of holomorphic functions [1, 2, 6, 15, 18, 19].

We now give a brief survey about the local polynomial convexity of real submanifolds in $\mathbb{C}^2$. We start with a very simple case: the graph of a linear function on $\mathbb{C}$ is either a totally-real subspace or a complex subspace that is of the form $M = \{(z, f(z)) \in \mathbb{C}^2 : f(z) = az + b\bar{z}\}$. Any compact subset of M is polynomially convex. M is totally real if and only if $b \neq 0$. Next, we consider $\mathcal{C}^2$ -smooth real surface in $\mathbb{C}^2$. For each point $a \in M$, there exists a $\delta > 0$ such that

$$M \cap B(a; \delta) = \{(z, f(z)) \in \mathbb{C}^2 : f \in \mathcal{C}^2(D(a; \delta))\}.$$

Such a surface M is totally real at $a \in M$ if and only if $\frac{\partial f}{\partial \bar{z}}(a) \neq 0$. Wermer [26] showed that a real surface M in $\mathbb{C}^2$ is locally polynomially convex at each of its totally-real points, which was then generalized to higher dimensions in [14]. Let $0 \in M$ be an isolated CR singularity of the surface M . Locally, near the origin, M is of the form

$$(1.1) \quad M \cap B(0; \delta) = \{(z, f(z)) \in \mathbb{C}^2 : f(z) = p_k(z, \bar{z}) + o(|z|^k)\},$$

where $p_k(z, \bar{z})$ is a homogeneous nonholomorphic polynomial of degree k , $k \geq 2$. We say that such a surface M has a CR singularity of order k . Bishop [6] considered the case $k = 2$, i.e., $p_2(z, \bar{z}) = az^2 + b\bar{z}^2 + cz\bar{z}$. Under Morse theoretic nondegeneracy condition (equivalently $c \neq 0$), Bishop [6]

showed that, by biholomorphic change of variables, for sufficiently small $\delta > 0$, $M \cap B(0; \delta)$ can be taken to now-famous Bishop's normal form:

$$(1.2) \quad M \cap B(0; \delta) = \{(z, f(z)) \in \mathbb{C}^2 : f(z) = \gamma(z^2 + \bar{z}^2) + |z|^2 + o(|z|^2)\},$$

where $\gamma \geq 0$ is a biholomorphic invariant. The origin is called the elliptic CR singularity if $\gamma < 1/2$, parabolic if $\gamma = 1/2$, and hyperbolic if $\gamma > 1/2$. For $\gamma < 1/2$, Bishop [6] showed, by constructing a one-parameter family of analytic discs with boundaries passing around the origin in M , that the surface is not locally polynomially convex at the origin. Kenig–Webster [19], for $\mathcal{C}^\infty$ -smooth surface M , showed that the local hull is a three dimensional $\mathcal{C}^\infty$ -smooth manifold. For $0 < \gamma < 1/2$, Moser–Webster [20] described complete local invariants of M in case M is real analytic near the origin and it is immediate from their normal form that the local hull is a three dimensional real analytic manifold in this case. Forstnerič and Stout [9] showed that the surface is locally polynomially convex at the hyperbolic CR singularities. Jöricke [16] studied the case $\gamma = 1/2$. For $k \geq 3$, Morse theory does not give any nondegeneracy condition. This type of CR singularity is called *degenerate* CR singularity. Before proceeding further with the discussion, we mention the following definition on nonparabolic CR singularity [5, 13] that makes sense in case of degenerate CR singularity as well as nondegenerate.

DEFINITION 1.3. — *A surface M of the form (1.1) is said to have a nonparabolic CR singularity of order k at the origin if there is an isolated CR singular point for the surface $\{(z, p_k(z, \bar{z})) : z \in \mathbb{C}\}$ at the origin, where p_k is as in (1.1).*

Harris [13] showed that having a nonparabolic singularity of order k at the origin is stable under $o(|z|^k)$ perturbation. We note that the quadratic terms in Bishop's normal form (1.2) are real-valued. In general, for $k \geq 3$, there is no biholomorphic change of coordinates under which the lowest order homogeneous term becomes real-valued. The question arises: *is it possible to characterize the local polynomial convexity of a surface with nonparabolic CR singularity of order k , $k \geq 3$?* Efforts [5, 13] have been made to achieve a Bishop-type dichotomy for nonparabolic points in case of higher order CR singular points, but one of the assumptions of the results in this directions is — up to a biholomorphic change of variables — the lowest order homogeneous term is real-valued. Maslov-type index (see Section 2.1 for a definition) plays a crucial role in case of higher order degeneracy (see [5]). The papers [3, 4] describe conditions, in terms of the coefficients bound of the lowest order homogeneous term (not necessarily real-valued), under which M is locally polynomially convex at the origin. Wiegerink [27],

on the other hand, demonstrated conditions under which the local hull is nontrivial (see Theorems 2.16, 2.17 and 2.18).

In this paper, we restrict our attention to surfaces M of the form (1.1) with $k = 3$ and, to simplify the notation, from now onwards we call the polynomial p_3 as p . A general form of p is:

$$p(z, \bar{z}) = a_1 z^2 \bar{z} + a_2 z \bar{z}^2 + a_3 \bar{z}^3,$$

where $a_1, a_2, a_3 \in \mathbb{C}$. We denote $S := \{(z, p(z, \bar{z})) \in \mathbb{C}^2 : z \in \mathbb{C}\}$. We impose the following geometric condition on M :

(*) *M is a real surface in $\mathbb{C}^2$ as in (1.1) with $k = 3$ and S is as defined above from M such that there exists a proper holomorphic map $\Phi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ with $\Phi^{-1}(S)$ is the union of three totally real planes in $\mathbb{C}^2$.*

It is interesting to note that condition (*) turns out to be a condition on the coefficients of the polynomial p . We consider the proper holomorphic map $\Phi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$(1.3) \quad \Phi(z, w) := (z, p(z, w)),$$

where $p(z, w) = a_3 w^3 + a_2 w^2 z + a_1 w z^2$. The following result, due to Thomas [24, Lemma 3], gives a necessary and sufficient condition in terms of the coefficients of p for $\Phi^{-1}(S)$ to be a union of three totally-real planes.

THEOREM 1.4 (Thomas). — *Assume $S = \{(z, w) \in \mathbb{C}^2 : w = p(z, \bar{z})\}$, where $p(z, w) = a_3 w^3 + a_2 w^2 z + a_1 w z^2$. Let $\Phi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be as above. Then $\Phi^{-1}(S)$ is the union of three totally-real planes if and only if $a_2^2 = 3a_1 a_3$ and $a_3 \neq 0$.*

In view of Theorem 1.4, we obtain that, under the conditions $a_2^2 = 3a_1 a_3$ and $a_3 \neq 0$, the preimage of S under the particular proper holomorphic map $\Phi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a union of three transverse totally-real planes. One is tempted to think that there might be different proper holomorphic map on $\mathbb{C}^2$ such that the preimage under that map is also a union of three transverse totally-real planes, possibly different. We show that, up to an invertible complex linear map, the proper holomorphic map Φ is unique (see Lemma 2.1). Hence, in view of Theorem 1.4, for each $a \in \mathbb{C} \setminus \{0\}$, the given surface is

$$(1.4) \quad S_a = \{(z, p_a(z, \bar{z})) \in \mathbb{C}^2 : z \in \mathbb{C}\},$$

where $p_a(z, \bar{z}) = z^2 \bar{z} + a z \bar{z}^2 + \frac{a^2}{3} \bar{z}^3$. The complex number a which appears in the coefficients of the cubic polynomial p_a is not a biholomorphic invariant.

Our search of a biholomorphic invariant for this family of surfaces leads us to certain normal form due to Harris [12].

THEOREM 1.5 (Harris). — *Let M be a $\mathcal{C}^\infty$ -smooth real surface in $\mathbb{C}^2$ with an isolated CR-singularity of order 3 at the origin. Then, locally, near the origin, up to biholomorphism of $\mathbb{C}^2$, M is of the form*

$$\{(z, p(z, \bar{z}) + o(|z|^3)) \in \mathbb{C}^2 : |z| < \delta\},$$

where p is one of the following degree 3 homogeneous polynomials: $z^2\bar{z}$, $z\bar{z}^2$, $\bar{z}^3$, $z^2\bar{z} + \gamma z\bar{z}^2$, $z^2\bar{z} + \gamma z\bar{z}^2 + c\bar{z}^3$ with $\gamma > 0$ and $c \in \mathbb{C}$. Moreover, γ and c are biholomorphic invariants.

Our geometric condition $(*)$ forces the normal form of p to be of the form $z^2\bar{z} + \gamma z\bar{z}^2 + c\bar{z}^3$. Therefore, by Theorem 1.5, the normal form of surfaces of our study is

$$(1.5) \quad \{(z, p_t(z, \bar{z})) \in \mathbb{C}^2 : z \in \mathbb{C}\}, \quad t \in (0, \infty),$$

where

$$p_t(z, \bar{z}) = z^2\bar{z} + tz\bar{z}^2 + \frac{t^2}{3}\bar{z}^3.$$

The real number t becomes a biholomorphic invariant. Therefore, it is enough to consider the surfaces S_t for $t \in (0, \infty)$.

We now provide a couple of argument supporting our claim that the family of surfaces M_t , $t \in (0, \infty)$, though the lowest degree homogeneous term of the graphing function is not necessarily real-valued, is a right class of surfaces that can provide the Bishop-type dichotomy. Firstly, condition $(*)$ is also hidden in Bishop's surfaces. For every surface with nonparabolic CR-singularity, the graph of the quadratic polynomial in Bishop's normal form can be pulled back by proper holomorphic map on $\mathbb{C}^2$ to the union of two totally-real planes. This leads to pulling back Bishop's surfaces with CR singularity to unions of two totally-real surfaces. Forstnerič–Stout [9] used this to show local polynomial convexity at hyperbolic CR singularity. Approaching to local polynomial convexity at CR singularity of higher order, in general, is difficult; this is the main reason of assuming “thin” or “flat” surfaces, i.e., the lowest order homogeneous terms in the graphing function to be real-valued in [5, 13]. In [3, 4], pulling back by proper holomorphic mapping on $\mathbb{C}^2$ to union of certain totally-real surfaces is used crucially. Hence, condition $(*)$ is a very natural condition to assume if we use this same approach for our surfaces of consideration. Secondly, we look at Bishop's normal form in a slightly different viewpoint, using the following

biholomorphic transformations $\tau_j: \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $j = 1, 2$, defined by

$$\begin{aligned}\tau_1(z, w) &:= (z, w - \gamma z^2), \\ \tau_2(z, w) &:= (z, 4\gamma w), \\ \tau_3(z, w) &:= (z, w + z^2).\end{aligned}$$

Define $\varphi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $\varphi(z, w) := \tau_3 \circ \tau_2 \circ \tau_1$. Therefore, we have

$$\varphi(z, f(z)) = (z + 2\gamma\bar{z})^2 + o(|z|^2).$$

By putting $t := 2\gamma$, we obtain that

$$(1.6) \quad \varphi(z, f(z)) = (z + t\bar{z})^2 + o(|z|^2),$$

where t is also a biholomorphic invariant. Surfaces with parabolic CR singularity are given by $t = 1$. From the relation of t with Bishop's invariant, we can say that the surface has an elliptic CR singularity if $t < 1$, and hyperbolic CR singularity if $t > 1$. We now apply the biholomorphic map $\sigma: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ on M_t defined by

$$\sigma(z, w) := \sigma_1 \circ \sigma_2(z, w),$$

where

$$\begin{aligned}\sigma_1(z, w) &:= (z, w + z^3), \\ \sigma_2(z, w) &:= (z, 3tw).\end{aligned}$$

We also see that the surface M_t is, locally near the origin, equivalent to a surface of the form:

$$\{(z, w) \in \mathbb{C}^2 : w = (z + t\bar{z})^3 + o(|z|^3)\}, \quad t \in (0, \infty).$$

Therefore, for obtaining a Bishop-type phenomenon, a right class of surfaces with isolated CR singularity of order k at the origin to consider are:

$$\{(z, w) \in \mathbb{C}^2 : w = (z + t\bar{z})^k + o(|k|^k)\}, \quad t \in (0, \infty).$$

In this paper, we will study the surfaces for $k = 3$, i.e., the surfaces M_t , $t \in (0, \infty)$. For each $t \in (0, \infty)$, we will consider the corresponding proper holomorphic map

$$(1.7) \quad \Phi_t: \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \quad \text{defined by} \quad \Phi_t(z, w) = (z, p_t(z, w)).$$

Thomas [24] considered a class of triples of totally-real planes that are the preimage of surfaces S_a of the form (1.4), where $a \in \mathbb{C}$ is sufficiently small, for demonstrating triples of totally-real planes whose pairwise unions are locally polynomially convex at the origin but the local hull of the whole union contains a ball centred at the origin and with positive radius. We note that, for every $t \in \mathbb{C}$, the map Φ_t , defined in (1.7), is a proper holomorphic

map on $\mathbb{C}^2$. For each $t \in \mathbb{C} \setminus \{0\}$, the preimage $\Phi_t^{-1}(S_t)$ is a union of three totally-real planes. Thanks to the proper map (1.7) and the normal form (1.5) of S_t , it is enough to consider the following triples of totally-real planes whose image is S_t , $t \in (0, \infty)$, under the proper holomorphic map (1.7).

$$(1.8) \quad \begin{aligned} P_0^t : \quad & w = \bar{z}, \\ P_1^t : \quad & w = -\frac{\sqrt{3}(\sqrt{3}-i)}{2t}z - \frac{1-i\sqrt{3}}{2}\bar{z}, \\ P_2^t : \quad & w = -\frac{\sqrt{3}(\sqrt{3}+i)}{2t}z - \frac{1+i\sqrt{3}}{2}\bar{z}. \end{aligned}$$

Our first couple of results are about the local polynomial convexity of compact subsets of $P_0^t \cup P_1^t \cup P_2^t$.

THEOREM 1.6. — *For each $t > \frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$, $P_0^t \cup P_1^t \cup P_2^t$ is locally polynomially convex at the origin.*

The first part of the following theorem gives a generalization of a result of Thomas [24], and the second part demonstrates a new phenomenon for the union of three totally-real planes in $\mathbb{C}^n$.

THEOREM 1.7. — *If $t < 1$, $P_0^t \cup P_1^t \cup P_2^t$ is not locally polynomially convex at the origin. Moreover, for every $\delta > 0$:*

- (i) *the polynomial hull of $(P_0^t \cup P_1^t \cup P_2^t) \cap \overline{B(0; \delta)}$ contains a neighborhood of the origin in $\mathbb{C}^2$ if $0 < t < \sqrt{3}/2$;*
- (ii) *the polynomial hull contains a one-parameter family of analytic varieties passing through the origin if $\sqrt{3}/2 \leq t < 1$.*

The next couple of results are about the local polynomial convexity of compact subsets of S_t at the origin.

THEOREM 1.8. — *For $t \in (0, \infty)$, let $S_t = \{(z, w) \in \mathbb{C}^2 : w = p_t(z, \bar{z})\}$, where $p_t(z, \bar{z}) = z^2\bar{z} + tz\bar{z}^2 + t^2\bar{z}^3/3$. Then:*

- (i) *the surface S_t is locally polynomially convex at the origin if $t > \frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$;*
- (ii) *S_1 is locally polynomially convex at the origin.*

A couple of new phenomena occur here which were not present in the case of Bishop's surfaces. Part (ii) of the next theorem describes the surfaces S_t whose local hull contains a nonempty open ball centred at the origin and part (iii) determines those surfaces which have analytic discs with boundary in S_t and pass through the CR singularity at the origin.

THEOREM 1.9. — *Let $S_t = \{(z, w) \in \mathbb{C}^2 : w = p_t(z, \bar{z})\}$, where $p_t(z, \bar{z}) = z^2\bar{z} + tz\bar{z}^2 + t^2\bar{z}^3/3$. Then:*

- (i) *for $t \in (0, 1)$, S_t is not locally polynomially convex at the origin;*
- (ii) *for every $\delta > 0$, the polynomial hull of $S_t \cap \mathbb{B}(0; \delta)$ contains a nonempty open ball centred at the origin if $t \in (0, \sqrt{3}/2)$;*
- (iii) *for every $\delta > 0$, the polynomial hull of $S_t \cap \mathbb{B}(0; \delta)$ contains a one-parameter family of analytic discs with boundary in S_t passing through the origin if $t \in [\sqrt{3}/2, 1)$.*

We now consider the surfaces $M_t \subset \mathbb{C}^2$ with an isolated CR singularity at the origin of order three, i.e. there exists $r > 0$ such that

$$(1.9) \quad M_t \cap \overline{B(0; r)} = \{(z, w) \in \mathbb{C}^2 : \phi(z, w) = 0\},$$

where $\phi(z, w) = p_t(z, \bar{z}) + F(z, \bar{z}) - w$ with $F(z) = o(|z|^3)$. Next, we state a couple of theorems for M_t analogous to Theorems 1.8 and 1.9.

THEOREM 1.10. — *For $t \in (0, \infty)$, let M_t be as in (1.9). Then M_t is locally polynomially convex at the origin if $t \geq \sqrt{\frac{3}{2}}$.*

THEOREM 1.11. — *For $t \in (0, \infty)$, let M_t be as in (1.9). Then:*

- (i) *M_t is not locally polynomially convex if $t \in (0, 1)$;*
- (ii) *for every $\delta > 0$, the polynomially convex hull of $M_t \cap \overline{\mathbb{B}(0; \delta)}$ contains a nonempty open ball centred at the origin if $t \in (0, \sqrt{3}/2)$;*
- (iii) *for every $\delta > 0$, the polynomial hull of $M_t \cap \mathbb{B}(0; \delta)$ contains a one-parameter family of analytic discs with boundary in M_t and passing through the origin if $\sqrt{3}/2 \leq t < 1$.*

Remark 1.12. — We make a couple of remarks here.

- (1) The number $\sqrt{3}/2$ has a crucial geometric meaning. For this number onward the boundary of any analytic disc attached to M_t has to pass through the CR singularity, the origin. It seems a hyperbolic sector is appearing in the picture. In another viewpoint, both the matrices corresponding to Weinstock's normal form (see Lemma 2.5) of the planes have nonreal eigenvalues if and only if $0 < t < \sqrt{3}/2$.
- (2) The numbers $\sqrt{\frac{3}{2}}$ and $\frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$ seem to have no special importance. It is just that our technique does not work. We expect that M_t will be locally polynomially convex at the origin for $t > 1$.

We now turn our discussion towards local polynomial convexity and the local hull of the unions of three totally-real surfaces. In view of condition (*), the unions of three totally-real surfaces have a very close connection with the surfaces M_t . One of the main approaches of showing local

polynomial convexity of surfaces at the isolated CR singularity comes from looking at the unions totally-real surfaces those arise as preimages under a proper holomorphic maps, and showing that the unions are locally polynomially convex at the origin. Forstnerič–Stout [9] used this to prove their theorem. We will also use this in our proofs of Theorem 1.8 and Theorem 1.10. If the totally-real surfaces are pairwise transverse, then locally the union of totally-real surfaces can be seen as a union of small perturbations of their tangent spaces at the origin. The study of the local polynomial convexity of the union of two totally-real subspaces began with Weinstock’s [25] (see Theorem 2.6) necessary and sufficient condition. The unions of three totally-real planes in $\mathbb{C}^2$ was first considered by Thomas [23, 24] showing that there exist triples of totally-real planes such that the local hull of the union contains a ball centred at the origin and, also, by demonstrating examples where there are no nontrivial hull. Sufficient conditions for local polynomial convexity of the unions of three totally-real planes in $\mathbb{C}^2$, in terms of the pair of matrices in Weinstock’s normal form (see Lemma 2.5), is given in [11] (see Theorem 2.7). In this paper, we provide some new results in this setting (see Theorem 3.1, Theorem 3.2, Theorem 3.3) that will also be used in the proof of Theorem 1.8. It is not trivial to pass the local polynomial convexity to the union of totally-real submanifolds intersecting at the origin from the union of their tangent spaces at the origin. The union of two totally-real submanifolds in $\mathbb{C}^n$ intersecting transversely at the origin is locally polynomially convex if the union of their tangent spaces is [10, 21]. These results use Weinstock’s result (Theorem 2.6) crucially. In our case Kallin’s lemma will allow us to pass local polynomial convexity from the union of tangent spaces to the union of totally-real surfaces for $t \geq \sqrt{\frac{3}{2}}$ as in [10]. This passage does not seem to work for

$$\frac{\sqrt{15 - \sqrt{33}}}{2\sqrt{2}} < t \leq \sqrt{\frac{3}{2}}.$$

Few comments about our proofs of the theorems.

- In the base of our approach towards local polynomial convexity of M_t , there lies the union of three pairwise transverse totally-real planes. Therefore, we first prove Theorem 1.6 with the help of Theorem 2.7 and theorems that are stated and proved in Section 3. The first part of Theorem 1.8 follow immediately from Theorem 1.6. For S_1 , the planes which we get are not pairwise transverse. We deal with that separately.

- To prove Theorem 1.10, we use Kallin's lemma to enable us to prove local polynomial convexity of union of three totally real surfaces which can be seen as small perturbation of the planes in Theorem 1.6.
- We use some results of Wiegerink (Theorems 2.16, 2.17 and 2.18) to prove Theorems 1.9 and 1.11. Theorem 1.7 follows from Theorem 1.9.

About the layout of the paper: in Section 2, we collect some results from the literature and prove a lemma about uniqueness of proper holomorphic map, up to invertible $\mathbb{C}$ -linear transformation. We also provide a discussion about Maslov-type index here. We state and prove three general results about local polynomial convexity of union of three totally-real planes in $\mathbb{C}^2$ in Section 3. In Section 4, we state and prove the local polynomial convexity of certain unions of three totally real surfaces. Proofs of Theorems 1.6, 1.8 and 1.10 are discussed in Section 5. We demonstrate the proofs of Theorems 1.7, 1.9 and 1.11 in Section 6. Some open questions are mentioned in Section 7.

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2. Preliminaries

The most part of this section is devoted to collect the results from the literature that will be used in our proofs. We begin the section by showing that the proper map in condition $(*)$ is unique up to invertible $\mathbb{C}$ -linear transformations.

LEMMA 2.1. — *Let $S = \{(z, p(z, \bar{z})) \in \mathbb{C}^2 : z \in \mathbb{C}\}$, where $p(z, \bar{z}) = \sum_{j=1}^3 a_j \bar{z}^j z^{3-j}$ with $a_3 \neq 0$ and $a_2^2 = 3a_1 a_3$. Let $\Psi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a proper holomorphic map of the form $\Psi(z, w) = (z, Q(z, w))$, where Q is a homogeneous polynomial. Assume further that $\Psi^{-1}(S)$ can be written as union of three pairwise transverse totally-real planes. Then $\Psi = \Phi$ as defined in (1.3) up to a $\mathbb{C}$ -linear transformation.*

Proof. — We first note that the degree of the polynomial Q must be 3, and, since Q is homogeneous, it can be written as:

$$Q(z, w) = c(\mu_1 z + w)(\mu_2 z + w)(\mu_3 z + w).$$

Then, if we designate $\Psi^{-1}(S) = \bigcup_{j=1}^3 \{(z, w) \in \mathbb{C}^2 : w = \alpha_j z + \beta_j \bar{z}\}$, for $\alpha_j, \beta_j \in \mathbb{C}$, $j = 1, 2, 3$, we must investigate the solutions (α, β) of the equation

$$(2.1) \quad c \prod_{j=1}^3 ((\mu_j + \alpha)z + \beta \bar{z}) = \sum_{j=1}^3 a_j \bar{z}^j z^{3-j}.$$

Comparing the coefficients, we get that $c = a_3$ and

$$(\mu_1 + \alpha)(\mu_2 + \alpha)(\mu_3 + \alpha) = 0,$$

$$(\mu_1 + \mu_2 + \mu_3 + 3\alpha) = \frac{a_2}{a_3 \beta^2},$$

$$(\mu_1 + \alpha)(\mu_2 + \alpha) + (\mu_2 + \alpha)(\mu_3 + \alpha) + (\mu_3 + \alpha)(\mu_1 + \alpha) = \frac{a_1}{a_3 \beta},$$

$$\beta^3 = 1.$$

Thus, we have $\beta \in \{1, \omega, \omega^2\}$, where $\omega = e^{2\pi i/3}$. Let us name the pairs of coefficients as (α_j, β_j) , $j = 1, 2, 3$ defining the three planes, where $\beta_1 = 1$, $\beta_2 = \omega$, and $\beta_3 = \omega^2$. We note that, for fixed $k \in \{1, 2, 3\}$, $\mu_j + \alpha_k$, $j = 1, 2, 3$, are the roots of the following equation:

$$\lambda^3 - \frac{a_2}{a_3} \left(\frac{\lambda}{\beta_k} \right)^2 + \frac{a_1 \lambda}{a_3 \beta_k} = 0.$$

Consider the equation

$$(2.2) \quad r^3 - \frac{a_2}{a_3} r^2 + \frac{a_1}{a_3} r = 0.$$

By previous computations, we see that each of the following triples are the roots of equation (2.2): $(\mu_1 + \alpha_1, \mu_2 + \alpha_1, \mu_3 + \alpha_1)$, $(\omega(\mu_1 + \alpha_2), \omega(\mu_2 + \alpha_2), \omega(\mu_3 + \alpha_2))$, $(\omega^2(\mu_1 + \alpha_3), \omega^2(\mu_2 + \alpha_3), \omega^2(\mu_3 + \alpha_3))$. If we assume that r_1, r_2 and r_3 are the roots of equation (2.2), then the polynomial Q exists such that Ψ is a proper map with the required properties if and only if there exist permutations $\pi_1, \pi_2 \in S_3$ such that

$$r_k = \omega^j r_{\pi_j(k)} + \alpha_1 - \alpha_{j+1}, \quad j = 1, 2 \quad \text{and} \quad k = 1, 2, 3.$$

We also note that one of the members in each triple is zero. Therefore, by looking at the first triple, we obtain that

$$\{(z, w) \in \mathbb{C}^2 : w = \bar{z}\} \subset \Psi^{-1}(S).$$

This implies that

$$Q(z, \bar{z}) = p(z, \bar{z}) \quad \forall z \in \mathbb{C}.$$

Since $\{(z, w) \in \mathbb{C}^2 : w = \bar{z}\}$ is maximally totally real, therefore, we conclude that

$$Q(z, w) = p(z, w) \quad \forall (z, w) \in \mathbb{C}^2.$$

Choosing any other triple will contribute a $\mathbb{C}$ -linear transformation to the map Φ . $\square$

The following result [22, Theorem 1.6.24] plays a vital role in this paper.

THEOREM 2.2. — *Let $\Phi: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a proper holomorphic map. A compact subset K of $\mathbb{C}^n$ is polynomially convex if and only if $\Phi^{-1}(K)$ is polynomially convex.*

Next, we state a lemma due to Kallin [17] (also see [7]), which will be used repeatedly in this sequel. It gives a condition under which the union of two polynomially convex sets is polynomially convex.

THEOREM 2.3 (Kallin). — *Let K_1 and K_2 be two compact polynomially convex subsets in $\mathbb{C}^n$. Suppose L_1 and L_2 are two compact polynomially convex subsets of $\mathbb{C}$ with $L_1 \cap L_2 = \{0\}$. Suppose further that there exists a holomorphic polynomial P satisfying the following conditions:*

- (i) $P(K_1) \subset L_1$ and $P(K_2) \subset L_2$;
- (ii) $P^{-1}\{0\} \cap (K_1 \cup K_2)$ is polynomially convex.

Then $K_1 \cup K_2$ is polynomially convex.

The next couple of lemmas are of linear algebraic flavour. The first one from [11] gives certain normal form for pair of matrices under similarity of matrices with real entries.

LEMMA 2.4. — *Let $A, B \in \mathbb{R}^{2 \times 2}$ such that A has two distinct real eigenvalues, say λ_1, λ_2 . Also assume that $\det[A, B] > 0$. Then there exists an invertible matrix $T \in \mathbb{R}^{2 \times 2}$ such that*

$$TAT^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad TBT^{-1} = \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix}$$

for $\lambda_j, s_j, q \in \mathbb{R}$, $j = 1, 2$.

The next result from [25] gives a normal form for tuple of totally-real maximal subspaces of $\mathbb{C}^n$.

LEMMA 2.5. — *Let $P_0, \dots, P_N$ be maximal totally-real subspaces in $\mathbb{C}^n$ such that $P_j \cap P_0 = \{0\}$ for all $j = 1, \dots, N$. Then the subspaces can be parametrized by a tuple of $n \times n$ matrices with real entries as follows:*

$$P_0 = \mathbb{R}^n, \quad P_j = (A_j + i\mathbb{I})\mathbb{R}^n, \quad j = 1, \dots, N,$$

where $A_j \in \mathbb{R}^{n \times n}$, $j = 1, \dots, N$.

We call this normal form as Weinstock's normal form. Weinstock [25] proved the following theorem about the union of two transverse totally-real subspaces in $\mathbb{C}^n$.

THEOREM 2.6 (Weinstock). — *Suppose P_1 and P_2 are two totally-real subspaces of $\mathbb{C}^n$ of maximal dimension intersecting only at $0 \in \mathbb{C}^n$. Denote the normal form for this pair as:*

$$P_1 := \mathbb{R}^n \quad \text{and} \quad P_2 := (A + i\mathbb{I})\mathbb{R}^n,$$

for some matrix A with real entries. Then the union $P_1 \cup P_2$ is locally polynomially convex at the origin if and only if A has no purely imaginary eigenvalue of modulus greater than 1.

In this paper we are concerned only with the case $n = 2$. We wish to transform the planes in (1.8) into Weinstock's normal form. In view of Lemma 2.5, Weinstock's normal form depends only on whether the intersection with a particular totally real subspace is trivial or not. In our case we consider that particular plane to be P_0 . To transform into Weinstock's normal form it is enough to consider a couple of totally real planes of the form:

$$(2.3) \quad \begin{aligned} P_0 : \quad w &= \bar{z}, \\ P_1 : \quad w &= \alpha z + \beta \bar{z}, \end{aligned}$$

where $\beta \neq 0$. We first apply the following invertible $\mathbb{C}$ -linear transformation

$$T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

defined by

$$T(z, w) = (z + w, -i(z - w)).$$

Clearly, $T(P_0) = \mathbb{R}^2$ and

$$T(P_1) = \{(z + \alpha z + \beta \bar{z}, -i(z - \alpha z - \beta \bar{z})) : z \in \mathbb{C}\}.$$

Putting $z = x + iy$, we note that

$$T(P_1) = (B + iC)(\mathbb{R}^2),$$

where B and C are two 2×2 matrices with real entries that we get as follows: putting $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$

$$B = \begin{pmatrix} 1 + \alpha_1 + \beta_1 & \beta_2 - \alpha_2 \\ -\alpha_2 - \beta_2 & 1 - \alpha_1 + \beta_1 \end{pmatrix}$$

and

$$C = \begin{pmatrix} \alpha_2 + \beta_2 & 1 + \alpha_1 - \beta_1 \\ \alpha_1 + \beta_1 - 1 & \beta_2 - \alpha_2 \end{pmatrix}.$$

Here we view the matrices as linear transformations. We have $P_0 \cap P_1 = \{0\}$ implies that the matrix C is invertible. The condition $P_0 \cap P_1 = \{0\}$ transfers into $|\beta|^2 - |\alpha|^2 - 2\Re\alpha\beta + 1 \neq 0$. Hence, we can write

$$T(P_1) = (BC^{-1} + i\mathbb{I})(C(\mathbb{R}^2)).$$

Since the matrix C is invertible matrix with real entries, we obtain that

$$T(P_1) = (BC^{-1} + i\mathbb{I})(\mathbb{R}^2).$$

Therefore, Weinstock's normal form is $P_0 \sim \mathbb{R}^2$ and $P_1 \sim (A + i\mathbb{I})(\mathbb{R}^2)$, where $A = BC^{-1}$ as before. Putting the values of α and β we obtain Weinstock's normal form for the triple of planes (P_0^t, P_1^t, P_2^t) in (1.8) as

$$P_0^t = \mathbb{R}^2 \quad \text{and} \quad P_j^t = (A_j^t + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2,$$

where the matrices A_1^t and A_2^t are:

$$(2.4) \quad A_1^t = \begin{pmatrix} \frac{t}{\sqrt{3}(1+t)} & \frac{1}{1+t} \\ -\frac{1}{1-t} & -\frac{t}{\sqrt{3}(1-t)} \end{pmatrix} \quad \text{and} \quad A_2^t = \begin{pmatrix} -\frac{t}{\sqrt{3}(1+t)} & \frac{1}{1+t} \\ -\frac{1}{1-t} & \frac{t}{\sqrt{3}(1-t)} \end{pmatrix}.$$

We will use these matrices in the proofs of our theorems.

Next, we mention a general result from [11] about local polynomial convexity of the union of three totally-real planes in $\mathbb{C}^2$. It gives a sufficient condition, in terms of the matrices involved in the Weinstock's normal form, for local polynomial convexity of union of three totally-real planes at $0 \in \mathbb{C}^2$. We need few notations from [11]:

$$\Omega := \{(A_1, A_2) \in (\mathbb{R}^{2 \times 2})^2 : \det[A_1, A_2] \neq 0, \# \sigma(A_1) = 2, i \notin \sigma(A_j) \forall j\},$$

$$\Theta(A_1, A_2) := \det A_1 (\text{Tr } A_2)^2 + \text{Tr } A_1 A_2 (\text{Tr } A_1 A_2 - \text{Tr } A_1 \text{Tr } A_2),$$

$$\Lambda(A_1, A_2) := 4 \det A_1 A_2 - \frac{1}{4} (\text{Tr } A_1 \text{Tr } A_2)^2.$$

THEOREM 2.7 (Gorai). — *Let P_0, P_1, P_2 be three totally-real planes in $\mathbb{C}^2$ such that $P_0 \cap P_j = \{0\}$ for $j = 1, 2$. Hence, let Weinstock's normal form for $\{P_0, P_1, P_2\}$ be*

$$P_0 = \mathbb{R}^2, \quad P_j = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2,$$

and assume (A_1, A_2) belongs to parameter domain Ω . Assume further that the pairwise unions of P_0, P_1, P_2 are locally polynomially convex at $0 \in \mathbb{C}^2$. Given $j \in \{1, 2\}$, let j^c denote the other element in $\{1, 2\}$. Then, the following hold.

- (i) Let $\sigma(A_j) \subset \mathbb{R}$, $j = 1, 2$. If:
 - either $\det A_j \det[A_1, A_2] > 0$, $j = 1, 2$;
 - or $\det A_j \det[A_1, A_2] < 0$ and $(\det A_j)\Theta(A_j, A_{j^c}) < 0$ for some $j \in \{1, 2\}$;
 then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at $0 \in \mathbb{C}^2$.
- (ii) Suppose $\sigma(A_1) \subset \mathbb{R}$ and $\sigma(A_2) \subset \mathbb{C} \setminus \mathbb{R}$. If:
 - either $\det A_1 \det[A_1, A_2] < 0$ and $(\det A_1)\Theta(A_1, A_2) < 0$;
 - or $\Theta(A_1, A_2) < \Lambda(A_1, A_2)$;
 then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at $0 \in \mathbb{C}^2$.
- (iii) Suppose $\sigma(A_j) \subset \mathbb{C} \setminus \mathbb{R}$, $j = 1, 2$. If $\Theta(A_j, A_{j^c}) < \Lambda(A_1, A_2)$ for some $j \in \{1, 2\}$, then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at $0 \in \mathbb{C}^2$.

2.1. Maslov type index

Let M be a $\mathcal{C}^1$ -smooth orientable totally-real submanifold of $\mathbb{C}^2$ and $\gamma: S^1 \rightarrow M$ be a curve. In this case, the pull back bundle γ^*TM over γ is a trivial bundle. Let X_1, X_2 be global sections of the above pull back bundle over γ such that

$$(2.5) \quad T_{\gamma(\zeta)}M = \text{Span}_{\mathbb{R}}\{X_1(\zeta), X_2(\zeta)\} \quad \forall \zeta \in S^1.$$

DEFINITION 2.8. — Let γ be a curve in a $\mathcal{C}^1$ -smooth totally-real submanifold of $\mathbb{C}^2$. The Maslov-type index of γ in M is denoted as $\text{Ind}_M(\gamma)$ and is defined as the winding number of the map $h_{(X_1, X_2)}: S^1 \rightarrow \mathbb{C} \setminus \{0\}$ defined by $h_{(X_1, X_2)}(\zeta) := \det(X_1(\zeta), X_2(\zeta))$, where X_1, X_2 are global sections of the above pull back bundle over γ satisfying (2.5).

Remark 2.9. — We note that, for any other choice of a pair of global sections $\{Y_1, Y_2\}$ of the pull back bundle γ^*TM over γ , there exist a smooth map $A: S^1 \rightarrow \text{GL}(n, \mathbb{R})$ such that $Y_j(\zeta) = A(\zeta)X_j(\zeta)$, $j = 1, 2$. Hence, the winding number of the map $h_{(X_1, X_2)}$ is equal to the winding number of the map $h_{(Y_1, Y_2)}$. Therefore, the Maslov-type index of γ in M is independent of the choice of the global sections of γ^*TM .

Remark 2.10. — We note that any two closed curves γ_1 and γ_2 which are homologous in M have the same Maslov-type index (see [8] for more discussions).

We now proceed to define the Maslov-type index of a surface in $\mathbb{C}^2$ with an isolated CR singularity.

DEFINITION 2.11. — *Let M be an oriented real submanifold of $\mathbb{C}^2$ with an isolated CR singularity at $p \in M$ and U_p be a contractible neighborhood of p in M such that $U_p \setminus \{p\}$ is an oriented totally-real submanifold of $\mathbb{C}^2$, where the orientation is induced by the orientation of M . The Maslov-type index of p is denoted by $\text{Ind}_M(p)$ and is defined by the Maslov-type index of a simple closed curve $\gamma: S^1 \rightarrow U_p \setminus \{p\}$ that winds around the point p .*

Remark 2.12. — Since $\text{Ind}_M(\gamma)$ depends on the homology class of γ and the neighborhood U_p is contractible, we see that the definition of the Maslov-type index of an isolated CR singularity is independent of the curve γ chosen.

We now mention a lemma from [8] which is pertinent to graphs over domains in $\mathbb{C}$ in $\mathbb{C}^2$.

THEOREM 2.13 ([8, Lemma 8]). — *Let Ω be a domain in $\mathbb{C}$ containing the origin and $f \in \mathcal{C}^1(\Omega)$ such that $(\frac{\partial f}{\partial \bar{z}})^{-1}\{0\} = \{0\}$. Suppose that the graph S_f has an isolated CR singularity at $0 \in \mathbb{C}^2$. Let $\gamma: S^1 \rightarrow \Omega \setminus \{0\}$ be a smooth, positively-oriented, simple closed curve that encloses $0 \in \mathbb{C}^2$. Then the Maslov-type index $\text{Ind}_{S_f}(0)$ equals to the winding number of the curve $\frac{\partial f}{\partial \bar{z}} \circ \gamma$ around the origin.*

Next, we mention a lemma due to Bharali [5], which gives an easy way to compute the Maslov-type index of a graph of homogeneous polynomial in z and $\bar{z}$ of degree k around an isolated CR singularity at the origin.

THEOREM 2.14 ([5, Lemma 2.5]). — *Let p be a homogeneous polynomial of degree k in z and $\bar{z}$ such that*

$$\left\{ z \in \mathbb{C} : \frac{\partial p}{\partial \bar{z}}(z, \bar{z}) = 0 \right\} = \{0\}.$$

Define a polynomial q in z by $q(z) = \frac{\partial p}{\partial \bar{z}}(z, \bar{z})|_{\bar{z}=1}$. Then the Maslov-type index of the graph S_p of p of the origin,

$$\text{Ind}_{S_p}(0) = 2 \left(\sum \{ \mu(\zeta) : \zeta \in q^{-1}\{0\} \cap \mathbb{D} \} \right) - (k - 1),$$

where $\mu(\zeta)$ denotes the multiplicity of ζ as a zero of the polynomial q .

The next theorem is due to Forstnerič [8] in $\mathbb{C}^2$, which says, in certain cases, the existence of one analytic disc attached to a totally-real surface M in $\mathbb{C}^2$ gives the existence of many analytic discs attached to it.

THEOREM 2.15 (Forstnerič). — *Let $\gamma \subset M$ be the boundary of an immersed analytic disc $F_0: \overline{\mathbb{D}} \rightarrow \mathbb{C}^2$ in a totally-real surface M of $\mathbb{C}^2$ with Maslov-type index $j \geq 1$. Then F_0 lies in a $2j - 1$ parameter family of analytic discs with boundary in M . If $\widetilde{M}$ is a sufficiently small $\mathcal{C}^2$ -perturbation of M , then there exists such a $2j - 1$ parameter family of analytic discs with boundary in $\widetilde{M}$. If $j = 1$, the union of discs form a Levi flat hypersurface. If $j > 1$, the union of the discs contains an open ball.*

Next, we state a series of theorems due to Wiegerinck [27] about the local polynomial hull of certain graphs. We will use these results in our proofs of describing local hulls.

THEOREM 2.16 (Wiegerinck). — *Let φ be a $\mathcal{C}^k$ -smooth, $k \geq 2$, function on a disc in $\mathbb{C}$ centred at the origin. Suppose that the graph of φ , denoted by S_φ , has an isolated CR singularity at the origin of Maslov-type index j , $0 < j < k$. If $\Re(\varphi(z, \bar{z})/z^{j-1})$ is strictly subharmonic on a punctured neighborhood of the origin, then there exist analytic discs with boundary in S_φ .*

THEOREM 2.17 (Wiegerinck). — *Let $F(z, \bar{z})$ be a homogeneous function of degree k in z and $\bar{z}$, and $\mathcal{C}^2$ -smooth away from origin in $\mathbb{C}$. Suppose that the origin is an isolated CR singularity of $S = \{w = F(z, \bar{z})\}$ and the Maslov-type index at $0 \in \mathbb{C}^2$ is j , $0 < j < k$. Assume that $\Re(F(z, \bar{z})/z^{j-1})$ is a subharmonic but nowhere harmonic function on $\mathbb{C} \setminus \{0\}$. Then, the following hold.*

- (i) S is not locally polynomially convex at $0 \in \mathbb{C}^2$.
- (ii) For every $r > 0$, the polynomial hull of $S \cap \overline{B(0; r)}$ contains a $(2j - 1)$ -parameter family of analytic discs with boundary in S passing around zero if and only if the curve $\mathcal{C}: S^1 \rightarrow \mathbb{C}$ defined by

$$\mathcal{C}(z) = \frac{F(z, \bar{z})}{z^k}$$

has the following property:

- if, for two different points $z_1 \neq z_2$ on the unit circle, $\mathcal{C}(z_1) = \mathcal{C}(z_2)$,
- (**) then z_1 and z_2 divide the unit circle into two segments of length at least $\frac{\pi}{k-j+1}$.

Moreover, if the Maslov-type index $j > 1$, this family will fill an open neighborhood of the origin in $\mathbb{C}^2$.

- (iii) If property (**) is not satisfied by the curve $\mathcal{C}$, then for every $r > 0$, the polynomial hull of $S \cap \overline{B(0; r)}$ contains at least a one-parameter family of analytic discs with boundary in S passing through the origin.

THEOREM 2.18 (Wiegerinck). — *Let $f(z, \bar{z}) = p_k(z, \bar{z}) + O(|z|^{k+1})$ be a smooth function of class $\mathcal{C}^k$ on a disc $D(0; r)$, where p_k is a homogeneous polynomial of degree k in z and $\bar{z}$. Suppose that the origin is an isolated CR singularity of $S = \{(z, w) \in \mathbb{C}^2 : w = p_k(z, \bar{z})\}$ and that the Maslov-type index at $0 \in \mathbb{C}^2$ is j , $0 < j < k$. Assume that $\Re(p_k(z, \bar{z})/z^{j-1})$ is subharmonic but nowhere harmonic on $\mathbb{C} \setminus \{0\}$ and the curve $\mathcal{C}: S^1 \rightarrow \mathbb{C}$ defined by*

$$\mathcal{C}(z) = \frac{p_k(z, \bar{z})}{z^k}$$

*has the property (**). Then, for every $r > 0$, the polynomial hull of $S \cap \overline{B(0; r)}$ contains a $(2j-1)$ -parameter family of analytic discs with boundary in S , whose union contains an open ball centred the origin in $\mathbb{C}^2$ if $j > 1$.*

3. The union of three totally-real planes

The next few theorems describe a class of triples of totally-real planes in $\mathbb{C}^2$ whose union is locally polynomially convex at the origin, but the pair of 2×2 matrices corresponding to these triples does not fall into the open set described by Theorem 2.7. We will use these theorems while proving Theorem 1.8. These theorems might also be of independent interest. In the statements of the next three theorems, we will assume that the planes are in Weinstock's normal form.

THEOREM 3.1. — *Let P_0, P_1, P_2 be three totally-real planes in $\mathbb{C}^2$ such that*

$$P_0 = \mathbb{R}^2 \quad \text{and} \quad P_j = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2.$$

Assume that the pairwise unions of P_0, P_1 , and P_2 are locally polynomially convex at $0 \in \mathbb{C}^2$. Assume further that A_1 and A_2 satisfy the following conditions:

- (i) $\det A_1 = 0$ and $\det A_2 \geq 0$;
- (ii) $\det[A_1, A_2] > 0$.

Then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at the origin.

THEOREM 3.2. — *Let P_0, P_1, P_2 be totally-real planes such that*

$$P_0 = \mathbb{R}^2, \quad P_j = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2.$$

Assume that the pairwise unions of P_0, P_1 , and P_2 are locally polynomially convex at $0 \in \mathbb{C}^2$. Further assume that

- (i) $|\det A_j| \leq 1$ and $\det A_j < 0$ for $j = 1, 2$;
- (ii) $\det[A_1, A_2] > 0$.

Then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at the origin.

THEOREM 3.3. — *Let P_0, P_1, P_2 be three totally-real planes such that*

$$P_0 = \mathbb{R}^2, \quad P_j = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2.$$

Assume that the pairwise unions of P_0, P_1 , and P_2 are locally polynomially convex at $0 \in \mathbb{C}^2$. Assume further that

- (i) $\det[A_1, A_2] > 0$;
- (ii) $\det A_j < 0, j = 1, 2$;
- (iii) $\beta(A_1, A_2) > \min\{\det A_2(\text{Tr } A_1)^2, \det A_1(\text{Tr } A_2)^2\}$;

where $\beta(A_1, A_2) := 4\det A_1 A_2 - \text{Tr}(A_1 A_2)(\text{Tr}(A_1 A_2) - \text{Tr } A_1 \text{Tr } A_2) - \frac{1}{4}(\text{Tr } A_1 \text{Tr } A_2)^2$. Then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at the origin.

Proof of Theorem 3.1. — Under the conditions (i) and (ii), thanks to Lemma 2.4, without loss of generality, we may take the matrices as:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix},$$

where $\lambda, s_1, s_2, q \in \mathbb{R}$. The planes are of the form

$$\begin{aligned} P_0 &= \{(x, y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}, \\ P_1 &= \{(ix, (\lambda + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}, \\ P_2 &= \{(s_1 + i)x + qy, qx + (s_2 + i)y : x, y \in \mathbb{R}\}. \end{aligned}$$

Consider $K_j = P_j \cap \overline{\mathbb{B}(0; 1)}$, $j = 0, 1, 2$. We consider the polynomial

$$p(z, w) = z^2 + w^2.$$

Clearly,

$$p(K_0) \subset \{z \in \mathbb{C} : \Im z = 0, \Re z \geq 0\}.$$

For $(z, w) \in K_1$,

$$p(z, w) = p(ix, (\lambda + i)y) = -x^2 + (\lambda^2 - 1)y^2 + 2i\lambda y^2.$$

We wish to look at the image of K_1 under p more carefully and prove that $\widehat{p(K_1)} \cap \widehat{p(K_0)} = \{0\}$, and $p^{-1}\{0\} \cap K_1 = \{0\}$. Our argument goes as follows.

Case I: $\lambda > 0$. — For $(z, w) \in K_1$, $p(z, w) \in \{z \in \mathbb{C} : \Im z \geq 0\}$. If $\Im p(z, w) = 0$, then the real part $\Re p(z, w) \leq 0$, and $\Re p(z, w) = 0 \iff (z, w) = 0$.

Case II: $\lambda = 0$. — In this case, $\Re p(z, w) < 0$ for all $(z, w) \in K_1 \setminus \{0\}$.

Case III: $\lambda < 0$. — For $(z, w) \in K_1$, $p(z, w) \in \{z \in \mathbb{C} : \Im z \leq 0\}$, and $\Im p(z, w) = 0$ implies the real part $\Re p(z, w) < 0$ for all $(z, w) \in K_1 \setminus \{0\}$.

Therefore, in each of the above cases,

$$(3.1) \quad \widehat{p(K_1)} \cap \widehat{p(K_0)} = \{0\} \quad \text{and} \quad p^{-1}\{0\} \cap K_1 = \{0\}.$$

For $(z, w) \in K_2$,

$$\begin{aligned} p(z, w) &= p((s_1 + i)x + qy, qx + (s_2 + i)y) \\ &= (s_1^2 + q^2 - 1)x^2 + (s_2^2 + q^2 - 1)y^2 + 2(s_1 + s_2)qxy \\ &\quad + 2i(s_1x^2 + s_2y^2 + 2qxy). \end{aligned}$$

We first assume that $\det A_2 > 0$. This gives us $s_1s_2 > 0$. If $s_1 > 0$, $\Im p(z, w) > 0$; if $s_1 < 0$, then $\Im p(z, w) < 0$ for all $(z, w) \in K_2 \setminus \{0\}$. Therefore, $\Im p(z, w) \neq 0$ for all $(z, w) \in K_2 \setminus \{0\}$.

We now assume that $\det A_2 = 0$, i.e., $q^2 = s_1s_2$. Hence, s_1 and s_2 have the same sign. If $s_1 > 0$, then we have, for $(z, w) \in K_2$,

$$\Im p(z, w) = 2(\sqrt{s_1}x + \sqrt{s_2}y)^2 \geq 0.$$

If $s_1 < 0$, then

$$\Im p(z, w) = 2(\sqrt{-s_1}x + \sqrt{-s_2}y)^2 \geq 0 \quad \forall (z, w) \in K_2.$$

We also have

$$\Im p(z, w) = 0 \implies \Re p(z, w) \leq 0.$$

Therefore, in both cases, we obtain that

$$(3.2) \quad \widehat{p(K_2)} \cap \widehat{p(K_0)} = \{0\} \quad \text{and} \quad p^{-1}\{0\} \cap K_2 = \{0\}.$$

By assumption, $K := K_1 \cup K_2$ is polynomially convex. We now obtain, from (3.1) and (3.2), that

$$\widehat{p(K)} \cap \widehat{p(K_0)} = \{0\} \quad \text{and} \quad p^{-1}\{0\} \cap (K \cup K_0) = \{0\}.$$

Hence, by Theorem 2.3, $K \cup K_0$ is polynomially convex. Therefore, $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at the origin. $\square$

Proof of Theorem 3.2. — Thanks to Lemma 2.4, without loss of generality, we may assume that

$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix},$$

where $\lambda_j, s_j, q \in \mathbb{R}$, $j = 1, 2$. The planes become

$$\begin{aligned} P_0 &= \{(x, y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}, \\ P_1 &= \{((\lambda_1 + i)x, (\lambda_2 + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}, \\ P_2 &= \{((s_1 + i)x + qy, qx + (s_2 + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}. \end{aligned}$$

Let $K_j := P_j \cap \overline{\mathbb{B}}$, $j = 0, 1, 2$, and let $K := K_1 \cup K_2$. Consider the polynomial

$$p(z, w) = z^2 + w^2.$$

We have $p(K_0) \subset \{z \in \mathbb{C} : z \geq 0\}$ and $p^{-1}\{0\} \cap K_0 = \{0\}$. We now divide the remaining proof into three cases.

Case I: $|\det A_j| < 1$ for $j = 1, 2$. — We now look carefully at the image of K_1 under the polynomial p :

$$(3.3) \quad p((\lambda_1 + i)x, (\lambda_2 + i)y) = (\lambda_1^2 - 1)x^2 + (\lambda_2^2 - 1)y^2 + 2i(\lambda_1 x^2 + \lambda_2 y^2).$$

The imaginary part is

$$(3.4) \quad \Im p((\lambda_1 + i)x, (\lambda_2 + i)y) = 2(\lambda_1 x^2 + \lambda_2 y^2).$$

Since $\det A_1 = \lambda_1 \lambda_2 < 0$, one of λ_1 or λ_2 must be positive and the other is negative. Assuming $\lambda_2 < 0$, we see that

$$(3.5) \quad \Im p((\lambda_1 + i)x, (\lambda_2 + i)y) = 0 \iff y^2 = -\frac{\lambda_1}{\lambda_2} x^2.$$

Then the real part of $p((\lambda_1 + i)x, (\lambda_2 + i)y)$ becomes, in view of (3.5),

$$(3.6) \quad \Re p((\lambda_1 + i)x, (\lambda_2 + i)y) = \frac{1}{\lambda_2} (\lambda_1 - \lambda_2) (\lambda_1 \lambda_2 + 1) x^2 \leq 0.$$

Since $|\det A_1| < 1$, we obtain that $\Re p((\lambda_1 + i)x, (\lambda_2 + i)y) = 0 \iff (x, y) = 0$. Thus, we obtain that $\widehat{p(K_0)} \cap \widehat{p(K_1)} = \{0\}$ and $p^{-1}\{0\} \cap K_1 = \{0\}$.

For $(z, w) \in K_2$,

$$(3.7) \quad p(z, w) = (s_1^2 + q^2 - 1)x^2 + (s_2^2 + q^2 - 1)y^2 + 2q(s_1 + s_2)xy + 2i(s_1 x^2 + 2qxy + s_2 y^2).$$

The imaginary part

$$(3.8) \quad \Im p((s_1 + i)x + qy, qx + (s_2 + i)y) = 2(s_1 x^2 + 2qxy + s_2 y^2).$$

We have

$$(3.9) \quad \Im p((s_1 + i)x + qy, qx + (s_2 + i)y) = 0 \iff 2qxy = -(s_1 x^2 + s_2 y^2).$$

The real part becomes, in view of (3.9),

$$(3.10) \quad \begin{aligned} \Re p((s_1 + i)x + qy, qx + (s_2 + i)y) &= (q^2 - s_1 s_2 - 1)(x^2 + y^2) \\ &= -(\det A_2 + 1)(x^2 + y^2). \end{aligned}$$

Since $|\det A_2| < 1$, $p(K_0) \cap p(K_2) = \{0\}$. Therefore,

$$\widehat{p(K_0)} \cap \widehat{p(K)} = \{0\} \quad \text{and} \quad p^{-1}\{0\} \cap (K_0 \cup K) = \{0\}.$$

Hence, by Theorem 2.3, $K_0 \cup K$ is polynomially convex.

The next two cases fall under the condition $|\det A_j| = 1$ for some $j = 1, 2$. Since $\det A_j < 0$, we only need to consider $\det A_j = -1$ for some $j = 1, 2$. We now consider the other two cases.

Case II: $\det A_j = -1$ for some $j \in \{1, 2\}$ and $|\det A_i| < 1$ for $i \neq j$. — We again consider the Kallin's polynomial $p(z, w) = z^2 + w^2$. We first look at the image of K_1 under the polynomial p . Equations (3.3), (3.4), (3.5), and (3.6) remain the same. Clearly, $p(K_0) \cap p(K_1) = \{0\}$. If $|\det A_1| < 1$ and (3.5) holds, then, by (3.6), we get that

$$\Re p((\lambda_1 + i)x, (\lambda_2 + i)y) = 0 \iff (x, y) = 0.$$

Therefore,

$$(3.11) \quad p^{-1}\{0\} \cap K_1 = \{0\}.$$

If $\det A_1 = -1$ and (3.5) holds, then, by (3.6), we obtain that

$$\Re p((\lambda_1 + i)x, (\lambda_2 + i)y) \equiv 0.$$

This implies that

$$(3.12) \quad p^{-1}\{0\} \cap K_1 = \left\{ ((\lambda_1 + i)x, (\lambda_2 + i)y) \in \mathbb{C}^2 : y^2 = -\frac{\lambda_1}{\lambda_2} x^2 \right\},$$

which is the union of two real line segments in $\mathbb{C}^2$ intersecting only at the origin.

We now look at the image of K_2 . Equations (3.7), (3.8), (3.9), and (3.10) remain the same. Clearly, $p(K_0) \cap p(K_2) = \{0\}$. If $|\det A_2| < 1$ and (3.9) holds, then, by (3.10), we get that

$$\Re p((s_1 + i)x + qy, qx + (s_2 + i)y) = 0 \iff (x, y) = 0.$$

Therefore,

$$(3.13) \quad p^{-1}\{0\} \cap K_2 = \{0\}.$$

If $\det A_2 = -1$ and (3.9) holds, then, by (3.10), we obtain that

$$\Re p((s_1 + i)x + qy, qx + (s_2 + i)y) \equiv 0.$$

This gives that $p^{-1}\{0\} \cap K_2$ is the union of two real line segments in $\mathbb{C}^2$ intersecting only at the origin. Therefore, in this case, we have $\widehat{p(K_0)} \cap \widehat{p(K)} = \{0\}$ and $p^{-1}\{0\} \cap (K_0 \cup K)$ is the union of two real line segments in $\mathbb{C}^2$ intersecting only at the origin. Hence, $p^{-1}\{0\} \cap (K_0 \cup K)$ is polynomially convex.

Case III: $\det A_j = -1$ for $j = 1, 2$. — We consider the same Kallin's polynomial $p(z, w) = z^2 + w^2$. We now need to consider carefully $p^{-1}\{0\} \cap K_j$ for $j = 1, 2$. If $\det A_1 = -1$, then we see, from the above computations, that, for each $z \in K_1$, the real part $\Re p(z)$ vanishes whenever $\Im p(z) = 0$. The similar is true when $\det A_2 = -1$. We also see that, if $\det A_j = -1$, then the set $p^{-1}\{0\} \cap K_j = \{z \in K_j : \Im p(z) = 0\}$ is a union of two real line segments in $\mathbb{C}^2$ intersecting at the origin. Therefore, $p^{-1}\{0\} \cap K$ is a union of four real line segments in $\mathbb{C}^2$ intersecting pairwise only at the origin, which is polynomially convex. Hence, by Kallin's lemma, $K_0 \cup K$ is polynomially convex. $\square$

Proof of Theorem 3.3. — Since $A_1, A_2 \in \mathbb{R}^{2 \times 2}$ and $\det A_j < 0$ for $j = 1, 2$, both matrices have real eigenvalues. Assume, without loss of generality (otherwise we will interchange the matrices), that

$$(3.14) \quad \beta(A_1, A_2) > \det A_2 (\text{Tr } A_1)^2.$$

Since $\det[A_1, A_2] > 0$, by applying Lemma 2.4, we get, up to a simultaneous similarity by real matrix, that

$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix}.$$

We first view (3.14) in terms of the entries of the matrices A_1 and A_2 .

$$A_1 A_2 = \begin{pmatrix} \lambda_1 s_1 & \lambda_1 q \\ \lambda_2 q & \lambda_2 s_2 \end{pmatrix}.$$

Therefore, we compute:

$$\begin{aligned} \beta(A_1, A_2) &= 4\lambda_1 \lambda_2 (s_1 s_2 - q^2) + (\lambda_1 s_1 + \lambda_2 s_2)(\lambda_1 s_2 + \lambda_2 s_1) \\ &\quad - \frac{1}{4}(\lambda_1 + \lambda_2)^2 (s_1 + s_2)^2 \\ &= 4\lambda_1 \lambda_2 (s_1 s_2 - q^2) - \frac{1}{4}(\lambda_1 - \lambda_2)^2 (s_1 - s_2)^2. \end{aligned}$$

Hence,

$$(3.15) \quad \beta(A_1, A_2) > \det A_2 (\text{Tr } A_1)^2 \quad \Longleftrightarrow \quad q^2 > \frac{(s_1 + s_2)^2}{4}.$$

We now consider compact sets

$$K_j = P_j \cap \overline{\mathbb{B}}, \quad j = 0, 1, 2,$$

and the polynomial

$$p(z, w) = zw.$$

Clearly,

$$(3.16) \quad p(K_0) \subset \mathbb{R}.$$

Since $p((\lambda_1 + i)x, (\lambda_2 + i)y) = (\lambda_1\lambda_2 - 1)xy + i(\lambda_1 + \lambda_2)xy$, hence, we obtain that

$$(3.17) \quad p(K_1) \subset \{u + iv \in \mathbb{C} : (\lambda_1 + \lambda_2)u = (\lambda_1\lambda_2 - 1)v\}.$$

We also get that

$$\begin{aligned} p((s_1 + i)x + qy, qx + (s_2 + i)y) &= ((s_1 + i)x + qy)(qx + (s_2 + i)y) \\ &= (s_1x + qy)(qx + s_2y) - xy \\ &\quad + i(qx^2 + (s_1 + s_2)xy + qy^2). \end{aligned}$$

The imaginary part of $p((s_1 + i)x + qy, qx + (s_2 + i)y)$ is $qx^2 + (s_1 + s_2)xy + qy^2$, which vanishes, thanks to (3.15), if and only if $(x, y) = (0, 0)$. Therefore, we have

$$(3.18) \quad p(K_2) \subset \{u + iv \in \mathbb{C} : v \neq 0\} \cup \{0\}.$$

We now divide the remaining part of the proof into two cases.

Case I: $\text{Tr } A_1 \neq 0$. — In this case, we consider $K := K_1 \cup K_2$. From equations (3.16), (3.17) and (3.18), we obtain that $\widehat{p(K_0)} \cap \widehat{p(K)} \subset \{0\}$. We also see that $p^{-1}\{0\} \cap (K_0 \cup K)$ is a union of two real line segments intersecting only at the origin, which is polynomially convex. Hence, by Kallin's lemma (Theorem 2.3), $K_0 \cup K = K_0 \cup K_1 \cup K_2$ is polynomially convex.

Case II: $\text{Tr } A_1 = 0$. — In this case, we consider $K := K_0 \cup K_1$. Since $\text{Tr } A_1 = 0$, from equation (3.17), we have

$$p(K_1) \subset \mathbb{R}.$$

Therefore, $p(K) \subset \mathbb{R}$. In this case also equation (3.18) holds. Therefore, We obtain $\widehat{p(K)} \cap \widehat{p(K_2)} = \{0\}$. Again, as before, $p^{-1}\{0\} \cap (K \cup K_2)$ is a union of two real line segments in $\mathbb{C}^2$ intersecting only at the origin. Hence, $p^{-1}\{0\} \cap (K \cup K_2)$ is polynomially convex. Therefore, by Kallin's lemma (Theorem 2.3) $K \cup K_2$ is polynomially convex. $\square$

4. The union of three totally-real surfaces

The next couple of theorems are the most important theorems of this section. Those will be crucially used in our proof of Theorem 1.10.

THEOREM 4.1. — *Let S_0, S_1 and S_2 be three totally-real surfaces in $\mathbb{C}^2$ intersecting transversely at the origin such that*

$$\begin{aligned} T_0 S_0 &= \mathbb{R}^2, \\ T_0 S_1 &= (A_1 + i\mathbb{I})\mathbb{R}^2, \\ T_0 S_2 &= (A_2 + i\mathbb{I})\mathbb{R}^2, \end{aligned}$$

where $A_1, A_2 \in M_2(\mathbb{R})$. Assume that the pairwise unions of $T_0 S_0, T_0 S_1$ and $T_0 S_2$ are locally polynomially convex at the origin. Assume further that $\det[A_1, A_2] > 0$. Then $S_0 \cup S_1 \cup S_2$ is locally polynomially convex at the origin if one of the following holds:

- (i) $\det A_j > 0$ for $j = 1, 2$;
- (ii) $\det A_j < 0$ and $|\det A_j| < 1$.

Proof. — Since $A_1, A_2 \in M_2(\mathbb{R})$ and $\det[A_1, A_2] > 0$, we obtain, in view of Lemma 2.4, that there exists an invertible matrix $T \in \mathbb{R}^{2 \times 2}$ such that

$$T A_1 T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad T A_2 T^{-1} = \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix}$$

for $\lambda, s_j, t_j \in \mathbb{R}$, $j = 1, 2$. Therefore, without loss of generality, we may assume that

$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix}.$$

Since S_0 is a totally-real surface passing through the origin and $T_0 S_0 = \mathbb{R}^2 \subset \mathbb{C}^2$, by implicit mapping theorem there exists a $\delta > 0$ such that

$$\begin{aligned} S_0 \cap B(0; \delta) &= \{(x + i f_0(x, y), y + i g_0(x, y)) \in \mathbb{C}^2 : \\ &\quad \text{for small enough } x, y \in \mathbb{R}\}, \end{aligned}$$

where f_0 and g_0 are real-valued functions. Locally around the origin we can view the totally-real surfaces as follows:

$$\begin{aligned} S_0(\delta) &:= \{(x + i f_0(x, y), y + i g_0(x, y)) \in \mathbb{C}^2 : (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}\}, \\ S_1(\delta) &:= \{((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + g_1(x, y)) \in \mathbb{C}^2 : \\ &\quad (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}\}, \\ S_2(\delta) &:= \{((s_1 + i)x + qy + f_2(x, y), qx + (s_2 + i)y + g_2(x, y)) \in \mathbb{C}^2 : \\ &\quad (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}\}, \end{aligned}$$

where f_0, g_0 are real-valued, and $f_j(x, y) \sim o(|(x, y)|)$ and $g_j(x, y) \sim o(|(x, y)|)$ for all $j = 0, 1, 2$.

We first assume (i), i.e., $\det A_j > 0$ for $j = 1, 2$. We will use Kallin's lemma to conclude the local polynomial convexity of $S_0(\delta) \cup S_1(\delta) \cup S_2(\delta)$ for some small enough $\delta > 0$. Consider the polynomial

$$p(z, w) = z^2 + w^2.$$

Clearly,

$$(4.1) \quad p(x + if_0(x, y), y + ig_0(x, y)) = x^2 + y^2 + o(|(x, y)|^2).$$

We also have

$$\begin{aligned} p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + g_1(x, y)) \\ = (\lambda_1 + i)^2 x^2 + (\lambda_2 + i)^2 y^2 + o(|(x, y)|^2) \\ = (\lambda_1^2 - 1)x^2 + (\lambda_2^2 - 1)y^2 + 2i(\lambda_1 x^2 + \lambda_2 y^2) + o(|(x, y)|^2). \end{aligned}$$

Since $\det A_1 > 0$, the imaginary part of $p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + g_1(x, y))$ is $2(\lambda_1 x^2 + \lambda_2 y^2) + o(|(x, y)|^2) \geq 0$ or ≤ 0 for $(x, y) \in \mathbb{R}^2 \cap B(0; \delta)$ and equal to zero only when $(x, y) = (0, 0)$. Therefore there exists a constant $C_1 > 0$ such that

$$(4.2) \quad p(S_1(\delta)) \subset \{u + iv \in \mathbb{C} : |u| < C_1|v|\}.$$

We also compute:

$$\begin{aligned} p((s_1 + i)x + qy + f_2(x, y), qx + (s_2 + i)y + g_2(x, y)) \\ = (s_1^2 + q^2 - 1)x^2 + (s_2^2 + q^2 - 1)y^2 + 2i(s_1 x^2 + s_2 y^2 + 2qxy) \\ + o(|(x, y)|^2). \end{aligned}$$

Since $\det A_2 > 0$, the imaginary part of the above equation $2(s_1 x^2 + s_2 y^2 + 2qxy) + o(|(x, y)|^2) \geq 0$ or ≤ 0 and equal to zero only when $(x, y) = (0, 0)$. Therefore, there exists a constant $C_2 > 0$ such that

$$(4.3) \quad p(S_2(\delta)) \subset \{u + iv \in \mathbb{C} : |u| < C_2|v|\}.$$

Consider $C := \max\{C_1, C_2\}$, and choose $\varepsilon > 0$ such that $\varepsilon < 1/C$. From equation (4.1) we see that the imaginary part of $p(x + if_0(x, y), y + ig_0(x, y))$ is $o(|z|^2)$. Therefore, shrinking δ if necessary, we obtain

$$(4.4) \quad p(S_0(\delta)) \subset \{u + iv \in \mathbb{C} : |v| < \varepsilon|u|\}.$$

From equations (4.2), (4.3) and (4.4) we have

$$p(\widehat{S_0(\delta)}) \cap p(\widehat{S_1(\delta) \cup S_2(\delta)}) = \{0\}.$$

We also have from above computations that

$$p^{-1}\{0\} \cap (S_0(\delta) \cup S_1(\delta) \cup S_2(\delta)) = \{0\}.$$

Hence, all the conditions of Kallin's lemma hold and we conclude that $S_0 \cup S_1 \cup S_2$ is locally polynomially convex at the origin.

Finally, we assume (ii), i.e., $\det A_j < 0$ and $|\det A_j| < 1$. We consider the matrices in the normal form

$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix}.$$

Consider the polynomial

$$p(z, w) = z^2 + w^2.$$

Let $D_\delta = \mathbb{R}^2 \cap \overline{B(0; \delta)}$ as before. We have

$$p(x, y) = x^2 + y^2.$$

Hence, $p^{-1}\{0\} \cap S_0(\delta) = \{0\}$. In fact, for given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(4.5) \quad p(S_0(\delta)) \subset \{w \in \mathbb{C} : \Re w \geq 0, |\Im w| \leq \varepsilon \Re w\}.$$

By using the computations in the proof of Theorem 3.2 we get that

$$(4.6) \quad p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + g_1(x, y)) \\ = (\lambda_1^2 - 1)x^2 + (\lambda_2^2 - 1)y^2 + 2i(\lambda_1 x^2 + \lambda_2 y^2) + o(|(x, y)|^2).$$

Equation (4.6) gives us that the imaginary part

$$(4.7) \quad \Im p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + f_2(x, y)) \\ = 2(\lambda_1 x^2 + \lambda_2 y^2) + o(|(x, y)|^2).$$

From (4.7) we see that $\Im p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + g_1(x, y)) = 0$ in a curve which is a small perturbation of the curve $y^2 = -\frac{\lambda_1}{\lambda_2}x^2$. Then the real part of $p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + g_1(x, y))$ becomes, shrinking δ if required,

$$(4.8) \quad \Re p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + g_1(x, y)) \\ = \frac{1}{\lambda_2}(\lambda_1 - \lambda_2)(\lambda_1 \lambda_2 + 1)x^2 + o(|(x, y)|^2) \\ \leq 0.$$

Since $|\det A_1| < 1$, we obtain that $\Re p((\lambda_1 + i)x + f_1(x, y), (\lambda_2 + i)y + f_2(x, y)) = 0 \iff (x, y) = 0$. Hence, there is a conical neighborhood V of $\{(z, w) \in S_1(\delta) : \Im p(z, w) = 0\}$ such that $\Re p(z, w) \leq 0$ for all $(z, w) \in V$. Thus, from (4.5) and (4.8), we obtain, shrinking $\delta > 0$ if necessary, that $p(\widehat{S_0(\delta)}) \cap p(\widehat{S_1(\delta)}) = \{0\}$ and $p^{-1}\{0\} \cap S_1(\delta) = \{0\}$.

For $(z, w) \in S_2(\delta)$,

$$p(z, w) = (s_1^2 + q^2 - 1)x^2 + (s_2^2 + q^2 - 1)y^2 + 2q(s_1 + s_2)xy \\ + 2i(s_1x^2 + 2qxy + s_2y^2) + o(|(x, y)|^2).$$

The imaginary part

$$\Im p(z, w) = 2(s_1x^2 + 2qxy + s_2y^2) + o(|(x, y)|^2).$$

We have the imaginary part vanishes in a curve which is a small perturbation of the curve

$$2qxy = -(s_1x^2 + s_2y^2).$$

Then, in view of (3.9), the real part becomes

$$\begin{aligned} \Re p((s_1 + i)x + qy + f_2(x, y), qx + (s_2 + i)y + g_2(x, y)) \\ = (q^2 - s_1s_2 - 1)(x^2 + y^2) + o(|(x, y)|^2) \\ = -(\det A_2 + 1)(x^2 + y^2) + o(|(x, y)|^2) \\ < 0 \text{ and } = 0 \text{ if and only if } (x, y) = 0. \end{aligned}$$

Hence, there is a conical neighborhood W of $\{(z, w) \in S_2(\delta) : \Im p(z, w) = 0\}$ such that $\Re p(z, w) \leq 0$ for all $(z, w) \in V$.

Therefore, shrinking $\delta > 0$ further, if required, we obtain that

$$p(\widehat{S_0(\delta)}) \cap p(\widehat{S_2(\delta)}) = \{0\} \quad \text{and} \quad p^{-1}\{0\} \cap (S_0(\delta) \cup S_2(\delta)) = \{0\}.$$

Hence, by Theorem 2.3, $S_0(\delta) \cup S_1(\delta) \cup S_2(\delta)$ is polynomially convex. Therefore, $S_0 \cup S_1 \cup S_2$ is locally polynomially convex at $0 \in \mathbb{C}^2$. $\square$

THEOREM 4.2. — *Let S_0, S_1 and S_2 be three totally-real surfaces in $\mathbb{C}^2$ intersecting transversely at the origin such that $T_0S_0 = \mathbb{R}^2$, $T_0S_1 = (A_1 + i\mathbb{I})\mathbb{R}^2$ and $T_0S_2 = (A_2 + i\mathbb{I})\mathbb{R}^2$. Assume that the pairwise unions of T_0S_0 , T_0S_1 and T_0S_2 are locally polynomially convex at the origin. Assume further that:*

- (i) $\det A_1 = 0$ and $\det A_2 \geq 0$;
- (ii) $\det[A_1, A_2] > 0$.

Then $S_0 \cup S_1 \cup S_2$ is locally polynomially convex at the origin.

Proof. — Under the conditions (i) and (ii), by Lemma 2.4, without loss of generality we assume that

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix}.$$

Locally around the origin we can view the totally-real surfaces as follows:

$$\begin{aligned} S_0(\delta) &:= \{(x + if_0(x, y), y + ig_0(x, y)) \in \mathbb{C}^2 : (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}\}, \\ S_1(\delta) &:= \{(ix + f_1(x, y), (\lambda + i)y + g_1(x, y)) \in \mathbb{C}^2 : \\ &\quad (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}\}, \\ S_2(\delta) &:= \{((s_{21} + i)x + t_2y + f_2(x, y), t_2x + (s_{22} + i)y + g_2(x, y)) \in \mathbb{C}^2 : \\ &\quad (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}\}, \end{aligned}$$

where f_0 and g_0 are real-valued, and

$$f_j(x, y) \sim o(|(x, y)|) \quad \text{and} \quad g_j(x, y) \sim o(|(x, y)|) \quad \text{for all } j = 0, 1, 2.$$

We now divide our argument in two cases.

Case I. — Assume $\det A_1 = 0$ and $\det A_2 = 0$.

We consider the polynomial

$$p(z, w) = z^2 + w^2.$$

Clearly,

$$(4.9) \quad p(x + if_0(x, y), y + ig_0(x, y)) = x^2 + y^2 + o(|(x, y)|^2).$$

Hence, $\Re p(x + if_0(x, y), y + ig_0(x, y)) \geq 0$, and $= 0$ if and only if $(x, y) = 0$.

For $(z, w) \in S_1(\delta)$,

$$\begin{aligned} p(z, w) &= p(ix + f_1(x, y), (\lambda + i)y + g_1(x, y)) \\ &= -x^2 + (\lambda^2 - 1)y^2 + 2i\lambda y^2 + o(|(x, y)|^2). \end{aligned}$$

We first note that on the set $S_1(\delta) \cap \{y = 0\}$ the real part $\Re p \leq 0$ and equal to zero if and only if $x = 0$. Therefore, by continuity, there exists $\varepsilon > 0$ such that $\Re p < 0$ on $\{(ix + f_1(x, y), (\lambda + i)y + g_1(x, y)) \in \mathbb{C}^2 : (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}, |y| < \varepsilon|x|\}$. On the other part of $S_1(\delta)$, we get the imaginary part $\Im p \geq 0$ and $= 0$ only when $(x, y) = 0$. Hence, there exists a constant $C_1 > 0$ such that $|\Re p(z)| \leq C_1 \Im p(z)$ for all $z \in \{(ix + f_1(x, y), (\lambda + i)y + g_1(x, y)) \in \mathbb{C}^2 : (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}, |y| \geq \varepsilon|x|\}$.

For $(z, w) \in K_2$,

$$\begin{aligned} p(z, w) &= p((s_{21} + i)x + t_2y + f_2(x, y), t_2x + (s_{22} + i)y + g_2(x, y)) \\ &= (s_{21}^2 + t_2^2 - 1)x^2 + (s_{22}^2 + t_2^2 - 1)y^2 + 2(s_{21} + s_{22})t_2xy \\ &\quad + 2i(s_{21}x^2 + s_{22}y^2 + 2t_2xy) + o(|(x, y)|^2). \end{aligned}$$

Since $\det A_2 = 0$, we have

$$\Im p(z, w) = 2(\sqrt{s_{21}}x + \sqrt{s_{22}}y)^2 + o(|(x, y)|^2).$$

On the set $S_2(\delta) \cap \{\sqrt{s_{21}}x + \sqrt{s_{22}}y = 0\}$ the real part

$$\Re p = -(1 + s_{22}/s_{21})y^2 + o(|y|^2) \leq 0,$$

and equal to zero if and only if $y = 0$. Again, by continuity of the real part, there exists $\varepsilon_1 > 0$ such that $\Re p < 0$ on $\{((s_{21} + i)x + t_2y + f_2(x, y), t_2x + (s_{22} + i)y + g_2(x, y)) \in \mathbb{C}^2 : (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}, |x| < (\varepsilon_1 - \sqrt{s_{22}}/\sqrt{s_{21}})|y|\}$. On the other part of $S_2(\delta)$, we get the imaginary part $\Im p \geq 0$ and $= 0$ only when $(x, y) = 0$. Hence, there exists a constant $C_2 > 0$ such that $|\Re p(z)| \leq C_2 \Im p(z)$ for all $z \in \{((s_{21} + i)x + t_2y + f_2(x, y), t_2x + (s_{22} + i)y + g_2(x, y)) \in \mathbb{C}^2 : (x, y) \in \mathbb{R}^2 \cap \overline{B(0; \delta)}, |x| \geq (\varepsilon_1 - \sqrt{s_{22}}/\sqrt{s_{21}})|y|\}$.

From (4.9) we get that the imaginary part

$$\Im p(x + if_0(x, y), y + ig_0(x, y)) \sim o(|(x, y)|^2).$$

Since

$$\Re p(x + if_0(x, y), y + ig_0(x, y)) \geq 0,$$

and $= 0$ if and only if $(x, y) = 0$, we get a sufficiently small $\varepsilon_2 > 0$ ($\varepsilon_2 < \min\{1/C_1, 1/C_2\}$) such that

$$p(S_0(\delta)) \subset \{z \in \mathbb{C} : |\Im z| \leq \varepsilon_2 |\Re z|\}.$$

Hence, all the conditions of Kallin's lemma are satisfied. Therefore, shrinking δ if necessary, we conclude that $S_0(\delta) \cup S_1(\delta) \cup S_2(\delta)$ is polynomially convex.

Case II. — Assume $\det A_1 = 0$ and $\det A_2 > 0$.

We will deal with the surface $S_1(\delta)$ as in Case I and for the other two surfaces we will use computations done in the proof of Theorem 4.1. Then, by using Kallin's lemma, we conclude that $S_0 \cup S_1 \cup S_2$ is locally polynomially convex. $\square$

5. Proofs of Theorems 1.6, 1.8 and 1.10

We begin this section by proving Theorem 1.6. This theorem plays a vital role in the proof of the other two theorems.

Proof of Theorem 1.6. — We will work with Weinstock's normal form of the triples of totally-real planes as in (2.4)

$$P_0^t = \mathbb{R}^2, \quad P_j^t = (A_j^t + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2,$$

where

$$A_1^t = \begin{pmatrix} \frac{t}{\sqrt{3}(1+t)} & \frac{1}{1+t} \\ -\frac{1}{1-t} & -\frac{t}{\sqrt{3}(1-t)} \end{pmatrix} \quad \text{and} \quad A_2^t = \begin{pmatrix} -\frac{t}{\sqrt{3}(1+t)} & \frac{1}{1+t} \\ -\frac{1}{1-t} & \frac{t}{\sqrt{3}(1-t)} \end{pmatrix}.$$

We have

$$\begin{aligned}\sigma(A_1^t) &= \left\{ \frac{-t^2 + \sqrt{4t^2 - 3}}{\sqrt{3}(1-t^2)}, \frac{-t^2 - \sqrt{4t^2 - 3}}{\sqrt{3}(1-t^2)} \right\}, \\ \sigma(A_2^t) &= \left\{ \frac{t^2 + \sqrt{4t^2 - 3}}{\sqrt{3}(1-t^2)}, \frac{t^2 - \sqrt{4t^2 - 3}}{\sqrt{3}(1-t^2)} \right\}.\end{aligned}$$

We note that the eigenvalues of A_1^t and A_2^t are not purely imaginary for $t \in (0, \infty)$. Hence, by Theorem 2.6, $P_0^t \cup P_1^t$ and $P_0^t \cup P_2^t$ are locally polynomially convex at $0 \in \mathbb{C}^2$. To show $P_1^t \cup P_2^t$ is polynomially convex, we need some computations. First, we apply an invertible $\mathbb{C}$ -linear transformation $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $z \mapsto (A_1^t - i\mathbb{I})z$. It maps P_1^t to $\mathbb{R}^2$, and P_2^t to $(B^t + i\mathbb{I})\mathbb{R}^2$, where $B^t = (A_1^t A_2^t + \mathbb{I})(A_1^t - A_2^t)^{-1}$. It suffices to show that $\text{Tr } B^t \neq 0$. We compute:

$$(5.1) \quad A_1^t A_2^t = \begin{pmatrix} -\frac{t^2}{3(1+t)^2} - \frac{1}{1-t^2} & \frac{t}{\sqrt{3}(1+t)^2} + \frac{t}{\sqrt{3}(1-t^2)} \\ \frac{t}{\sqrt{3}(1-t)^2} + \frac{t}{\sqrt{3}(1-t^2)} & -\frac{t^2}{3(1-t)^2} - \frac{1}{1-t^2} \end{pmatrix},$$

$$(5.2) \quad A_2^t A_1^t = \begin{pmatrix} -\frac{t^2}{3(1+t)^2} - \frac{1}{1-t^2} & -\frac{t}{\sqrt{3}(1+t)^2} - \frac{t}{\sqrt{3}(1-t^2)} \\ -\frac{t}{\sqrt{3}(1-t)^2} - \frac{t}{\sqrt{3}(1-t^2)} & -\frac{t^2}{3(1-t)^2} - \frac{1}{1-t^2} \end{pmatrix},$$

$$(5.3) \quad (A_1^t - A_2^t)^{-1} = \frac{\sqrt{3}(1-t^2)}{2t} \begin{pmatrix} \frac{1}{1-t} & 0 \\ 0 & -\frac{1}{1+t} \end{pmatrix},$$

$$(5.4) \quad (A_1^t A_2^t + \mathbb{I}) = \begin{pmatrix} 1 - \frac{3+3t+t^2-t^3}{3(1+t)(1-t^2)} & \frac{2t}{\sqrt{3}(1+t)(1-t^2)} \\ \frac{2t}{\sqrt{3}(1-t)(1-t^2)} & 1 - \frac{3-3t+t^2+t^3}{3(1-t)(1-t^2)} \end{pmatrix}.$$

Thus, we obtain that

$$\begin{aligned}\text{Tr}(B^t) &= \text{Tr}((A_1^t A_2^t + \mathbb{I})(A_1^t - A_2^t)^{-1}) \\ &= \sqrt{3} \left(1 - \frac{3-t^2}{3(1-t^2)} \right) \\ &\neq 0 \quad \forall t \in (0, \infty).\end{aligned}$$

Hence, the pairwise union of the totally-real planes P_0, P_1 and P_2 is polynomially convex. From (5.1) and (5.2), we obtain that

$$(5.5) \quad \det A_1^t = \det A_2^t = \frac{3-t^2}{3(1-t^2)},$$

$$(5.6) \quad \text{Tr } A_1^t = -\frac{2t^2}{\sqrt{3}(1-t^2)},$$

$$(5.7) \quad \text{Tr } A_2^t = \frac{2t^2}{\sqrt{3}(1-t^2)},$$

$$(5.8) \quad \text{Tr } A_1^t A_2^t = -\frac{2(t^4 - 2t^2 + 3)}{3(1 - t^2)^2},$$

$$(5.9) \quad \det[A_1^t, A_2^t] = -\frac{16t^2}{3(1 - t^2)^3}.$$

We now divide the remaining part of the proof into four cases.

Case I: $t^2 > 3$. — In this case, we note that

$$\det[A_1^t, A_2^t] > 0 \quad \text{and} \quad \det A_j^t > 0, \quad j = 1, 2.$$

Hence, we have $\det A_j^t \det[A_1^t, A_2^t] > 0$ for $j = 1, 2$. Therefore, in view of Theorem 2.7(i), we get that $P_0^t \cup P_1^t \cup P_2^t$ is locally polynomially convex at the origin.

Case II: $t^2 = 3$. — We note that

$$\det[A_1^t, A_2^t] > 0 \quad \text{and} \quad \det A_j^t = 0, \quad j = 1, 2.$$

Hence, by Theorem 3.1, we obtain that $P_0^t \cup P_1^t \cup P_2^t$ is locally polynomially convex at the origin.

Case III: $\frac{3}{2} \leq t^2 < 3$. — For $\sqrt{\frac{3}{2}} < t < \sqrt{3}$, we see that

$$\det[A_1^t, A_2^t] > 0, \quad \det A_j^t < 0, \quad j = 1, 2, \quad \text{and} \quad |\det A_j^t| = \left| \frac{3 - t^2}{3(1 - t^2)} \right| \leq 1.$$

Hence, all the conditions of Theorem 3.2 are satisfied. Therefore, $P_0^t \cup P_1^t \cup P_2^t$ is locally polynomially convex at the origin.

Case IV: $\frac{15 - \sqrt{33}}{8} < t^2 < 2$. — In this case, we have

$$\det[A_1^t, A_2^t] > 0 \quad \text{and} \quad \det A_j^t < 0, \quad j = 1, 2.$$

We now show that $\beta(A_1^t, A_2^t) < \det A_2(\text{Tr } A_1)^2$. We compute:

$$\begin{aligned} \text{Tr } A_1^t A_2^t - \text{Tr } A_1^t \text{Tr } A_2^t &= \frac{2(t^4 + 2t^2 - 3)}{3(1 - t^2)^2}, \\ \text{Tr } A_1^t A_2^t (\text{Tr } A_1^t A_2^t - \text{Tr } A_1^t \text{Tr } A_2^t) &= -\frac{4(t^8 - (2t^2 - 3)^2)}{9(1 - t^2)^4}, \\ \beta(A_1^t, A_2^t) &= \frac{4t^2}{9(1 - t^2)^4} [t^6 - 8t^4 + 18t^2 - 12]. \end{aligned}$$

Thus, we have

$$\begin{aligned}
 \det A_2^t (\text{Tr } A_1^t)^2 - \beta(A_1^t, A_2^t) &= \frac{4t^4(3-t^2)}{9(1-t^2)^3} - \frac{4t^2}{9(1-t^2)^4}(t^6 - 8t^4 + 18t^2 - 12) \\
 &= \frac{4t^2}{9(1-t^2)^4}(t^2(3-4t^2+t^4) - t^6 + 8t^4 - 18t^2 + 12) \\
 &= \frac{4t^2}{9(1-t^2)^4}(4t^4 - 15t^2 + 12).
 \end{aligned}$$

Since, by assumption, $\frac{15-\sqrt{33}}{8} < t^2 < 2$, we get that $4t^4 - 15t^2 + 12 < 0$. This implies $\det A_2^t (\text{Tr } A_1^t)^2 < \beta(A_1^t, A_2^t)$. Hence, using Theorem 3.3, we conclude that $P_0^t \cup P_1^t \cup P_2^t$ is locally polynomially convex at the origin.

Therefore, combining all the cases, we conclude that $P_0^t \cup P_1^t \cup P_2^t$ is locally polynomially convex at the origin for every $t > \frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$. $\square$

Proof of Theorem 1.8.

(i). — Assume $t > \frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$. We will show that S_t is locally polynomially convex at $0 \in \mathbb{C}^2$. Given $S_t = \{(z, p_t(z, \bar{z})) : z \in \mathbb{C}\}$, where $p_t(z, \bar{z}) = z^2\bar{z} + tz\bar{z}^2 + \frac{t^2}{3}\bar{z}^3$. We now use the proper holomorphic map $\Phi_t: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $\Phi_t(z, w) = (z, p_t(z, w))$. In view of Theorem 1.4, we obtain

$$\Phi_t^{-1}(S_t) = P_0^t \cup P_1^t \cup P_2^t.$$

We recall that the pair of matrices (2.4) corresponding to Weinstock's normal form of the triple (P_0^t, P_1^t, P_2^t) is

$$A_1^t = \begin{pmatrix} \frac{t}{\sqrt{3}(1+t)} & \frac{1}{1+t} \\ -\frac{1}{1-t} & -\frac{t}{\sqrt{3}(1-t)} \end{pmatrix} \quad \text{and} \quad A_2^t = \begin{pmatrix} -\frac{t}{\sqrt{3}(1+t)} & \frac{1}{1+t} \\ -\frac{1}{1-t} & \frac{t}{\sqrt{3}(1-t)} \end{pmatrix}.$$

By Theorem 1.6, we get that, for each $t > \frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$, $P_0^t \cup P_1^t \cup P_2^t$ is locally polynomially convex at the origin. Therefore, in view of Lemma 2.2, for $t > \frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$, the surface S_t is locally polynomially convex at the origin.

(ii). — We use the proper map Φ_1 so that

$$\Phi_1^{-1}(S_1) = P_0 \cup P_1 \cup P_2,$$

where the planes are:

$$\begin{aligned} P_0^1 : \quad w &= \bar{z}, \\ P_1^1 : \quad w &= -\frac{\sqrt{3}(\sqrt{3}-i)}{2}z - \frac{1-i\sqrt{3}}{2}\bar{z}, \\ P_2^1 : \quad w &= -\frac{\sqrt{3}(\sqrt{3}+i)}{2}z - \frac{1+i\sqrt{3}}{2}\bar{z}. \end{aligned}$$

Since $P_j^1 \cap P_k^1 = \{(z, w) \in P_j^1 : z = -\bar{z}\}$ for $0 \leq j < k \leq 2$, by [10, Lemma 1.3], the pairwise unions of P_0 , P_1 and P_2 are locally polynomially convex at the origin. We now use the change of variables $\Psi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$\Psi(z, w) = (z + w, i(z - w)).$$

We denote $\widetilde{P}_j := \Psi(P_j)$, $j = 0, 1, 2$, where

$$\begin{aligned} \widetilde{P}_0 &= \{(2x, -2y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}, \\ \widetilde{P}_1 &= \{(-(1-i\sqrt{3})x, -2y + (\sqrt{3}+3i)x) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}, \\ \widetilde{P}_2 &= \{(-(1+i\sqrt{3})x, -2y - (\sqrt{3}-3i)x) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}. \end{aligned}$$

Let $K_j = \widetilde{P}_j \cap \overline{\mathbb{B}(0; 1)}$, $j = 0, 1, 2$, and $K = K_1 \cup K_2$. We consider the polynomial $F(z, w) = z$. Clearly, we have:

$$\begin{aligned} F(K_1) &\subset \mathbb{R} \subset \mathbb{C}, \\ F(K_2) &\subset (1-i\sqrt{3})\mathbb{R} \subset \mathbb{C}, \\ F(K_3) &\subset (1+i\sqrt{3})\mathbb{R} \subset \mathbb{C}. \end{aligned}$$

Thus, we have $\widehat{K_0} \cap \widehat{K} = \{0\}$. We also obtain that $F^{-1}\{0\} \cap K_0 = \{(0, -y) \in \mathbb{C}^2 : y \in \mathbb{R}\} \cap K_0$, which is polynomially convex, and $F^{-1}\{0\} \cap K = \{(0, -y) \in \mathbb{C}^2 : y \in \mathbb{R}\} \cap K$. Therefore, by Kallin's lemma (Theorem 2.3), we conclude that $K_0 \cup K (= K_0 \cup K_1 \cup K_2)$ is polynomially convex. $\square$

Proof of Theorem 1.10. — We know that there exists $r > 0$ such that

$$M_t \cap \overline{B(0; r)} = \{(z, w) \in \mathbb{C}^2 : \phi(z, w) = 0\},$$

where $\phi(z, w) = p_t(z, \bar{z}) + F(z, \bar{z}) - w$ with $F(z) = o(|z|^3)$. We first prove that, for $t \in (0, 1) \cup (1, \infty)$, the preimage of M_t , near the origin, under the proper holomorphic map Φ_t , is a union of three pairwise transverse totally-real surfaces intersecting only at the origin. More precisely, we will show, shrinking r , if needed, that

$$\Phi_t^{-1}\{M_t \cap \overline{B(0; r)}\} = S_0^t \cup S_1^t \cup S_2^t,$$

where

$$\begin{aligned} S_0^t &= \{(\zeta, \bar{\zeta} + f(\zeta)) \in \mathbb{C}^2 : \zeta \in \mathbb{C}, |\zeta| \leq r\}, \\ S_1^t &= \left\{ \left(\zeta, -\frac{3-i\sqrt{3}}{2t}\zeta - \frac{1-i\sqrt{3}}{2}\bar{\zeta} + g(\zeta) \right) \in \mathbb{C}^2 : \zeta \in \mathbb{C}, |\zeta| \leq r \right\}, \\ S_2^t &= \left\{ \left(\zeta, -\frac{3+i\sqrt{3}}{2t}\zeta - \frac{1+i\sqrt{3}}{2}\bar{\zeta} + h(\zeta) \right) \in \mathbb{C}^2 : \zeta \in \mathbb{C}, |\zeta| \leq r \right\}, \end{aligned}$$

with $f(\zeta) \sim o(|\zeta|)$, $g(\zeta) \sim o(|\zeta|)$ and $h(\zeta) \sim o(|\zeta|)$ near the origin.

We see that

$$\Phi_t(S_0^t) \subset M_t \implies \phi(\Phi(\zeta, \bar{\zeta} + f(\zeta))) = 0.$$

This implies

$$\begin{aligned} \phi(\zeta, p_t(\zeta, \bar{\zeta} + f(\zeta))) &= 0 \\ \implies t^2(f(\zeta))^3 + 3t(\zeta + t\bar{\zeta})(f(\zeta))^2 + 3(\zeta + t\bar{\zeta})^2 f(\zeta) - 3F(\zeta, \bar{\zeta}) &= 0. \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} (5.10) \quad (tf(\zeta) + (\zeta + t\bar{\zeta}))^3 &= 3tF(\zeta) + (\zeta + t\bar{\zeta})^3 \\ \implies tf(\zeta) + (\zeta + t\bar{\zeta}) &= (\zeta + t\bar{\zeta}) \left(1 + \frac{3tF(\zeta)}{(\zeta + t\bar{\zeta})^3} \right)^{1/3}. \end{aligned}$$

Since our requirement is $f(\zeta) \sim o(|\zeta|)$ near the origin, we have chosen the cube root of unity as 1 in (5.10). Since $F(\zeta) \sim o(|\zeta|^3)$ and $(\zeta + t\bar{\zeta})^3 \sim O(|\zeta|^3)$,

$$\left| \frac{3tF(\zeta)}{(\zeta + t\bar{\zeta})^3} \right| \longrightarrow 0 \quad \text{as } \zeta \rightarrow 0.$$

Hence, in a small neighborhood of the origin, the function $\left(1 + \frac{3tF(\zeta)}{(\zeta + t\bar{\zeta})^3} \right)^{1/3}$ is well defined and continuous. Expanding the right-hand side of (5.10), we get that f is $\mathcal{C}^1$ -smooth near the origin and $f(\zeta) \sim o(|\zeta|)$.

Since $\Phi_t(S_1^t) \subset M_t$, we get that

$$\phi \left(\Phi \left(\zeta, -\frac{3-i\sqrt{3}}{2t}\zeta - \frac{1-i\sqrt{3}}{2}\bar{\zeta} + g(\zeta) \right) \right) = 0.$$

This implies

$$\begin{aligned} t^2(g(\zeta))^3 - 3te^{2i\pi/3}(\zeta + t\bar{\zeta})(g(\zeta))^2 + 3e^{4i\pi/3}(\zeta + t\bar{\zeta})g(\zeta) - 3F(\zeta) &= 0 \\ \implies (tg(\zeta) + e^{i\pi/3}(\zeta + t\bar{\zeta}))^3 &= (\zeta + t\bar{\zeta})^3 \left(1 + \frac{3tF(\zeta)}{(\zeta + t\bar{\zeta})^3} \right). \end{aligned}$$

Thus, we have

$$\left| \frac{3tF(\zeta)}{(\zeta + t\bar{\zeta})^3} \right| \longrightarrow 0 \quad \text{as } \zeta \longrightarrow 0.$$

We now choose $e^{i\pi/3}$ as the cube root of unity to get a $\mathcal{C}^1$ -smooth g in a small neighborhood of the origin and

$$g(\zeta) \sim o(|\zeta|).$$

We also have $\Phi_t(S_2^t) \subset M_t$. This implies that

$$\phi\left(\zeta, -\frac{3+i\sqrt{3}}{2t}\zeta - \frac{1+i\sqrt{3}}{2}\bar{\zeta} + h(\zeta)\right) = 0,$$

and hence,

$$\begin{aligned} (5.11) \quad & t^2(h(\zeta))^3 + 3te^{4i\pi/3}(\zeta + t\bar{\zeta})(h(\zeta))^2 \\ & + 3e^{2i\pi/3}(\zeta + t\bar{\zeta})^2h(\zeta) - 3F(\zeta) = 0 \\ \implies & (th(\zeta) + e^{2i\pi/3}(\zeta + t\bar{\zeta}))^3 = (\zeta + t\bar{\zeta})^3 \left(1 + \frac{3tF(\zeta)}{(\zeta + t\bar{\zeta})^3}\right). \end{aligned}$$

In this case, choosing $e^{2i\pi/3}$ as the cube root of unity, we get from (5.11) that h is $\mathcal{C}^1$ -smooth in a small neighborhood of the origin and $h(\zeta) \sim o(|\zeta|)$.

The tangent spaces of S_0^t, S_1^t and S_2^t at the origin are:

$$\begin{aligned} T_0S_0^t &= \{(\zeta, \bar{\zeta}) \in \mathbb{C}^2 : \zeta \in \mathbb{C}\}, \\ T_0S_1^t &= \left\{ \left(\zeta, -\frac{3-i\sqrt{3}}{2t}\zeta - \frac{1-i\sqrt{3}}{2}\bar{\zeta} \right) \in \mathbb{C}^2 : \zeta \in \mathbb{C} \right\}, \\ T_0S_2^t &= \left\{ \left(\zeta, -\frac{3+i\sqrt{3}}{2t}\zeta - \frac{1+i\sqrt{3}}{2}\bar{\zeta} \right) \in \mathbb{C}^2 : \zeta \in \mathbb{C} \right\}. \end{aligned}$$

Using a $\mathbb{C}$ -linear change of coordinates, in view of (2.4), we get Weinstock's normal form for the tangent spaces as

$$T_0(S_0^t) = \mathbb{R}^2, \quad T_0S_j^t = (A_j^t + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2,$$

where

$$A_1^t = \begin{pmatrix} \frac{t}{\sqrt{3(1+t)}} & \frac{1}{1+t} \\ -\frac{1}{1-t} & -\frac{t}{\sqrt{3(1-t)}} \end{pmatrix} \quad \text{and} \quad A_2^t = \begin{pmatrix} -\frac{t}{\sqrt{3(1+t)}} & \frac{1}{1+t} \\ -\frac{1}{1-t} & \frac{t}{\sqrt{3(1-t)}} \end{pmatrix}.$$

Since $t > \frac{\sqrt{15-\sqrt{33}}}{2\sqrt{2}}$, by Theorem 1.6, any compact subset of the union $\mathbb{R}^2 \cup (A_1^t + i\mathbb{I})\mathbb{R}^2 \cup (A_2^t + i\mathbb{I})\mathbb{R}^2$ is polynomially convex. We now divide the rest of the proof in few cases.

Case I: $t^2 > 3$. — In this case, we note that

$$\det[A_1^t, A_2^t] > 0 \quad \text{and} \quad \det A_j^t > 0, \quad j = 1, 2.$$

Therefore, in view of Theorem 4.1(i), we get that $S_0^t \cup S_1^t \cup S_2^t$ is locally polynomially convex at the origin.

Case II: $\frac{3}{2} < t^2 < 3$. — For $\sqrt{\frac{3}{2}} < t < \sqrt{3}$, we see that

$$\det[A_1^t, A_2^t] > 0, \quad \det A_j^t < 0, \quad j = 1, 2, \quad \text{and} \quad |\det A_j^t| = \left| \frac{3 - t^2}{3(1 - t^2)} \right| < 1.$$

Hence, by Theorem 4.1(ii), we conclude that $S_0^t \cup S_1^t \cup S_2^t$ is locally polynomially convex at the origin.

Case III: $t^2 = 3$. — We note that

$$\det[A_1^t, A_2^t] > 0 \quad \text{and} \quad \det A_j^t = 0, \quad j = 1, 2,$$

Hence, by Theorem 4.2, we obtain that $S_0^t \cup S_1^t \cup S_2^t$ is locally polynomially convex at the origin.

Therefore, combining all the cases, we conclude that $S_0^t \cup S_1^t \cup S_2^t$ is locally polynomially convex at the origin for every $t > \sqrt{\frac{3}{2}}$. Thus, in view of Theorem 2.2, we conclude that M_t is locally polynomially convex at $0 \in \mathbb{C}^2$ if $t > \sqrt{\frac{3}{2}}$. $\square$

6. Proofs of Theorems 1.7, 1.9 and 1.11

In this section, we first prove Theorem 1.9 and we apply it to prove Theorem 1.7. Finally, we give a proof of Theorem 1.11. We begin with a lemma in one variable complex analysis.

LEMMA 6.1. — *Let $g_t(z) = z^2 + tz^4 + \frac{t^2}{3}z^6$, $0 \leq t < 1$. For each point $a \in \partial\mathbb{D}$, g_t has only two preimage of $g_t(a)$ on $\partial\mathbb{D}$ if and only if $0 \leq t < \sqrt{3}/2$.*

Proof of Lemma 6.1. — Consider a point $a \in \partial\mathbb{D}$. The equation $g_t(z) = g_t(a)$ gives us

$$z^2 + tz^4 + \frac{t^2}{3}z^6 = a^2 + ta^4 + \frac{t^2}{3}a^6.$$

If $t = 0$, then there are only two roots $a, -a$ of the above equation. Therefore, we assume $t \neq 0$. The above equation gives us

$$\begin{aligned} \implies & (z + a)(z - a) + t(z^4 - a^4) + \frac{t^2}{3}(z^6 - a^6) = 0 \\ \implies & (z - a)(z + a) \left(1 + t(z^2 + a^2) + \frac{t^2}{3}(z^4 + a^2z^2 + a^4) \right) = 0. \end{aligned}$$

Hence, the set $g_t^{-1}\{g_t(a)\}$ consists of $a, -a$, and the roots of the following quartic equation:

$$\frac{t^2}{3}z^4 + t\left(1 + \frac{ta^2}{3}\right)z^2 + 1 + ta^2 + \frac{t^2}{3}a^4 = 0.$$

To find the roots of the above quartic we make a change of variable $w = z^2$, where w is the new variable. The quartic now reduces to a quadratic equation:

$$\frac{t^2}{3}w^2 + t\left(1 + \frac{ta^2}{3}\right)w + 1 + ta^2 + \frac{t^2}{3}a^4 = 0.$$

The roots are

$$w = -\frac{3}{2t}\left(1 + \frac{ta^2}{3}\right) \pm \frac{\sqrt{3}i}{2t}(1 + ta^2).$$

Since $a \in \partial\mathbb{D}$, $a = e^{i\psi}$ for some $\psi \in [0, 2\pi]$. The roots of the above quadratic are:

$$\begin{aligned}\lambda_1 &:= -\frac{3}{2t}\left(1 + \frac{t}{3}e^{2i\psi}\right) + \frac{\sqrt{3}i}{2t}(1 + te^{2i\psi}), \\ \lambda_2 &:= -\frac{3}{2t}\left(1 + \frac{t}{3}e^{2i\psi}\right) - \frac{\sqrt{3}i}{2t}(1 + te^{2i\psi}).\end{aligned}$$

The real and the imaginary parts of λ_1 are:

$$\begin{aligned}\Re\lambda_1 &= -\frac{3}{2t} - \cos(2\psi - \pi/3), \\ \Im\lambda_1 &= \frac{\sqrt{3}}{2t} - \sin(2\psi - \pi/3).\end{aligned}$$

Hence,

$$|\lambda_1|^2 = \frac{3}{t^2} + \frac{3}{t}\cos(2\psi - \pi/3) - \frac{\sqrt{3}}{t}\sin(2\psi - \pi/3) + 1.$$

Thus, we have

$$|\lambda_1| = 1 \iff \frac{3}{t} + 3\cos(2\psi - \pi/3) - \sqrt{3}\sin(2\psi - \pi/3) = 0.$$

We now consider a function $g: [0, 2\pi] \rightarrow \mathbb{R}$ defined by

$$g(x) := 3\cos x - \sqrt{3}\sin x.$$

We need to find the minimum value of g in $[0, 2\pi]$. We compute $g'(x) = -3\sin x - \sqrt{3}\sin x$ and $g'(x) = 0$ implies $x = \pi - \pi/6$ and $x = 2\pi - \pi/6$. The function g attains its minimum at $x = \pi - \pi/3$, and the minimum value is $-2\sqrt{3}$. Hence, $|\lambda| = 1$ if and only if

$$3/t \leq 2\sqrt{3} \iff t \geq \frac{\sqrt{3}}{2}.$$

By a similar argument, we obtain that $|\lambda_2|^2 = 1$ if and only if

$$\frac{3}{t} + 3\cos(2\psi + \pi/3) + \sqrt{3}\sin(2\psi + \pi/3) = 0.$$

That is, $|\lambda_2| = 1$ if and only if $t \geq \frac{\sqrt{3}}{2}$. Hence, the number of preimages of $g_t(a)$ under g_t is 2 if and only if $0 \leq t < \frac{\sqrt{3}}{2}$. $\square$

We now present our proof of Theorem 1.9.

Proof of Theorem 1.9. — We first compute the Maslov-type index for S_t at $0 \in \mathbb{C}^2$. Given $p_t(z, \bar{z}) = z^2\bar{z} + t z\bar{z}^2 + \frac{t^2}{3}\bar{z}^3$. Putting $z = x + iy$, we get

$$\frac{\partial p_t}{\partial \bar{z}}(z, \bar{z}) = (1+t)^2 x^2 - (1-t)^2 y^2 + 2i(1-t^2)xy.$$

For $t \neq 1$, clearly, $(\partial p_t / \partial \bar{z})^{-1}\{0\} = \{0\}$. In this case, we have

$$q_t(z) = \frac{\partial p_t}{\partial \bar{z}}(z, 1) = (z+t)^2.$$

Therefore, using Theorem 2.14, we obtain that the Maslov-type index:

$$(6.1) \quad \text{Ind}_{S_t}(0) = \begin{cases} 2, & \text{if } t < 1, \\ -2, & \text{if } t > 1. \end{cases}$$

Next, we show that the function $\phi_t(z, \bar{z}) := \Re\left(\frac{p_t(z, \bar{z})}{z}\right)$ is subharmonic in $\mathbb{C} \setminus \{0\}$ for $t < 1$. We have

$$\frac{p_t(z, \bar{z})}{z} = |z|^2 + t\bar{z}^2 + \frac{t^2}{3} \frac{\bar{z}^3}{z}.$$

Hence,

$$(6.2) \quad \phi_t(z, \bar{z}) = z\bar{z} + \frac{t}{2}(z^2 + \bar{z}^2) + \frac{t^2}{6} \left(\frac{z^4 + \bar{z}^4}{|z|^2} \right),$$

$$(6.3) \quad \begin{aligned} \frac{\partial^2 \phi_t}{\partial z \partial \bar{z}}(z, \bar{z}) &= 1 - \frac{t^2(z^4 + \bar{z}^4)}{|z|^4} \\ &= 1 - t^2 \Re\left(\frac{z^2}{\bar{z}^2}\right). \end{aligned}$$

Hence, in view of (6.3), we get that ϕ_t is subharmonic in $\mathbb{C} \setminus \{0\}$ for all $t \in (0, 1)$.

(i). — Assume that $t < 1$. In view of (6.1), the Maslov-type index $\text{Ind}_{S_t}(0) = 2$ and $\Re\left(\frac{p_t(z, \bar{z})}{z}\right)$ is subharmonic in $\mathbb{C} \setminus \{0\}$. Therefore, applying Theorem 2.17(i), we conclude that S_t is not locally polynomially convex at the origin for $t < 1$.

(ii). — By part (i), we already know that the local hull of S_t is nontrivial. We will use Theorem 2.17(ii). For that, we consider the curve defined on the unit circle:

$$(6.4) \quad \begin{aligned} \mathcal{C}_t(z) &= \frac{p_t(z, \bar{z})}{z^3} \\ &= \bar{z}^2 + t\bar{z}^4 + \frac{t^2}{3}\bar{z}^6, \quad |z| = 1. \end{aligned}$$

Since $t \in \mathbb{R}$, we see that $\overline{\mathcal{C}_t(z)} = \mathcal{C}_t(\bar{z})$. We now consider the polynomial

$$(6.5) \quad g_t(z) := z^2 + tz^4 + \frac{t^2}{3}z^6.$$

By Lemma 6.1, we obtain, under the condition $0 \leq t < \sqrt{3}/2$, the preimage of $g_t(a)$ under g_t has exactly two points and the points are a and $-a$ for any $a \in \partial\mathbb{D}$. Hence, if for any two distinct points $z_1, z_2 \in \partial\mathbb{D}$, $\mathcal{C}_t(z_1) = \mathcal{C}_t(z_2)$, then $z_1 = -z_2$. Therefore, z_1, z_2 divides the unit circle in two segments of length $\pi > \pi/2$. Hence, Theorem 2.17(ii) applies in this situation. Therefore, for $0 \leq t < \sqrt{3}/2$, the polynomial hull of $S_t \cap \overline{B(0; \delta)}$ contains a ball centred at the origin and with positive radius.

(iii). — We assume $\sqrt{3}/2 \leq t < 1$. In this case, using Lemma 6.1, we obtain that the set $g_t^{-1}\{g_t(a)\}$ has more than two points in the unit circle. The proof of Lemma 6.1 shows that it has exactly four points in the set $g_t^{-1}\{g_t(a)\} \cap \partial\mathbb{D}$ if $\sqrt{3}/2 \leq t < 1$, and the set is not of the form $\{a, -a, ia, -ia\}$. Hence, the curve $\mathcal{C}_t(z)$ defined as in (6.4) does not satisfy property (**) in Theorem 2.17. Hence, by Theorem 2.17(iii), we obtain, at least, a one-parameter family of analytic discs with boundary lying in S_t passing through the origin. $\square$

Proof of Theorem 1.7. — Let us fix a $\delta > 0$ and let $K = (P_0^t \cup P_1^t \cup P_2^t) \cap \Phi_t^{-1}(B(0; \delta))$. We note that $\Phi_t^{-1}(\Phi_t(K)) = K$. Clearly, $\Phi_t(K) \subset S_t \cap B(0; \delta)$ is a compact subset containing a neighborhood the origin in S_t . By Theorem 1.9(i), for $t < 1$, $\Phi_t(K)$ is not polynomially convex. Therefore, K is not polynomially convex. Since $\delta > 0$ is chosen arbitrarily, $P_0^t \cup P_1^t \cup P_2^t$ is not locally polynomially convex at the origin.

(i). — We consider K as before. By Theorem 2.2, we get that $\Phi_t(\widehat{K})$ is polynomially convex. Since $\Phi_t(K)$ contains a neighborhood of the origin in the relative topology of S_t , using Theorem 1.9(ii) we obtain that $\widehat{\Phi_t(K)}$ contains a ball $B(0; r)$ for some $r > 0$. Since $\Phi_t(\widehat{K})$ is polynomially convex and contains $\Phi_t(K)$. Hence,

$$(6.6) \quad \widehat{\Phi_t(K)} \subset \Phi_t(\widehat{K}).$$

Therefore, $\Phi_t^{-1}(B(0; r)) \subset \widehat{K}$, which contains a neighborhood of the origin in $\mathbb{C}^2$.

(ii). — Using same K as in part (i), we see that, for $\frac{\sqrt{3}}{2} \leq t < 1$, $\widehat{\Phi_t(K)}$ contains at least a one-parameter family of analytic discs passing through the origin. We also have $\Phi_t^{-1}\{0\} = \{0\}$. Therefore, in view of (6.6), the preimages of the analytic discs lie in $\widehat{K}$. The boundary of each of the preimages of analytic discs passes through the origin. The preimage of a disc is an analytic variety with boundary in K . $\square$

Proof of Theorem 1.11.

(i). — We assume $t < 1$. In view of Lemma 2.13, we get that

$$\text{Ind}_{S_t}(0) = \text{Ind}_{M_t}(0).$$

In the proof of Theorem 1.9(i), the Maslov-type index for S_t is computed. Hence, for $t \in (0, 1)$, the Maslov-type index

$$\text{Ind}_{M_t}(0) = 2.$$

We also know that $\Re \frac{p_t(z, \bar{z})}{z}$ is strictly subharmonic in $\mathbb{C} \setminus \{0\}$. Since a small $\mathcal{C}^2$ -perturbation of strictly subharmonic function is again strictly subharmonic and $F(z, \bar{z}) \sim o(|z|^3)$ near the origin, we obtain that $\Re \frac{p_t(z, \bar{z}) + F(z, \bar{z})}{z}$ is strictly subharmonic in a small deleted neighborhood of the origin. Therefore, by Theorem 2.16, we conclude that there is an analytic disc with boundary in M_t . Hence, M_t is not locally polynomially convex at the origin.

(ii). — We will use Theorem 2.18 to prove this. We know that

$$\text{Ind}_{M_t}(0) = 2, \quad \forall 0 < t < 1,$$

and $\Re \frac{p_t(z, \bar{z})}{z}$ is subharmonic but nowhere harmonic in $\mathbb{C} \setminus \{0\}$. Since $0 < t < \sqrt{3}/2$, by Lemma 6.1, S_t satisfies property (**) (see Theorem 1.9(ii) for computations). Hence all the conditions of Theorem 2.18 are satisfied. Thus, for every $r > 0$, the polynomial hull of $M_t \cap \overline{B(0; r)}$ contains a ball centred at the origin and of positive radius.

(iii). — Assume $\frac{\sqrt{3}}{2} < t < 1$. We know that the Maslov-type index of S_t , the graph of lowest order polynomial of the local graphing function near the origin, is 2. By part (i), we already know that there is an analytic disc with boundary on M_t . Actually, the proof of Theorem 2.16 in [27] gives a one-parameter family. If the disc encloses the CR singularity at the origin, then, by using Theorem 2.15, we obtain a three parameter family of analytic discs that will fill a neighborhood of the origin. Otherwise, the boundaries of the discs must pass through the origin. $\square$

7. Questions

In this section we mention a few questions whose answers will make the picture clearer. The questions seem quite interesting (at least to the author) and classical. We begin with a conjecture.

CONJECTURE 7.1. — *The surface M_t is locally polynomially convex if $t > 1$.*

Theorem 1.10 proves above for $t > \sqrt{\frac{3}{2}} = 1.2247\dots$ The only part that remains open is for $1 < t \leq \sqrt{\frac{3}{2}}$.

QUESTION 7.2. — *Does there exist an example of a surface M_t for $\sqrt{3}/2 \leq t < 1$ such that the local polynomial hull at the origin contains a nonempty open ball centred at the origin? Or does the disc attached to it always have boundaries passing through the origin?*

Wiegerinck's result just gives a one-parameter family on analytic discs. We do not know whether an if-and-only-if condition for obtaining a three parameter family of analytic discs, analogous to Theorem 2.17, is possible when the higher order terms are present. In any case, it will be interesting to know the local hull of M_t for $\sqrt{3}/2 \leq t < 1$.

QUESTION 7.3. — *What can one say about the fine structure of the local hull of S_t ? Will it contain an interior point if $\frac{\sqrt{3}}{2} \leq t < 1$?*

QUESTION 7.4. — *What can one say about the local polynomial convexity of M_t for $t = 1$, i.e., at the parabolic CR singularity of higher order?*

QUESTION 7.5. — *What can one say about the local hulls and local polynomial convexity about the surfaces with cubic lowest order homogeneous term that do not belong to this family? Does there exist any other one-parameter family exhibiting Bishop-type dichotomy?*

QUESTION 7.6. — *Can one achieve analogous results for surfaces with CR singularity of higher order, i.e. when M_t , locally at the origin, is of the form $\{(z, w) \in \mathbb{C}^2 : w = (z + t\bar{z})^k + o(|z|^k)\}$, $k > 3$?*

In this case, one cannot apply Theorem 2.16 to get nontrivial hull for $t < 1$ as the subharmonicity condition that appears in Theorems 2.16, 2.17, and 2.18 is not automatic, which was the case for $k = 3$.

BIBLIOGRAPHY

- [1] E. BEDFORD & B. GAVEAU, “Envelopes of holomorphy of certain 2-spheres in $\mathbb{C}^2$ ”, *Am. J. Math.* **105** (1983), no. 4, p. 975-1009.
- [2] E. BEDFORD & W. KLINGENBERG, “On the envelope of holomorphy of a 2-sphere in $\mathbb{C}^2$ ”, *J. Am. Math. Soc.* **4** (1991), no. 3, p. 623-646.
- [3] G. BHARALI, “Surfaces with degenerate CR singularities that are locally polynomially convex”, *Mich. Math. J.* **53** (2005), no. 2, p. 429-445.
- [4] ———, “Polynomial approximation, local polynomial convexity, and degenerate CR singularities”, *J. Funct. Anal.* **236** (2006), no. 1, p. 351-368.
- [5] ———, “The local polynomial hull near a degenerate CR singularity: Bishop discs revisited”, *Math. Z.* **271** (2012), no. 3-4, p. 1043-1063.
- [6] E. BISHOP, “Differentiable manifolds in complex Euclidean space”, *Duke Math. J.* **32** (1965), p. 1-21.
- [7] P. DE PAEPE, “Eva Kallin’s lemma on polynomial convexity”, *Bull. Lond. Math. Soc.* **33** (2001), no. 1, p. 1-10.
- [8] F. FORSTNERIČ, “Analytic disks with boundaries in a maximal real submanifold of $\mathbb{C}^2$ ”, *Ann. Inst. Fourier* **37** (1987), no. 1, p. 1-44.
- [9] F. FORSTNERIČ & E. L. STOUT, “A new class of polynomially convex sets”, *Ark. Mat.* **29** (1991), no. 1, p. 51-62.
- [10] S. GORAI, “Local polynomial convexity of the union of two totally-real surfaces at their intersection”, *Manuscr. Math.* **135** (2011), no. 1-2, p. 43-62.
- [11] ———, “On the polynomial convexity of the union of three totally real planes in $\mathbb{C}^2$ ”, *Int. Math. Res. Not.* (2013), no. 21, p. 4985-5001.
- [12] G. A. HARRIS, “Lowest order invariants for real-analytic surfaces in $\mathbb{C}^2$ ”, *Trans. Am. Math. Soc.* **288** (1985), no. 1, p. 413-422.
- [13] ———, “Degenerate surfaces in $\mathbb{C}^2$ ”, in *Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989)* (E. Bedford, J. P. D’Angelo, R. E. Greene & S. G. Krantz, eds.), Proceedings of Symposia in Pure Mathematics, vol. 52(3), American Mathematical Society, 1991, p. 179-189.
- [14] L. V. HÖRMANDER & J. WERMER, “Uniform approximation on compact sets in $\mathbb{C}^n$ ”, *Math. Scand.* **23** (1968), p. 5-21.
- [15] L. R. HUNT & R. O. WELLS, JR., “The envelope of holomorphy of a two-manifold in $\mathbb{C}^2$ ”, *Rice Univ. Stud.* **56** (1970), no. 2, p. 51-62.
- [16] B. JÖRICKE, “Local polynomial hulls of discs near isolated parabolic points”, *Indiana Univ. Math. J.* **46** (1997), no. 3, p. 789-826.
- [17] E. KALLIN, “Fat polynomially convex sets”, in *Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965)* (F. T. Birtel, ed.), Scott Foresman, 1966, p. 149-152.
- [18] H. KANG & B. STENSØNES, “Envelope of holomorphy of certain two-manifolds in $\mathbb{C}^2$ ”, *Int. J. Math.* **22** (2011), no. 5, p. 713-730.
- [19] C. E. KENIG & S. M. WEBSTER, “The local hull of holomorphy of a surface in the space of two complex variables”, *Invent. Math.* **67** (1982), no. 1, p. 1-21.
- [20] J. K. MOSER & S. M. WEBSTER, “Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations”, *Acta Math.* **150** (1983), no. 3-4, p. 255-296.
- [21] R. SHAFIKOV & A. SUKHOV, “Polynomially convex hulls of singular real manifolds”, *Trans. Am. Math. Soc.* **368** (2016), no. 4, p. 2469-2496.
- [22] E. L. STOUT, *Polynomial convexity*, Progress in Mathematics, vol. 261, Birkhäuser, 2007.

- [23] P. J. THOMAS, “Enveloppes polynomiales d’unions de plans réels dans $\mathbb{C}^n$ ”, *Ann. Inst. Fourier* **40** (1990), no. 2, p. 371-390.
- [24] ———, “Unions minimales de n -plans réels d’enveloppe égale à $\mathbb{C}^n$ ”, in *Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989)* (E. Bedford, J. P. D’Angelo, R. E. Greene & S. G. Krantz, eds.), Proceedings of Symposia in Pure Mathematics, vol. 52(1), American Mathematical Society, 1991, p. 233-244.
- [25] B. M. WEINSTOCK, “On the polynomial convexity of the union of two maximal totally real subspaces of $\mathbb{C}^n$ ”, *Math. Ann.* **282** (1988), no. 1, p. 131-138.
- [26] J. WERMER, “Approximation on a disk”, *Math. Ann.* **155** (1964), p. 331-333.
- [27] J. WIEGERINCK, “Local polynomially convex hulls at degenerated CR singularities of surfaces in $\mathbb{C}^2$ ”, *Indiana Univ. Math. J.* **44** (1995), no. 3, p. 897-915.

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