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TAUTOLOGICAL CLASSES OF STRATA OF EXACT DIFFERENTIALS

by Frederik BENIRSCHKE

ABSTRACT. — Strata of exact differentials are moduli spaces for differentials on Riemann surfaces with vanishing absolute periods. Our main result is that classes of closures of strata of exact differentials inside the moduli space of multi-scale differentials lie in the divisorial tautological ring. By relating exact differentials to rational functions we obtain a new proof that classes of Hurwitz spaces are tautological and a new method for computations

RÉSUMÉ. — Les strates des différentielles exactes sont des espaces de modules pour les différentielles sur les surfaces de Riemann avec des périodes absolues nulles. Notre résultat principal est que les classes des adhérences des strates de différentielles exactes à l'intérieur de l'espace de modules des différentielles multi-échelles appartiennent à l'anneau tautologique divisoriel. En reliant les différentielles exactes aux fonctions rationnelles, nous obtenons une nouvelle preuve que les classes des espaces de Hurwitz sont tautologiques et une nouvelle méthode pour les calculs.

1. Introduction

1.1. Motivation from algebraic geometry: cycle classes of Hurwitz spaces

A classical way of producing subvarieties of the moduli space of curves is using Hurwitz spaces. The moduli space $\text{Hur}_{\underline{\lambda}}$ of Hurwitz covers of $\mathbb{P}^1$ parametrizes finite degree morphisms $f : X \rightarrow \mathbb{P}^1$ with prescribed ramification. A *ramification profile* for degree d covers is a tuple $\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$ of ordered partitions of d . The Hurwitz space $\text{Hur}_{\underline{\lambda}}$ parametrizes branched covers $f : X \rightarrow \mathbb{P}^1$ of degree d branched over m points $x_1, \dots, x_m$ in $\mathbb{P}^1$ with ramification described by λ_i in the fiber over x_i , and no other branch points.

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A modular compactification $\overline{\text{Hur}}_{\underline{\lambda}}$ of Hurwitz spaces using admissible covers was first constructed by Harris and Mumford. Forgetting the branched cover yields a morphism $\phi : \overline{\text{Hur}}_{\underline{\lambda}} \rightarrow \overline{\mathcal{M}}_{g,n}$ and pushing forward the fundamental class produces natural *admissible cover cycles* $\text{Adm}_{\underline{\lambda}} \in \text{CH}^*(\overline{\mathcal{M}}_{g,n})$. The computation of admissible cover cycles has attracted a lot of research, see [13, 14, 15, 27] for a collection of results. While in general no closed formulas for admissible cover cycles are known, it was shown in [15] that admissible cover cycles lie in the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$.

In order to study admissible cover cycles we will now change our point of view and work with differential forms instead of rational functions. This perspective has appeared already in [4, 5], where we constructed a smooth compactification of Hurwitz spaces using exact differentials. Similar ideas were also used in [22, 25].

Given a rational function $f : X \rightarrow \mathbb{P}^1$ from a Riemann surface X of genus g , the *associated exact differential* is $df = f^*dz$ and we can recover f , up to additive constants, by integrating. The key observation is that exact differentials can be characterized by the vanishing of all absolute periods

$$(1.1) \quad \int_{\gamma} df = 0, \quad \forall \gamma \in H_1(X \setminus P(df); \mathbb{C}),$$

where $P(df) \subseteq X$ is the set of poles of df . (1.1) can be rephrased as lying in the zero locus of a section of a vector bundle over a suitable moduli space of differentials, which will allow us to perform intersection theory. We will now setup the moduli spaces in question.

1.2. Strata of exact differentials

The stratum $\mathcal{H}(\mu)$ parametrizes tuples

$$(X, Q = (q_1, \dots, q_n), \omega)$$

of marked Riemann surfaces X together with a meromorphic differential ω vanishing to order μ_i at q_i . Most of the time we only consider differentials up to rescaling, which are parametrized by the projectivized stratum $\mathcal{P}(\mu) = \mathcal{H}(\mu)/\mathbb{C}^*$.

Given a ramification profile $\underline{\lambda}$ we let μ be the partition recording the zero and pole orders of df . For example if we consider hyperelliptic curves the ramification profile is $\underline{\lambda} = (2^{2g+2})$ and the associated partition is $\mu = (-3, 1^{2g+1})$, where we used exponential notation to denote repetitions.

The *stratum of exact differentials*

$$\mathrm{Exc}(\mu) = \left\{ (X, \omega) \left| \int_{\gamma} \omega = 0 \text{ for all } \gamma \in H_1(X \setminus P(\omega); \mathbb{Z}) \right. \right\} \subseteq \mathcal{P}(\mu)$$

is either empty or a codimension $2g + |P(\omega)| - 1$ subvariety of the projectivized stratum.

Differentials in $\mathrm{Exc}(\mu)$ correspond to rational functions with prescribed ramification multiplicities, but not necessarily with ramification profile $\underline{\lambda}$, since so far we have no requirement on which ramification points lie in the same fiber. In Section 2 we will construct the *stratum of $\underline{\lambda}$ -exact differentials* $\mathrm{Exc}_{\underline{\lambda}} \subset \mathrm{Exc}(\mu)$ consisting of exact differentials coming from rational functions with ramification profile $\underline{\lambda}$. In order to perform intersection theory we will work on a compactification of the stratum $\mathcal{P}(\mu)$.

Recently in [3] the authors construct the *moduli space of multi-scale differentials* $\overline{\mathcal{P}}(\mu)$, a smooth compactification of the stratum $\mathcal{P}(\mu)$. We will recall its main features in Section 3. The boundary of $\overline{\mathcal{P}}(\mu)$ is a normal crossing divisor, which can be decomposed into the divisor D_h consisting of irreducible curves with horizontal nodes and divisors $D_{\overline{\Gamma}}$ parametrized by non-horizontal two level graphs. Similar to $\overline{\mathcal{M}}_{g,n}$, the moduli space of multi-scale differentials has a natural tautological ring $R^*(\overline{\mathcal{P}}(\mu))$, which was first defined in [10]. For example the ring $R^*(\overline{\mathcal{P}}(\mu))$ contains all boundary divisors $D_{\overline{\Gamma}}$ for non-horizontal two-level graphs and also the first Chern class

$$\xi = c_1(\mathcal{O}(-1)) \in \mathrm{CH}^1(\overline{\mathcal{P}}(\mu))$$

of the tautological line bundle, whose fiber over a twisted differential is the line generated by the differential.

Our main result is the computation of the class of the closure $\overline{\mathrm{Exc}}(\mu)$ in the Chow ring of $\overline{\mathcal{P}}(\mu)$.

THEOREM 1.1. — *Let μ be a partition of $2g - 2$. The class of the stratum of exact differentials $\overline{\mathrm{Exc}}(\mu)$ lies in the divisorial tautological ring $\mathrm{div}R^*(\overline{\mathcal{P}}(\mu)) \subseteq \mathrm{CH}^*(\overline{\mathcal{P}}(\mu))$ generated by ξ and $D_{\overline{\Gamma}}$ where $\overline{\Gamma}$ is a non-horizontal boundary divisor. Furthermore, there exists an explicit algorithm computing the class $\overline{\mathrm{Exc}}(\mu)$. The same is true for the class of $\underline{\lambda}$ -exact strata $\overline{\mathrm{Exc}}_{\underline{\lambda}}$ for any ramification profile $\underline{\lambda}$.*

In particular the class of $\overline{\mathrm{Exc}}(\mu)$ behaves like the class of a complete intersection, although we do not expect $\overline{\mathrm{Exc}}(\mu)$ to be a complete intersection unless $g = 0$.

We now circle back to the connection between rational functions and exact differentials. The moduli space $\overline{\mathcal{P}}(\mu)$ comes with a forgetful map

$\rho : \bar{\mathcal{P}}(\mu) \rightarrow \bar{\mathcal{M}}_{g,n}$ and by construction the pushforward of $[\overline{\text{Exc}}_\lambda]$ is the admissible cover cycle Adm_λ .

COROLLARY 1.2 ([15]). — *Admissible cover cycles $\text{Adm}_\lambda \in \text{CH}^*(\bar{\mathcal{M}}_{g,n})$ are tautological.*

Proof. — The pushforward $\rho_* : \text{CH}^*(\bar{\mathcal{P}}(\mu)) \rightarrow \text{CH}^*(\bar{\mathcal{M}}_{g,n})$ sends tautological classes to tautological classes (see Proposition 3.7) and the statement now follows directly from Theorem 1.1 for strata of λ -exact differentials. $\square$

The proof of Theorem 1.1 is explicit and together with the algorithm for the computation of the pushforward ρ_* in [10], we obtain an explicit approach for calculating admissible cover cycles.

The algorithm for the computation of strata of λ -exact differentials has been implemented in **sage** using the packages **admcycles** [12] and **diffstrata** [11] and can be used to cross check our formulas, see Section 6 for examples.

1.3. Closed formulas in low genus

While in general our formula only produces a recursive algorithm to compute the class of strata of exact differentials, in genus $g = 0, 1$ we can use Theorem 1.1 to extract closed formulas for the class of the stratum of λ -exact differentials.

PROPOSITION 1.3. — *Let Exc_λ be a stratum λ -exact differentials in $g = 0, 1$. Then*

$$[\overline{\text{Exc}}_\lambda] = (-1)^{\text{codim } \text{Exc}_\lambda} \prod_i \left(\xi + \sum_{\bar{\Gamma} \in W_i(\lambda)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \cdot \alpha_\lambda \in \text{CH}^*(\bar{\mathcal{P}}(\mu))$$

for some explicit collection of two-level graphs $W_i(\lambda)$ and an explicit class α_λ . If $g = 0$, then $\alpha_\lambda = 1$ and if $g = 1$, then $\alpha_\lambda \in \text{divR}^2(\bar{\mathcal{P}}(\mu))$. We describe $W_i(\lambda)$ and α_λ explicitly in Propositions 6.2 and 6.4.

The coefficients $\ell_{\bar{\Gamma}}$ will be defined in Section 3.2.

Lifting admissible cover cycles to the log Chow ring

The log Chow ring $\text{logCH}^*(\bar{\mathcal{M}}_{g,n})$ of $\bar{\mathcal{M}}_{g,n}$ can be constructed via logarithmic geometry or alternatively as the colimit of the Chow rings of

iterated blowups of boundary strata of $\overline{\mathcal{M}}_{g,n}$, see for example [21] for an introduction. Sometimes tautological classes become expressible more easily in the log Chow ring. For example it was shown in [21] that the top λ -class λ_g does not lie in the subalgebra of $\mathrm{CH}^*(\overline{\mathcal{M}}_g)$ generated by divisorial classes for $g \geq 2$ but if lifted to the log Chow ring λ_g lies in the divisorial subring $\mathrm{divlogCH}^*(\overline{\mathcal{M}}_g)$.

In [7, Theorem 5.13] the moduli space of multi-scale differentials $\overline{\mathcal{P}}(\mu)$ was related to an iterated blowup coming from log geometry. We can thus use closures of strata of λ -exact differentials Exc_{λ} to define a lift of admissible cover cycles to the log Chow ring. We have shown that the class of the closure of Exc_{λ} lies in the divisorial subring of $\mathrm{CH}^*(\overline{\mathcal{P}}(\mu))$ and it seems an interesting question to determine whether the lift lies in the divisorial subring $\mathrm{divlogCH}^*(\overline{\mathcal{M}}_g)$.

1.4. Motivation from Teichmüller dynamics and bialgebraic geometry

Strata of exact differentials are examples of a more general class of subvarieties of strata $\mathcal{P}(\mu)$. The stratum $\mathcal{H}(\mu)$ has a linear structure given by local period coordinates and *linear subvarieties* of $\mathcal{H}(\mu)$ are algebraic subvarieties which are locally given by linear equations in period coordinates.

If the linear equations are defined over the real numbers, as it happens for $\mathrm{Exc}(\mu)$, then a linear subvariety is invariant for the natural $\mathrm{GL}_+(2, \mathbb{R})$ -action on strata. In a holomorphic stratum $\mathrm{GL}_+(2, \mathbb{R})$ -invariant varieties are unions of finite orbit closures and the intersection theory of an orbit closure is (conjecturally) closely related to the dynamics of the $\mathrm{GL}_+(2, \mathbb{R})$ -action (see for example [8, Conjecture 4.3]). Additionally, in [9] the Euler characteristic of a linear subvariety is expressed as intersection number on the moduli space of multi-scale differentials.

If the class of an orbit closure can be explicitly computed in the Chow ring of $\overline{\mathcal{P}}(\mu)$ these formulas can be evaluated effectively. This motivates the question whether classes of orbit closures lie in the tautological ring. One way to produce an orbit closure from a given one is using covering constructions. But tautological classes are not preserved under branched coverings, for example the bielliptic locus (locus of double covers of elliptic curves) is in general not tautological, see [18] and can be realized as a covering construction of a stratum in genus one. In [20], Lian constructs an extension of the tautological ring, the *H-tautological ring*, which is stable under pulling back classes along covering maps and thus is a natural place to study classes of affine invariant submanifolds.

QUESTION 1.4. — *Are pushforwards of classes of affine invariant submanifolds H -tautological in $\overline{\mathcal{M}}_{g,n}$ (in the sense of [20])?*

In this paper we deal with two classes of linear subvarieties, strata of exact differentials and residue subspaces, both of which only occur in strata of meromorphic differentials. In both cases the classes happen to lie in the (classical) tautological ring.

Linear subvarieties are examples of bialgebraic subvarieties of strata (see [19] for an introduction to these circles of ideas). Since bialgebraic varieties are very special and rare, one expects their classes to have special numerical properties as well and it seems interesting to investigate when these classes are tautological.

1.5. Residueless differentials

Our methods not only work for strata of exact differentials but also for linear subvarieties defined by linear equations among residues. These can be defined as follows. On a stratum of meromorphic differentials $\mathcal{H}(\mu)$ there exists a residue map

$$R : \mathcal{H}(\mu) \longrightarrow \mathbb{C}^r,$$

where r is the number of poles of $\mathcal{H}(\mu)$, sending a differential to the residues at the marked poles. A *residue subspace* is an algebraic variety of the form $R^{-1}(V)$, where $V \subseteq \mathbb{C}^r$ is some linear subspace. If V is defined over the real numbers, $R^{-1}(V)$ is a $\mathrm{GL}_+(2, \mathbb{R})$ -invariant subvariety. In the case where V is the zero subspace we call the resulting variety the stratum of residueless differentials. Their classes in the cohomology ring of $\overline{\mathcal{M}}_{g,n}$ have been recently considered in [6] to construct a partial cohomological field theory related to the KP hierarchy.

Theorem 4.1 implies that Chow classes of residue subspaces lie in the divisorial tautological ring but in this special case the proof is easier and we also obtain a closed formula.

PROPOSITION 1.5. — *Let Z be the closure of a residue subspace in $\overline{\mathcal{P}}(\mu)$. Then there exists a closed formula*

$$Z = (-1)^{\mathrm{codim} Z} \prod_i \left(\xi + \sum_{\bar{\Gamma} \in W_i(Z)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \in \mathrm{CH}^*(\overline{\mathcal{P}}(\mu))$$

in the divisorial tautological ring. Here $W_i(Z)$ is a collection of two-level graphs that we describe explicitly in Proposition 4.6.

1.6. Outline

We briefly sketch the proof idea for Theorem 1.1, highlighting the technical difficulties that we encounter along the way. On the stratum $\mathcal{P}(\mu)$ we can find a vector bundle $\mathcal{E}$ with a section s whose zero locus is exactly the stratum of exact differentials $\text{Exc}(\mu)$. In particular the class of $\text{Exc}(\mu)$ is given by the top Chern class of $\mathcal{E}$. The first step in extending the computation to the compactification $\overline{\mathcal{P}}(\mu)$ is to extend $\mathcal{E}$ and s to $\overline{\mathcal{P}}(\mu)$. This is done in Section 4.1. The extended section $\bar{s}$ contains the closure of $\text{Exc}(\mu)$ but it also contains other extraneous components. The extraneous components can be described explicitly and are all related to exact differentials on nodal curves. The main technical difficulty is that some extraneous components are not of expected codimension. We deal with this difficulty by blowing up all extraneous components in a carefully chosen order. On the blowup the extraneous components become divisorial and the extended section $\bar{s}$ vanishes with a certain multiplicity. After dividing by the defining equation of the divisors we obtain a new section of a different vector bundle and it turns out that now the zero locus only consists of the proper transform of $\overline{\text{Exc}(\mu)}$. We then need to blow down all the exceptional divisors and use intersection theory of blowups to obtain the class of $\overline{\text{Exc}(\mu)}$. Another technical difficulty is that some of the extraneous components are not smooth but instead have non-reduced stack structures and therefore the resulting blowup might not be smooth as well. By a careful analysis of the local defining equations we show that the singularities are mild enough so that the Chow groups still have an intersection product and the classes of proper transforms can be computed.

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2. Preliminaries

2.1. Exact differentials

Given a rational function $f : X \rightarrow \mathbb{P}^1$ from a Riemann surface X of genus g , the *associated exact differential* is $df = f^*dz$ and we can recover

f by integrating. By acting on $\mathbb{P}^1$ with Möbius transformations we can produce isomorphic rational functions with different exact differentials. In order to associate to the isomorphism class of f a unique exact differential, at least up to rescaling, we need to normalize f suitably. By choosing a marked point p_1 on X , we can normalize f such that $f(p_1) = \infty$. This determines f up to rescaling.

Exact differentials are characterized in terms of their periods. A meromorphic differential ω on X is exact if and only if all its absolute periods $\int_\gamma \omega$ for $\gamma \in H^1(X \setminus P(\omega); \mathbb{Z})$ vanish, where $P(\omega)$ denotes the set of poles of ω . In that case the rational function is given by $f(x) = \int_{x_0}^x \omega$ for some choice of base point x_0 and some choice of path from x_0 to x . Later we will see that the vanishing of absolute periods can be described as the zero locus of a section of a vector bundle on the stratum of meromorphic differentials, which will be the basis of our computation.

Now we address how to determine the ramification of a rational function from its exact differential. The ramification multiplicities of f are determined by the orders of zeros and poles of df . In particular we have

$$\text{mult}_p f = \begin{cases} \text{ord}_p df + 1 & \text{if } f(p) \neq \infty, \\ -\text{ord}_p df - 1 & \text{otherwise.} \end{cases}$$

A *ramification profile* $\underline{\lambda} = (d; \lambda_1, \dots, \lambda_k)$ consists of a natural number d and a tuple of partitions $\lambda_i = (\lambda_{i,1}, \dots, \lambda_{i,l(\lambda_i)})$ satisfying the Riemann–Hurwitz condition

$$\sum_{i,j} (\lambda_{i,j} - 1) = 2d + 2g(X) - 2.$$

A marked curve $(X, Q = (q_{i,j}), f)$ with a rational function f of degree d is said to have ramification profile $\underline{\lambda}$ if at all marked points we have $\text{mult}_{q_{i,j}} f = \lambda_{i,j}$ and if two points $q_{i,j}, q_{i,k}$ correspond to the same partition, they lie in the same fiber of f , i.e., $f(q_{i,j}) = f(q_{i,k})$. In terms of periods of df this translates to

$$\int_{q_{i,j}}^{q_{i,k}} df = 0.$$

The integration here is performed on any path on X connecting $q_{i,j}$ and $q_{i,k}$ and is independent of the choice of such a path, since we already know that all absolute periods of df are zero.

2.2. Strata of exact differentials

By passing to exact differentials, rational functions with a given ramification profile $\underline{\lambda}$ can be described in terms of meromorphic differentials with prescribed orders and zeros, together with additional linear constraints on certain periods.

We let μ be a tuple of order of zeros and poles of an exact differential with ramification profile $\underline{\lambda}$. In general there is ambiguity because of the choice of a pole of f . However, we will always assume that the first partition λ_1 corresponds to the poles of f . With this convention $\underline{\lambda}$ determines a unique partition μ , which we call the *partition associated to* $\underline{\lambda}$. For example if we consider hyperelliptic curves the ramification profile is $\underline{\lambda} = (2^{2g+2})$ and the associated partition is $\mu = (-3, 1^{2g+1})$, where we used exponential notation to denote repetitions.

Our goal now is to define moduli spaces for exact differentials. The stratum $\mathcal{H}(\mu)$ parametrizes tuples $(X, Q = (q_1, \dots, q_n), \omega)$ of marked Riemann surfaces X of genus g together with a meromorphic differential ω vanishing to order μ_i at p_i . Most of the time we only consider differentials up to rescaling, which are parametrized by the projectivized stratum $\mathcal{P}(\mu) = \mathcal{H}(\mu)/\mathbb{C}^*$. The *stratum of exact differentials*

$$\text{Exc}(\mu) = \left\{ (X, \omega) \left| \int_{\gamma} \omega = 0 \text{ for all } \gamma \in H_1(X \setminus P(\omega); \mathbb{Z}) \right. \right\} \subseteq \mathcal{P}(\mu)$$

is a codimension $2g + |P(\omega)| - 1$ subvariety of the projectivized stratum.

Differentials in $\text{Exc}(\mu)$ are exact and have prescribed ramification multiplicities. To further specify the ramification profile $\underline{\lambda}$ we thus need to pass to a further subvariety, taking into account which points lie in the same fiber. We write $\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i = (\lambda_{i,1}, \dots, \lambda_{i,l(\lambda_i)})$. Sometimes it will be useful to re-index the marked points $Q = (q_1, \dots, q_n)$ by $(q_{1,1}, \dots, q_{1,l(\lambda_1)}, \dots, q_{m,1}, \dots, q_{m,l(\lambda_m)})$ such that the marked point $q_{i,j}$ corresponds to the entry $\lambda_{i,j}$ in the partition λ_i . We say that $(q_{i,1}, \dots, q_{i,l(i)})$ are the *marked points lying in the i^{th} fiber of $\underline{\lambda}$* .

We can now define the *stratum of $\underline{\lambda}$ -exact differentials*

$$\text{Exc}_{\underline{\lambda}} := \left\{ (X, Q, \omega) \in \text{Exc}(\mu) \left| \int_{q_{i,j}}^{q_{j,k}} \omega = 0 \text{ for all } 1 \leq i \leq m, 1 \leq j \leq l(\lambda_i) \right. \right\}.$$

The expected codimension of $\text{Exc}_{\underline{\lambda}}$ inside the stratum is

$$d_{\underline{\lambda}} = 2g + |P(\omega)| - 1 + \sum_{i=1}^m (l(\lambda_i) - 1).$$

3. Local systems on strata

Over the stratum $\mathcal{P}(\mu)$ the relative homology groups $H_1(X \setminus P(\omega), Z(\omega); \mathbb{C})$ assemble to a $\mathbb{C}$ -local system. Here $P(\omega)$ and $Z(\omega)$ denote the set of poles and zeros of ω , respectively. We let $\mathcal{H}$ be the associated flat vector bundle, which comes with the *evaluation morphism*

$$\mathrm{ev} : \mathcal{O}(-1) \otimes \mathcal{H} \longrightarrow \mathcal{O}, \quad \omega \otimes \alpha \longmapsto \int_{\alpha} \omega.$$

Our goal is to study subvarieties of strata defined by the vanishing of periods $\int_{\alpha} \omega$. In order to formalize this idea, let $\mathcal{V} \subseteq \mathcal{H}$ be a sub-local system. Here and in the following we do not distinguish between a local system and the associated flat vector bundle. We let

$$\mathrm{ev}_{\mathcal{V}} : \mathcal{O}(-1) \otimes \mathcal{V} \longrightarrow \mathcal{O}$$

be the restriction of the evaluation section to cycles in $\mathcal{V}$ and then define the *annihilator of $\mathcal{V}$* to be

$$\begin{aligned} \mathrm{Ann}_{\mathcal{P}(\mu)}(\mathcal{V}) &:= Z(\mathrm{ev}_{\mathcal{V}}) \\ &= \left\{ (X, \omega) \in \mathcal{P}(\mu) \mid \int_{\alpha} \omega = 0 \text{ for all } \alpha \in \mathcal{V}_{(X, \omega)} \right\} \subseteq \mathcal{P}(\mu). \end{aligned}$$

The zero locus $\mathrm{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})$ is either empty or of codimension $\mathrm{rank}(\mathcal{V})$ in the stratum $\mathcal{P}(\mu)$. This can be checked for example in local period coordinates.

In the sequel we will mainly focus on three examples of local systems $\mathcal{V}$, which we will now discuss.

The first one is the local system of absolute homology $\mathcal{H}_{\mathrm{abs}}$ whose fiber over a point (X, ω) is $H_1(X \setminus P(\omega); \mathbb{C})$. The zero locus $\mathrm{Ann}_{\mathcal{P}(\mu)}(\mathcal{H}_{\mathrm{abs}})$ coincides with the stratum of exact differentials $\mathrm{Exc}(\mu)$. The expected codimension of $\mathrm{Exc}(\mu)$ is $\mathrm{rank}(\mathcal{H}_{\mathrm{abs}}) = 2g + |P(\omega)| - 1$.

If we want to take the ramification profile of a rational function into consideration, we need to require certain relative periods to vanish and thus need to pass a local system containing $\mathcal{H}_{\mathrm{abs}}$. This will be our second example. Given a ramification profile $\underline{\lambda}$ we define

$$H_{\underline{\lambda}}(X, \omega) \subseteq H_1(X \setminus P(\omega), Z(\omega); \mathbb{C})$$

to be the smallest subspace containing the absolute homology $H_1(X \setminus P(\omega); \mathbb{C})$ as well as all paths such that both endpoints q, q' lying in the same fiber, i.e., there exist i, k, k' such that $q = q_{i,k}, q' = q_{i,k'}$. In particular we have $\mathcal{H}_{\mathrm{abs}} \subseteq \mathcal{H}_{\underline{\lambda}} \subseteq \mathcal{H}$. The annihilator of $\mathcal{H}_{\underline{\lambda}}$ consists of $\underline{\lambda}$ -exact differentials, in other words

$$\mathrm{Ann}(\mathcal{H}_{\underline{\lambda}}) = \mathrm{Exc}_{\underline{\lambda}}.$$

The expected codimension in this case is $2g + |P(\omega)| - 1 + \sum_{i=1}^m (l(\lambda_i) - 1)$, where we recall that $l(\lambda_i)$ is the number of parts in the partition λ_i .

The last example we will consider is related to linear relations among residues. First consider the local system $\mathcal{H}_{\text{res}}$ with fiber $\ker(H_1(X \setminus P(\omega); \mathbb{C}) \rightarrow H_1(X; \mathbb{C}))$. The bundle $\mathcal{H}_{\text{res}}$ is trivial and for any subspace $R \subseteq (\mathcal{H}_{\text{res}})_{(X, \omega)}$ of a fiber we consider the associated vector bundle $\mathcal{R} \subseteq \mathcal{H}_{\text{res}}$. The annihilator of $\mathcal{R}$ consists of all differentials whose residues satisfy the linear equations given by $\mathcal{R}$. A special case is the case $\mathcal{R} = \mathcal{H}_{\text{res}}$ which yields the *stratum of residueless differentials* $\overline{\mathcal{P}}(\mu)^{\text{res}}$.

The annihilator of any local system $\mathcal{V} \subseteq \mathcal{H}$ is the zero locus of a vector bundle and has the expected codimension. From this we can determine the class of $\text{Ann}(\mathcal{V})$ as the top Chern class of $\mathcal{V}^* \otimes \mathcal{O}(-1)$. The Chern classes of a flat vector bundle vanish in cohomology by Chern–Weil theory. For the examples $\mathcal{H}_{\text{abs}}, \mathcal{H}_{\underline{\lambda}}, \mathcal{R}$, which we will focus on the sequel, the same is true also in the Chow ring.

DEFINITION 3.1. — *A local system $\mathcal{V} \subseteq \mathcal{H}$ on $\mathcal{P}(\mu)$ is globally defined if either $\mathcal{V} \subseteq \mathcal{H}_{\text{res}}$ or $\mathcal{H}_{\text{abs}} \subseteq \mathcal{V}$.*

In particular $\mathcal{H}_{\text{abs}}, \mathcal{H}_{\underline{\lambda}}, \mathcal{R}$ are all globally defined local systems.

Remark 3.2. — If $\mathcal{V} \subseteq \mathcal{H}_{\text{res}}$ then $\mathcal{V}$ is a trivial local system. On the other hand if $\mathcal{H}_{\text{abs}} \subseteq \mathcal{V}$, then $\frac{\mathcal{V}}{\mathcal{H}_{\text{abs}}}$ is trivial.

LEMMA 3.3. — *Let $\mathcal{V}$ be any globally defined local system. Then $\mathcal{V}$ has vanishing Chern classes in the Chow ring of $\mathcal{P}(\mu)$.*

Proof. — Firstly, if the bundle is trivial, it has zero Chern classes. Otherwise we can write $\mathcal{V}$ as an extension

$$0 \longrightarrow \mathcal{H}_{\text{abs}} \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}^{n+1} \longrightarrow 0$$

of $\mathcal{H}_{\text{abs}}$ by a trivial bundle.

Similarly, $\mathcal{H}_{\text{abs}}$ is an extension of $\mathcal{H}_{\text{res}}$ by the bundle $\mathcal{H}_1$ with fiber $H_1(X; \mathbb{C})$ by a trivial bundle. Finally, the bundle $\mathcal{H}_1$ sits in a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{H}_1 \longrightarrow \mathcal{E}^* \longrightarrow 0,$$

where $\mathcal{E}$ is the Hodge bundle over $\mathcal{M}_{g,n}$. It then follows from Mumfords computation [23] that in the Chow ring of $\mathcal{M}_{g,n}$ the identity $\text{ch}(\mathcal{H}_1) = \text{ch}(E) + \text{ch}(E^*) = 0$ holds. The same thus holds for the pullback to $\mathcal{P}(\mu)$. $\square$

PROPOSITION 3.4. — *Suppose that $\mathcal{V} \subseteq \mathcal{H}$ is a globally defined local system. Then the class of $\text{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})$ in the Chow ring of $\mathcal{P}(\mu)$ is given by*

$$[\text{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})] = c_1(\mathcal{O}(1))^r \in \text{CH}^r(\mathcal{P}(\mu)),$$

where $r := \text{rank } \mathcal{V}$.

Proof. — Since $\text{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})$ is the zero locus of a section of $\mathcal{H}om(\mathcal{O}(-1) \otimes \mathcal{V}, \mathcal{O}) \simeq \mathcal{V}^* \otimes \mathcal{O}(1)$ of the expected codimension, the class $[\text{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})]$ agrees with the top Chern class $c_r(\mathcal{V}^* \otimes \mathcal{O}(1))$. Thus

$$[\text{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})] = c_r(\mathcal{V}^* \otimes \mathcal{O}(1)) = \mathcal{O}(1)^r,$$

where in the last equality we used that $\mathcal{V}$ has zero Chern classes. $\square$

3.1. Moduli space of multi-scale differentials

The moduli space of multi-scale differentials $\overline{\mathcal{P}}(\mu)$ was constructed in [3] and compactifies the projectivized stratum $\mathcal{P}(\mu)$. The space $\overline{\mathcal{P}}(\mu)$ is a smooth projective Deligne–Mumford stack and the boundary is a normal crossing divisor. Furthermore, the boundary $\overline{\mathcal{P}}(\mu) \setminus \mathcal{P}(\mu)$ is stratified by level graphs, which we now recall.

3.2. Level graphs

A level graph is a stable graph with a total order on the vertices (allowing equality) together with an integer for each half-leg, the order of vanishing, and an additional non-negative integer for each edge, called the prong at e . Our notation slightly differs from [3], where the above is called an “enhanced level graph”. In this paper we exclusively deal with enhanced level graphs and thus refer to them simply as level graphs. We follow the convention from [10] and index the levels of $\bar{\Gamma}$ by negative integers $\{0, -1, \dots, -L\}$. The i^{th} level passage is a horizontal line above level $-i$ and below level $-i + 1$ and thus level passages are indexed by $\{1, \dots, L\}$. An edge between vertices of the same level is called *horizontal* and *vertical* otherwise. A level graph without horizontal edges is called *non-horizontal*. The *codimension* of a level graph is the number of levels minus one, plus the number of horizontal nodes.

We denote LG the set of isomorphism classes of all level graphs and LG_k the subset of codimension k level graphs. We also denote $\text{BIC} \subseteq \text{LG}_1$ the set of isomorphism classes of non-horizontal two level-graphs. For $\bar{\Gamma} \in \text{BIC}$ we also define the factor $\ell_{\bar{\Gamma}}$ to be the least common multiple of all prongs.

3.3. The boundary of $\bar{\mathcal{P}}(\mu)$

The boundary of $\bar{\mathcal{P}}(\mu)$ is stratified by level graphs. The generic point of a boundary stratum $D_{\bar{\Gamma}}$ corresponding to a level graph $\bar{\Gamma}$ consists of *twisted differentials* (X, Q, ω) which are compatible with $\bar{\Gamma}$. The codimension of $D_{\bar{\Gamma}}$ equals the codimension of $\bar{\Gamma}$, i.e. the number of level of $\bar{\Gamma}$ minus one, plus the number of horizontal edges. A feature of $\bar{\mathcal{P}}(\mu)$, first noticed in [10], is that the non-horizontal boundary $\bigcup_{\bar{\Gamma} \in \text{BIC}} D_{\bar{\Gamma}}$ is a *simple* normal crossing divisor.

3.4. Tautological rings

CONVENTION 3.5. — *All Chow rings in this paper are considered with rational coefficients.*

The tautological ring $R^*(\bar{\mathcal{M}}_{g,n})$ is the smallest subring of the Chow ring which is closed under forgetful maps and gluing maps. A set of additive generators for $R^*(\bar{\mathcal{M}}_{g,n})$ consists of decorated strata classes, which can be described as follows. Given a stable graph Γ set $\bar{\mathcal{M}}_{\Gamma} = \prod \mathcal{M}_{g_v, n_v}$ where g_v and n_v are the genus and the number of half-legs adjacent to v , respectively. For every polynomial α in ψ -classes in $\text{CH}^*(\bar{\mathcal{M}}_{\Gamma})$ the decorated strata class $[\Gamma, \alpha]$ is defined by

$$[\Gamma, \alpha] := \zeta_{\Gamma, *}\alpha,$$

where $\zeta_{\Gamma} : \bar{\mathcal{M}}_{\Gamma} \rightarrow \bar{\mathcal{M}}_{g,n}$ is the clutching morphism.

The tautological ring of the moduli space of multi-scale differentials $\bar{\mathcal{P}}(\mu)$ was first introduced in [10] and has a similar description. Given a level graph $\bar{\Gamma}$ we have induced partitions μ_v for every vertex $v \in V(\bar{\Gamma})$. Additionally there are induced residue conditions $\mathfrak{R}_i$ for every level coming from the matching residue condition at horizontal nodes and the global residue condition. We can then form the disconnected stratum

$$\mathcal{H}(\mu_{[i]}) := \prod_{\ell(v)=i} \mathcal{H}(\mu_v)$$

and the *generalized stratum* is the subspace $\mathcal{H}(\mu_{[i]})^{\mathfrak{R}_i} \subseteq \mathcal{H}(\mu_{[i]})$ satisfying all residue conditions in $\mathfrak{R}_i$. The projectivized disconnected stratum $\mathcal{P}(\mu_{[i]})^{\mathfrak{R}_i}$ is the quotient by the diagonal $\mathbb{C}^*$ -action. As remarked in [10, Proposition 4.2], the construction for multi-scale differentials can

also be applied to generalized strata to obtain a compactification $B_{\bar{\Gamma}}^{[i]} := \mathbb{P} \Xi \overline{\mathcal{M}}_{g[i], n[i]}(\mu_{[i]})$ of $\mathcal{P}(\mu_{[i]})^{\mathfrak{M}_i}$. We then set

$$B_{\bar{\Gamma}} = \prod_{i=-L}^0 B_{\bar{\Gamma}}^{[i]}.$$

The product $B_{\bar{\Gamma}}$ comes with a forgetful map $f_{\bar{\Gamma}} : B_{\bar{\Gamma}} \rightarrow \overline{\mathcal{M}}_{\Gamma}$, where Γ is the underlying stable graph of $\bar{\Gamma}$. In [10] the authors construct a commutative diagram

$$(3.1) \quad \begin{array}{ccccc} & D_{\bar{\Gamma}}^s & & & \\ p_{\bar{\Gamma}} \swarrow & & \searrow c_{\bar{\Gamma}} & & \\ B_{\bar{\Gamma}} & & D_{\bar{\Gamma}} & \longrightarrow & \overline{\mathcal{P}}(\mu) \\ f_{\bar{\Gamma}} \downarrow & & & & \downarrow \rho \\ \overline{\mathcal{M}}_{\Gamma} & \xrightarrow{\zeta_{\Gamma}} & & & \overline{\mathcal{M}}_{g,n} \end{array}$$

where $c_{\bar{\Gamma}}$ and $p_{\bar{\Gamma}}$ are finite maps and $D_{\bar{\Gamma}}^s$ is Cohen–Macaulay. The tautological ring $R^*(\overline{\mathcal{P}}(\mu))$ is the smallest subring of $\mathrm{CH}^*(\overline{\mathcal{P}}(\mu))$ containing the fundamental classes of all generalized strata and stable under clutching maps $c_{\bar{\Gamma},*} p_{\bar{\Gamma}}^*$ for all non-horizontal level graphs. It was shown in [10, Theorem 1.5] that an additive generating set of $R^*(B)$ is given by

$$c_{\bar{\Gamma},*} p_{\bar{\Gamma}}^* f_{\bar{\Gamma}}^* \alpha,$$

where $\alpha \in \mathrm{CH}^*(\overline{\mathcal{M}}_{\bar{\Gamma}})$ is a polynomial in ψ -classes and $\bar{\Gamma}$ runs over all non-horizontal level graphs.

Remark 3.6. — In [10, Section 8] various tautological rings are constructed. In this paper we only consider the non-horizontal version, the smallest of the subrings in (loc. cit.).

PROPOSITION 3.7. — *The forgetful map $\rho : \overline{\mathcal{P}}(\mu) \rightarrow \overline{\mathcal{M}}_{g,n}$ preserves tautological rings, i.e.,*

$$\rho_* R^*(\overline{\mathcal{P}}(\mu)) \subseteq R^*(\overline{\mathcal{M}}_{g,n}).$$

Proof. — Since $R^*(\overline{\mathcal{P}}(\mu))$ is generated by classes of the form $c_{\bar{\Gamma},*} p_{\bar{\Gamma}}^* f_{\bar{\Gamma}}^* \alpha$, it suffices to show that

$$\rho_* c_{\bar{\Gamma},*} p_{\bar{\Gamma}}^* f_{\bar{\Gamma}}^* \alpha$$

is tautological. This follows from the commutativity of didagram (3.1) and the fact that strata classes are tautological [2]. $\square$

We stress that the inclusion is strict in general for dimension reasons.

3.5. Profiles and boundary strata

The boundary of a generalized stratum B is naturally stratified by level graphs, but we will organize the boundary strata somewhat differently to have a better behaved intersection theory. The set of non-horizontal two level graphs admits a partial ordering, see [10, Section 5], which we now recall. Given any level graph $\bar{\Lambda}$ we define $\delta_i(\bar{\Lambda})$ to be the 2-level graph, which is obtained by smoothing all edges, except the ones crossing the i^{th} level passage. Let $\bar{\Gamma}, \bar{\Gamma}' \in \text{BIC}$ be two different non-horizontal 2-level graphs. We say that $\bar{\Gamma} \succ \bar{\Gamma}'$ if there exists a non-horizontal 3-level graph $\bar{\Lambda}$ with $\delta_1(\bar{\Lambda}) = \bar{\Gamma}, \delta_2(\bar{\Lambda}) = \bar{\Gamma}'$. This is well defined by [10, Proposition 5.1].

A profile $P = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_k)$ is an ordered tuple of non-horizontal two-level graphs ordered such that $\bar{\Gamma}_i \succ \bar{\Gamma}_{i+1}$. We call $L(P) := k + 1$ the number of levels of P and define the factor $\ell_P := \prod_{i=1}^k \ell_{\bar{\Gamma}_i}$, where we recall that $\ell_{\bar{\Gamma}_k}$ is the product of all prongs of $\bar{\Gamma}_k$. The collection of all profiles of B is denoted by $\Sigma(B)$. Every $(k + 1)$ -level-graph $\bar{\Lambda}$ has an associated profile $\delta(\bar{\Lambda}) = (\delta_1(\bar{\Lambda}), \dots, \delta_k(\bar{\Lambda}))$.

For a non-horizontal two level graph $\bar{\Gamma}$ we usually write $\bar{\Gamma}$ instead of $(\bar{\Gamma})$. The empty profile $\star$ corresponds to the unique level graph with one vertex.

The boundary stratum associated to a profile is

$$D_P = \bigcup_{\bar{\Lambda} : \delta(\bar{\Lambda}) = P} D_{\bar{\Lambda}}.$$

The codimension of D_P is $L(P) - 1$. It was shown in [10] that for a profile $P = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_k)$ we have $D_P = \bigcap_{i=1}^k D_{\bar{\Gamma}_i}$. In particular D_P is a complete intersection of codimension i and $[D_P] = \prod_{i=1}^k [D_{\bar{\Gamma}_i}] \in R^*(B)$.

Given two profiles P_1, P_2 we say that P_1 and P_2 are comparable if the intersection $D_{P_1} \cap D_{P_2}$ is non-empty and for two comparable profiles we define the sum $P_1 + P_2$ to be the profile of some (or equivalently any) level graph in $D_{P_1} \cap D_{P_2}$. We say $P_1 = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_k)$ and $P_2 = (\Delta_1, \dots, \Delta_l)$ are complementary if the sets $\{\bar{\Gamma}_1, \dots, \bar{\Gamma}_k\}$ and $\{\Delta_1, \dots, \Delta_l\}$ are disjoint and P_1, P_2 are comparable. In this case we have

$$\ell_{P_1 + P_2} = \ell_{P_1} \ell_{P_2}.$$

If P_1, P_2 are complementary we write $P_1 \oplus P_2 = P_1 + P_2$ and say that $P_1 \oplus P_2$ is a direct sum. For example if $P = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_k)$ then $P = \bigoplus_{i=1}^k \bar{\Gamma}_i$. If $Q = P_1 \oplus P_2$ is a direct sum, then the intersection

$$D_Q = D_{P_1} \cap D_{P_2}$$

is a transversal intersection and in particular $[D_Q] = [D_{P_1}] \cdot [D_{P_2}] \in \text{divR}^*(B)$.

We say P degenerates to P' or P' is a *degeneration* of P if $D_{P'} \subseteq D_P$. This is equivalent to being able to write $P' = P \oplus R$ for some profile R . We also say that P is an *undegeneration* of P' and we write $P \rightsquigarrow P'$. For a level graph $\bar{\Lambda}$ we say that $\bar{\Lambda}$ is a degeneration of a profile P' if the profile $P(\bar{\Lambda})$ of $\bar{\Lambda}$ is a degeneration of P' .

Similarly, we say that P' is a degeneration of P by splitting level i , if we have $P = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_n)$ and $P' = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_{-i}, \Delta_1, \dots, \Delta_k, \bar{\Gamma}_{-i+1}, \dots, \bar{\Gamma}_n)$. We let $\Sigma(P, i)$ be the collection of all profiles, which are degenerations of P by splitting level i .

3.6. The divisorial tautological ring

Let B be a generalized boundary stratum. Instead of the whole tautological ring we will mostly work in the *divisorial tautological ring* $\text{divR}^*(B)$, the subring of the tautological ring $R^*(B)$ generated by codimension 1 classes in $R^*(B)$. For us the most important divisorial classes are ψ -classes, boundary divisors $D_{\bar{\Gamma}}$ where $\bar{\Gamma} \in \text{BIC}$ and the first Chern class $\xi = c_1(\mathcal{O}_B(-1))$ of the tautological line bundle $\mathcal{O}_B(-1)$ whose fibers is spanned by the differential ω .

PROPOSITION 3.8. — *If the generalized boundary stratum B contains no simple poles, then a multiplicative generating set for $\text{divR}^*(B)$ is given by*

$$\xi \text{ and } D_{\bar{\Gamma}} \text{ for } \bar{\Gamma} \in \text{BIC}.$$

On the other hand, if B contains simple poles, then a multiplicative generating set for $\text{divR}^(B)$ is given by $D_{\bar{\Gamma}}$ for $\bar{\Gamma} \in \text{BIC}$ and ψ -classes ψ_j where j runs over all marked simple poles.*

Proof. — By [10, Theorem 1.5] $\text{divR}^*(B)$ is generated by $D_{\bar{\Gamma}}$ for $\bar{\Gamma} \in \text{BIC}$ and ψ_j where j runs over all marked points. It then follows from [10, Proposition 8.2] that ψ_j can be expressed in term of boundary divisors and ξ , as long as the j^{th} marked point is not a simple pole. The same relation also shows that ξ is a linear combination of boundary divisors if B contains simple poles. $\square$

In the sequel we will mostly be interested in the case of exact differentials, in which case simple poles cannot appear.

For a boundary stratum D_P we let $\text{divR}^*(D_P)$ be the pullback of $\text{divR}^*(B)$ along the inclusion $D_P \hookrightarrow B$. On D_P there is a collection of

line bundles $\mathcal{O}_{D_P}^{[i]}(-1)$, which are generated by the differentials ω_i on each level, see also [10, Section 4.3]. We write

$$\xi_P^{[i]} := c_1\left(\mathcal{O}_{D_P}^{[i]}(-1)\right).$$

The line bundle $\mathcal{O}_{D_P}^{[0]}(-1)$ is the restriction of $\mathcal{O}_B(-1)$ and thus $\xi_P^{[0]}$ is contained in $\text{divR}^*(D_P)$. We will need later that in fact all Chern classes $\xi_P^{[i]}$ are pullbacks of tautological divisorial classes on B .

PROPOSITION 3.9. — *Let P be a profile. Then*

$$\xi_P^{[i]} \in \text{divR}^1(D_P)$$

for $i = -L(P), \dots, 0$.

Proof. — Fix $i < 0$ and let $\bar{\Gamma} := \delta_{-i}(P)$. Then by [10, Lemma 7.6] we have

$$\xi_P^{[i]} = j_{P, \bar{\Gamma}}^* \left(\xi_{\bar{\Gamma}}^{[-1]} \right),$$

where $j_{P, \bar{\Gamma}} : D_P \hookrightarrow D_{\bar{\Gamma}}$ denotes the inclusion. It thus suffices to show the claim for a two-level graph $\bar{\Gamma} \in \text{BIC}$. In this case it follows from [10, Lemma 7.1] that

$$\ell_{\bar{\Gamma}} j_{\bar{\Gamma}}^* [D_{\bar{\Gamma}}] = \ell_{\bar{\Gamma}} c_1(\mathcal{N}_{D_{\bar{\Gamma}}/B}) = -\xi_{\bar{\Gamma}}^{[0]} + \xi_{\bar{\Gamma}}^{[-1]} - \sum_{\bar{\Gamma}'} \ell_{\bar{\Gamma}'} j_{\bar{\Gamma}'}^* [D_{\bar{\Gamma}'}]$$

where the sum runs over all two level graphs with $\bar{\Gamma}' \succ \bar{\Gamma}$. □

3.7. Generalized boundary strata

The boundary strata D_P are smooth substacks but we will be forced to consider infinitesimal thickenings of the boundary strata D_P . Let $P = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_m)$ and $Q = (\bar{\Gamma}_{m+1}, \dots, \bar{\Gamma}_n)$ be complementary profiles. Local equations for the boundary stratum D_{P+Q} are given by

$$t_{\bar{\Gamma}_1} = \dots = t_{\bar{\Gamma}_n} = 0,$$

where $t_{\bar{\Gamma}_i}$ is a transversal coordinate for the divisor $D_{\bar{\Gamma}_i}$.

DEFINITION 3.10. — *We define the generalized boundary stratum $D_{P,Q}$ to be the thickening of D_{P+Q} , which is locally defined by the ideal*

$$\left(t_{\bar{\Gamma}_1}, \dots, t_{\bar{\Gamma}_m}, t_{\bar{\Gamma}_{m+1}}^{\ell_{\bar{\Gamma}_{m+1}}}, \dots, t_{\bar{\Gamma}_n}^{\ell_{\bar{\Gamma}_n}} \right).$$

In particular $D_{P,Q}$ is a complete intersection and

$$[D_{P,Q}] = \ell(Q) \prod_{i=1}^n [D_{\bar{\Gamma}_i}] \in \text{divR}^*(B).$$

3.8. Comparing the cohomology at the boundary

Our goal is extend the computation of $[\text{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})]$ to the Chow ring of $\overline{\mathcal{P}}(\mu)$. We will do this by extending both the vector bundle $\mathcal{V}$ and the evaluation section. As preparation we need a way to compare the relative homology at a twisted differential in some boundary stratum $D_P \subseteq \overline{\mathcal{P}}(\mu)$ to the homology at a nearby smooth Riemann surface in $\mathcal{P}(\mu)$. This has been carried out in general in [4], here we only recall the basic notions we need, with the notation adapted to our situation.

Let $x = (X, \omega)$ be a twisted differential compatible with a level graph $\overline{\Lambda}$, i.e., x lies in the interior $D_{\overline{\Lambda}}^{\circ}$ and denote by $\tilde{X}$ the normalization of the stable curve X together with a marking of all preimages of nodes. We can then filter all connected components of $\tilde{X}$ as well as the marked zeros and half-legs by level. We let $\tilde{X}_{[i]}$ be the union of connected components corresponding to level i and $\tilde{Z}_{[i]}, \tilde{P}_{[i]}$ the marked points of level i , sorted into zeros and poles. Thus for example $\tilde{Z}_{[0]}$ consists of all marked zeros on the top level together with the preimages of all nodes on the top level which do not correspond to simple poles.

We define $H_1^{[i]}(X, \omega) := H_1(\tilde{X}_i \setminus \tilde{P}_{[i]}, \tilde{Z}_{[i]}; \mathbb{C})$. From now on all cohomology groups are considered with complex coefficients. Note that the dimension $\dim H_1^{[i]}(X, \omega)$ only depends on the level graph $\overline{\Lambda}$.

Let $x' = (\Sigma, \omega') \in \mathcal{P}(\mu)$ be a nearby flat surface with a collection of simple closed curves $\{\sigma_e, e \in E(\overline{\Lambda})\}$, called *vanishing cycles*, such that X is obtained by contracting the vanishing cycles. Away from the nodes we can realize X as a subsurface of Σ and thus consider $P = P(\omega), Z = Z(\omega)$ as a subset of Σ . Vanishing cycles can be sorted into *horizontal* and *vertical* vanishing cycles depending on whether the corresponding node is horizontal or vertical.

In order to relate periods along a degeneration with periods of the limiting twisted differential, we need to compare suitable relative homology groups of Σ and X . There are natural restriction maps from the relative homology on Σ to the relative homology of X , with special care taken at homology classes that intersect horizontal vanishing cycles. We will only work with complex coefficients but all constructions in this section can also be carried out with integral coefficients.

We define the *non-horizontal cycles at level 0* to be

$$W_{[0]} := \left\{ \gamma \in H_1(\Sigma \setminus P, Z; \mathbb{C}) \left| \begin{array}{l} \langle \gamma, [\sigma_e] \rangle = 0 \\ \text{for any horizontal node } e \text{ of level 0} \end{array} \right. \right\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the intersection pairing

$$H_1(\Sigma \setminus P, Z; \mathbb{C}) \times H_1(\Sigma \setminus Z, P; \mathbb{C}) \longrightarrow \mathbb{C}$$

and $[\sigma_e] \in H_1(\Sigma \setminus Z, P; \mathbb{C})$ denotes the corresponding cohomology class.

On $W_{[0]}$ every cohomology class can be represented by a sum of simple closed curves disjoint from all horizontal vanishing cycles in top level, which we can then restrict a cohomology class in $H_1^{[0]}(X, \omega)$. This yields the specialization morphism

$$\text{sp}_{[0]} : W_{[0]} \longrightarrow H_1^{[0]}(X, \omega).$$

The kernel $\ker \text{sp}_{[0]}$ consists of homology class that can be represented by classes in $X_{[-1]}$. We set

$$W_{[-1]} := \left\{ \gamma \in H_1(\Sigma \setminus P, Z; \mathbb{C}) \left| \begin{array}{l} \langle \gamma, [\sigma_e] \rangle = 0 \\ \text{for any horizontal node } e \text{ of level } -1 \end{array} \right. \right\}.$$

On $W_{[-1]}$ we can now restrict a homology class to $X_{[-1]}$ and thus obtain the specialization morphism in level -1

$$\text{sp}_{[-1]} : W_{[-1]} \longrightarrow H_1^{[-1]}(X, \omega).$$

We thus have a three step filtration

$$\{0\} \subseteq W_{[-1]} \subseteq \ker \text{sp}_{[0]} \subseteq W_{[0]} \subseteq H_1(\Sigma \setminus P, Z)$$

with graded pieces

$$\begin{aligned} W_{[0]} / \ker \text{sp}_{[0]} &\simeq H_1^{[0]}(X, \omega), \\ W_{[-1]} &\simeq H_1^{[-1]}(X, \omega). \end{aligned}$$

DEFINITION 3.11. — Suppose $x = (X, \omega)$ is a twisted differential compatible with a level graph $\bar{\Lambda}$. We define

$$\begin{aligned} \mathcal{V}_{\bar{\Lambda}, x}^{[0]} &:= \text{sp}_{[0]}(\mathcal{V}_{x'} \cap W_{[0]}), \\ \mathcal{V}_{\bar{\Lambda}, x}^{[-1]} &:= \text{sp}_{[-1]}(\mathcal{V}_{x'} \cap W_{[-1]}), \end{aligned}$$

where $\mathcal{V}_{x'}$ is the fiber of $\mathcal{V}$ at x' .

The spaces $\mathcal{V}_{\bar{\Lambda}, x}^{[0]}$ and $\mathcal{V}_{\bar{\Lambda}, x}^{[-1]}$ can be interpreted as limits of homology classes as Σ degenerates to X .

So far we have only treated the case of a two level graph, but everything can be generalized to an arbitrary profile P . In that case one has two filtrations

$$0 \subseteq W_{[-L]} \subseteq \ker \text{sp}_{[-L+1]} \subseteq \cdots \subseteq W_{[-1]} \subseteq \ker \text{sp}_{[0]} \subseteq W_{[0]} \subseteq H_1(\Sigma \setminus P, Z),$$

where $\mathrm{sp}_{[i]}$ now restricts a cohomology class in $W_{[i]}$ to the i^{th} level subsurface. We define

$$\mathcal{V}_{P,x}^{[i]} := \mathrm{sp}_{[i]}(\mathcal{V}_{x'} \cap W_{[i]}).$$

On the interior D_P° of a boundary stratum D_P , the vector spaces $\mathcal{V}_{P,x}^{[i]}$ glue together to a flat vector bundle $\mathcal{V}_P^{[i]}$.

Remark 3.12. — The boundary stratum D_P can have several components, so in theory $\mathcal{V}_P^{[i]}$ could have different ranks on each component. To see that the rank is indeed independent of the component, observe first that for any level graph $\bar{\Lambda}$ the rank of $\mathcal{V}_{\bar{\Lambda}}^{[0]}$ can be computed from $\bar{\Gamma}_1 = \delta_1(\bar{\Lambda})$, since we only need to know which paths are in level 0 and which paths are in lower level. In the next step we consider $\bar{\Gamma}_2 := \delta_2(\bar{\Lambda})$. The paths on top level of $\bar{\Gamma}_2$ consists of all paths in $\mathcal{V}_{\bar{\Lambda}}^{[0]} \oplus \mathcal{V}_{\bar{\Lambda}}^{[-1]}$, thus the dimension of $\mathcal{V}_{\bar{\Lambda}}^{[-1]}$ only depends on $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. By induction it follows that the dimension of $\mathcal{V}_{\bar{\Lambda}}^{[i]}$ only depends on the profile $\delta(\bar{\Lambda}) = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_{L(\bar{\Lambda}-1)})$.

4. The class of strata of exact differentials

For the rest of this section we fix a globally defined local system $\mathcal{V} \subseteq \mathcal{H}$ on a projectivized stratum $\mathcal{P}(\mu)$. We are now setting up our computation of the closure $\overline{\mathrm{Ann}(\mathcal{V})}$ in the Chow ring of $\bar{\mathcal{P}}(\mu)$. The goal is the following, which has Theorem 1.1 as a corollary.

THEOREM 4.1. — *Let $\mathcal{V}$ be a globally defined local system on a generalized boundary stratum B . Then the class of the closure $[\overline{\mathrm{Ann}_B(\mathcal{V})}]$ is contained in the divisorial tautological ring $\mathrm{divR}^*(\bar{\mathcal{P}}(\mu))$. Furthermore, the class can be computed explicitly.*

The outline for this section is as follows. We will only work with a stratum $\bar{\mathcal{P}}(\mu)$ instead of a generalized boundary stratum, but the argument goes through verbatim in the more general case. We start by extending both the vector bundle $\mathcal{V}$ as well as the evaluation section to $\bar{\mathcal{P}}(\mu)$ by passing to the Deligne extension $\bar{\mathcal{V}}$ as carried out in [10, Section 6.3]. The zero locus of the extended section contains $\overline{\mathrm{Ann}_{\bar{\mathcal{P}}(\mu)}(\mathcal{V})}$ as well as several extraneous component of varying dimensions. We determine the extraneous components explicitly as zero loci of sections of vector bundles on a generalized boundary stratum; similar to the original bundle $\mathcal{V}$. This sets up the beginning of a recursive computation.

4.1. The Deligne extension of $\mathcal{V}$

The local system $\mathcal{V}$ has unipotent monodromy along the normal crossing divisor $\overline{\mathcal{P}}(\mu) \setminus \mathcal{P}(\mu)$. In this situation the Deligne extension $\overline{\mathcal{V}}$ is an extension of $\mathcal{V}$ to $\overline{\mathcal{P}}(\mu)$ to a vector bundle on $\overline{\mathcal{P}}(\mu)$ admitting a connection with regular singularities along the boundary. We will need a local description of $\overline{\mathcal{V}}$. By choosing a local analytic coordinate system for $\overline{\mathcal{P}}(\mu)$ near the boundary, the complement $\overline{\mathcal{P}}(\mu) \setminus \mathcal{P}(\mu)$ is isomorphic to $(\Delta^*)^k \times \Delta^l$. Let T_i be the monodromy of $\mathcal{V}$ corresponding to the i^{th} generator of $(\Delta^*)^k$. We have $(I - T_i)^2 = 0$ and thus $N_i := -\log T_i = I - T_i$. We choose a base point $x' \in (\Delta^*)^k \times \Delta^l$ and set $V = \mathcal{V}_{x'}$. Any cycle $v \in V$ can be extended to a flat section in a small neighborhood of x . In order to extend v to a section on all of $(\Delta^*)^k \times \Delta^l$ we need to make v monodromy invariant. Consider the universal cover

$$\begin{aligned} \pi : \mathbb{H}^k \times \Delta^l &\longrightarrow (\Delta^*)^k \times \Delta^l, \\ (z_1, \dots, z_k, q_1, \dots, q_l) &\longmapsto (e^{2\pi i z_1}, \dots, e^{2\pi i z_k}, q_1, \dots, q_l). \end{aligned}$$

The section $e^{2\pi i \sum_{i=1}^k z_i N_i} v$ of $\pi^* \mathcal{V}$ is monodromy invariant and thus descends to a section $\bar{v}$ for $\mathcal{V}_{|(\Delta^*)^k \times \Delta^l}$. The Deligne extension $\overline{\mathcal{V}}$ is the unique extension of $\mathcal{V}$ that is locally over $\Delta^k \times \Delta^l$ trivialized by the sections $\bar{v}$ for $v \in \mathcal{V}_{x'}$. We call $\bar{v}$ the Deligne extension of the cycle v and stress that it depends on the choice of local coordinates for $\overline{\mathcal{P}}(\mu)$ near the boundary.

The following was proven in [10] for the local system $\mathcal{H}$ and then follows for any sub-local system $\mathcal{V} \subseteq \mathcal{H}$ by restriction.

PROPOSITION 4.2. — *The evaluation morphism $\text{ev}_{\mathcal{V}} : \mathcal{V} \otimes \mathcal{O}_{\mathcal{P}(\mu)}(-1) \rightarrow \mathcal{O}_{\mathcal{P}(\mu)}$ extends to a morphism*

$$\overline{\text{ev}}_{\mathcal{V}} : \overline{\mathcal{V}} \otimes \mathcal{O}_{\overline{\mathcal{P}}(\mu)}(-1) \longrightarrow \mathcal{O}_{\overline{\mathcal{P}}(\mu)}.$$

Suppose $x = (X, \omega) \in D_{\mathbb{P}}^{\circ}$, $x' \in \mathcal{P}(\mu)$ a nearby point and $v \in \mathcal{V}_{x'} \cap W_{[0]}$. Then

$$\overline{\text{ev}}_{\mathcal{V}}(\bar{v} \otimes \omega) = \int_{\text{sp}_{[0]}(v)} \omega.$$

In particular, $\overline{\text{ev}}_{\mathcal{V}}$ vanishes if $v \in \mathcal{V}_{x'} \cap W_{[-1]}$.

It follows that the evaluation morphism vanishes identically at x if and only if the periods over all cycles in $\mathcal{V}_{\mathbb{P}, x}^{[0]}$ vanishes.

Everything so far can be repeated for the vector bundles $\mathcal{V}_{\mathbb{P}}^{[i]}$ on $D_{\mathbb{P}}^{\circ}$. The boundary $D_{\mathbb{P}} \setminus D_{\mathbb{P}}^{\circ}$ is a normal crossing divisor and both $\mathcal{V}_{\mathbb{P}}^{[i]}$ as well as the

natural evaluation morphism

$$\mathrm{ev}_{\mathcal{P},[i]} : \mathcal{V}_{\mathcal{P}}^{[i]} \otimes \mathcal{O}_{D_{\mathcal{P}}}^{[i]}(-1) \longrightarrow \mathcal{O}_{D_{\mathcal{P}}} : \gamma \otimes \omega_i \longmapsto \int_{\gamma} \omega_i,$$

can be extended to $D_{\mathcal{P}}$ using the Deligne extension $\bar{\mathcal{V}}_{\mathcal{P}}^{[i]}$. Here we recall that ω_i is the restriction of the multi-scale differential to the i^{th} level and $\mathcal{O}_{D_{\mathcal{P}}}^{[i]}(-1)$ is the line bundle whose fiber over (X, ω) is defined by ω_i .

PROPOSITION 4.3. — *Let $\mathcal{V}$ be a globally defined local system on a stratum $\mathcal{P}(\mu)$. The Deligne extension $\bar{\mathcal{V}}_{\mathcal{P}}^{[i]}$ has vanishing Chern classes in $\mathrm{CH}^*(D_{\mathcal{P}})$.*

Proof. — We first note that the statement is immediate in rational cohomology since the Deligne extension has a connection with nilpotent curvature. To see the statement in the Chow ring we first consider the case of $B = \mathcal{P}(\mu)$ an ordinary stratum and $\mathcal{P}$ the empty profile. In this case $\mathcal{V}_{\star}^{[0]}$ is an extension of a trivial bundle by the bundle of absolute cohomology, which has trivial Chern classes by Mumfords computation [23] of the Chern classes of the Hodge bundle.

Now for the general case $D_{\mathcal{P}}$, it suffices to show that the statement for every connected component $D_{\bar{\Lambda}}$ of $D_{\mathcal{P}}$. We consider the finite covering $\widetilde{D_{\bar{\Lambda}}}$ by marking all edges. On $\widetilde{D_{\bar{\Lambda}}}$ the bundle $\mathcal{V}_{\mathcal{P}}^{[i]}$ is the pullback of the Deligne extension from $\mathcal{M}_{\bar{\Lambda}}$, which has zero Chern classes by Mumfords computation again. $\square$

Thus far we have only worked with a projectivized stratum but all the local systems $\mathcal{H}_{\mathrm{abs}}, \mathcal{R}, \mathcal{H}_{\bar{\Lambda}}$ can be defined in an analogous way on the smooth part of a generalized stratum B . For any flat vector bundle $\mathcal{V}$ on B we can define $\mathcal{V}_{\mathcal{P}}^{[i]}$ on the smooth part of a boundary stratum $D_{\mathcal{P}}$ and $\mathcal{V}_{\mathcal{P}}^{[i]}$ can be extended to all of $D_{\mathcal{P}}$ using the Deligne extensions. The computation of Proposition 4.3 for globally defined local systems extends to this more general situation and shows that $c_i(\mathcal{V}_{\mathcal{P}}^{[j]}) = 0$ for all i and j .

4.2. Coordinates near the boundary and log periods

Using the Deligne extension of $\mathcal{H}$ we can construct a local analytic (orbifold) coordinate system $\bar{\mathcal{P}}(\mu)$ near a boundary point $(X, \omega) \in D_{\bar{\Lambda}}^{\circ}$ for some level graph $\bar{\Lambda}$.

First assume that $\bar{\Lambda}$ has no horizontal nodes.

We write $\omega = (\omega_i)_{i \in 0, \dots, -L(\bar{\Gamma})}$. For every level i we choose a basis $a_i = (a_{i,1}, \dots, a_{i,l_i})$ for the homology $\mathcal{H}_{\bar{\Lambda}}^{[i]}$ of level i . For each cycle $a_{i,k}$ on X we let $\bar{a}_{i,k}$ be a cycle on the Deligne extension $\bar{\mathcal{H}}$ that specializes to $a_{i,k}$. We also need the rescaling factor of level i

$$T_{[i]} := \prod_{j=-L(\bar{\Lambda})}^{-1} t_j^{\ell_j},$$

where $\ell_i = \ell_{\delta_i(\bar{\Lambda})}$ and t_i is a coordinate transversal to the divisor $D_{\delta_i(\bar{\Lambda})}$. In the literature on multi-scale differentials $T_{[i]}$ is sometimes denoted as $t_{[i]}$. We stress that $T_{[i]}$ does depend on the level graph $\bar{\Lambda}$ we are working with. In order to not overload the notation, we usually omit the level graph $\bar{\Lambda}$ as it should be clear from the context.

We then define the *log periods of level i* $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,l_i})$ by the formula

$$\alpha_{i,k} = \frac{1}{T_{[i]}} \int_{\bar{a}_{i,k}} \eta,$$

where η is the universal family of differentials over the stratum $\mathcal{P}(\mu)$.

It was shown in [4] that log-periods extend holomorphically to the boundary and that

$$(t_1, \dots, t_{L(\bar{\Gamma})}, \alpha_0, \dots, \alpha_{L(\bar{\Gamma})})$$

is a local coordinate system near $(X, \omega) \in \bar{\mathcal{P}}(\mu)$ and that furthermore on the boundary

$$\alpha_{i,k}(X, \omega) = \int_{a_{i,k}} \omega,$$

where the integral is over the *truncated cycle* $a_{i,k}$ obtained by cutting of all parts of $a_{i,k}$ supported on lower levels. Since $a_{i,k}$ is supported on the i^{th} level, the integral only depends on the differential ω_i at level i . By abuse of notation we will denote the truncated cycle by the same symbol, since it is clear from the context whether we integrate over a boundary point or a point in $\mathcal{P}(\mu)$.

Let P be some profile and $\bar{\Lambda}$ a level graph, which is a degeneration of P . We now describe the evaluation morphism

$$\bar{\text{ev}}_P^{[i]} : \bar{\mathcal{V}}_P^{[i]} \otimes \mathcal{O}_{D_P}^{[i]}(-1) \longrightarrow \mathcal{O}_{D_P}$$

near a point in $D_{\bar{\Lambda}}^-$ in terms of log periods. First we need a description of the universal family η_i for differentials on level i of D_P . If η_i is a family of differential on level i of D_P that degenerates to a multi-scale differential

on $D_{\bar{\Lambda}}$, then on each level $j - k \leq l \leq j$ that undegenerates to i , the differential η_i grows like $\top_l \omega_l$, where ω_l is the universal differential on level l of $D_{\bar{\Lambda}}$ and $\top_l := \frac{T_{[l]}}{T_{[j]}} = \prod_{s=l}^{j-1} t_s^{\ell_s}$.

The description is analogous to the discussion in [10, Section 6.2], the main difference is that we are working with the line bundle $\mathcal{O}_{D_P}^{[i]}(-1)$, whose fiber over a boundary point (X, ω) is generated by the differential ω_i at level i . Assume that on $\bar{\Lambda}$ the i^{th} level has split into levels $j, \dots, j - k$ for some j, k and we choose a basis of cycles a_l for $\mathcal{V}_{\bar{\Lambda}}^{[l]}$, $j \geq l \geq j - k$. In order to compute the evaluation morphism $\overline{\text{ev}}_P^{[i]}$ we need to work with a frame for $\bar{\mathcal{V}}_P^{[i]}$. On the other hand to compute log periods we use a frame for $\bar{\mathcal{V}}$. For a cycle $a_{l,m}$ of level l , the different extensions to cycles on the Deligne extensions $\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}_P$ only differ in levels lower than l and thus $(X, \omega) \in D_{\bar{\Lambda}}^{\circ}$ has the same periods over both possible extensions. We use η for a differential on D_P° and ω for a limiting multi-scale differential on $D_{\bar{\Lambda}}$.

In a local frame near $D_{\bar{\Lambda}}$ the evaluation morphism $\overline{\text{ev}}_P^{[i]}$ is given by

$$(4.1) \quad \bar{a}_{l,m} \otimes \eta_i \longmapsto \top_l \alpha_{l,m}(X, \eta),$$

where $\top_l := \frac{T_{[l]}}{T_{[j]}} = \prod_{s=l}^{j-1} t_s^{\ell_s}$. In particular, on $D_{\bar{\Lambda}}$ the evaluation morphism vanishes on all levels below j and for $(X, \omega) \in D_{\bar{\Lambda}}$ we have

$$\overline{\text{ev}}_P^{[i]}(\bar{a}_{j,k} \otimes \omega_j) = \int_{a_{j,k}} \omega_j = \int_{a_{j,k}} \omega.$$

Extension to horizontal nodes

The evaluation morphism can also be extended if the level graph $\bar{\Lambda}$ has horizontal nodes. For details we refer the reader to [10, Section 6.2]. It will not be important in the sequel, since we are exclusively interested in the vanishing of the evaluation morphism. Since the period of the vanishing cycle around a horizontal node has always a non-zero period (corresponding to the residue at the simple pole of the limiting multi-scale differential), the evaluation morphism never vanishes on a multi-scale differential with a horizontal node in level i .

4.3. The zero locus of the evaluation section

Our next goal is to describe the zero locus of the evaluation morphism $\overline{\text{ev}}_{P,[i]} : \mathcal{V}_P^{[i]} \otimes \mathcal{O}_P^{[i]}(-1) \rightarrow \mathcal{O}$. To lighten the notation we fix a projectivized stratum $B = \mathcal{P}(\mu)$ and a globally defined flat vector bundle $\mathcal{V}$.

We first need to introduce a few more spaces and infinitesimal thickenings, which will appear in the computation. We define

$$A_{P,Q}^{[i]} := \overline{\left\{ (X, \omega) \in D_{P,Q}^\circ \mid \omega \in \text{Ann}\left(\mathcal{V}_{P+Q}^{[i]}\right) \right\}},$$

where $D_{P,Q}^\circ$ is the open set corresponding to differentials with profile $P+Q$. Here we implicitly pushed $\mathcal{V}_{P+Q}^{[i]}$ forward under the inclusion $D_{P+Q} \subseteq D_{P,Q}$, where we recall the generalized boundary stratum $D_{P,Q}$ from Section 3.7.

If Q is the empty profile, we simply write $A_P^{[i]}$. Rephrased in this notation our main goal is to compute $A_\star^{[0]}$ but it turns out that in our recursive computation we will need to compute the classes of all subspaces $A_{P,Q}^{[i]}$.

Our next goal is to describe the local equations for $A_{P,Q}^{[i]}$ inside $D_{P,Q}$. For this we will employ the theory of closures of linear subvarieties as developed in [5] as follows. The subvariety $A_{P+Q}^{[i]}$ is a linear subvariety of D_{P+Q} in the sense of [5], since it is given by the vanishing of absolute and relative periods. Thus [5, Section 4] allows to describe the equations of $A_{P+Q}^{[i]}$ near a boundary point explicitly. The results do not apply directly to $A_{P,Q}^{[i]}$ as a substack of the non-reduced stack $D_{P,Q}$. Since the methods of [5] are purely local and we can use log periods to describe local analytic equations on $A_{P,Q}^{[i]}$, the description of the boundary is still valid.

In our situation the equations are particularly simple, since if a cycle in $\mathcal{V}_P^{[i]}$ crosses a horizontal node, the corresponding vanishing cycle is also contained in $\mathcal{V}_P^{[i]}$. Thus if on the open part $D_{P,Q}^\circ$ the defining equations for $A_{P,Q}^{[i]}$ are given by the vanishing of a collection of periods $a_1, \dots, a_n$, the local equations near the boundary are given by the corresponding log period $\alpha_1, \dots, \alpha_n$.

Let P, Q two complementary profiles and consider $A_{P,Q}^{[i]} \subseteq D_{P,Q}$. Fix $(X, \omega) \in D_{\bar{\Lambda}}$ for some level graph $\bar{\Lambda}$, which is a degeneration of $P+Q$ and let $I \subseteq \{0, \dots, -L(\bar{\Lambda})\}$ be the collection of all levels of $\bar{\Lambda}$ that undegenerate to level i , when undegenerating $\bar{\Lambda}$ to the profile $P+Q$. We choose $\alpha = (\alpha_1, \dots, \alpha_n)$ to be the collection of all log periods over a basis of $\bigoplus_{j \in I} \mathcal{V}_{\bar{\Lambda}}^{[j]}$. Then local equations for $A_{P,Q}^{[i]} \subseteq D_{P,Q}$ near (X, ω) are given by

$$(4.2) \quad \alpha_1 = \dots = \alpha_n = 0.$$

Since the log periods α are part of a local coordinate system, $A_{P,Q}^{[i]}$ is regularly embedded in $D_{P,Q}$.

Given a subset $J \subseteq \{0, \dots, -L(P)\}$. We also define

$$A_{P,Q}^{[J]} = \bigcap_{i \in J} A_{P,Q}^{[i]}.$$

Note the intersection is algebraically transverse inside $D_{P,Q}$ (see Definition A.7 for the definition of algebraically transverse).

If Q is non-empty, then $A_{P,Q}^{[i]}$ is non-reduced and $A_{P+Q}^{[i]}$ is the induced reduced substack. Furthermore, $A_{P,Q}^{[i]}$ has geometric multiplicity $\ell(Q)$ along $A_{P+Q}^{[i]}$ and $A_{P,Q}^{[i]}$ is regularly embedded in $D_{P,Q}$ with normal bundle $\mathcal{N}_{A_{P,Q}^{[i]}/D_{P,Q}}$. We thus have

$$\left[A_{P,Q}^{[i]} \right] = \ell(Q) \left[A_{P+Q}^{[i]} \right] \in \mathrm{CH}^*(D_P),$$

$$\begin{aligned} c\left(\mathcal{N}_{A_{P,Q}^{[i]}/D_{P,Q}}\right) \cap \left[A_{P,Q}^{[i]} \right] \\ = \ell(Q) c\left(\mathcal{N}_{A_{P+Q}^{[i]}/D_{P+Q}}\right) \cap \left[A_{P+Q}^{[i]} \right] \in \mathrm{CH}^*\left(A_{P+Q}^{[i]}\right). \end{aligned}$$

For later use we notice that the collection of substacks $A_{P,Q}^{[i]}$ behaves well with respect to intersection.

PROPOSITION 4.4. — *Let P be a profile and $Q, R \in \Sigma(P, i)$. Suppose*

$$I \subseteq \{i, \dots, i - L(Q) + 1\}, \quad J \subseteq \{i, \dots, i - L(R) + 1\}$$

are subsets. Then

$$A_{P,Q}^{[I]} \cap A_{P,R}^{[J]} = A_{P,Q+R}^{[K]}$$

for some subset $K \subseteq \{i, \dots, i - L(Q+R) + 1\}$.

Proof. — We have the containment $A_{P,Q}^{[I]} \cap A_{P,R}^{[J]} \subseteq D_{P,Q+R}$. At a point (X, ω) in $D_{P,Q+R}$ the local defining equations for $A_{P,Q}^{[I]} \cap A_{P,R}^{[J]} \subseteq D_{P,Q+R}$ are given by log periods $\alpha = 0$ for $\mathcal{V}_{P+Q+R}$ over a basis of cycles corresponding to levels that undegenerate to a level in I under $P+Q+R \rightsquigarrow P+Q$ or to a level in J under $P+Q+R \rightsquigarrow P+R$. If we denote K the set of all these levels of $P+Q+R$, then

$$A_{P,Q}^{[I]} \cap A_{P,R}^{[J]} = A_{P,Q+R}^{[K]}. \quad \square$$

PROPOSITION 4.5. — *The zero locus $Z(\overline{\mathrm{ev}}_{P,i}) \subseteq D_P$ is the scheme-theoretic union of*

$$Z(\overline{\mathrm{ev}}_{P,i}) = A_P^{[i]} \cup \bigcup_{\bar{\Gamma} \in \Sigma^1(P,i)} A_{P,\bar{\Gamma}}^{[i]},$$

where we recall that $\Sigma^1(P, i)$ is the collection of non-horizontal two-level graphs $\bar{\Gamma}$ such that the degeneration $P \rightsquigarrow P + \bar{\Gamma}$ is obtained by splitting the i^{th} level.

Proof. — Let $(X, \omega) \in D_{\bar{\Lambda}}^{\circ}$ for some level graph $\bar{\Lambda}$. On $\bar{\Lambda}$ the i^{th} level of P has split into levels $j, \dots, j - k$ for some choice of j, k .

We first deal with the case that $\bar{\Lambda}$ has no horizontal nodes. In order to compute the evaluation section in local coordinates we choose a tuple $\alpha = (\alpha_j, \dots, \alpha_{j-k})$, where α_l is a basis for log periods for the bundle $\mathcal{V}_{\bar{\Lambda}}^{[l]}$. In (4.1) we computed the local equations for $\overline{\text{ev}}_{P,i}$, thus the ideal $\mathcal{I}$ of the zero locus of $\overline{\text{ev}}_{P,i}$ evaluation section is given by

$$\mathcal{I} = (\top_j \alpha_j, \dots, \top_{j-k} \alpha_{j-k}),$$

where $\top_l := \frac{T_{[l]}}{T_{[j]}} = \prod_{s=l}^{j-1} t_s^{\ell_s}$. The ideal can be rewritten as an intersection of ideals

$$\begin{aligned} \mathcal{I} = & \left(\alpha_j, t_{j-1}^{\ell_{j-1}} \right) \cap \left(\alpha_j, \alpha_{j-1}, t_{j-2}^{\ell_{j-2}} \right) \\ & \cap \dots \cap \left(\alpha_j, \dots, \alpha_{j-k-1}, t_{j-k}^{\ell_{j-k}} \right) \cap (\alpha_j, \dots, \alpha_{j-k}). \end{aligned}$$

The last ideal $(\alpha_j, \dots, \alpha_{j-k})$ is the ideal of $A_{P,Q}^{[i]}$. The remaining ideals are the defining ideals of $A_{P,\bar{\Gamma}}^{[i]}$ for some $\bar{\Gamma} \in \text{BIC}$, where $\bar{\Gamma}$ is constructed as follows. Fix some ideal $\mathcal{J} = (\alpha_j, \dots, \alpha_{j-l+1}, t_{j-l}^{\ell_{j-l}})$. The two-level graph $\bar{\Gamma}$ is obtained by only keeping the level passage between the level $j - l + 1$ and $j - l$ and contracting all other levels, in other words $\bar{\Gamma} = \delta_{l-j+1}(\bar{\Lambda})$. By comparing $\mathcal{J}$ with the local defining equations for $A_{P,\bar{\Gamma}}^{[i]} \subseteq D_P$ from (4.2), we see that the ideals coincide.

We now address the remaining case where $\bar{\Lambda}$ has horizontal nodes. As before let $I = \{j, \dots, j - k\}$ be the levels undegenerating to level i . First note that if $\bar{\Lambda}$ has some horizontal node in a level l not contained in I , we can undegenerate level l and still stay inside the zero locus of $\mathcal{I}$. We also claim that $\bar{\Lambda}$ cannot have a horizontal node e in level j . To see this note that the vanishing cycle λ_e has a non-zero period and is contained in $\mathcal{V}_P^{[i]}$, since we assumed $\mathcal{V}_P^{[i]}$ to be a globally defined vector bundle. Thus the evaluation morphism cannot vanish on the j^{th} level. The remaining case is that $\bar{\Lambda}$ has a horizontal node e contained in a level in $I \setminus \{j\}$. In this case all periods appearing in $\overline{\text{ev}}_i$ involving the horizontal node e are multiplied by some rescaling factor $t_l^{\ell_k}$. Thus we can undegenerate the horizontal node while staying inside $Z(\overline{\text{ev}}_P^{[i]})$. Hence we reduced to the case of non-horizontal level graphs. $\square$

4.4. Residue subspaces

We can now give a quick proof of Proposition 1.5. Consider the residue map

$$\text{Res} : \mathcal{O}_{\mathcal{P}(\mu)}(-1) \longrightarrow \mathcal{O}_{\mathcal{P}(\mu)}^r,$$

where r is the number of poles. Let $W \subseteq \mathcal{O}^r$ be a subbundle. A residue subspace $Z \subseteq \overline{\mathcal{P}}(\mu)$ is the closure of the zero locus of the composition $\mathcal{O}_{\mathcal{P}(\mu)}(-1) \rightarrow \mathcal{O}_{\mathcal{P}(\mu)}^r \rightarrow \mathcal{O}_{\mathcal{P}(\mu)}^r/W$. Thus a generic point in Z consists of differentials whose residues are contained in W . Recall for later that $\xi = c_1(\mathcal{O}_{\mathcal{P}(\mu)}(-1))$.

Dually we can also describe a residue subspace using vanishing of periods. For this let $V = (\mathcal{O}^r/W)^*$ be such that $\text{Ann}(V) = W$. Recall the trivial local system $\mathcal{H}_{\text{res}}$ which is generated by small loops around the marked poles and let $\mathcal{V} \subseteq \mathcal{H}_{\text{res}}$ be the sub local system corresponding to V . Here any element in V defines a linear equation among residues, which we consider as an element of the dual of cohomology and thus as a relative homology class. Then $\mathcal{V}$ is a globally defined local system on the stratum and $Z = \overline{\text{Ann}_{\mathcal{P}(\mu)}(\mathcal{V})}$. We say Z is a *residue subspace defined by* V .

Let $k = \dim V$ and choose a complete flag

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V.$$

This induces a flag of subbundles $\mathcal{V}_\bullet : \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \cdots \subsetneq \mathcal{V}_k = \mathcal{V}$.

PROPOSITION 4.6. — *Let Z be a residue subspace defined by a k -dimensional linear subspace $V \subseteq \mathbb{C}^r$. The class of Z in the Chow ring of $\overline{\mathcal{P}}(\mu)$ is given by*

$$[Z] = (-1)^k \prod_{i=1}^k \left(\xi + \sum_{\bar{\Gamma} \in W_i(\mathcal{V}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \in \text{divR}^k(\overline{\mathcal{P}}(\mu)),$$

where $W_i(\mathcal{V}_\bullet) \subseteq \text{BIC}$ is the collection of all non-horizontal two level graphs $\bar{\Gamma}$ such that the restrictions $(\mathcal{V}_i)_{\bar{\Gamma}}^{[0]} = (\mathcal{V}_{i-1})_{\bar{\Gamma}}^{[0]}$ agree.

Proof. — We define a chain of subspaces by letting $Z_i = \overline{\text{Ann}_{\mathcal{P}(\mu)} \mathcal{V}_i}$. In particular we have $\mathcal{P}(\mu) = Z_0, Z = Z_r$. Note that $Z_i \subseteq Z_{i-1}$ is a divisor, since it equals the closure of the annihilator of $(\mathcal{V}_i/\mathcal{V}_{i-1})|_{Z_{i-1} \cap \mathcal{P}(\mu)}$.

The argument in [10, Proposition 8.3] (which follows closely [26, Proposition 7.6]) is only stated for linear subspaces $V \subseteq \mathbb{C}^r$ of a special form induced by global residue conditions but can be modified to show

$$[Z_i] = - \left(\xi + \sum_{\bar{\Gamma} \in W_{i-1}(\mathcal{V}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \in \text{CH}^1(Z_{i-1}).$$

To modify the argument we note that the evaluation section $\mathcal{V}_i/\mathcal{V}_{i-1} \otimes \mathcal{O}(-1) \rightarrow \mathcal{O}$ vanishes on Z_i with multiplicity one, which can be seen in period coordinates. Furthermore it vanishes identically on a boundary divisor $D_{\bar{\Gamma}} \cap Z_{i-1}$ if and only if all the residues in V_i are on the lower level of $D_{\bar{\Gamma}}$, at least modulo $(\mathcal{V}_{i-1})_{\bar{\Gamma}}^{[0]}$. On the lower level the universal family of multi-scale differentials is given by the differential $t^{\ell_{\bar{\Gamma}}} \omega$ and thus the residue of a marked point in the bottom level vanishes with multiplicity $\ell_{\bar{\Gamma}}$. The proof now follows by recursively using push-pull. $\square$

5. The arrangement of exact differentials

For the rest of this section we fix a globally defined local system $\mathcal{V}$ on a generalized stratum B as well as a profile P and a level i . Our goal is to compute the class of $A_P^{[i]}$.

We order all non-horizontal two-level graphs $\bar{\Gamma}_1, \dots, \bar{\Gamma}_n \in \Sigma^1(P, i)$ in a way that respects the partial ordering $\succ$ on BIC. Recall the definition of $\Sigma^1(P, i)$ from Proposition 4.5. We then blow B iteratively up in

$$A_{P, \bar{\Gamma}_1}^{[i]}, \dots, A_{P, \bar{\Gamma}_n}^{[i]}.$$

On the final blowup $\tilde{Z}$ there will be a vector bundle $\mathcal{E}$ and a regular section whose zero locus is the proper transform $\widetilde{A_P^{[i]}}$. By pushing forward along $\tilde{Z} \rightarrow B$ we then obtain a formula for the class and the Chern class of the normal bundle of $A_P^{[i]}$ in terms of the classes and normal bundles of the blowup centers $A_{P, \bar{\Gamma}_j}^{[i]}$. We then repeat the procedure for $A_{P, \bar{\Gamma}_j}^{[i]}$ and the globally defined vector bundle $\mathcal{V}_{P + \bar{\Gamma}_j}^{[i]}$. At each step the codimension of the underlying boundary stratum D_P increases and therefore the procedure has to terminate at some point, allowing us to compute the class $A_P^{[i]}$ recursively.

For the remainder of the section we will freely make use of the definitions and statements from the appendix.

Remark 5.1. — It might be possible to choose a different order in which to perform the blow-ups. The reason why we decided to choose the ordering that respects the partial ordering on BICs is to reduce the amount of bookkeeping that is necessary. In the sequel we will often have to consider intersections of

$$A_{P, \bar{\Gamma}_j}^{[i]} \quad \text{and} \quad A_{P, \bar{\Gamma}_k}^{[i]}.$$

The effect of the first degeneration from P to $P + \bar{\Gamma}_j$ is obtained by splitting the i^{th} level into level i and $i - 1$. The subsequent degeneration $P + \bar{\Gamma}_i$ to

$P + \bar{\Gamma}_j + \bar{\Gamma}_k$ then splits the lower level $i - 1$ into two levels. Here we used that $\bar{\Gamma}_i > \bar{\Gamma}_k$. If one reorders the blow-ups, this changes the order in which the levels are split and makes some of the following arguments more cumbersome.

5.1. The blowup construction

We start with the explicit construction of $\tilde{Z}$. We now write $Z = B$. For brevity set $W_k := A_{\mathbf{P}, \bar{\Gamma}_k}^{[i]}$ and $A := A_{\mathbf{P}}^{[i]}$, $\text{ev} = \overline{\text{ev}}_{\mathbf{P}}^{[i]}$. We then define the first blowup $\tilde{Z}_1 := \text{Bl}_{W_1} Z$. Afterwards we let $\widetilde{W}_i^{(1)}$ be the proper transform of W_i for $i > 1$ and $\widetilde{W}_1^{(1)}$ the exceptional divisor. Define the second blowup $\tilde{Z}_2 := \text{Bl}_{\widetilde{W}_2^{(1)}} \tilde{Z}_1$. We then proceed inductively. If we have already constructed $\tilde{Z}_i$, we let $\widetilde{W}_j^{(i)}$ be the iterated proper transform of W_j for $j > i$ and $\widetilde{W}_j^{(i)}, j \leq i$ the pullback of the exceptional divisor. We then define the $(i + 1)^{\text{th}}$ blowup by

$$\tilde{Z}_{i+1} := \text{Bl}_{\widetilde{W}_{i+1}^{(i)}} \tilde{Z}_i.$$

We write $\tilde{Z} := \tilde{Z}_n$ for the final blowup.

Recall that on Z we have the vector bundle $\mathcal{E} := (\mathcal{V}_{\mathbf{P}}^{[i]})^* \otimes \mathcal{O}_Z^{[i]}(1)$ and the evaluation morphism $\text{ev} : \mathcal{E}^* \rightarrow \mathcal{O}_Z$, which we consider a global section of $\mathcal{E}$. We can pullback $\mathcal{E}$ and ev to all partial blowups $\tilde{Z}_i$ and usually omit the pullbacks in our notation. Since ev vanishes on $A_{\mathbf{P}, \bar{\Gamma}_1}$, the pullback section vanishes with some multiplicity $m(\bar{\Gamma}_1)$ on the exceptional divisor $\widetilde{W}_1^{(1)}$. Dividing ev by the defining equation for $\widetilde{W}_1^{(1)}$, we obtain a section $\tilde{\text{ev}}_1$ of the vector bundle $\mathcal{E} \otimes \mathcal{O}(\widetilde{W}_1^{(1)})^{\otimes -m(\bar{\Gamma}_1)}$.

We now claim that

$$Z(\tilde{\text{ev}}_1) = \tilde{A}^{(1)} \bigcup_{j \geq 2} \widetilde{W}_j^{(1)}.$$

In particular we removed one of the exceptional components.

This will be proven as part of Proposition 5.2. From here we proceed inductively. We now blow $\widetilde{W}_2^{(1)}$ up, divide the pullback of $\tilde{\text{ev}}_1$ by the defining equation for the exceptional divisor and obtain a section $\tilde{\text{ev}}_2$ of $\mathcal{E} \otimes \mathcal{O}(\widetilde{W}_1^{(2)})^{\otimes -m(\bar{\Gamma}_1)} \otimes \mathcal{O}(\widetilde{W}_2^{(2)})^{\otimes -m(\bar{\Gamma}_2)}$. The vanishing locus of $\tilde{\text{ev}}_2$ will then be

$$Z(\tilde{\text{ev}}_2) = \tilde{A}^{(2)} \bigcup_{j \geq 3} \widetilde{W}_j^{(2)}.$$

After the final blowup we will thus have constructed a section $\tilde{e}v = \tilde{e}v_n$ of

$$\mathcal{E} \otimes \bigotimes_{i=1}^n \mathcal{O}(\widetilde{W}_i)^{\otimes -m(\bar{\Gamma}_i)}$$

whose zero locus is

$$Z(\tilde{e}v) = \tilde{A}$$

the proper transform of $A = A_P^{[i]}$. Since $\text{codim}_{\tilde{Z}} \tilde{A} = \text{rank } \mathcal{E}$ has the expected codimension, the section $\tilde{e}v$ is regular and thus we can compute both the class $[\tilde{A}]$ and the Chern classes of the normal bundle of $\tilde{A}$. All this information is summarized in the total Chern class $P_{\tilde{A}/\tilde{Z}}$ (see Definition A.9). The key feature of the total Chern class $P_{\tilde{A}/\tilde{Z}}$ is that the top degree part is the fundamental class of $\tilde{Z}$.

To simplify the notation in the next proposition, let $\mathcal{O}(1) \in \text{Pic}(\tilde{Z})$ be the pullback of the line bundle $\mathcal{O}_Z^{[i]}(1)$.

PROPOSITION 5.2. — *On the final blowup $\tilde{Z}$ we have*

$$P_{\tilde{A}/\tilde{Z}} = c \left(\mathcal{O}^r \otimes \mathcal{O}(1) \otimes \bigotimes_{i=1}^n \mathcal{O}(-\widetilde{W}_i) \right),$$

where r is the rank of $\mathcal{E}$.

Proof. — First note that $\mathcal{V}_P^{[i]}$ has vanishing Chern classes by Proposition 4.3 so for the purpose of Chern class computation we replace it by the trivial bundle. We now address the multiplicity $m(\bar{\Gamma}_k)$. It can be computed as the lowest degree term of $\tilde{e}v^{(k-1)}$ along a generic point of $\widetilde{W}_k^{(k-1)}$. Since $\widetilde{W}_k^{(k-1)}$ and W_k are birational it suffices to compute the multiplicity of e along W_k . A generic point of W_k has profile $P + \bar{\Gamma}_k$, where the degeneration is obtained by splitting the i^{th} level into two levels. There are now two possibilities. Either there exists some period of $\mathcal{V}$ in level i , in which case the multiplicity along that period is 1. Therefore the evaluation section vanishes with multiplicity 1 in this case. In the remaining case all periods of $\mathcal{V}$ are in lower level, thus the evaluation section can be written as

$$z_k = t_{\bar{\Gamma}_k}^{\ell_k} \alpha,$$

where α denotes the log periods of $\mathcal{V}$ in the lower level, $\ell_k = \ell(\bar{\Gamma}_k)$ and $t_{\bar{\Gamma}_k}$ is the coordinate transversal to the boundary $D_{P+\bar{\Gamma}_k} \subseteq D_P$. Generically some lower level period is non-zero. Thus the multiplicity is 1, since we are working with the non-reduced space $A_{P, \bar{\Gamma}_k}^{[i]}$, where in the blowup z_k is a uniformizer.

It remains to show the claim that at each step

$$Z(\tilde{\mathbf{e}}\mathbf{v}_k) = \tilde{A}^{(k)} \cup \bigcup_{j=k+1}^n \tilde{A}_{\mathbf{P}, \bar{\Gamma}_j}^{(k)}.$$

Before addressing the general case, we explain the argument in a simplified situation first. Suppose $(X, \omega) \in W_2$ with underlying 3-level graph $\bar{\Lambda}$. Additionally suppose the profile of $\bar{\Lambda}$ is $(\bar{\Gamma}_1, \bar{\Gamma}_2)$ and we have $W_i = A_{\mathbf{P}, \bar{\Gamma}_i}^{[i]}$. Let α, β, γ be a basis of the restriction $\mathcal{V}_{\bar{\Lambda}}^{[i]}$ for levels 0, -1 and -2 respectively and let t_1 and t_2 be the level parameters for level -1 and -2 , respectively. For simplicity we assume $i = 0$ and $\ell(\bar{\Gamma}_1) = \ell(\bar{\Gamma}_2) = 1$. The ideal sheaves for W_1 and W_2 are locally generated by

$$\mathcal{I}(W_1) = (\alpha, t_1), \mathcal{I}(W_2) = (\alpha, \beta, t_2),$$

while the evaluation section is given by

$$\mathbf{e}\mathbf{v} = (\alpha, t_1\beta, t_1t_2\gamma).$$

On the exceptional divisor we have to introduce projective coordinates α' and t'_1 . At a point supported on the exceptional divisor of the blowup $\tilde{Z}_1$ the equation for the proper transform of W_2 are given by a regular sequence (α', β, t_2) where α' denotes the projective coordinates corresponding to α' . On the chart where $\alpha'_j \neq 0$, the evaluation section $\tilde{\mathbf{e}}\mathbf{v}_{(1)}$ is given by

$$\tilde{\mathbf{e}}\mathbf{v}^{(1)} = (\hat{\alpha}, t'_1\beta, t'_1t_2\gamma),$$

where $\hat{\alpha} = (\alpha'_1, \dots, \alpha'_{j-1}, 1, \alpha'_{j+1}, \dots, \alpha'_d)$. Thus $\tilde{\mathbf{e}}\mathbf{v}_{(1)}$ is non-vanishing. On the chart where $t'_1 \neq 0$, we have

$$\tilde{\mathbf{e}}\mathbf{v}_{(1)} = (\alpha', \beta, t_2\gamma).$$

Thus the zero locus is the union of

$$Z(\alpha', \beta, t_2) = \tilde{W}_2^{(1)} \quad \text{and} \quad Z(\alpha', \beta, \gamma) = \tilde{A}^{(1)}.$$

Now return to the general case and argue by induction. The case $k = 0$ follows from Proposition 4.5. For the induction step we assume the statement has been proven for some k . We claim that on $\tilde{Z}_k$ local equations for $\tilde{W}_{k+1}^{(k)} = \tilde{A}_{\mathbf{P}, \bar{\Gamma}_{k+1}}^{(k)}$ are given by a regular sequence

$$(\alpha, \tilde{z}_{k+1})$$

where α is a tuple corresponding to the periods specializing to level i and $\tilde{z}_{k+1}$ corresponds to the transversal coordinate z_{k+1} . This can be seen by induction, using the observation that local equations for the proper transform are obtained by replacing some terms in the regular sequences by corresponding projective coordinates; see Appendix A.2 for a description

in local coordinates. Similarly the evaluation section can now be written in the form

$$\tilde{e}v_k = (\alpha, \tilde{z}_{k+1} \cdot \beta),$$

where β is a tuple such that $(\alpha, \beta, \tilde{z}_{k+1})$ is a regular sequence.

Thus on the blowup, after dividing by the exceptional divisor, the zero locus of $\tilde{e}v_{k+1}$ is given by $(\tilde{\alpha}, \beta) = 0$, where $\tilde{\alpha}$ denotes the corresponding projective coordinates. In particular there is no component of the zero substack of the evaluation section that is supported entirely on the new exceptional divisor $\tilde{W}_{k+1}^{(k+1)}$ and the remaining components are of the form $\tilde{A}_{P, \tilde{\Gamma}_j}^{(k)}$. $\square$

We now have all the ingredients to finish the proof of the main theorem.

Proof of Theorem 1.1. — In our new language we are trying to compute the total Chern class of $A_\star^{[0]}$, where $\star$ denotes the unique level graph with a single vertex.

We will show more generally that the total Chern class for $A_P^{[I]}$ in D_P can be computed explicitly and lies in the divisorial tautological ring $\text{divR}^*(D_P)$. As we have already remarked the intersection

$$A_P^{[I]} = \bigcap_{i \in I} A_P^{[i]} \subseteq D_P,$$

is algebraically transverse. Thus it suffices to compute the total Chern class of $A_P^{[i]}$ for all profiles and all levels i .

CLAIM 5.3. — *Suppose that we have already shown*

$$[A_{P+Q}^{[j]}] \in \text{divR}^*(D_{P+Q})$$

for any $Q \in \Sigma(P, i)$ and any level j that undegenerates to the i^{th} level. Then $[A_P^{[i]}] \in \text{divR}^(D_P)$ and can be computed explicitly.*

Before we prove the claim we explain how it finishes the proof of the theorem. For a profile P with no degenerations that split level i , it follows from Proposition 4.5 that

$$Z(\overline{e}v_{P,i}) = A_P^{[i]}$$

and thus $\overline{e}v_{P,i}$ is a regular section. In particular the total Chern class of $A_P^{[i]}$ is given by the total Chern class of $(\mathcal{V}_P^{[i]})^* \otimes \mathcal{O}_{D_P}^{[i]}(1)$ and thus

$$P_{A_P^{[i]}/D_P} = c\left(\mathcal{O}_{D_P}^r \otimes \mathcal{O}_{D_P}^{[i]}(1)\right) \in \text{divR}^*(B),$$

where $r = \text{rank } \mathcal{V}_P^{[i]}$. The general case follows now from the claim by induction on the maximal length of a profile in $\Sigma(P, i)$. Since the length of profiles is bounded the process terminates.

We now prove the claim. Our goal is to apply Theorem A.19 with the system of regular embeddings given by

$$\mathcal{S} = \{D_P\} \cup \left\{ A_{P,Q}^{[I]} \left| \begin{array}{l} Q \in \Sigma(P, i), L(Q) \geq 2 \\ \text{and any level in } I \text{ undegenerates to level } i \end{array} \right. \right\},$$

the building set $B = \{A_{P,\bar{\Gamma}}^{[i]} \mid \bar{\Gamma} \in \Sigma^1(P, i)\}$ and $A = A_P^{[i]}, R^* = \text{div}R^*$. To see that $\mathcal{S}$ is indeed a system of regular embeddings, we need to check that $\mathcal{S}$ is closed under finite intersections and if $A_{P,Q}^{[I]} \subseteq A_{P,R}^{[J]}$ is a containment, then the inclusion is a regular embedding. Closedness under finite intersection follows from Proposition 4.4. By inspecting the local equations for $A_{P,Q}^{[I]}$ and $A_{P,R}^{[J]}$ inside B , we see that $A_{P,Q}^{[I]}$ is defined by the additional vanishing of log periods in the levels in I that do not undegenerate to J and some additional level scaling parameters $t_i^{\ell_i}$, which are part of a regular sequence and thus showing that inclusion is a regular embedding. We also need to verify that all $A_{P,Q}^{[I]}$ are Alexander stacks. By Lemma A.2(2) it suffices to show the claim for the reduced substack $A_{P+Q}^{[I]}$. But $A_{P+Q}^{[I]}$ is smooth, since the local equations are part of a system of parameters for B and any smooth Deligne–Mumford stack is Alexander, see Lemma A.2(3).

We now address the total Chern classes. First, we show that for any $A_{P+Q}^{[I]} \in \mathcal{S}$ the total Chern class $P_{A_{P+Q}^{[I]}/D_P}$ is defined over $\text{div}R^*$. By the assumption of the claim we know

$$P_{A_{P+Q}^{[I]}/D_{P+Q}} = \prod_{j \in I} P_{A_{P+Q}^{[j]}/D_{P+Q}}$$

is defined over $\text{div}R^*$ for all $A_{P+Q}^{[I]} \in \mathcal{S}$ and hence

$$P_{A_{P,Q}^{[I]}/D_{P,Q}} = \cdot P_{A_{P+Q}^{[I]}/D_{P+Q}} \in \text{div}R^*(D_{P+Q}).$$

On the other hand $P_{D_{P+Q}/D_P} \in \text{div}R^*(D_P)$. We then apply Proposition A.12 to the inclusions $A_{P,Q}^{[I]} \subseteq D_{P,Q} \subseteq D_P$.

It only remains to show that for any inclusion $A_{P,Q}^{[I]} \subseteq A_{P,S}^{[K]}$ with $A_{P,Q}^{[I]} \in \mathcal{S}$, $A_{P,S}^{[K]} \in \mathcal{S} \cup \{A_P^{[i]}\}$ the total Chern class $P_{A_{P,Q}^{[I]}/A_{P,S}^{[K]}}$ is defined over $\text{div}R^*$.

We can write $Q = R + S$ for some profile R and then

$$A_{P,S}^{[K]} \cap D_{P,Q} = A_{P,S}^{[K]} \cap D_{P,R} = A_{P,Q}^{[J]}$$

for some set $J \subseteq I$. Since R and S are disjoint, the intersection $A_{P,Q}^{[J]} = A_{P,S}^{[K]} \cap D_{P,R}$ is algebraically transverse inside D_P . In particular we see

$$(5.1) \quad P_{A_{P,Q}^{[J]}/A_{P,S}^{[K]}} = P_{D_{P,R}/D_P} = \ell(R)P_{D_{P+R}/D_P} \in R^*\left(A_{P,S}^{[K]}\right).$$

Additionally,

$$A_{P,Q}^{[I]} = A_{P,Q}^{[J]} \cap \bigcap_{j \in I \setminus J} A_{P,Q}^{[j]} \subseteq D_{P,Q}$$

is an algebraically transverse intersection and thus

$$(5.2) \quad P_{A_{P,Q}^{[I]}/A_{P,Q}^{[J]}} = \prod_{i \in I \setminus J} P_{A_{P,Q}^{[i]}/D_{P,Q}} = \prod_{i \in I \setminus J} P_{A_{P+Q}^{[i]}/D_{P+Q}} \in R^*(A_{P,Q}^{[J]}),$$

omitting evident pullbacks.

The situation is summarized in the following diagram of Cartesian squares.

$$\begin{array}{ccccc} A_{P,Q}^{[I]} & \longrightarrow & A_{P,Q}^{[J]} & \longrightarrow & A_{P,S}^{[K]} \\ \downarrow & & \downarrow & & \downarrow \\ \bigcap_{j \in I \setminus J} A_{P,Q}^{[j]} & \longrightarrow & D_{P,Q} & \longrightarrow & D_{P,S} \\ & & \downarrow & & \downarrow \\ & & D_{P,R} & \longrightarrow & D_P \end{array}$$

By (5.2) and (5.1), it now follows from Proposition A.12(2) that

$$P_{A_{P,Q}^{[I]}/A_{P,S}^{[K]}} \in \operatorname{div} R^*(A_{P,S}^{[K]}).$$

This finishes the proof of the induction step. $\square$

6. Low genus formulas and crosschecks

The iterative procedure for the computation of the class of $\overline{[\operatorname{Exc}_\lambda]}$ has been implemented in **SAGE**, building on the packages **admcycles** and **diffs-trata**. The code is available on the authors website. For the algorithm we need to list all two-level graphs as well as the partial ordering between them. The number of level graphs grows exponentially and thus slows down the computation. In this section we explain a few cases where we can avoid the complicated recursion using blowups and obtain more explicit formulas.

Imposing ramification

If one has already computed the class $\overline{[\operatorname{Exc}(\mu)]}$ inside the Chow ring of $\overline{\mathcal{P}}(\mu)$, then imposing additional ramification imposes divisorial conditions.

In particular we can obtain a closed formula for the class of $[\overline{\text{Exc}}_\lambda] \in \text{CH}^*(\overline{\text{Exc}}(\mu))$ as follows.

Let $\underline{\lambda}$ be a ramification profile. Recall the sublocal system $\mathcal{H}_{\underline{\lambda}} \subseteq \mathcal{H}$ which is spanned by the absolute homology bundle $\mathcal{H}_{\text{abs}}$ whose fiber over a point (X, ω) is $H_1(X - P(\omega))$, together with all relative cycles whose end points are contained in a fiber corresponding to a partition λ_i of $\underline{\lambda}$. Restricted to $\text{Exc}(\mu)$ the local systems $\mathcal{H}_{\underline{\lambda}}/\mathcal{H}_{\text{abs}}$ has trivial monodromy and we can thus choose a flag of subbundles

$$\mathcal{U}_\bullet : \mathcal{H}_{\text{abs}} = \mathcal{U}_0 \subsetneq \cdots \subsetneq \mathcal{U}_l = \mathcal{H}_{\underline{\lambda}}.$$

In fact, we can choose the flag such that $\mathcal{U}_i/\mathcal{U}_{i-1}$ is generated by a single cycle with endpoints in the same fiber. We now let $W_i(\mathcal{U}_\bullet) \subseteq \text{BIC}$ be the collection of two level graphs such that for the top level restrictions we have $\mathcal{U}_{i,\bar{\Gamma}}^{[0]} = \mathcal{U}_{i-1,\bar{\Gamma}}^{[0]}$. Recall the definition of the top level restriction from Definition 3.11. In other words, $W_i(\mathcal{U}_\bullet)$ consists of the level graphs where the generator of $\mathcal{U}_i/\mathcal{U}_{i-1}$ is on the bottom level modulo cycles in $\mathcal{U}_{i-1}$. Or slightly restated, if we pick cycles $\gamma_1, \dots, \gamma_n$ such that γ_i generates $\mathcal{U}_i/\mathcal{U}_{i-1}$, then $\gamma_1, \dots, \gamma_i$ are independent and $W_i(\mathcal{U}_\bullet)$ consists of the level graphs where the restriction of γ_i becomes a linear combination of $\gamma_1, \dots, \gamma_{i-1}$. In general, describing the sets $W_i(\mathcal{U}_\bullet)$ is a complicated combinatorial problem, but in case $i = 1$ we can be more explicit. Let p_1, p_2 be the endpoints of a generator of $\mathcal{U}_1$. The set $W_1(\mathcal{U}_\bullet)$ then consists of all 2-level graphs where both points p_1, p_2 lie on the lower level.

We stress that $W_i(\mathcal{U}_\bullet)$ depends on the choice of flag $\mathcal{U}_\bullet$. Exactly as in the proof of Proposition 4.6 we obtain:

PROPOSITION 6.1. — *Let $\underline{\lambda}$ be a ramification profile. The class of the closure of Exc_λ inside the Chow ring of $\overline{\text{Exc}}(\mu)$ is given by*

$$[\overline{\text{Exc}}_\lambda] = (-1)^l \prod_{i=1}^l \left(\xi + \sum_{\bar{\Gamma} \in W_i(\mathcal{U}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \in \text{divR}^l(\overline{\text{Exc}}(\mu)),$$

where $l = \text{codim}_{\text{Exc}(\mu)} \text{Exc}_\lambda$.

In particular we have

$$[\overline{\text{Exc}}_\lambda] = (-1)^l \prod_{i=1}^l \left(\xi + \sum_{\bar{\Gamma} \in W_i(\mathcal{U}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \cdot [\overline{\text{Exc}}(\mu)] \in \text{CH}^*(\overline{\mathcal{P}}(\mu)).$$

Genus zero and realizability of branched covers of the sphere

Every stratum of $\underline{\lambda}$ -exact differentials can be decomposed as

$$\overline{\text{Exc}_{\underline{\lambda}}} \subseteq \overline{\text{Exc}(\mu)} \subseteq \overline{\mathcal{P}(\mu)^{\text{res}}} \subseteq \overline{\mathcal{P}(\mu)},$$

where $\overline{\mathcal{P}(\mu)^{\text{res}}}$ is the closure of the stratum of residueless differentials. We have closed formulas for the class of $\text{Exc}_{\underline{\lambda}}$ inside $\text{Exc}(\mu)$ and for $\overline{\mathcal{P}(\mu)^{\text{res}}} \subseteq \overline{\mathcal{P}(\mu)}$. Thus if we let $\iota : \overline{\mathcal{P}(\mu)^{\text{res}}} \subseteq \overline{\mathcal{P}(\mu)}$ be the inclusion and $\alpha \in \text{divR}^*(\overline{\mathcal{P}(\mu)})$ be a tautological class such that $\iota^* \alpha = [\overline{\text{Exc}(\mu)}] \in \text{divR}^* \overline{\mathcal{P}(\mu)^{\text{res}}}$, then

$$\begin{aligned} [\overline{\text{Exc}_{\underline{\lambda}}}] &= (-1)^{k+l} \prod_{i=1}^l \left(\xi + \sum_{\bar{\Gamma} \in W_i(\mathcal{U}_{\bullet})} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \\ &\quad \prod_{j=1}^k \left(\xi + \sum_{\bar{\Gamma} \in W_j(\mathcal{V}_{\bullet})} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \cdot \alpha \in \text{divR}^*(\overline{\mathcal{P}(\mu)}), \end{aligned}$$

where we combined Proposition 4.6 and Proposition 6.1.

For general g , we can only use the recursive blowup procedure to compute α but for $g = 0, 1$ the codimension is small enough that we can obtain closed formulas.

For genus zero, every residueless differential is exact and thus $\overline{\mathcal{P}(\mu)^{\text{res}}} = \overline{\text{Exc}(\mu)}$, i.e., we have $\alpha = 1$.

PROPOSITION 6.2. — *Let $\text{Exc}_{\underline{\lambda}}$ be a stratum of $\underline{\lambda}$ -exact differentials in genus $g = 0$. Then*

$$\begin{aligned} [\overline{\text{Exc}_{\underline{\lambda}}}] &= (-1)^{k+l} \prod_{i=1}^l \left(\xi + \sum_{\bar{\Gamma} \in W_i(\mathcal{U}_{\bullet})} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \\ &\quad \prod_{j=1}^k \left(\xi + \sum_{\bar{\Gamma} \in W_j(\mathcal{V}_{\bullet})} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \in \text{divR}^{k+l}(\overline{\mathcal{P}(\mu)}), \end{aligned}$$

where

- $k = \text{codim}_{\text{Exc}(\mu)} \text{Exc}_{\underline{\lambda}}$, $l = \text{codim}_{\overline{\mathcal{P}(\mu)}} \text{Exc}(\mu)$,
- $\mathcal{V}_{\bullet}$ is a complete flag of subbundles of the residue local system $\mathcal{H}_{\text{res}} = \mathcal{H}_{\text{abs}}$,
- $\mathcal{U}_{\bullet}$ is a complete flag of subbundles of $(\mathcal{H}_{\underline{\lambda}}/\mathcal{H}_{\text{abs}})|_{\overline{\mathcal{P}(\mu)^{\text{res}}}}$.

It is still an open question which ramification profiles $\underline{\lambda}$ can be realized as branched covers of $\mathbb{P}^1 \rightarrow \mathbb{P}^1$, see for example [24] for the current state.

A necessary condition is that $\underline{\lambda}$ satisfies the conditions of the Riemann–Hurwitz theorem but in general this is not sufficient. For example in degree $d = 4$, the ramification profile $\underline{\lambda} = (3, 1), (2, 2), (2, 2)$ cannot be realized by a branched covering. We can verify this by computing that the class of the corresponding stratum of $\underline{\lambda}$ -exact differentials has zero class in $\mathrm{CH}^*(\overline{\mathcal{P}}(\mu))$. We do not have an efficient way of checking that a class is zero in the Chow ring of $\overline{\mathcal{P}}(\mu)$ and instead it is easier to pushforward along $\rho : \overline{\mathcal{P}}(\mu) \rightarrow \overline{\mathcal{M}}_{0,n}$. Using the **SAGE** package **admcycles** and Proposition 6.2 we can verify that indeed for $\underline{\lambda} = (3, 1), (2, 2), (2, 2)$ the class $[\overline{\mathrm{Exc}}_{\underline{\lambda}}]$ is zero in $\mathrm{CH}^*(\overline{\mathcal{M}}_{0,6})$. Similarly we can check for the remaining degree $d = 4$ ramification profiles that the class is non-zero.

Since for a projective variety a subvariety is non-empty if and only if its class in the Chow ring is zero, we get the following criterion for when a branched cover can be realized.

COROLLARY 6.3. — *Let $\underline{\lambda}$ be a ramification profile in genus zero. Then $\underline{\lambda}$ can be realized by a branched covering $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ if and only if the class*

$$\rho_* \left[\prod_{i=1}^l \left(\xi + \sum_{\bar{\Gamma} \in W_i(\mathcal{U}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \prod_{j=1}^k \left(\xi + \sum_{\bar{\Gamma} \in W_j(\mathcal{V}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \right] \neq 0 \in \mathrm{CH}^*(\overline{\mathcal{M}}_{0,n}).$$

Genus one

For genus 1 the codimension of $\overline{\mathrm{Exc}}(\mu)$ inside $\overline{\mathcal{P}}(\mu)^{\mathrm{res}}$ is 2. In particular the zero locus of the evaluation section has expected codimension, after we remove divisorial components.

We recall the local systems $\mathcal{H}_{\mathrm{abs}} \supseteq \mathcal{H}_{\mathrm{res}}$ of absolute homology and residues, respectively. We let $N_i \subseteq \mathrm{BIC}$ be all the two level graphs such that

$$\mathrm{rank} \mathcal{H}_{\mathrm{abs}, \bar{\Gamma}}^{[0]} - i = \mathrm{rank} \mathcal{H}_{\mathrm{res}, \bar{\Gamma}}^{[0]}, \quad i = 0, 1.$$

Here $\mathcal{H}_{\mathrm{abs}, \bar{\Gamma}}^{[0]}, \mathcal{H}_{\mathrm{res}, \bar{\Gamma}}^{[0]}$ are the images of the specialization morphism of the Deligne extension of $\mathcal{H}_{\mathrm{abs}}$ and $\mathcal{H}_{\mathrm{res}}$, respectively. Then N_i corresponds exactly to all two level graphs such that $A_{\bar{\Gamma}}^{[0]}$ has codimension $1+i$ inside $\overline{\mathcal{P}}(\mu)$.

PROPOSITION 6.4. — *Let Exc_λ be a stratum of λ -exact differentials in genus $g = 1$. Then*

$$\begin{aligned} [\overline{\text{Exc}_\lambda}] &= (-1)^d \prod_{i=1}^l \left(\xi + \sum_{\bar{\Gamma} \in W_i(\mathcal{U}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \prod_{j=1}^k \left(\xi + \sum_{\bar{\Gamma} \in W_j(\mathcal{V}_\bullet)} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right) \\ &\quad \times \left[\left(\xi + \sum_{\bar{\Gamma} \in N_0} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \right)^2 + \sum_{\bar{\Gamma} \in N_1} \ell_{\bar{\Gamma}} D_{\bar{\Gamma}} \left(\xi + \sum_{\bar{\Gamma}' \in N_1(\bar{\Gamma})} \ell_{\bar{\Gamma}'} D_{\bar{\Gamma}'} \right) \right] \\ &\quad \in \text{divR}^*(\bar{\mathcal{P}}(\mu)), \end{aligned}$$

where

- $d = \text{codim}_{\bar{\mathcal{P}}(\mu)} \text{Exc}_\lambda$,
- $N_i \subseteq \text{BIC}$ is the set of two level graphs such that

$$\text{rank } \mathcal{H}_{\text{abs}, \bar{\Gamma}}^{[0]} - i = \text{rank } \mathcal{H}_{\text{res}, \bar{\Gamma}}^{[0]},$$

- $N_1(\bar{\Gamma})$ is the set of all two level graphs $\bar{\Gamma}'$ with $\bar{\Gamma}' \succ \bar{\Gamma}$ and

$$\text{rank } \mathcal{H}_{\text{abs}, \bar{\Gamma} + \bar{\Gamma}'}^{[0]} - 1 = \text{rank } \mathcal{H}_{\text{res}, \bar{\Gamma} + \bar{\Gamma}'}^{[0]}.$$

Proof. — We recall that the zero locus of the evaluation section is supported on $A_{\star}^{[0]}$ and $A_{\star, \bar{\Gamma}}^{[0]}$ for $\bar{\Gamma} \in \text{BIC}$. If $A_{\star, \bar{\Gamma}}^{[0]}$ is non-empty, then the codimensions are either one or two. In the first case $A_{\star, \bar{\Gamma}}^{[0]} = D_{\bar{\Gamma}}^\bullet$ and $\bar{\Gamma} \in N_0$, where we recall that $D_{\bar{\Gamma}}^\bullet$ is an infinitesimal thickening of $D_{\bar{\Gamma}}$ with multiplicity $\ell_{\bar{\Gamma}}$. The latter case corresponds to $\bar{\Gamma} \in N_1$. In particular the zero locus of the evaluation section ev , twisted by $\otimes_{\bar{\Gamma} \in N_0} \mathcal{O}(D_{\bar{\Gamma}})^{-\ell_{\bar{\Gamma}}}$ has expected codimension and thus

$$\begin{aligned} [\overline{\text{Exc}}(\mu)] &= [A_{\star}^{[0]}] = c_2 \left(\mathcal{O}(1) \otimes \bigotimes_{\bar{\Gamma} \in N_0} \mathcal{O}(D_{\bar{\Gamma}})^{-\ell_{\bar{\Gamma}}} \otimes \mathcal{O}^{\oplus 2} \right) \\ &\quad - \sum_{\bar{\Gamma} \in N_1} [A_{\star, \bar{\Gamma}}^{[0]}] \in \text{CH}^*(\bar{\mathcal{P}}(\mu)^{\text{res}}). \end{aligned}$$

To compute the class of $A_{\star, \bar{\Gamma}}^{[0]}$ we will instead compute the class of $[A_{\star, \bar{\Gamma}}^{[0]}] \in \text{CH}^1(\bar{\mathcal{P}}(\mu)^{\text{res}} \cap D_{\bar{\Gamma}}^\bullet)$ and then use push-pull. Similarly to before we have an evaluation section defined on $\bar{\mathcal{P}}(\mu)^{\text{res}} \cap D_{\bar{\Gamma}}^\bullet$ vanishing on $A_{\star, \bar{\Gamma}}^{[0]}$ as well as

$$D_{\bar{\Gamma}'}^\bullet \cap \bar{\mathcal{P}}(\mu)^{\text{res}} \cap D_{\bar{\Gamma}}^\bullet \quad \text{for } \bar{\Gamma} \in N_1(\bar{\Gamma}').$$

And we have

$$\left[D_{\bar{\Gamma}}^{\bullet} \cap \bar{\mathcal{P}}(\mu)^{\text{res}} \cap D_{\bar{\Gamma}}^{\bullet} \right] = \ell_{\bar{\Gamma}'} D_{\bar{\Gamma}'} \in \text{CH}^1 \left(\bar{\mathcal{P}}(\mu)^{\text{res}} \cap D_{\bar{\Gamma}}^{\bullet} \right). \quad \square$$

For example we can consider the ramification profile $\underline{\lambda} = ((2), (2), (2), (2))$, i.e., we consider

$$Z := \left\{ (E, p, q, r, s) \in \mathcal{M}_{1,4} \mid \exists f : E \xrightarrow{2:1} \mathbb{P}^1 \text{ ramified at } p, q, r, s \right\},$$

or in other words elliptic curves with a full 2-torsion package. Pushing forward the class of $[\bar{\text{Exc}}_{\underline{\lambda}}]$ forward we obtain

$$\frac{75}{2} \kappa_3 - \frac{45}{4} \kappa_1 \kappa_2 + \frac{3}{4} \kappa_1^3 + \frac{3}{2} \kappa_2 (\psi_1 + \psi_2 + \psi_3 + \psi_4) \in \text{CH}^3(\bar{\mathcal{M}}_{1,4}),$$

which agrees with the formula computed by `admcycles`.

With additional work one can also obtain a formula for $g = 2$ or even higher genus. The main difference is the for higher genus one has extraneous components which are neither divisorial nor of expected dimension and one cannot avoid the blowup procedure, which makes the formulas much more complicated.

Appendix A. Intersection theory on Iterated blowups

Alexander stacks

Let X, Y be separated Deligne–Mumford stacks over $\mathbb{C}$. For a representable morphism $f : X \rightarrow Y$ we let $\text{CH}_{\text{op}}^*(X \rightarrow Y)$ be the bivariant Chow groups. An element $c \in \text{CH}_{\text{op}}^*(X \rightarrow Y)$ is an operator $c : \text{CH}^*(Z) \rightarrow \text{CH}^*(X \times_Y Z \rightarrow Z)$ for any morphism $f : Z \rightarrow Y$ from a scheme Z , which is compatible with proper pushforward, flat pullback and Gysin homomorphisms for regular embeddings (see [16, Chapter 17] for a precise definition). As explained in Vistoli, an operational Chow class $c \in \text{CH}_{\text{op}}^*(X \rightarrow Y)$ also defines an operator $c \in \text{CH}_{\text{op}}^*(F \rightarrow X \times_Y F)$ for any representable morphism $F \rightarrow Y$ of Deligne–Mumford stacks. Furthermore we have an evaluation map

$$\text{ev}_Y : \text{CH}_{\text{op}}^*(F \rightarrow Y) \longrightarrow \text{CH}^*(F), \quad c \longmapsto c \cap [Z].$$

We denote by $\text{CH}_{\text{op}}^*(X) := \text{CH}_{\text{op}}^*(X \xrightarrow{\text{id}} X)$ the operational rational Chow ring. The ring structure on $\text{CH}_{\text{op}}^*(X)$ is induced from the product

$$\text{CH}_{\text{op}}^*(X \rightarrow Y) \otimes \text{CH}_{\text{op}}^*(Y \rightarrow Z) \longrightarrow \text{CH}_{\text{op}}^*(X \rightarrow Z)$$

and is commutative.

We call a Deligne–Mumford stack X an *Alexander stack* if X is equidimensional and for every representable morphism $f : F \rightarrow X$ the evaluation $\mathrm{ev}_X : \mathrm{CH}_{\mathrm{op}}^*(F \rightarrow X) \rightarrow \mathrm{CH}^*(F)$ is an isomorphism.

Remark A.1. — In [29] an additional commutative condition is required. For applications in this paper we are working over a field of characteristic zero, so that every Deligne–Mumford stack has a resolution of singularities and thus the commutativity is automatically satisfied.

We will use the following properties from [30]. Note that in (loc. cit.) everything is stated for schemes, but the proofs work analogously for Deligne–Mumford stacks.

LEMMA A.2. — *Let X, Y, Z be Deligne–Mumford stacks and $f : X \rightarrow Y$ a representable morphism.*

- (1) *Suppose f is smooth of constant fiber dimension. If Y is an Alexander stack, so is X .*
- (2) *Suppose f is a universal homeomorphism, then X is an Alexander stack if and only if Y is an Alexander stack. In particular, X is Alexander if and only if the reduced stack X^{red} is Alexander.*
- (3) *Every smooth Deligne–Mumford stack is an Alexander stack.*

Proof. — (1) is [29, Proposition 2.2], (2) is [29, Proposition 2.7], and (3) is [30, Proposition 5.6]. $\square$

The following is most likely well known to experts but we could not find a reference.

PROPOSITION A.3. — *Suppose that $X \subseteq Y$ are Alexander stacks and X is regularly embedded in Y . Then $\tilde{Y} := \mathrm{Bl}_Y X$ is an Alexander stack.*

Proof. — Let $\tilde{X}$ be the exceptional divisor. For any representable morphism $Y' \rightarrow Y$ we set $X' = X \times_Y Y'$, $\tilde{X}' = X' \times_X \tilde{X}$, $\tilde{Y}' = Y' \times_Y \tilde{Y}$.

We have short exact sequences

$$\begin{array}{ccccccc}
 0 \rightarrow \mathrm{CH}_{\mathrm{op}}^*(X' \rightarrow X) & \xrightarrow{\tilde{\alpha}} & \mathrm{CH}_{\mathrm{op}}^*(\tilde{X}' \rightarrow \tilde{X}) \oplus \mathrm{CH}_{\mathrm{op}}^*(Y' \rightarrow Y) & \xrightarrow{\tilde{\beta}} & \mathrm{CH}_{\mathrm{op}}^*(\tilde{Y}' \rightarrow \tilde{Y}) & \rightarrow 0 \\
 \downarrow \mathrm{ev}_X & & \downarrow \mathrm{ev}_{\tilde{X}} \oplus \mathrm{ev}_Y & & \downarrow \mathrm{ev}_{\tilde{Y}} & \\
 0 \longrightarrow \mathrm{CH}^*(X') & \xrightarrow{\alpha} & \mathrm{CH}^*(\tilde{X}') \oplus \mathrm{CH}^*(Y') & \xrightarrow{\beta} & \mathrm{CH}^*(\tilde{Y}') & \longrightarrow 0
 \end{array}$$

where the vertical maps are the evaluation maps.

The horizontal maps are given by

$$\begin{aligned}
 \tilde{\alpha}(c) &= (c_{d-1}(E) \cdot g^*c, i_*(c \cdot [i])), & \tilde{\beta}(r, s) &= -j_*(r \cdot [j]) + f^*s, \\
 \alpha(x) &= c_{d-1}(E) \cap g^*x, & \beta(\tilde{x}, y) &= j_*\tilde{x} + f^*y
 \end{aligned}$$

and the diagram is commutative. (See [16, Chapter 17] for the operations on operational Chow groups.) Since $\tilde{X}$ is a projective bundle, $\tilde{X}$ is Alexander if X is Alexander. By the 4-lemma it follows that the evaluation morphism $\mathrm{ev}_{\tilde{Y}}$ is an isomorphism. $\square$

From the view point of intersection theory Alexander stacks behave very similar to smooth Deligne–Mumford stacks. In particular Chow rings have an intersection product and satisfy a projection formula. Given an Alexander stack Z and $\alpha \in \mathrm{CH}^*(Z)$, we let T_α be the unique operator with $T_\alpha \cap [Z] = \alpha$. The intersection product on $\mathrm{CH}^*(Z)$ takes the form

$$\alpha \cdot \beta = T_\alpha \cap \beta = \alpha \cap T_\beta = T_\alpha \cap T_\beta \cap [Z].$$

For any proper, representable morphism $f : X \rightarrow Z$ there is a projection formula

$$(A.1) \quad f_*(f^*\beta \cdot \alpha) = \beta \cdot f_*\alpha.$$

induced by the projection formula on operational Chow groups.

For any representable morphism $f : X \rightarrow Y$ we have a pullback on operational Chow groups, on the hand if f is additionally a local complete intersection morphism we also have a pullback on Chow groups.

LEMMA A.4. — *Let $f : X \rightarrow Y$ be a representable local complete intersection morphism of Alexander stacks. Then*

$$f^*T_\alpha = T_{f^*\alpha}.$$

Proof. — Since f is a local complete intersection morphism, we can decompose it into a regular embedding and a smooth morphism. In particular we have $f^*[Y] = [X]$ by [30, Lemma 5.5]. Thus

$$f^*T_\alpha \cap [Y] = f^*T_\alpha \cap f^*[X] = f^*(T_\alpha \cap [X]) = f^*\alpha.$$

Since X is an Alexander stack, we conclude $f^*T_\alpha = T_{f^*\alpha}$. $\square$

CONVENTION A.5. — *We assume for the rest of the appendix that all Deligne–Mumford stacks are Alexander stacks and from now on we identify $\mathrm{CH}_{\mathrm{op}}^*$ and CH^* via the evaluation map.*

Tautological rings

Suppose Y is an Alexander stack and $X \subseteq Y$ a closed substack. Recall that X is called regularly embedded if etale locally Y is defined by a regular sequence. It follows that the normal sheaf $\mathcal{N}_{X/Y}$ is a vector bundle of rank $\mathrm{codim}_Y X$.

Given a subring $R^*(Z) \subseteq \mathrm{CH}^*(Z)$ and a regular embedding $i : W \hookrightarrow Z$ we define $R^*(W) = i^* R^*(Z)$. Additionally, let $\pi : \tilde{Z} \rightarrow Z$ be the blowup of Z along W . We define $R^*(\tilde{Z}) = \pi^* R^*(Z)[E]$, where E denotes the exceptional divisor.

For a substack $\tilde{Y}$ of $\tilde{Z}$ we then also let $R^*(\tilde{Y})$ be the pullback of $R^*(\tilde{Z})$.

Thus for every regularly embedded subscheme of a blowup of Z along a regularly embedded substack we have defined a ring $R^*(Y)$. We call the assignment $Y \mapsto R^*(Y)$ a *tautological ring*.

Remark A.6. — Suppose $R^* : Y \subseteq Z$ is regularly embedded and let $\tilde{Z} = \mathrm{Bl}_W Z, \tilde{Y} = \mathrm{Bl}_{W \cap Y} Y$. Then the inclusion $j : \tilde{Y} \rightarrow \tilde{Z}$ is a regular embedding (see Appendix A.2) and we have a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{Z} \\ g \downarrow & & \downarrow f \\ Y & \xrightarrow{i} & Z \end{array}$$

Furthermore,

$$R^*(\tilde{Y}) = j^* R^*(\tilde{Z}) = g^*(R^*(Y))[E_Y],$$

where E is the exceptional divisor in $\tilde{Z}$ and $E_Y = j^* E$ the exceptional divisor in $\tilde{Y}$. In particular there is no ambiguity in defining R^* .

Algebraically clean and transverse intersection

DEFINITION A.7. — We say two substacks $W, Y \subseteq Z$ intersect algebraically cleanly inside Z if every morphism in the Cartesian diagram

$$\begin{array}{ccc} X = W \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W & \longrightarrow & Z \end{array}$$

is a regular embedding. If furthermore

$$\mathrm{codim}_Z(X) = \mathrm{codim}_Z(W) + \mathrm{codim}_Z(Y),$$

we say the intersection is algebraically transverse.

Remark A.8. — The analogy to clean and transverse intersection of smooth varieties is as follows. If X and W have algebraically clean intersection, then étale-locally we can find regular sequences such that the ideal sheaves $\mathcal{I}(W)$ and $\mathcal{I}(Y)$ are generated by

$$\mathcal{I}(W) = (a_1, \dots, a_n), \quad \mathcal{I}(Y) = (a_1, \dots, a_m, b_1, \dots, b_k), \quad m \leq n.$$

Here we allow $m = 0$, in which case the intersection is algebraically transverse.

Total Chern classes

DEFINITION A.9. — Suppose $i : V \hookrightarrow Z$ is regularly embedded of codimension k . We say that V has a total Chern class if there exists a polynomial

$$P_{V/Z}(t) = \bar{c}_0(\mathcal{N}_{V/Z}) + \bar{c}_1(\mathcal{N}_{V/Z}) + \cdots + \bar{c}_{k-1}(\mathcal{N}_{V/Z}) + \bar{c}_k(\mathcal{N}_{V/Z}) \in \mathrm{CH}^*(Z)$$

such that

- $\bar{c}_0(\mathcal{N}_{V/Z}) \cap [Z] = [Z]$, $\bar{c}_k(\mathcal{N}_{V/Z}) \cap [Z] = [V]$.
- $i^* \bar{c}_i(\mathcal{N}_{V/Z}) = c_i(\mathcal{N}_{V/Z})$ for all i .

Note that total Chern classes are unique only up to elements in $\ker(i^* : \mathrm{CH}^*(Z) \rightarrow \mathrm{CH}^*(X))$. We stress that our convention differs from [17] where instead a polynomial is used and also the indexing is different.

PROPOSITION A.10. — The total Chern class has the following properties.

- (1) If $D \subseteq Z$ is a divisor, then $P_{D/Z} = 1 + [D]$.
- (2) Suppose X and Y intersect algebraically transversely inside Z and X has a total Chern class $P_{X/Z}$ in Z . Then

$$P_{X \cap Y/Y} = P_{X/Z},$$

as well as

$$P_{X \cap Y/Z} = P_{X/Z} \cdot P_{Y/Z},$$

omitting evident pullbacks.

Proof. — [17, Lemma 5.1] □

Our main goal is to determine whether the total Chern class can be chosen to be contained in some subring $R^*(Z) \subseteq \mathrm{CH}^*(Z)$. For example R^* could be the tautological ring of the moduli space of curves.

DEFINITION A.11. — Let $R^*(Z) \subseteq \mathrm{CH}^*(Z)$ be a tautological subring and $V \subseteq Z$ regularly embedded.

We say that V has a total Chern class in Z defined over R^* if there exists a total Chern classes $P_{V/Z} \in R^*(Z)$.

PROPOSITION A.12. — Suppose $X \subseteq Y \subseteq Z$ are regularly embedded substacks.

- (1) If $P_{X/Y}, P_{Y/Z}$ are defined over R^* , then $P_{X/Z}$ is defined over R^* .
- (2) If $X = Y \cap W \subseteq Z$ is an algebraically transverse intersection for some substack $W \subseteq Z$ and $P_{W/Z} \in R^*(Z)$, then $P_{X/Y} = i^* P_{W/Z} \in R^*(Y)$, where $i : Y \hookrightarrow Z$ is the inclusion.

A.1. Intersection theory of blowups

Let $W \subseteq Z$ be a regular embedding of Deligne–Mumford stacks and consider the blowup $\widetilde{Z}$ of Z in W with exceptional divisor $\widetilde{W}$.

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{j} & \widetilde{Z} \\ \pi' \downarrow & & \downarrow \pi \\ W & \xrightarrow{i} & Z \end{array}$$

Let $s(W, Z) = \sum_i s_i(\mathcal{N}_{W/Z})$ be the total Segre class of the normal bundle $\mathcal{N}_{W/Z}$. By the birational invariance of Segre classes we have

$$s(W, Z) = \pi_{\widetilde{W},*} s(\widetilde{W}, \widetilde{Z}) = \sum_{i \geq 0} \pi'_* \left(\mathcal{O}(-1)^i \cap [\widetilde{W}] \right) = \sum_{i \geq 0} s_{i-r+1}(\mathcal{N}_{W/Z})$$

If W has a total Chern class in Z , then there exists classes $\bar{s}_i(\mathcal{N}_{W/Z}) \in \mathrm{CH}^*(Z)$ which restrict to $s_i(\mathcal{N}_{W/Z})$ on X . We can then write

$$(A.2) \quad \pi_* [\widetilde{W}]^i = (-1)^{i-1} \bar{s}_{i-\mathrm{codim}(W)}(\mathcal{N}_{W/Z}) \cap [W] \text{ for } i \geq 1.$$

We have the following useful consequence.

PROPOSITION A.13. — Let $R^*(Z) \subseteq \mathrm{CH}^*(Z)$ be a tautological ring and $W \subseteq Z$ regularly embedded. Furthermore, suppose $P_{W/Z} \in R^*(Z)$. Consider the blowup $\pi : \widetilde{Z} \rightarrow Z$ of Z in W . Then π_* preserves R^* , i.e.,

$$\pi_*(R^*(\widetilde{Z})) = R^*(Z).$$

Proof. — From $\pi_* \pi^* = \mathrm{id}$ it follows that $\pi_*(R^*(\widetilde{Z})) \supseteq R^*(Z)$. On the other hand we have by definition that $R^*(\widetilde{Z})[\widetilde{W}]$, thus it suffices to show that $\pi_* [\widetilde{W}^i] \in R^*(Z)$, which is the content of (A.2). $\square$

A.2. Local equations for proper transforms

Suppose $W, V \subseteq Z$ intersect algebraically cleanly in Z . Let $\tilde{Z} = \text{Bl}_W Z$ and $X = W \cap V$. Locally we can find a regular sequence $x_1, \dots, x_n, y_1, \dots, y_m$ such that the ideals of X and V are $\mathcal{I}(W) = (x_1, \dots, x_n)$, $\mathcal{I}(V) = (x_1, \dots, x_d, y_1, \dots, y_m)$ with $d \leq n$.

Furthermore, the blowup $\tilde{Z} = \text{Bl}_X Z$ is locally defined in $Z \times \mathbb{P}^{n-1}$ by the equations

$$\{x_i T_j = x_j T_i \quad \text{for } 1 \leq i < j \leq n\}.$$

The proper transform $\tilde{V}$ of V can be identified with $\text{Bl}_{W \cap V} V$. In the affine chart where $T_i \neq 0$, the equation for the exceptional divisor $\tilde{W}$ is $z_i = 0$. Local equations for the proper transform $\tilde{V}$ in $Z \times \mathbb{P}^{n-1}$ are given by

$$(A.3) \quad \mathcal{I}(\tilde{V}) = (T_1, \dots, T_d, y_1, \dots, y_m),$$

where $T_1, \dots, T_d$ are projective coordinates. In particular $\tilde{V} \subset \tilde{Z}$ is regularly embedded. (For details see [1, Lemma 4.1]).

A.3. Total Chern classes of proper transforms

Our next goal is the computation of classes and normal bundles for proper transforms along a blowup. In the case $W \subseteq V \subseteq Z$ a formula is found in [16], and was generalized to the general situation in [1].

The main result is that the class of the proper transform $\tilde{V}$ as well as its normal bundle can be computed from the classes of $W, V, V \cap W$ as well as their normal bundles.

COROLLARY A.14. — *Suppose V and W intersect algebraically cleanly in Z . Let $\tilde{Z} = \text{Bl}_W Z$ and $\tilde{V} = \text{Bl}_X V, X = V \cap W$.*

Furthermore assume that

$$P_{X/V}, P_{X/W}, P_{W/Z}$$

are defined over R^ .*

Then $P_{V/Z} \in R^(Z)$ if and only if $P_{\tilde{V}/\tilde{Z}} \in R^*(\tilde{Z})$.*

In fact we will prove an explicit formula for the total Chern class $P_{\tilde{V}/\tilde{Z}}$ from which Corollary A.14 follows.

We start by setting up the notation for Aluffi's formula [1]. As before we consider the blowup $\tilde{Z}$ of Z in W with exceptional divisor $\tilde{W}$. Furthermore V and W intersect cleanly in Z and we set $X = V \cap W$. The exceptional

divisor is $\widetilde{W}$ and $\widetilde{V}$ is the proper transform of V is $\widetilde{V}$. The situation is summarized in the following diagram.

$$\begin{array}{ccccc}
 & \widetilde{X} & \xrightarrow{\tilde{i}} & \widetilde{V} & \\
 \pi_X \swarrow & \downarrow & & \swarrow \pi_V & \\
 X & \xrightarrow{i} & V & & \\
 \downarrow j & \downarrow \tilde{j} & \downarrow k & \downarrow \tilde{k} & \\
 & \widetilde{W} & \xrightarrow{\tilde{l}} & \widetilde{Z} & \\
 \pi_W \swarrow & & & \swarrow \pi_Z & \\
 W & \xrightarrow{l} & Z & &
 \end{array}$$

We define the classes

$$\begin{aligned}
 C &= \sum_m C_m := j^* c(\mathcal{N}_{W/Z}) c(\mathcal{N}_{X/V})^{-1} \in \mathrm{CH}^*(X), \\
 N &= \sum_n N_n := c(\mathcal{N}_{X/W}) \in \mathrm{CH}^*(X).
 \end{aligned}$$

If we formally assume that C and N are the pullbacks of the total Chern class of vector bundles $\mathbb{C}, \mathbb{N}$ on V , then we can expand

$$c(\mathbb{N}) \cdot c(\mathbb{C} \otimes \mathcal{O}(-\widetilde{X})) = c(\mathbb{N}) \cdot c(\mathbb{C}) + Q(c_m(\mathbb{N}), c_n(\mathbb{C}), [\widetilde{X}]) \cdot \widetilde{X} \in \mathrm{CH}^*(V).$$

for a polynomial $Q = \sum_k Q_k(c_m(\mathbb{N}), c_n(\mathbb{C}))[\widetilde{X}]^k$. While in general we cannot ensure the existence of a vector bundle, the polynomial Q still appears in the following formula.

In the next proposition we omit some pullback maps or sometimes write $\alpha|_V$ for the pullback along an inclusion $V \subset V'$, $\alpha \in \mathrm{CH}^*(V')$.

PROPOSITION A.15. — *Let Z be an Alexander stack. Assume $V, W \subseteq Z$ intersect cleanly and let $X = V \cap W$. Furthermore, suppose that $P_{X/W}, P_{X/V}, P_{W/Z}$, are all defined over R^* .*

Let $\overline{C}, \overline{N}, \overline{X} \in R^(Z)$ such that*

$$\overline{N}|_X = N, \quad \overline{C}|_X = C \in R^*(X), \quad \overline{X}|_W = X \in R^*(W).$$

The class of the proper transform $\widetilde{V}$ of V has the following form

$$[\widetilde{V}] - \pi_Z^*[V] = - \sum_{k=1}^{d-e} \overline{C}_{d-e-k} \cdot \overline{X} \cap \left(-[\widetilde{W}]^k \right) \in R^*(\widetilde{Z})$$

where $d = \mathrm{codim}_Z V$, $e = \mathrm{codim}_W X$.

For the normal bundle of the proper transform we have

$$c(\mathcal{N}_{\tilde{V}/\tilde{Z}}) - \pi_V^* c(\mathcal{N}_{V/Z}) = \left[Q(\bar{N}_m, \bar{C}_n, [\tilde{W}]) \cap \tilde{W} \right]_{|\tilde{V}} \in R^*(\tilde{V}).$$

Proof. — By [16, Theorem 6.7] we have

$$(A.4) \quad [\tilde{V}] = \pi_Z^*[V] + \tilde{l}_* \left\{ \frac{c(\pi_W^* \mathcal{N}_{W/Z})}{c(\mathcal{O}(\tilde{W}))} \cap \pi_W^*(c^{-1}(\mathcal{N}_{X/Y}) \cap [X]) \right\}^{d-1}$$

By assumption we have $c(\mathcal{N}_{W/Z})c^{-1}(\mathcal{N}_{X/Y}) = j^*l^*\bar{C}, [X] = l^*\bar{X}$ for $\bar{C}, \bar{X} \in R^*(Z)$.

We can now rewrite (A.4) as

$$\begin{aligned} [\tilde{V}] &= \pi_Z^*[V] + \tilde{l}_* \left\{ \frac{\tilde{l}^* \pi_Z^* \bar{C}}{c(\mathcal{O}(\tilde{W}))} \cap \tilde{l}^* \pi_Z^* \bar{X} \right\}^{d-1} \\ &= \pi_Z^*[V] + \tilde{l}_* \sum_{k=1}^{d-e} \tilde{l}^* \bar{C}_{d-e-k} \cdot \tilde{l}^* \bar{X} \cap \left(-c_1(\mathcal{O}(\tilde{W}))^{k-1} \right) \\ &= \pi_Z^*[V] - \sum_{k=1}^{d-e} \bar{C}_{d-e-k} \cdot \bar{X} \cap (-[\tilde{W}])^k \in \text{CH}^d(\tilde{Z}) \end{aligned}$$

where $e = \text{codim}_W X$. In particular the difference $[\tilde{V}] - \pi_Z^*[V]$ lies in $R^*(\tilde{Z})$.

We now proceed with the normal bundle of the proper transform. By [1, Theorem 4.2] we can write

$$\begin{aligned} c(\mathcal{N}_{\tilde{V}/\tilde{Z}}) \cap \alpha - \pi_V^*(\mathcal{N}_{V/Z}) \cap \alpha &= \tilde{i}_* \left[Q(\pi_X^* c_m(\mathcal{N}_{X/W}), \pi_X^* C_n, c_1(\mathcal{O}_{\tilde{X}}(-1)) \cap \tilde{i}^* \alpha) \right] \\ &= \sum_{k=0} \tilde{i}_* \left[Q_k(\bar{N}_m, \bar{C}_n) \cdot c_1(\mathcal{O}_{\tilde{X}}(-1))^k \cap \tilde{i}^* \alpha \right] \\ &= \sum_{k=0} Q_k(\bar{N}_m, \bar{C}_n) \cdot [\tilde{X}]^{k+1} \cap \alpha \\ &= \left[\sum_{k=0} \left(Q_k(\bar{N}_m, \bar{C}_n) \cdot [\tilde{W}]^{k+1} \right) \cap \alpha \right]_{|\tilde{V}}. \quad \square \end{aligned}$$

Proof of Corollary A.14. — By Proposition A.15 the differences

$$[\tilde{V}] - \pi_Z^*[V], \quad c(\mathcal{N}_{\tilde{V}/\tilde{Z}}) - \pi_V^* \mathcal{N}_{V/Z}$$

are both defined over R^* and can both be expressed solely in terms of the total Chern classes $P_{X/V}, P_{X/W}, P_{W/Z}$ and the exceptional divisor $\widetilde{W}$. In particular, if $P_{V/Z}$ is defined over R^* the same is true for $P_{\widetilde{V}/\widetilde{Z}}$.

For the second claim we recall that $\pi_{Z,*}\pi_Z^* = \text{id}$, $\pi_{V,*}\pi_V^* = \text{id}$ and furthermore that $\pi_{Z,*}[\widetilde{W}]^k$ can be computed in terms of the Segre classes $s(W, Z) = c^{-1}(\mathcal{N}_{W/Z}) \cap [W]$. Similarly $\pi_{V,*}[\widetilde{X}]^k$ can be computed in terms of the Segre classes $s(X, V) = c^{-1}(\mathcal{N}_{X/V}) \cap [X]$. All these classes can be read off from the total Chern classes $P_{W/Z}$ and $P_{X/V}$. $\square$

A.4. Systems of regular embeddings

So far we have discussed total Chern classes of proper transforms for a single blowup, we now generalize the computation of for a sequence of blowups. Roughly speaking, if we blowup Z iteratively in a sequence of subspaces $W_1, \dots, W_n$ and can compute the total Chern classes of W_i as well as the total Chern class of all finite intersections of different W_j , then we can compute the total Chern class of the proper transform of some subspace V , provided we also know the total Chern class of all intersections $V \cap W$ where W is any finite intersection of varieties W_i .

DEFINITION A.16. — *A collection S of substacks of Z is called a system of regular embeddings in Z if:*

- every stack $Y \in S$ is an Alexander stack,
- $Z \in S$ and S is closed under finite intersections,
- for any $A, B, C \in S$ with $A, B \subseteq C$ the intersection of A and B inside C is clean.

A building set $B = (W_1, \dots, W_n)$, $W_i \in S$ is an ordered list of subspaces such that W_i and W_j are not contained in each other for $i \neq j$.

Starting with a system S of regular embeddings in Z and a building set $B \subseteq S$ we construct an iterative blowup $\widetilde{Z}$ as follows. We start by setting

$$\widetilde{Z}_0 = Z, \quad S_0 = S, \quad B_0 = B, \quad \widetilde{Z}_0 := Z_0, \quad \widetilde{W}^{(0)} = W \text{ for all } W \in S_0.$$

Suppose we have already constructed $\widetilde{Z}_i, \widetilde{W}^{(i)}, S_i, B_i$. We then set $\widetilde{Z}_{i+1} = \text{Bl}_{W_{i+1}^{(i)}} \widetilde{Z}_i$. For any subspace $W^{(i)} \in S_i$ we let $W^{(i+1)}$ be the proper transform of $W^{(i)}$ if $W^{(i)}$ is not contained in $W_{i+1}^{(i)}$ and the preimage under the blowup otherwise. We call $W^{(i+1)}$ the i^{th} iterated transform of W .

Afterwards we define

$$S_{i+1} := \left\{ \widetilde{W}^{(i+1)} \mid \widetilde{W}^{(i)} \in S_i, \ W \not\subseteq W_{i+1} \right\},$$

$$B_{i+1} = \left(\widetilde{W}_{i+2}^{(i+1)}, \dots, \widetilde{W}_n^{(i+1)} \right).$$

We call the resulting space $\widetilde{Z} := \widetilde{Z}_n$ the iterated blowup associated to B .

PROPOSITION A.17. — Suppose S is system of regular embeddings in Z with building set B .

For any i the equality

$$\widetilde{A}_1^{(i)} \cap \widetilde{A}_2^{(i)} = \widetilde{A_1 \cap A_2}^{(i)}$$

holds for all $\widetilde{A}_1, \widetilde{A}_2 \in S$. If A_1 and A_2 intersect algebraically transversely, then the same is true for $\widetilde{A}_1^{(i)}, \widetilde{A}_2^{(i)}$.

Furthermore, for every i the collection of subspaces S_i is a system of regular embeddings in $\widetilde{Z}_i$ and B_i is a building set.

Proof. — The proof is by induction on i . The case $i = 0$ is true by definition.

For the rest of the proof we set $X_1 = \widetilde{A}_1^{(i-1)}$, $X_2 = \widetilde{A}_2^{(i-1)}$, $X_3 = W_i^{(i-1)}$.

If we have already proven that S_{i-1} is a system of regular embeddings, then it follows that S_i is also a system of regular embeddings since we can explicitly compute regular sequences defining the ideal of the proper transform (see (A.3)) and the preimage.

We can reduce the local case where $Z = \text{Spec } R$ and $X_i, i = 1, 2, 3$ are affine. Since outside X_3 the blowup is an isomorphism, it suffices to check the equality near $x \in X_3$. Let $\mathcal{I}_i$ be the ideal defining X_i in a neighborhood of x . We claim that there exists tuples of regular sequences $a, b_{12}, b_{13}, b_{23}, c_1, c_2, c_3$ such that the concatenation $(a, b_{12}, b_{13}, b_{23}, c_1, c_2, c_3)$ is a regular sequence in R and

$$\begin{aligned} \mathcal{I}_1 &= (a, b_{12}, b_{13}, c_1), \\ \mathcal{I}_2 &= (a, b_{12}, b_{23}, c_2), \\ \mathcal{I}_3 &= (a, b_{13}, b_{23}, c_3), \\ \mathcal{I}_1 + \mathcal{I}_2 &= (a, b_{12}, b_{13}, b_{23}, c_1, c_2), \\ \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 &= (a, b_{12}, b_{13}, b_{23}, c_1, c_2, c_3). \end{aligned}$$

Let $x \in Z$. To construct such a tuple of regular sequence in a neighborhood of x we can lift a suitable basis for the $\kappa(x)$ -vector space $(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3) / \mathfrak{m}_x(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)$. By [28, Tag 067N] the lift is a regular sequence.

Next we can compute the ideal sheaves of the proper transform following Appendix A.2. For every element $\alpha \in \mathcal{I}_3$ we introduce a corresponding projective coordinate $\tilde{\alpha}$. Let $\tilde{a}, \tilde{b}_{13}, \tilde{b}_{23}, \tilde{c}_3$ be the corresponding tuples of projective coordinates. As before the sequence $(\tilde{a}, \tilde{b}_{12}, \tilde{b}_{13}, \tilde{b}_{23}, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$ is a regular sequence on the blowup $\tilde{Z}_i$ and for the ideal sheaves $\tilde{\mathcal{I}}_i$ of the proper transforms we find

$$\begin{aligned}\tilde{\mathcal{I}}_1 &= (\tilde{a}, \tilde{b}_{12}, \tilde{b}_{13}, \tilde{c}_1), \\ \tilde{\mathcal{I}}_2 &= (\tilde{a}, \tilde{b}_{12}, \tilde{b}_{23}, \tilde{c}_2), \\ \tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2 &= (\tilde{a}, \tilde{b}_{12}, \tilde{b}_{13}, \tilde{b}_{23}, \tilde{c}_1, \tilde{c}_2).\end{aligned}$$

Therefore $\tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2 = \widetilde{\mathcal{I}_1 + \mathcal{I}_2}$. □

DEFINITION A.18. — *Let S a system of regular embeddings and B a building set. We say that S has enough total Chern classes in R^* if for $V, W \in S$ with $V \subseteq W$ the total Chern class $P_{V/W}$ exists and is defined over R^* .*

THEOREM A.19. — *Let S be a system of regular embeddings that has enough total Chern classes and B a building set with iterated blowup $\tilde{Z} \rightarrow Z$.*

Let $A \subseteq Z$, not necessarily contained in S . We assume that for any $W \in S$ the intersection $W \cap A$ is clean and that all total Chern classes

$$P_{A \cap W/A}, P_{A \cap W/W}, P_{\tilde{A}/\tilde{Z}}$$

are defined over R^ .*

Then A has a total Chern class defined over $R^(Z)$ and it can be computed explicitly.*

Proof. — By induction and Proposition A.17 we can reduce to the case of a single blowup, i.e., $B = \{W_1\}$. Since all total Chern classes

$$P_{A \cap W_1/A}, P_{A \cap W_1/W_1}, P_{W_1/Z}, P_{\tilde{A}/\tilde{Z}}$$

are defined over R^* , the claim follows from Corollary A.14. □

BIBLIOGRAPHY

- [1] P. ALUFFI, “Chern classes of blow-ups”, *Math. Proc. Camb. Philos. Soc.* **148** (2010), no. 2, p. 227-242.
- [2] Y. BAE, D. HOLMES, R. PANDHARIPANDE, J. SCHMITT & R. SCHWARZ, “Pixton’s formula and Abel–Jacobi theory on the Picard stack”, 2021, <https://arxiv.org/abs/2004.08676>.
- [3] M. BAINBRIDGE, D. CHEN, Q. GENDRON, S. GRUSHEVSKY & M. MÖLLER, “The moduli space of multi-scale differentials”, 2019, <https://arxiv.org/abs/1910.13492>.

- [4] F. BENIRSCHKE, “The boundary of linear subvarieties in strata of differentials”, *J. Eur. Math. Soc.* **25** (2023), no. 11, p. 4521-4582.
- [5] F. BENIRSCHKE, B. DOZIER & S. GRUSHEVSKY, “Equations of linear subvarieties of strata of differentials”, *Geom. Topol.* **26** (2022), no. 6, p. 2773-2830.
- [6] A. BURYAK, P. ROSSI & D. ZVONKINE, “Moduli spaces of residueless meromorphic differentials and the KP hierarchy”, 2021, <https://arxiv.org/abs/2110.01419>.
- [7] D. CHEN, S. GRUSHEVSKY, D. HOLMES, M. MÖLLER & J. SCHMITT, “A tale of two moduli spaces: logarithmic and multi-scale differentials”, 2022, <https://arxiv.org/abs/2212.04704>.
- [8] D. CHEN, M. MÖLLER & A. SAUVAGET, “Masur-Veech volumes and intersection theory: the principal strata of quadratic differentials”, 2019, <https://arxiv.org/abs/1912.02267>.
- [9] M. COSTANTINI, M. MÖLLER & J. SCHWAB, “Chern classes of linear submanifolds with application to spaces of k -differentials and ball quotients”, 2023, <https://arxiv.org/abs/2303.17929v1>.
- [10] M. COSTANTINI, M. MÖLLER & J. ZACHHUBER, “The Chern classes and the Euler characteristic of the moduli space of Abelian differentials”, 2020, <https://arxiv.org/abs/2006.12803>.
- [11] ———, “diffstrata – a Sage package for calculations in the tautological ring of the moduli space of Abelian differentials”, 2020, <https://arxiv.org/abs/2006.12815>.
- [12] V. DELECROIX, J. SCHMITT & J. VAN ZELM, “admcycles – a Sage package for calculations in the tautological ring of the moduli space of stable curves”, *J. Softw. Algebra Geom.* **11** (2021), no. 1, p. 89-112.
- [13] D. EISENBUD & J. HARRIS, “The Kodaira dimension of the moduli space of curves of genus ≥ 3 ”, *Invent. Math.* **90** (1987), no. 2, p. 359-387.
- [14] C. FABER & N. PAGANI, “The class of the bielliptic locus in genus 3”, *Int. Math. Res. Not.* **2015** (2015), no. 12, p. 3943-3961.
- [15] C. FABER & R. PANDHARIPANDE, “Relative maps and tautological classes”, *J. Eur. Math. Soc.* **7** (2005), no. 1, p. 13-49.
- [16] W. FULTON, *Intersection theory*, 2nd ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 2*, Springer, 1998.
- [17] W. FULTON & R. MACPHERSON, “A Compactification of Configuration Spaces”, *Ann. Math.* **139** (1994), no. 1, p. 183-225.
- [18] T. GRABER & R. PANDHARIPANDE, “Constructions of nontautological classes on moduli spaces of curves”, *Mich. Math. J.* **51** (2003), no. 1, p. 93-109.
- [19] B. KLINGLER & L. A. LERER, “Abelian differentials and their periods: the bialgebraic point of view”, 2022, <https://arxiv.org/abs/2202.06031>.
- [20] C. LIAN, “The $\mathcal{H}$ -tautological ring”, *Sel. Math., New Ser.* **27** (2021), article no. 96 (74 pages).
- [21] S. MOLCHO, R. PANDHARIPANDE & J. SCHMITT, “The Hodge bundle, the universal 0-section, and the log Chow ring of the moduli space of curves”, *Compos. Math.* **159** (2023), no. 2, p. 306-354.
- [22] S. MULLANE, “Strata of differentials of the second kind, positivity, and irreducibility of certain Hurwitz spaces”, *Ann. Inst. Fourier* **72** (2022), no. 4, p. 1379-1416.
- [23] D. MUMFORD, “Towards an enumerative geometry of the moduli space of curves”, in *Arithmetic and geometry. Papers dedicated to I. R. Shafarevich on the occasion of his sixtieth birthday. Vol. II: Geometry*, Progress in Mathematics, vol. 36, Birkhäuser, 1983, p. 271-328.
- [24] E. PERVOVA & C. PETRONIO, “On the existence of branched coverings between surfaces with prescribed branch data. I”, *Algebr. Geom. Topol.* **6** (2006), no. 4, p. 1957-1985.

- [25] A. SAUVAGET, “Théorie de l’intersection sur les espaces de différentielles holomorphes et méromorphes”, PhD Thesis, Université Paris 6, Paris, France, 2017.
- [26] ———, “Cohomology classes of strata of abelian differentials”, *Geom. Topol.* **23** (2019), no. 3, p. 1085-1171.
- [27] J. SCHMITT & J. VAN ZELM, “Intersections of loci of admissible covers with tautological classes”, *Sel. Math., New Ser.* **26** (2020), article no. 79 (69 pages).
- [28] THE STACKS PROJECT AUTHORS, “Stacks”, <https://stacks.math.columbia.edu>, 2023.
- [29] A. VISTOLI, “Alexander duality in intersection theory”, *Compos. Math.* **70** (1989), no. 3, p. 199-225.
- [30] ———, “Intersection theory on algebraic stacks and on their moduli spaces”, *Invent. Math.* **97** (1989), no. 3, p. 613-670.

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