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RELATIVE ÉTALE SLICES AND COHOMOLOGY OF MODULI SPACES

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ABSTRACT. — We use techniques of Alper–Hall–Rydh to prove a local structure theorem for smooth morphisms between smooth stacks around points with linearly reductive stabilizers. This implies that the good moduli space of a smooth stack over a base has equisingular fibers. As an application, we show that any two fibers have isomorphic ℓ -adic cohomology rings and intersection cohomology groups. If we work over the complex numbers, we show that the family is topologically locally trivial on the base, and that the intersection cohomology groups of the fibers fit into a polarizable variation of pure Hodge structures. We apply these results to derive some consequences for the moduli spaces of G -bundles on smooth projective curves, and for certain moduli spaces of sheaves on del Pezzo surfaces.

RÉSUMÉ. — On utilise des techniques d’Alper–Hall–Rydh pour démontrer un théorème de structure locale pour des morphismes lisses entre deux champs lisses autour de points ayant des stabilisateurs linéairement réductifs. Cela implique que le bon espace de modules d’un champ lisse sur une base a des fibres équisingulières. Comme application, nous montrons que les anneaux de cohomologie ℓ -adique et les groupes de cohomologie d’intersection de deux fibres quelconques sont isomorphes. Si l’on travaille sur le corps des nombres complexes, nous montrons que la famille est localement triviale sur la base dans le sens topologique, et que les groupes de cohomologie d’intersection des fibres s’inscrivent dans une variation polarisable de structure de Hodge pure. Nous appliquons ces résultats pour en déduire certaines conséquences pour les espaces de modules de fibrés principaux sur des courbes projectives lisses, ainsi que pour certains espaces de modules de faisceaux sur des surfaces de del Pezzo.

1. Introduction

The cohomology of smooth projective varieties is locally constant in families. More precisely, if $X \rightarrow S$ is a smooth projective family of varieties

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over a connected base S , then the (ℓ -adic) cohomology groups of any two fibers are non-canonically isomorphic via specialization. If we work over the complex numbers, then the singular cohomology groups of the family $X \rightarrow S$ fit into a polarizable variation of pure Hodge structures. In particular, certain invariants, such as the Hodge numbers, are locally constant in smooth projective families.

In moduli theory, it is common to encounter moduli problems that naturally vary over a base S . Such moduli problems are often represented by an algebraic stack $\mathcal{X} \rightarrow S$. One can then employ the techniques of Geometric Invariant Theory [44], or more modern stack-theoretic methods [4], to construct a good moduli space $M \rightarrow S$ for the stack $\mathcal{X}$. Even when the stack $\mathcal{X}$ is smooth over S , the moduli space $M \rightarrow S$ might have singular fibers. In practice, this arises when there are strictly semistable objects in our moduli problem.

The main observation in this paper is that the desirable behavior of the cohomology of smooth projective families also holds for projective morphisms $M \rightarrow S$ that are obtained as good moduli spaces of smooth stacks $\mathcal{X} \rightarrow S$.

To that effect, we first prove a local structure theorem for smooth morphisms of smooth stacks at points with linearly reductive stabilizers (Proposition 2.1). It states that the étale local tubular neighborhoods of smooth points of stacks obtained in [2, Theorem 1.2] are functorial with respect to smooth morphisms. Proposition 2.1 should be thought of as a relative version of Luna étale slice theorem in the spirit of [3]. Alper–Janda have given an alternative proof of Proposition 2.1 [5, Corollary 5.1].

As a consequence of Proposition 2.1, we obtain the following (see Corollary 2.4 for a more precise formulation).

THEOREM 1.1. — *Let k be an algebraically closed field. Let $\mathcal{X} \rightarrow S$ be a smooth stack with affine relative diagonal over a smooth algebraic space S over k . If $\mathcal{X}$ admits a good moduli space M , then the fibers of $M \rightarrow S$ are equisingular.*

This equisingularity result, combined with the machinery of vanishing cycles, allows us to relate the cohomology of such moduli spaces as they vary in a family. For instance, we obtain the following simple description of the decomposition theorem, generalizing the classical theorem of Deligne on decomposition for projective smooth morphisms [14].

THEOREM 1.2 (Theorem 3.11). — *Let $f: M \rightarrow S$ be a proper morphism of schemes such that S is smooth over an algebraically closed field. Suppose that M is the good moduli space of a smooth algebraic stack $\mathcal{X} \rightarrow S$*

with affine relative diagonal. Then the pushforward $Rf_*\mathcal{IC}_M$ of the topologist's intersection complex $\mathcal{IC}_M$ (as in Section 3.1) admits a direct sum decomposition

$$Rf_*\mathcal{IC}_M \simeq \bigoplus_{i \geq 0} (R^i f_*\mathcal{IC}_M)[-i],$$

and the higher pushforward sheaves $R^i f_*\mathcal{IC}_M$ are lisse sheaves on S for all i .

Theorem 1.2 is stated in the context of ℓ -adic cohomology with coefficients in $\overline{\mathbb{Q}}_\ell$. It implies in particular that the ℓ -adic cohomology (resp. intersection cohomology) groups of any two geometric fibers of $M \rightarrow S$ are non-canonically isomorphic if S is connected; see Corollary 3.10.

If the ground field is $\mathbb{C}$, then the same results hold for singular cohomology (resp. intersection cohomology) with rational coefficients for the classical topology.

THEOREM 1.3 (Corollary 2.4; Propositions 3.12 and 3.13; Theorem 3.15). *Let $f: M \rightarrow S$ be a projective morphism of $\mathbb{C}$ -varieties such that S is smooth and connected. Suppose that M is the good moduli space of a smooth algebraic stack $\mathcal{X} \rightarrow S$ with affine relative diagonal. Then, the following hold.*

- (a) *The family $f: M \rightarrow S$ is topologically locally trivial. In particular, all S -fibers are homeomorphic.*
- (b) *The pushforward $Rf_*\mathcal{IC}_M$ of the topologist's intersection complex $\mathcal{IC}_M$ with rational coefficients admits a direct sum decomposition in the category of mixed Hodge modules*

$$Rf_*\mathcal{IC}_M \simeq \bigoplus_{i \geq 0} (R^i f_*\mathcal{IC}_M)[-i].$$

Every higher pushforward sheaf $R^i f_\mathcal{IC}_M$ is a local system underlying a polarizable variation of Hodge structures. The Hodge numbers $h^{p,q}(IH^\bullet)$ associated with the intersection cohomology of any two S -fibers agree.*

- (c) *The higher pushforwards $R^i f_*\mathbb{Q}_M$ of the constant sheaf $\mathbb{Q}_M$ are local systems underlying an admissible variation of mixed Hodge structures. The mixed Hodge numbers $h^{p,q} \operatorname{Gr}^k(H^\bullet)$ associated with the singular cohomology of any two S -fibers agree.*

In Section 4, we derive three cohomological consequences as applications of our results.

For our first application, in Section 4.1 we fix a complex reductive group G and an element of the algebraic fundamental group $d \in \pi_1(G)$. We show

that, for any two complex smooth projective curves C_1, C_2 of the same genus, the corresponding moduli spaces $M_G^d(C_1), M_G^d(C_2)$ of semistable G -bundles of degree d have the same mixed Hodge numbers (resp. Hodge numbers for intersection cohomology). The same holds for the moduli of parabolic G -bundles on pointed curves of the same genus for any fixed choice of parabolic weights.

In Section 4.2 we show a similar result for moduli of Gieseker semistable torsion-free sheaves on smooth surfaces X , where semistability is taken with respect to a polarization $\mathcal{L}$ satisfying $\mathcal{L} \cdot \omega_X < 0$ (we call these ω -negatively polarized surfaces). As we vary the surface and polarization on connected families, the cohomological and Hodge-theoretic invariants of the corresponding moduli spaces of sheaves do not vary, even in the case when the stability parameters are not generic (i.e. in the presence of strictly semistable sheaves). In the case of sheaves of pure dimension 1, we show that the induced isomorphisms on intersection cohomology are compatible with the perverse filtrations induced by the Fitting support morphism (also known as the Le Potier morphism).

Notation

We work over a fixed algebraically closed ground field k . All of the stacks in this paper are algebraic stacks over k , and all morphisms are k -morphisms. Given stacks $\mathcal{X}, \mathcal{Y}$ over k , we write $\mathcal{X} \times \mathcal{Y}$ to denote the fiber product over $\mathrm{Spec}(k)$. For an algebraic space X endowed with an action of an algebraic group G , we write X/G for the quotient stack (which is denoted by $[X/G]$ in works of other authors). We use cohomological grading conventions for all triangulated categories. If $\mathcal{X}$ is a stack of finite type over k , by its derived category of quasicoherent sheaves we mean the derived category of the Grothendieck Abelian category of quasicoherent sheaves on $\mathcal{X}$. In fact, every time that we employ the quasicoherent derived category, [21, Corollary 9.2(5)] applies, so that there should be no ambiguity with other versions of the derived category.

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2. Relative étale slices

2.1. Preliminaries and notation

Let $\mathcal{X}$ be an algebraic stack, quasi-separated and locally of finite type over k . Let $x \in \mathcal{X}(k)$ be a closed point in $\mathcal{X}$, and let $i: BG_x \rightarrow \mathcal{X}$ be the residual gerbe at x , where G_x is the stabilizer group of x . The closed immersion i is cut out by an ideal $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$. The *normal space* N_x to x is the k -vector space $(\mathcal{I}/\mathcal{I}^2)^\vee = (i^*\mathcal{I})^\vee$ with the structure of G_x -representation. If x is a smooth point and G_x is smooth over k , then N_x is the tangent space $T_{\mathcal{X},x}$ of the stack at x . We denote $\mathcal{N}_x := N_x/G_x = \mathrm{Spec}_{BG_x}(\mathrm{Sym}(i^*\mathcal{I}))$ and call it the *normal stack* of $\mathcal{X}$ at x . Let $0 \in \mathcal{N}_x(k)$ denote the point corresponding to the origin 0 in the vector space N_x . If $\mathcal{Y}$ is another algebraic stack which is quasi-separated and locally of finite type over k , and $f: \mathcal{X} \rightarrow \mathcal{Y}$ maps the closed point $x \in \mathcal{X}(k)$ to a closed point $y \in \mathcal{Y}(k)$, then there is an induced map $\mathcal{N}_x \rightarrow \mathcal{N}_y$ on normal stacks.

If $X = \mathrm{Spec}(A)$ is an affine scheme over k and G is a linearly reductive algebraic group, then $X/G \rightarrow \mathrm{Spec}(A^G)$ is a good moduli space. We use the notation $X//G := \mathrm{Spec}(A^G)$ for the affine GIT quotient.

If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic stacks admitting good moduli spaces $\mathcal{X} \rightarrow X$ and $\mathcal{Y} \rightarrow Y$, then a morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ is said to be *strongly étale* if the induced morphism $X \rightarrow Y$ is étale and the square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is Cartesian (see [3, Definition 3.13]).

2.2. Slices for smooth morphisms of smooth stacks

THEOREM 2.1. — *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a smooth k -morphism between quasi-separated smooth algebraic stacks over k with affine stabilizers. Let $x \in \mathcal{X}(k)$ be a closed point such that $y := f(x)$ is closed. Suppose the stabilizers G_x and G_y are linearly reductive and that G_y is smooth. Then there is a commutative diagram of pointed stacks*

$$(2.1) \quad \begin{array}{ccccc} (\mathcal{N}_x, 0) & \xleftarrow{a_2} & (\mathcal{U}, u) & \xrightarrow{a_1} & (\mathcal{X}, x) \\ h \downarrow & & \downarrow & & \downarrow f \\ (\mathcal{N}_y, 0) & \xleftarrow{b_2} & (\mathcal{V}, v) & \xrightarrow{b_1} & (\mathcal{Y}, y), \end{array}$$

where

- (1) the maps a_2, b_2 are affine and strongly étale,
- (2) the maps a_1, b_1 are étale and induce isomorphisms $G_u \xrightarrow{\sim} G_x$ and $G_v \xrightarrow{\sim} G_y$ of stabilizer groups, and
- (3) the map h is the one induced by f on normal spaces.

In addition, if $\mathcal{X}$ (resp. $\mathcal{Y}$) has affine diagonal, then a_1 (resp. b_1) can be arranged to be affine. Moreover, if $\mathcal{X}$ (resp. $\mathcal{Y}$) has affine diagonal and a good moduli space, then it can be arranged that a_1 (resp. b_1) is affine and strongly étale.

Remark 2.2. — Note that $\mathcal{U}$ and $\mathcal{V}$ have good moduli spaces because $\mathcal{N}_x$ and $\mathcal{N}_y$ do and a_2 and b_2 are affine [1, Lemma 4.14].

Proof. — Since the result is local at the source and target, we may assume that $\mathcal{X}$ and $\mathcal{Y}$ are quasi-compact. By [2, Theorem 1.2], we can find $(\mathcal{V}, v)$, b_1 and b_2 as in (2.1), with b_2 affine and strongly étale. Forming the fiber product $\mathcal{Z} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{V}$, we get a diagram

$$\begin{array}{ccc} (\mathcal{Z}, z) & \xrightarrow{b'_1} & (\mathcal{X}, x) \\ f' \downarrow & \lrcorner & \downarrow f \\ (\mathcal{N}_y, 0) & \xleftarrow{b_2} (\mathcal{V}, v) \xrightarrow{b_1} & (\mathcal{Y}, y), \end{array}$$

where b'_1 is étale and induces an isomorphism of stabilizer groups at z . Replacing $\mathcal{X}$ by $\mathcal{Z}$ and f by the composition $\mathcal{Z} \rightarrow \mathcal{V} \rightarrow \mathcal{N}_y$, we may assume that $(\mathcal{Y}, y) = (\mathcal{N}_y, 0)$. In order to construct diagram (2.1), we are reduced to producing a dashed étale roof $(\mathcal{U}, u)$ fitting into the following commutative diagram

$$\begin{array}{ccc} & (\mathcal{U}, u) & \\ a_2 \swarrow \text{dashed} & & \searrow \text{dashed} a_1 \\ (\mathcal{N}_x, 0) & & (\mathcal{X}, x) \\ h \searrow & & \swarrow f \\ & (\mathcal{N}_y, 0) & \end{array}$$

with a_2 affine and strongly étale and with a_1 étale and inducing an isomorphism of stabilizer groups at u and x .

The point x defines a closed immersion $BG_x \rightarrow \mathcal{X}$ cut out by an ideal sheaf $\mathcal{I}$. Let us denote by $\mathcal{X}^{[n]}$ the n -th thickening of x , that is the closed substack of $\mathcal{X}$ with ideal sheaf $\mathcal{I}^{n+1}$. Similarly, we denote by $\mathcal{N}_x^{[n]}$ and $\mathcal{N}_y^{[n]}$

the n -th order thickenings of 0 in $\mathcal{N}_x$ and $\mathcal{N}_y$, respectively. We begin by showing the following

CLAIM. — *There is an isomorphism $\mathcal{N}_x^{[1]} \cong \mathcal{X}^{[1]}$ fitting in a commutative diagram*

$$(2.2) \quad \begin{array}{ccc} \mathcal{N}_x^{[1]} & \xrightarrow{\sim} & \mathcal{X}^{[1]} \\ \downarrow & & \downarrow \\ \mathcal{N}_x & & \mathcal{X} \\ & \searrow h & \swarrow f \\ & \mathcal{N}_y & \end{array}$$

Proof of the claim. — First note that we have canonical isomorphisms $BG_x \cong \mathcal{N}_x^{[0]}$ and $BG_x \cong \mathcal{X}^{[0]}$ over $\mathcal{N}_y$. Now consider the commutative diagram of solid arrows

$$(2.3) \quad \begin{array}{ccccc} & & \mathcal{X}^{[0]} & & \\ & \swarrow & \downarrow & \searrow \sim & \\ \mathcal{X}^{[1]} & \xrightarrow{\quad \quad} & BG_x & & \\ \downarrow & & \downarrow & & \downarrow c \\ & \swarrow & \mathcal{N}_y^{[0]} & \searrow \sim & \\ \mathcal{N}_y^{[1]} & \xrightarrow{\quad \quad} & BG_y & & \end{array}$$

where all vertical arrows are the ones induced by f . We shall produce a dashed arrow making the diagram commutative. By [36, Theorem 8.5], the obstruction to the existence of such a dashed arrow lives in $\mathrm{Ext}_{BG_x}^1(\mathbb{L}_c, \mathcal{I}/\mathcal{I}^2)$. Since G_y is smooth, the morphism c is smooth, so the cotangent complex $\mathbb{L}_c$ is concentrated in degrees $[0, 1]$. Moreover, since G_y is linearly reductive, the category $\mathrm{QCoh}(BG_y)$ of G_y -representations is semisimple. It follows that

$$\mathrm{Ext}_{BG_x}^1(\mathbb{L}_c, \mathcal{I}/\mathcal{I}^2) = \mathrm{Hom}_{BG_x}(\mathbb{L}_c, \mathcal{I}/\mathcal{I}^2[1]) = 0,$$

and a desired dashed arrow as in (2.3) exists.

Since $\mathcal{X}^{[0]} \rightarrow \mathcal{X}^{[1]}$ is a surjective closed immersion and $\mathcal{X}^{[0]} \rightarrow BG_x$ is representable, we have that $\mathcal{X}^{[1]} \rightarrow BG_x$ is representable too. Since $\mathcal{X}^{[0]} \rightarrow \mathcal{X}^{[1]}$ is a square-zero extension and $\mathcal{X}^{[0]} \rightarrow BG_x$ is affine, by dévissage the map $\mathcal{X}^{[1]} \rightarrow BG_x$ is cohomologically affine. By Serre's criterion for algebraic spaces [46, Tag 07V6], we have that $\mathcal{X}^{[1]} \rightarrow BG_x$ is affine.

Diagram (2.3) induces compatible splittings d_1 and d_2 in the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N_x^\vee & \longrightarrow & \mathcal{O}_{\mathcal{X}^{[1]}} & \xleftarrow{d_1} & \mathcal{O}_{BG_x} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & c^* N_y^\vee & \longrightarrow & c^* \mathcal{O}_{\mathcal{N}_y^{[1]}} & \xleftarrow{d_2} & c^* \mathcal{O}_{BG_y} \longrightarrow 0
 \end{array}$$

of short exact sequences in $\mathrm{QCoh}(BG_x)$. This gives an isomorphism $\mathcal{X}^{[1]} \cong \mathcal{N}_x^{[1]} = \mathrm{Spec}_{BG_x}(\mathcal{O}_{BG_x} \oplus \varepsilon N_x^\vee)$, where $\varepsilon^2 = 0$, commuting with the maps into $\mathcal{N}_y^{[1]}$. The claim is thus established. $\square$

By [3, Proposition 7.18(2)], there is an affine and strongly étale neighborhood

$$a_2: (\mathcal{U}, u) \longrightarrow (\mathcal{N}_x, 0)$$

and a morphism $a_1: (\mathcal{U}, u) \rightarrow (\mathcal{X}, x)$ over $\mathcal{N}_y$ extending the map $\mathcal{N}_x^{[1]} \xrightarrow{\sim} \mathcal{X}^{[1]} \rightarrow \mathcal{X}$ from the claim.

Arguing as in the proof of [2, Theorem 1.2], by smoothness of $\mathcal{X}$ at x and since a_1 induces an isomorphism $\mathcal{U}^{[1]} \cong \mathcal{X}^{[1]}$ of first order thickenings at u and x , it follows from [2, Proposition A.8] that a_1 also induces isomorphisms $\mathcal{U}^{[n]} \rightarrow \mathcal{X}^{[n]}$ of n th order thickenings at u and x for all n . Therefore a_1 is étale at u and thus there is an open neighborhood $\mathcal{U}_1$ of u in $\mathcal{U}$ such that $a_1|_{\mathcal{U}_1}$ is étale. Since u is a closed point of $\mathcal{U}$, there is another open neighborhood $\mathcal{U}_2$ of u that is saturated with respect to the good moduli space of $\mathcal{U}$ and is contained in $\mathcal{U}_1$. Therefore, after shrinking $\mathcal{U}$ we may assume that a_1 is étale, while keeping a_2 affine and strongly étale. The map a_1 induces isomorphism of stabilizer groups at u because $\mathcal{U}^{[0]} \rightarrow \mathcal{X}^{[0]}$ is an isomorphism of residual gerbes. We have proved the first part of the theorem.

If $\mathcal{X}$ has affine diagonal, then by [2, Proposition 3.2] there is an open substack $\mathcal{U}' \subset \mathcal{U}$ containing u , saturated with respect to the good moduli space $\pi: \mathcal{U} \rightarrow U$ of $\mathcal{U}$, such that the composition $\mathcal{U}' \rightarrow \mathcal{U} \rightarrow \mathcal{X}$ is affine. Since $\pi^{-1}\pi(\mathcal{U}') = \mathcal{U}'$, the map $\mathcal{U}' \rightarrow \mathcal{U}$ is in particular strongly étale. If moreover $\mathcal{X}$ has a good moduli space, then by Luna's fundamental lemma for stacks [3, Theorem 3.14] there is a π -saturated open neighborhood $\mathcal{U}''$ of u in $\mathcal{U}'$ such that $\mathcal{U}'' \rightarrow \mathcal{X}$ is strongly étale. The same arguments work for $\mathcal{Y}$. $\square$

Remark 2.3. — We thank David Rydh for pointing out that [3, Proposition 7.18(2)] can be used to shorten our original proof of Proposition 2.1. The original argument used Tannaka duality and Artin approximation,

modifying the arguments in [2] to ensure that the relevant morphisms commute with the maps to the base algebraic stack $\mathcal{N}_y$. We refer to the first preprint version of this article [10, Theorem 2.1] for details.

In the setting of Proposition 2.1, suppose that the base $\mathcal{Y} = S$ is an algebraic space. Let $N_{f,x}$ denote the kernel of $N_x \rightarrow N_{f(x)}$. By the linear reductivity of G_x , we have an isomorphism of G_x -representations $N_x \cong N_{f,x} \oplus T_{S,f(x)}$, where G_x acts trivially on the tangent space $T_{S,f(x)}$.

COROLLARY 2.4 (Equisingularity of good moduli spaces of smooth stacks). — *Let $f: \mathcal{X} \rightarrow S$ be a smooth algebraic stack with affine diagonal over a smooth k -algebraic space S . Suppose that $\mathcal{X}$ admits a good moduli space $\bar{f}: M \rightarrow S$. Let $x \in \mathcal{X}(k)$ be a closed point with corresponding point $x \in M(k)$. Then there is a commutative diagram of pointed algebraic spaces*

$$(2.4) \quad \begin{array}{ccccc} (T_{S,\bar{f}(x)} \times (N_{f,x} // G_x), (0,0)) & \xleftarrow{a_2} & (U,u) & \xrightarrow{a_1} & (M,x) \\ \downarrow h & & \downarrow g & & \downarrow \bar{f} \\ (T_{S,\bar{f}(x)}, 0) & \xleftarrow{b_2} & (V,v) & \xrightarrow{b_1} & (S,\bar{f}(x)) \end{array}$$

where a_1, a_2 and b_1, b_2 are étale.

Proof. — The hypothesis that $\mathcal{X}$ admits a good moduli space implies that the closed k -point $x \in \mathcal{X}(k)$ has linearly reductive stabilizer [1, Proposition 12.4]. Hence, the corollary follows by taking good moduli spaces in diagram (2.1) of Proposition 2.1, where the horizontal morphisms can be arranged to be strongly étale because f has affine diagonal. $\square$

In plain words, Corollary 2.4 says that the singularities of a family $\bar{f}: M \rightarrow S$ obtained as the good moduli space of a smooth S -stack are étale locally constant.

Remark 2.5. — David Rydh has explained to us in a private communication that it is also possible to deduce Corollary 2.4 from [3, Corollary 8.7(8)].

3. Cohomology in families

In this section, we shall prove results about the (intersection) cohomologies of fibers of a locally isotrivial morphism in the following sense.

DEFINITION 3.1. — *Let $f: \mathcal{M} \rightarrow \mathcal{S}$ be a morphism of finite type algebraic stacks over k . We say that f is locally isotrivial if there is a smooth surjective morphism $\mathcal{Y} \rightarrow \mathcal{S}$ such that the base-change $f_{\mathcal{Y}}: \mathcal{M}_{\mathcal{Y}} \rightarrow \mathcal{Y}$ fits into a commutative diagram*

$$(3.1) \quad \begin{array}{ccccc} \mathcal{T} \times \mathcal{F} & \xleftarrow{a_2} & \mathcal{U} & \xrightarrow{a_1} & \mathcal{M}_{\mathcal{Y}} \\ p_{\mathcal{T}} \downarrow & & \downarrow & & \downarrow f_{\mathcal{Y}} \\ \mathcal{T} & \xleftarrow{b_2} & \mathcal{V} & \xrightarrow{b_1} & \mathcal{Y}, \end{array}$$

where $p_{\mathcal{T}}$ is the first projection, a_1, b_1 are surjective étale morphisms, and a_2, b_2 are étale.

Corollary 2.4 says that good moduli spaces of smooth stacks over a base are locally isotrivial.

3.1. Preliminaries on ℓ -adic sheaves

For the next subsections, we fix a prime number ℓ distinct from the characteristic of the algebraically closed ground field k , and we work with $\overline{\mathbb{Q}}_{\ell}$ -sheaves for the étale topology.

Let $\mathcal{T}$ be an algebraic stack which is quasi-separated and of finite type over an algebraically closed field, or over the spectrum of a strictly Henselian discrete valuation ring. Let $D_c^b(\mathcal{T}, \overline{\mathbb{Q}}_{\ell})$ be the $\overline{\mathbb{Q}}_{\ell}$ -constructible derived category endowed with the usual six functors $(f^*, Rf_*, Rf_!, f^!, \otimes^L, R\mathcal{H}om)$. Standard references for the construction of the “ ℓ -adic constructible derived” category and of the six functors are [8, 18] in the context of schemes, and [31] in the context of quasi-separated algebraic stacks.

Let $\mathcal{M}$ be a quasi-separated finite type algebraic stack over the algebraically closed field k . We use the notion of perverse t -structure defined in [32]. Let $IC_{\mathcal{M}}$ be the perverse intersection cohomology complex of $\mathcal{M}$ with coefficients in $\overline{\mathbb{Q}}_{\ell}$, defined as the direct sum of the perverse intersection cohomology complexes of the irreducible components of $\mathcal{M}$. If $\mathcal{M}$ is irreducible, then we define the topologist’s intersection complex $\mathcal{IC}_{\mathcal{M}}$ of $\mathcal{M}$ by setting $\mathcal{IC}_{\mathcal{M}} := IC_{\mathcal{M}}[-\dim(\mathcal{M})]$. If $\mathcal{M}$ is not irreducible with irreducible components $\mathcal{M}_i$, then we define the topologist’s intersection complex of $\mathcal{M}$ by setting $\mathcal{IC}_{\mathcal{M}} := \bigoplus_i \mathcal{IC}_{\mathcal{M}_i}$.

Notation 3.2. — We denote by $H^{\bullet}(\mathcal{M}, \overline{\mathbb{Q}}_{\ell})$ the cohomology ring of $\mathcal{M}$. Similarly, we write $IH^{\bullet}(\mathcal{M}, \overline{\mathbb{Q}}_{\ell})$ for the intersection cohomology of $\mathcal{M}$.

We will also make use of the theory of nearby and vanishing cycle functors, denoted ψ and ϕ respectively, for morphisms of schemes. Standard references for nearby and vanishing cycle functors, ψ and ϕ , are [16, Exposé XIII] and [27, 28]. See also [13, Appendix Section 5.2], and [9, Section 2.1] for a partial list of the properties concerning nearby and vanishing cycle functors. The facts about nearby and vanishing cycle functors we use freely in this paper are:

- (1) the existence of the functorial distinguished triangles of functors $i^*[-1] \rightarrow \psi[-1] \rightarrow \phi \rightsquigarrow$, and $\phi \rightarrow \psi[-1] \rightarrow i^![1] \rightsquigarrow$; for the first one, see [16, Exposé XIII, 2.1.2]; for the second one, apply Verdier Duality to the first one: see [27, Théorème 4.2] for the compatibility of nearby cycles with duality; see [34, Corollary 0.2] for the compatibility of vanishing cycles with duality;
- (2) the identities $Rf_*\psi = \psi Rf_*$ and $Rf_*\phi = \phi Rf_*$ for a proper morphism f (cf. [16, Exposé XIII, 2.1.7]);
- (3) the identities $f^*\psi = \psi f^*$ and $f^*\phi = \phi f^*$ for a smooth morphism f (cf. [16, Exposé XIII, 2.1.7]);
- (4) the compatibility of ψ with cup products: $\psi(F) \otimes^L \psi(G) \xrightarrow{\sim} \psi(F \otimes^L G)$ (cf. [27, Théorème 4.7]).

Remark 3.3. — While the six-functor formalism is available for algebraic spaces and stacks, we could not locate a suitable reference for the nearby and vanishing cycle functors. In view of this, we do not state Theorem 3.4 in the generality of proper DM morphisms, and instead we restrict ourselves to proper schematic morphisms.

3.2. Étale cohomology of locally isotrivial families

In this subsection, we work with $\overline{\mathbb{Q}}_\ell$ -sheaves. Let $f: \mathcal{M} \rightarrow \mathcal{S}$ be a proper morphism between algebraic stacks which are of finite type and with affine diagonal over the algebraically closed field k .

By the Decomposition Theorem [7, 47] there is a decomposition

$$(3.2) \quad Rf_*(\mathcal{IC}_{\mathcal{M}}) \cong \bigoplus_i {}^p\mathcal{H}^i(Rf_*(\mathcal{IC}_{\mathcal{M}}))[-i]$$

of the direct image into the finite direct sum of its shifted perverse cohomology sheaves. Up to shift, the non-trivial summands are semisimple perverse sheaves on $\mathcal{S}$ whose simple components are intersection complexes $\mathcal{IC}_{\mathcal{Z}}(L)[\dim(\mathcal{Z})]$ of closed irreducible substacks $\mathcal{Z} \subset \mathcal{S}$ (the supports) with

coefficients in irreducible lisse sheaves L (the simple coefficients) on a suitable open dense substack $\mathcal{Z}^\circ \subseteq \mathcal{Z}$.

In general, the determination of the supports and the coefficients is a difficult task. The following is one of the main theorems of this subsection, which provides some conditions that ensure that the only support is the base $\mathcal{S}$ and the coefficients are the lisse sheaves associated with the (intersection) cohomology groups of the fibers.

THEOREM 3.4 (Supports and coefficients). — *Let $f: \mathcal{M} \rightarrow \mathcal{S}$ be a proper schematic morphism between algebraic stacks which are of finite type and with affine diagonal over k . Suppose that f is locally isotrivial in the sense of Definition 3.1, and that $\mathcal{S}$ is smooth and connected. Then the following hold:*

- (a) *For every i , the direct image sheaves $R^i f_*(\overline{\mathbb{Q}}_\ell)_{\mathcal{M}}$ and $R^i f_* \mathcal{IC}_{\mathcal{M}}$ are lisse on $\mathcal{S}$.*
- (b) *We have the identity of perverse sheaves on $\mathcal{S}$*

$$(R^i f_*(\mathcal{IC}_{\mathcal{M}}))[\dim(\mathcal{S})] = {}^p\mathcal{H}^{\dim(\mathcal{S})+i}(Rf_*(\mathcal{IC}_{\mathcal{M}})).$$

- (c) *We have an isomorphism in $D_c^b(\mathcal{S}, \overline{\mathbb{Q}}_\ell)$*

$$Rf_* \mathcal{IC}_{\mathcal{M}} \simeq \bigoplus_{i \geq 0} R^i f_* \mathcal{IC}_{\mathcal{M}}[-i].$$

Our next goal in this subsection is to prove Theorem 3.4 and the upcoming Corollary 3.10. We need the following seemingly standard three lemmas, for which we could not find suitable references.

LEMMA 3.5. — *The formation of the intersection complex $\mathcal{IC}_{X_E}$ of a scheme of finite type X_E over a field E is compatible with field extensions L/E , i.e. we have a natural isomorphism $\mathcal{IC}_{X_L} = u^* \mathcal{IC}_{X_E}$, where $u: X_L := X_E \times_E L \rightarrow X_E$ is the natural projection.*

Proof. — This follows from the definition of intersection complexes as intermediate extensions (cf. [7, Définition 1.4.22]) and from the standard fact that the functor $u^*: D_c^b(X_E, \overline{\mathbb{Q}}_\ell) \rightarrow D_c^b(X_L, \overline{\mathbb{Q}}_\ell)$ is t -exact for the respective middle-perversity t -structures.

In order to establish this t -exactness, one can argue as follows. Recall that the perverse t -structure $({}^pD_c^b(X_E)^{\leq 0}, {}^pD_c^b(X_E)^{\geq 0})$ on X_E is obtained by gluing the perverse t -structures on any partition $X_E = U_E \cup F_E$ into complementary open and closed subsets (cf. [7, Théorème 1.4.10]). We need to show that the functor u^* preserves the subcategories of type $({}^pD_c^b(X_?)^{\leq 0}, {}^pD_c^b(X_?)^{\geq 0})$. By Noetherian induction, we can assume that the

desired t -exactness of u^* holds when X_E is replaced by U_E and by Z_E , respectively. Consider the Cartesian diagram in [7, p. 155] (with $U = S = \operatorname{Spec}(E)$ in loc. cit.) and its Conséquence in loc. cit., i.e. the compatibility of u^* with the formation of the six functors (we only need j^* for j an open immersion, and i^* and $i^!$ for i a closed immersion). This compatibility yields the desired preservation under u^* of the subcategories of type $({}^pD_c^b(X_?)^{\leq 0}, {}^pD_c^b(X_?)^{\geq 0})$. $\square$

LEMMA 3.6. — *Let $f: M \rightarrow S$ be a morphism of finite type schemes over k . Suppose that f is locally isotrivial in the sense of Definition 3.1 and that S is smooth. For every geometric point $s \rightarrow S$, let $\iota_s: M_s \rightarrow M$ denote the corresponding geometric fiber. Then, we have a canonical isomorphism $\iota_s^* \mathcal{IC}_M = \mathcal{IC}_{M_s}$.*

Proof. — Since the intersection complex $\mathcal{IC}_M$ is compatible with extension of the ground field (Lemma 3.5), we can replace k with the residue field of s and assume without loss of generality that s is a closed k -point in S . Since f is locally isotrivial, it is in particular flat. Therefore every irreducible component of M restricts to a union of irreducible components of M_s . By the definition of $\mathcal{IC}_M$, we may restrict to each irreducible component of M and assume without loss of generality that M is integral.

Consider the local model commutative diagram (3.1), where we may assume that everything is a scheme, and so we shall use roman letters V, U, F, T for the schemes appearing in the diagram. Choose k -points $0 \in T(k)$ and $v \in V(k)$ such that $b_1(v) = s$ and $b_2(v) = 0$. We take the fibers over the points $0, v$ and s and obtain the commutative diagram (with some decorations omitted)

$$(3.3) \quad \begin{array}{ccccc} F & \xleftarrow{a'_2} & U_v & \xrightarrow{a'_1} & M_s \\ \downarrow \iota_0 & & \downarrow \iota_v & & \downarrow \iota_s \\ T \times F & \xleftarrow{a_2} & U & \xrightarrow{a_1} & M \end{array}$$

where a_1, a_2 and a'_1, a'_2 are étale, and a_1 and a'_1 are surjective. By working with the commutative square on the left-hand side of (3.3), we obtain the following chain of natural isomorphisms

$$\iota_v^* \mathcal{IC}_U = \iota_v^* a_2^* \mathcal{IC}_{T \times F} = (a'_2)^* \iota_0^* \mathcal{IC}_{T \times F} = (a'_2)^* \mathcal{IC}_F = \mathcal{IC}_{U_v},$$

where: the first natural identification is because a_2 is étale; the second is because the diagram is commutative; the third is because we can factor the identity as $\operatorname{id}_F: F \xrightarrow{\iota_0} T \times F \xrightarrow{p_F} F$, where the projection morphism

satisfies $p_F^* \mathcal{IC}_F = \mathcal{IC}_{T \times F}$ because it is smooth; the fourth is because a'_2 is étale.

By working with the commutative square on the right-hand side of (3.3) the same way we have done on the left-hand side, we see that

$$(a'_1)^* \iota_s^* \mathcal{IC}_M = \iota_v^* a_1^* \mathcal{IC}_M = \iota_v^* \mathcal{IC}_U = \mathcal{IC}_{U_v}.$$

Since a'_1 is étale, we also have that $(a'_1)^* \mathcal{IC}_{M_s} = \mathcal{IC}_{U_v}$.

By our reduction, we are assuming that M is integral, and therefore every connected component of $T \times F$ is integral. This in particular implies that every connected component of F is integral, and therefore the same holds for M_s . By considering the perverse versions of the intersection complexes, we have a natural isomorphism of perverse shaves

$$(a'_1)^* \mathcal{IC}_{M_s} = \mathcal{IC}_{U_v} = (a'_1)^* \iota_s^* \mathcal{IC}_M[\dim S - \dim M].$$

Since being a perverse intersection complex is a local condition in the étale topology, and the morphism a'_1 is étale and surjective, we deduce that $\iota_s^* \mathcal{IC}_M[\dim S - \dim M]$ is an intersection complex on M_s with possibly twisted coefficients on M_s . By restricting to the smooth locus of M where the intersection complex is the constant sheaf, we see that the identification sends the unit section to the unit section, so that the twisted coefficients in question are in fact constant of rank one. The conclusion of the lemma follows. $\square$

Remark 3.7. — For Lemma 3.8 below, we use O. Gabber's rectified middle perversity structures for schemes of finite type over a DVR T ; see [27, 28]; for a summary and complements, see [13, Section 5.2]. In short, one glues the middle-perversity t -structure on the closed fiber with the middle-perversity t -structure on the generic fiber shifted by 1. Just as in the case of schemes of finite type over a field, one has, in the context of schemes of finite type over a DVR, the notion intermediate extension of a perverse sheaf from an open set to a bigger open set, and thus the usual notion of intersection complex on such schemes.

The following lemma, for which we could not locate a reference, is standard (Künneth Formula for intersection cohomology) when T is a scheme of finite type over the field k .

LEMMA 3.8. — *Let N be a scheme of finite type over an algebraically closed field k . Let T be the spectrum of a strictly Henselian DVR over k and with residue field k . Then $\mathcal{IC}_{N \times T} = p_N^* \mathcal{IC}_N$, where $p_N: N \times T \rightarrow N$ is the projection.*

Proof. — Without loss of generality, we assume that N is integral. We use the perverse version IC of intersection cohomology. Let $i: s \rightarrow T \leftarrow \eta: j$ be the closed and open point, respectively. Let $p_T: N \times T \rightarrow T$ be the projection and let $N = N_s$ and N_η be the special and generic fiber respectively. We have the following commutative diagram with Cartesian squares

$$(3.4) \quad \begin{array}{ccccc} & & N & & \\ & \nearrow id & \uparrow p_N & \nwarrow u & \\ N & \xrightarrow{i} & N \times T & \xleftarrow{j} & N_\eta \\ \downarrow & & \downarrow p_T & & \downarrow \\ s & \xrightarrow{i} & T & \xleftarrow{j} & \eta \end{array}$$

We have morphisms

$$i^*[-1](p_N^*IC_N[1]) \longrightarrow \psi[-1](p_N^*IC_N[1]) \longrightarrow i^![1](p_N^*IC_N[1]).$$

A cone for either morphism is $\phi(p_N^*IC_N[1])$, which is zero in view of [15, p. 243, Corollaire 2.16]. Hence, we have the natural identification

$$i^*[-1](p_N^*IC_N[1]) = i^![1](p_N^*IC_N[1]).$$

Since $i^*(p_N^*IC_N) = i^!(p_N^*IC_N)[2] = IC_N$ and $j^*(p_N^*IC_N) = IC_{N_\eta}$ (cf. Lemma 3.5), one verifies (cf. [13, (55)]) that $p_N^*(IC_N[1])$ is perverse on $N \times T$ for the rectified t -structure.

We have a distinguished attaching triangle

$$\begin{aligned} i_*IC_N[-1] &= i_*i^!p_N^*(IC_N[1]) \longrightarrow p_N^*(IC_N[1]) \longrightarrow \\ &\longrightarrow Rj_*j^*p_N^*(IC_N[1]) = Rj_*IC_{N_\eta}[1] \rightsquigarrow \end{aligned}$$

Note that all three complexes $p_N^*IC_N[1]$, $i_*(IC_N)$ and $Rj_*(IC_{N_\eta}[1])$ are perverse, the last two by the t -exactness of the functors i_* , $Rj_*[1]$ and the perversity of IC_N, IC_{N_η} . By taking the long exact sequence of perverse cohomology sheaves, we get the short exact sequence of perverse sheaves on $N \times T$

$$0 \longrightarrow p_N^*(IC_N[1]) \longrightarrow Rj_*(IC_{N_\eta}[1]) \longrightarrow i_*IC_N \longrightarrow 0.$$

Since $IC_{N \times T}$ is the smallest sub perverse sheaf of $Rj_*(IC_{N_\eta}[1])$ extending $IC_{N_\eta}[1]$ (viewed as a rectified perverse sheaf on the open $N_\eta \subseteq N \times T$), we see that we have a natural short exact sequence of perverse sheaves on $N \times T$

$$(3.5) \quad 0 \longrightarrow IC_{N \times T} \longrightarrow p_N^*(IC_N[1]) \longrightarrow i_*Q_N \longrightarrow 0,$$

where Q_N is a perverse sheaf on N . Since Q_N is supported on N , we have that $\phi(i_*Q_N) = Q_N$. We apply the exact functor ϕ to the short exact sequence (3.5) and, in view of the earlier-established vanishing $\phi(p_N^*IC_N[1]) = 0$, we deduce that $Q_N = \phi(i_*Q_N) = 0$. $\square$

FACT 3.9. — *Let S be a smooth irreducible scheme over the algebraically closed field k , and let K be a complex with lisse cohomology sheaves. Then we have a natural identification $\mathcal{H}^i(K)[\dim(S)] = {}^p\mathcal{H}^{\dim(S)+i}(K)$. In fact, by the gluing construction [7, Section 1.4] of t -structures, on the dense stratum the t -structure is the standard one, up to dimensional shift.*

Now we are ready to conclude the proof of Theorem 3.4.

Proof of Theorem 3.4. — The conclusions of the theorem can be checked smooth locally on $\mathcal{S}$. After base-changing to a smooth atlas $S \rightarrow \mathcal{S}$, we are immediately reduced to the case when $f: M \rightarrow S$ is a proper morphism of schemes. In view of Fact 3.9, part (b) follows from part (a). In view of the Decomposition Theorem (3.2), part (c) follows from part (b). We are left with proving part (a).

In what follows, we use implicitly the Proper Base Change Theorem for the morphism f , so that the formation of the direct image sheaves in question, simply denoted by E in what follows, are compatible with base change. According to [46, Tag 0GKC], we need to show that for every specialization $t \rightsquigarrow s$ of Zariski points in S , and for any choice of geometric points $\bar{s}$ over s and $\bar{t}$ over t , the resulting specialization morphism $E_{\bar{s}} \rightarrow E_{\bar{t}}$ is an isomorphism. Without loss of generality, we can assume that $s \in S$ is a closed point and take $\bar{s} = s$. By [46, Tag 0GJ6], we can furthermore replace S with its strict localization $\hat{S}$ at s . Consider the spectrum of a strictly Henselian DVR $\tilde{S}$ mapping to $\hat{S}$, and mapping its closed point $\tilde{s}$ to s and its generic point $\tilde{t}$ to t . We have the corresponding morphisms $\tilde{S} \rightarrow \hat{S} \rightarrow S$ which give rise to the corresponding morphisms obtained from f via base change: $\tilde{f}: \tilde{M} \rightarrow \tilde{S}$ and $\hat{f}: \hat{M} \rightarrow \hat{S}$. Let $\tilde{t}$ be a geometric point over $\tilde{t}$. The specialization morphism $E_{\tilde{s}} \rightarrow E_{\tilde{t}}$ coincides with the morphism $i^*E \rightarrow \psi(E)$ associated with $\tilde{S}$. We are thus reduced to showing the triviality of the following vanishing cycle complexes

$$\phi(R\tilde{f}_*(\overline{\mathbb{Q}}_\ell)_{\tilde{M}}) = \phi(R\tilde{f}_*\mathcal{IC}_{\tilde{M}}) = 0.$$

We have $\phi R\tilde{f}_* = R(\tilde{f}_*)_*\phi$ because $\tilde{f}$ is proper. Hence, we are reduced to proving that

$$(3.6) \quad \phi((\overline{\mathbb{Q}}_\ell)_{\tilde{M}}) = \phi(\mathcal{IC}_{\tilde{M}}) = 0.$$

We have the local model expressed by diagram (3.1) for $f: M \rightarrow S$, where we may assume that all of the objects T, U, V, F are schemes. The morphisms b_1 and b_2 in said diagram induce isomorphisms of strict localizations at chosen corresponding points $\widehat{T} \xleftarrow{\widehat{b_2} \simeq} \widehat{V} \xrightarrow{\widehat{b_1} \simeq} \widehat{S}$. It follows that to give a morphism $\widetilde{S} \rightarrow \widehat{S}$ is the same thing as giving a morphism to either $\widehat{T}$ or $\widehat{V}$. We base change the local model diagram first via the morphism $\widehat{S} \rightarrow S$ and then via $\widetilde{S} \rightarrow \widehat{S}$ and obtain the following commutative diagram:

$$(3.7) \quad \begin{array}{ccccc} \widetilde{T} \times F & \xleftarrow{\widetilde{a_2}} & \widetilde{U} & \xrightarrow{\widetilde{a_1}} & \widetilde{M} \\ \downarrow p_{\widetilde{T}} & & \downarrow g & & \downarrow \widetilde{f} \\ \widetilde{T} & \xleftarrow[\simeq]{\widetilde{b_2}} & \widetilde{V} & \xrightarrow[\simeq]{\widetilde{b_1}} & \widetilde{S}, \end{array}$$

where $\widetilde{a_1}$ and $\widetilde{a_2}$ are étale and $\widetilde{b_1}$ and $\widetilde{b_2}$ are isomorphisms.

In view of Lemma 3.8, the intersection complex $\mathcal{IC}_F$ pulls back via the projection morphism p_F to the intersection complex $\mathcal{IC}_{\widetilde{T} \times F}$, which, in turn, pulls back via the étale morphism $\widetilde{a_2}$ to the intersection complex $\mathcal{IC}_{\widetilde{U}}$. The intersection complex $\mathcal{IC}_{\widetilde{M}}$ also pulls back to $\mathcal{IC}_{\widetilde{U}}$.

In order to simplify the notation, for a scheme Z we let K_Z denote either the intersection complex $\mathcal{IC}_Z$, or the constant sheaf $(\overline{\mathbb{Q}}_\ell)_Z$. We then have:

$$\widetilde{a_2}^*(K_{\widetilde{T} \times F}) = K_{\widetilde{U}} = \widetilde{a_1}^*(K_{\widetilde{M}})$$

Since $K_{\widetilde{T} \times F} = p_{\widetilde{T}}^* K_F$, by virtue of [15, p. 243, Corollaire 2.16] for the morphism $F \rightarrow \text{Spec}(k)$, we have that $\phi_{p_{\widetilde{T}}}(K_{\widetilde{T} \times F}) = 0$.

We use freely that the formation of vanishing cycles “commutes” with smooth pull-back [16, Exposé XIII, (2.1.7.2)]. We apply this repeatedly to the étale morphisms of type $\widetilde{a}$. It follows that

$$0 = \widetilde{a_2}^* \phi_{p_{\widetilde{T}}}(K_{\widetilde{T} \times F}) = \phi_{p_{\widetilde{T}} \widetilde{a_2}}(\widetilde{a_2}^* K_{\widetilde{T} \times F}) = \phi_{\widetilde{b_2} g}(\widetilde{a_2}^* K_{\widetilde{T} \times F}) = \phi_{\widetilde{b_2} g}(K_{\widetilde{U}}),$$

so that, since $\widetilde{b_2}$ is an isomorphism, we have $\phi_g(K_{\widetilde{U}}) = 0$.

We have

$$\widetilde{a_1}^*(\phi_{\widetilde{f}}(K_{\widetilde{M}})) = \phi_{\widetilde{f} \widetilde{a_1}}(\widetilde{a_1}^* K_{\widetilde{M}}) = \phi_{\widetilde{b_1} g}(\widetilde{a_1}^* K_{\widetilde{M}}) = \phi_{\widetilde{b_1} g}(K_{\widetilde{U}}) = 0,$$

where: the first and second equalities hold because $\widetilde{a_1}$ is étale; the third equality holds because $\widetilde{b_1}$ is an isomorphism and we already know that $\phi_g(K_{\widetilde{U}}) = 0$. We have thus shown that the desired triviality of vanishing cycles complexes (3.6) holds locally on $\widetilde{M}$, so that it holds on $\widetilde{M}$. This concludes the proof of part (a). $\square$

COROLLARY 3.10 ((Intersection) cohomology and fibers). — *Let $f: \mathcal{M} \rightarrow \mathcal{S}$ be a proper schematic morphism between quasi-separated finite*

type algebraic stacks over k . Suppose that f is locally isotrivial (Definition 3.1), and that $\mathcal{S}$ is connected. Then the cohomology rings of any two geometric fibers of f are non-canonically isomorphic. Their intersection cohomology groups, endowed with their natural module over the respective cohomology rings structures, and with their natural intersection pairings, are also non-canonically isomorphic.

Proof. — After base-changing to connected components of the atlas $S \rightarrow \mathcal{S}$, we are immediately reduced to the case when $f: M \rightarrow S$ is a proper morphism of schemes. We begin by showing the statement for the fibers over two closed points $s_1, s_2 \in S$. By connectedness of S , there exists a connected reduced, possibly reducible algebraic curve on S connecting the two points (to see this we may reduce to the case when S is an affine and integral, where [35, Lemma, p. 56] applies). By taking the normalization of such a curve, we reduce to the case when S is a smooth and connected curve. Ignoring the ring, module and pairing structures, the additive part of the corollary now follows from Theorem 3.4, proper base-change for f and Lemma 3.6.

As to the ring, module and pairing structures, it is enough to prove the desired conclusion by replacing s_2 with the generic point η of the smooth curve, and replacing the curve by its strict localization $\tilde{S}$ at $s := s_1$. In this case, we need to prove that the morphism induced in cohomology by $i^* \rightarrow \psi$ is compatible with the ring and module structures.

The nearby cycle functor ψ preserves cup products (see e.g. [27, Section 4.3]). Since f is proper, upon taking stalks (see [16, Exposé XIII, (2.1.8.5)]) the morphism $i^* \rightarrow \psi$ induces the specialization morphism on stalks as defined in [46, Tag 0GJ2]. By the description of the specialization morphism in terms of pulling back sections as in [46, Tag 0GJ3], we see that the morphism $i^* \rightarrow \psi$ preserves cup products.

As to the module structure, we note that it can be described as follows. Let $\tilde{S}$ be the strict localization of S at s , with a choice of geometric generic point denoted by $\bar{\eta}$. We then have that $i^* \rightarrow \psi$ induces isomorphisms $H^*(M_s) = H^*(\tilde{M}) \simeq H^*(M_{\bar{\eta}})$, and similarly for intersection cohomology. Let $\alpha \in H^a(\tilde{M})$ and let $\beta \in IH^b(\tilde{M})$. We view α as a morphism $\alpha': (\mathbb{Q}_\ell)_{\tilde{M}} \rightarrow (\mathbb{Q}_\ell)_{\tilde{M}}[a]$; this morphism induces a morphism $\alpha'': \mathcal{IC}_{\tilde{M}} \rightarrow \mathcal{IC}_{\tilde{M}}[a]$ such that the cup product $\alpha \cdot \beta = H^b(\alpha'')(\beta)$ is obtained by taking cohomology. By plugging this construction into the morphism of functors $i^* \rightarrow \psi$, we see that the product $\alpha_s \cdot \beta_s$ is sent to the product $\alpha_{\bar{\eta}} \cdot \beta_{\bar{\eta}}$.

Finally, we consider the pairings on intersection cohomology. We place ourselves again over $\tilde{S}$. The pairing on intersection cohomology arises from the map in cohomology induced by the self-duality isomorphism $\iota: IC \xrightarrow{\sim} \mathbb{D}IC$ (here, $\mathbb{D}$ denotes the Verdier Duality Functor and, in order to avoid dimensional shifts, we consider perverse intersection cohomology complexes IC , which are self-dual). We start with $\iota_M: IC_M \rightarrow \mathbb{D}_M IC_M$ and apply $i^* \rightarrow \psi$. In order to conclude, it remains to observe that:

- (1) $i^* IC_M = IC_{M_s}[1]$ and $\psi(IC_M) = IC_{M_{\bar{\eta}}}[1]$;
- (2) $i^* \mathbb{D}_M IC_M = \mathbb{D}_{M_s} i^! IC_M = \mathbb{D}_{M_s} (i^* IC_M[-2]) = \mathbb{D}_{M_s} (IC_{M_s}[-1]) = (\mathbb{D}_{M_s}(IC_{M_s}))[1]$;
- (3) similar to (2), but for ψ instead of i^* .

We thus get that ι_M induces an isomorphism of pairings between $(\iota_M)|_{M_s}$ and $(\iota_M)|_{M_{\bar{\eta}}}$ via $i^* \rightarrow \psi$. These pairings are the respective duality pairings ι_{M_s} and $\iota_{M_{\bar{\eta}}}$. Indeed, the duality pairing isomorphisms at the level of intersection complexes are uniquely determined by their restriction to the smooth dense open subsets, where we see that they all preserve the unit section.

To conclude the statement for any two geometric fibers, we can find specializations to closed points, choose DVRs realizing the specializations, and repeat the argument above. $\square$

3.3. Étale cohomology of families of good moduli spaces

For the following result, we work with $\overline{\mathbb{Q}}_\ell$ -sheaves for the étale topology.

THEOREM 3.11. — *Let $f: \mathcal{X} \rightarrow \mathcal{S}$ be a smooth morphism between finite type algebraic stacks with affine diagonal over k . Suppose that $\mathcal{X}$ admits a relative good moduli space $\bar{f}: \mathcal{M} \rightarrow \mathcal{S}$ such that $\bar{f}$ is proper and schematic. Suppose that $\mathcal{S}$ is connected. Then the cohomology rings of any two geometric fibers of $\bar{f}$ are non-canonically isomorphic. Their intersection cohomology groups, endowed with their natural module over the respective cohomology rings structures, and with their natural intersection pairings, are also non-canonically isomorphic.*

If, furthermore, $\mathcal{S}$ is smooth, then the following hold.

- (a) *For every i , the direct image sheaves $R^i \bar{f}_*(\overline{\mathbb{Q}}_\ell)_{\mathcal{M}}$ and $R^i \bar{f}_* \mathcal{IC}_{\mathcal{M}}$ are lisse on $\mathcal{S}$.*
- (b) *We have the identity of perverse sheaves on $\mathcal{S}$*

$$(R^i \bar{f}_*(\mathcal{IC}_{\mathcal{M}}))[\dim(\mathcal{S})] = {}^p \mathcal{H}^{\dim(\mathcal{S})+i}(R \bar{f}_*(\mathcal{IC}_{\mathcal{M}})).$$

- (c) *We have an isomorphism*

$$R \bar{f}_* \mathcal{IC}_{\mathcal{M}} \simeq \bigoplus_{i \geq 0} R^i \bar{f}_* \mathcal{IC}_{\mathcal{M}}[-i].$$

Proof. — In view of Corollary 2.4, the morphism $\bar{f}: \mathcal{M} \rightarrow \mathcal{S}$ is locally isotrivial as in Definition 3.1. Therefore, the theorem follows directly from Theorem 3.4 and Corollary 3.10. $\square$

3.4. Singular cohomology in families over $\mathbb{C}$ and topological local triviality

In this section the ground field is $k = \mathbb{C}$. The relevant categories for a complex algebraic variety X are:

- (1) the $\overline{\mathbb{Q}}_\ell$ -constructible derived category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ (étale topology);
- (2) the algebraically constructible bounded derived categories $\widetilde{D}_c^b(-, \mathbb{Q})$ with rational (or other) coefficients (classical topology);
- (3) M. Saito's bounded derived category $D^b(\mathrm{MHM}_{\mathrm{alg}}(-, \mathbb{Q}))$ of algebraic mixed Hodge modules with rational coefficients (cf. [41, 43]).

They are related as follows:

$$(3.8) \quad D^b(\mathrm{MHM}_{\mathrm{alg}}(X, \mathbb{Q})) \xrightarrow{\mathrm{rat}} \widetilde{D}_c^b(X, \mathbb{Q}) \xrightarrow{\otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell} \widetilde{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \xleftarrow{\epsilon^*} D_c^b(X, \overline{\mathbb{Q}}_\ell).$$

Here, ϵ^* is the Artin comparison fully faithful functor in [7, Section 6.1.2]; the symbol $\otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$ denotes the change-of-coefficients functor; and rat is the faithful functor in the main result in [41, Theorem 0.1].

These categories are equipped with their natural nine functors: f^* , Rf_* , $Rf_!$, $f^!$, $R\mathcal{H}om$, $\otimes^{\mathbb{L}}$, the Verdier Duality functor, and the vanishing and nearby functors. The functors in (3.8) are compatible (commute) with these nine functors. Note that the following hold:

- (1) ϵ^*L is locally constant sheaf if and only if L is lisse;
- (2) there is a direct sum decomposition $u: \epsilon^*A \simeq B \oplus C$ if and only if there is a direct sum decomposition $u': A \simeq B' \oplus C'$ with $u = \epsilon^*(u')$;
- (3) taking direct images commutes with changing coefficients;
- (4) a sheaf of $\mathbb{Q}$ -vector spaces is locally constant if and only if it is locally constant after changing coefficients to $\overline{\mathbb{Q}}_\ell$;
- (5) changing coefficients and ϵ^* preserve the constant sheaf and the intersection complex.

In particular, if a direct image sheaf of the constant sheaf or of the intersection complex is lisse, then the corresponding direct image is locally constant.

The aforementioned nine functors on the constructible derived categories $\tilde{D}^b(-, \mathbb{Q})$ admit canonical lifts along the functor rat to M. Saito's categories $D^b(\text{MHM}_{\text{alg}}(-, \mathbb{Q}))$. If we denote the lift of any of these functors F by $\underline{F}$, then we have identities of the form $\text{rat} \circ \underline{F} = F \circ \text{rat}$. For example, for a morphism $f: X \rightarrow Y$ of complex algebraic varieties, we denote by $\underline{Rf_*}: D^b(\text{MHM}_{\text{alg}}(X, \mathbb{Q})) \rightarrow D^b(\text{MHM}_{\text{alg}}(Y, \mathbb{Q}))$ the canonical lift of the functor $Rf_*: D_c^b(X, \mathbb{Q}) \rightarrow D_c^b(Y, \mathbb{Q})$; then we have that $Rf_*(\text{rat}(A)) = \text{rat}(\underline{Rf_*}(A))$ for every complex of mixed Hodge modules A on X .

By construction, the standard t -structure on $D^b(\text{MHM}_{\text{alg}}(X, \mathbb{Q}))$ corresponds to the (middle)-perversity t -structure on $D_c^b(X, \mathbb{Q})$. In particular, given a complex A of mixed modules on X , a morphism $f: X \rightarrow Y$ of complex algebraic varieties, we have that ${}^p\mathcal{H}^i(Rf_*(\text{rat}(A))) = \text{rat}(\mathcal{H}^i(Rf_*A))$. The functors in (3.8) are t -exact for the corresponding t -structures (standard for mixed Hodge modules, perverse for the others).

For an arbitrary complex algebraic variety X , the sheaf $\mathbb{Q}_X$ and the intersection complex $\mathcal{IC}_X$ admit canonical lifts to $D^b(\text{MHM}_{\text{alg}}(X, \mathbb{Q}))$ denoted by $\underline{\mathbb{Q}}_X$ and $\underline{\mathcal{IC}}_X$.

Smooth mixed Hodge modules. A mixed Hodge module A on a complex variety X is smooth on X if $L := \text{rat}(A)[- \dim(X)]$ is locally constant

on X (concentrated in cohomological degree zero) and X is smooth of pure dimension. In this case, L is an admissible variation of mixed Hodge structures in the sense of Zucker and of Kashiwara; the converse holds (cf. [41, Theorem 0.2 and the paragraph preceding it]). The constant sheaf-type object $\underline{\mathbb{Q}}_X$ is a complex of mixed Hodge modules of weights at most 0 (cf. [41, Section 4.5]). Having weights at most w is preserved by pushforward via a proper morphism [41, Section 4.5.2].

Smooth polarizable pure Hodge modules. A polarizable pure Hodge module A of weight w on a complex variety X is smooth on X if $L := \text{rat}(A)[- \dim(X)]$ is lisse on X (concentrated in cohomological degree zero) and X is smooth of pure dimension. In this case, L is a polarizable variation of pure Hodge structures of weight $w - \dim(X)$; the converse holds (cf. [40, Lemma 3 and Theorem 2]). The intersection complex-type object $\underline{\mathcal{IC}}_X[\dim(X)]$ is a polarizable pure Hodge module of weight $\dim(X)$.

M. Saito's Decomposition Theorem. Bounded complexes of polarizable pure Hodge modules are preserved via pushforward by a projective morphism (cf. [40, Theorem 1]) so that the M. Saito's Decomposition Theorem holds: if A is a complex of polarizable pure Hodge modules on a complex variety X and $f: X \rightarrow Y$ is a morphism of complex algebraic varieties, then $Rf_*A \simeq \bigoplus_i \mathcal{H}^i(Rf_*A)[-i]$ splits into the direct sum of shifted polarizable pure Hodge modules.

Cohomology mixed Hodge modules/sheaves and smooth mixed Hodge modules. If X is smooth of pure dimension, A is a smooth complex of mixed Hodge modules, i.e. $\text{rat}(A)$ has lisse cohomology sheaves $\mathcal{H}^i(\text{rat}(A))$ on X , then we have the identity of perverse sheaves

$${}^p\mathcal{H}^{\dim(X)+i}(\text{rat}(A)) = \mathcal{H}^j(\text{rat}(A))[\dim(X)],$$

so that the cohomology mixed Hodge modules $\mathcal{H}^{\dim(X)+i}(A)$ is smooth on X and the lisse sheaf $\mathcal{H}^i(\text{rat}(A))$ underlies an admissible variation of mixed Hodge structures (pure, if A is a complex of polarizable pure Hodge modules).

PROPOSITION 3.12. — *Let $f: M \rightarrow S$ be a locally isotrivial morphism (Definition 3.1). Assume that M and S are complex algebraic varieties, that f is projective, and that S is smooth and connected. The evident mixed Hodge modules analogue of Theorem 3.4 holds, namely we have the following.*

- (a) For every $i \in \mathbb{Z}^{\geq 0}$, the direct image sheaves $R^i f_*(\overline{\mathbb{Q}}_\ell)_M$ and $R^i f_* \mathcal{IC}_M$ are locally constant on S . For each i , the sheaf $R^i f_*(\overline{\mathbb{Q}}_\ell)_M$ underlies a smooth mixed Hodge module $L[\dim(S)]$ on S , so that the local system L carries a natural admissible mixed Hodge structure of weights at most i ; similarly, for the sheaf $R^i f_* \mathcal{IC}_M$, except that in this case it underlies a polarizable variation of pure Hodge structures of weight i .

- (b) We have the identity of perverse sheaves on S

$$(R^i f_*(\mathcal{IC}_M))[\dim(S)] = {}^p \mathcal{H}^{\dim(S)+i}(Rf_*(\mathcal{IC}_M)),$$

as well as the identity in the category $\mathrm{MHM}_{\mathrm{alg}}(S, \mathbb{Q})$

$$(R^i f_*(\mathcal{IC}_M))[\dim(S)] = \mathcal{H}^{\dim(S)+i}(\underline{Rf_*}(\underline{\mathcal{IC}_M})).$$

- (c) We have an isomorphism in $\widetilde{D}_c^b(S, \mathbb{Q})$

$$Rf_*(\mathcal{IC}_M) \simeq \bigoplus_{i \geq 0} R^i f_*(\mathcal{IC}_M)[-i],$$

as well as the corresponding isomorphism in $D^b(\mathrm{MHM}_{\mathrm{alg}}(S, \mathbb{Q}))$

$$\underline{Rf_*}(\underline{\mathcal{IC}_M}) \simeq \bigoplus_{i \geq 0} \mathcal{H}^{\dim(S)+i}(\underline{Rf_*}(\underline{\mathcal{IC}_M}))[-i].$$

Proof. — Theorem 3.4, implies, via application of the comparison functor ϵ^* , the following statements at the level of $\widetilde{D}_c^b(S, \overline{\mathbb{Q}}_\ell)$, and thus at the level of $\widetilde{D}_c^b(S, \mathbb{Q})$: the local constancy assertions in (a); the identity of perverse sheaves in (b); the isomorphism in $\widetilde{D}_c^b(S, \mathbb{Q})$ appearing in (c).

Part (a) follows from the characterizations of smooth mixed Hodge modules and of smooth polarizable pure Hodge modules. Part (b) follows from the structure of the cohomology of a smooth complex of mixed Hodge modules. Part (c) follows from part (b) and from M. Saito's Decomposition Theorem. $\square$

Next, we prove the following topological local triviality result.

PROPOSITION 3.13 (Local triviality of families). — *Let $f: M \rightarrow S$ be a locally projective morphism of $\mathbb{C}$ varieties. Suppose that f is locally isotrivial (Definition 3.1) and that S is smooth. The family $f: M \rightarrow S$ is topologically locally trivial over the base S .*

Proof. — We work with the classical topology. Without loss of generality, we may assume that the morphism f factors as a closed embedding into $\mathbb{P}_S^n$, followed by the restriction of the projection onto S . The desired conclusion is a consequence of the First Thom Isotopy Lemma [19, p. 41, Section 1.5,

Theorem] if we can produce a Whitney stratification $\mathfrak{S}$ of M all of whose strata T map submersively onto S .

It is a deep result of B. Teissier [48] (see [12, p. 8] for a short discussion) that every complex variety admits a canonical algebraic Whitney stratification, which we call the WT-stratification. In order to prove the proposition, it is enough that the strata of the WT-stratification of M map submersively onto S . We do so by using diagram (3.1), which we may arrange to be a diagram of varieties T, F, U, V in this case.

The WT-stratification is preserved via local complex analytic isomorphisms. It follows that: if $U \rightarrow M$ is étale, then the pre-image of the WT-stratification of M is the WT-stratification of U ; if $U \rightarrow M$ is an étale cover, then a stratification of M is the WT-stratification of M if and only if its pre-image in U is the WT-stratification of U ; the WT-stratification of a product $T \times F$, with T smooth, is the pre-image of the WT-stratification of F via the projection.

It is clear that the WT-stratification of $T \times F$ has strata that map submersively onto T . In view of Corollary 2.4, the discussion in the previous paragraph implies that the WT-stratification of M has strata that map submersively onto S . $\square$

Notation 3.14. — Let M be a finite type scheme over $\mathbb{C}$. We write $H^\bullet(M, \mathbb{Q})$ and $IH^\bullet(M, \mathbb{Q})$ for the corresponding singular (intersection) cohomology groups with rational coefficients. We shall omit the coefficients $\mathbb{Q}$ from the notation whenever it is clear from context.

We end this section with the following result. We denote by $h^{p,q}(\mathrm{Gr}_a^W H)$ the dimension of the (p, q) -part of the weight a -graded piece of a rational mixed Hodge structure H .

THEOREM 3.15. — *Let $f: \mathcal{M} \rightarrow \mathcal{S}$ be a schematic locally projective morphism of quasi-separated stacks of finite type over $\mathbb{C}$. Suppose that f is locally isotrivial and that $\mathcal{S}$ is connected. Then, for any two $\mathbb{C}$ -points $s, s' \in \mathcal{S}(\mathbb{C})$, the following hold.*

- (a) *The projective $\mathbb{C}$ -varieties $\mathcal{M}_s$ and $\mathcal{M}_{s'}$ are homeomorphic.*
- (b) *There is a noncanonical isomorphism $H^\bullet(\mathcal{M}_s) \xrightarrow{\sim} H^\bullet(\mathcal{M}_{s'})$ that preserves the weight filtrations. Such noncanonical isomorphism can be arranged to come from a homeomorphism as in (a).*
- (c) *There is a noncanonical isomorphism $IH^\bullet(\mathcal{M}_s) \xrightarrow{\sim} IH^\bullet(\mathcal{M}_{s'})$ that is compatible with the respective intersection pairings and with the structures of modules over the respective cohomology rings intertwined by an isomorphism as in (b).*

(d) For any indexes p, q we have

$$h^{p,q}(\mathrm{Gr}_a^W H^\bullet(\mathcal{M}_s)) = h^{p,q}(\mathrm{Gr}_a^W H^\bullet(\mathcal{M}_{s'})),$$

and

$$h^{p,q}(IH^\bullet(\mathcal{M}_s)) = h^{p,q}(IH^\bullet(\mathcal{M}_{s'})).$$

In particular, the corresponding E polynomials of the mixed Hodge structures $H^\bullet(-)$ and of the pure Hodge structures $IH^\bullet(-)$ of the two fibers agree.

Proof. — After pulling back to a resolution of singularities $S' \rightarrow S$ of a connected component of an atlas $S \rightarrow \mathcal{S}$, we may assume that $f: M \rightarrow S$ is a locally projective morphism of schemes with S smooth. In that case, part (a) follows from Proposition 3.13. Our proof of the local constancy of the direct image sheaves $R^i f_* \overline{\mathbb{Q}}_M$, conjoined with [7, Proposition 6.2.3], implies that the weight filtrations on the stalks give rise to a locally constant filtration of the locally constant $R^i f_* \overline{\mathbb{Q}}_M$. This implies part (b). Part (c) follows from part (a). Finally, (d) is a consequence of Proposition 3.12. $\square$

4. Applications

In this section, we assume that the algebraically closed field k has characteristic 0 and we fix a prime number ℓ . We will exhibit several examples of pairs of schemes M_1, M_2 of finite type over k having the same cohomology rings and intersection cohomology groups.

For the sake of precision, we introduce the following property.

PROPERTY 4.1 (For M_1 and M_2). — *There exist:*

- (1) a $\overline{\mathbb{Q}}_\ell$ -algebra isomorphism $H^\bullet(M_1, \overline{\mathbb{Q}}_\ell) \cong H^\bullet(M_2, \overline{\mathbb{Q}}_\ell)$ between ℓ -adic cohomology rings;
- (2) a $H^\bullet(M_1, \overline{\mathbb{Q}}_\ell)$ -module isomorphism $IH^\bullet(M_1, \overline{\mathbb{Q}}_\ell) \cong IH^\bullet(M_2, \overline{\mathbb{Q}}_\ell)$, compatible with the intersection pairing in intersection cohomology.

Here, the $H^\bullet(M_1, \overline{\mathbb{Q}}_\ell)$ -module structure on $IH^\bullet(M_2, \overline{\mathbb{Q}}_\ell)$ is induced from its natural $H^\bullet(M_2, \overline{\mathbb{Q}}_\ell)$ -module structure by transport along the isomorphism of cohomology rings in (1).

When working over the complex numbers, $k = \mathbb{C}$, we consider the analogous property for singular cohomology, intersection cohomology and (mixed) Hodge structures on them.

PROPERTY 4.2 (For M_1 and M_2 , assuming $k = \mathbb{C}$). — *There exist:*

- (1) *a homeomorphism $M_1^{\text{an}} \cong M_2^{\text{an}}$ between the underlying topological spaces of the analytifications of M_1 and M_2 ;*
- (2) *an isomorphism of rational cohomology rings*

$$H^\bullet(M_1, \mathbb{Q}) \cong H^\bullet(M_2, \mathbb{Q})$$

induced by the homeomorphism $M_1^{\text{an}} \cong M_2^{\text{an}}$ above, that preserves the weight filtrations for the corresponding mixed Hodge structures;

- (3) *an $H^\bullet(M_1, \mathbb{Q})$ -module isomorphism*

$$IH^\bullet(M_1, \mathbb{Q}) \cong IH^\bullet(M_2, \mathbb{Q})$$

between rational intersection cohomology groups, where the module structure on $IH^\bullet(M_2, \mathbb{Q})$ is induced by the isomorphism of cohomology rings in (2).

Moreover, the dimensions

$$h^{p,q}(\text{Gr}_a^W H^j(M_1, \mathbb{Q})) \quad \text{and} \quad h^{p,q}(\text{Gr}_a^W H^j(M_2, \mathbb{Q}))$$

coincide; in particular, the corresponding E-polynomial coincide. Likewise, the dimensions $h^{p,q}(IH^\bullet(M_1, \mathbb{Q}))$ and $h^{p,q}(IH^\bullet(M_2, \mathbb{Q}))$ coincide.

4.1. Moduli space of G -bundles

Fix a connected reductive group G over k . Let C be a smooth connected projective curve over k . The moduli stack $\text{Bun}_G(C)$ of G -bundles on C is a smooth algebraic stack with affine diagonal over k ([24, Proposition 1]). The connected components of $\text{Bun}_G(C)$ are in correspondence with elements of the algebraic fundamental group $\pi_1(G)$ [25, Theorem 5.8]; for any $d \in \pi_1(G)$ we denote by $\text{Bun}_G^d(C)$ the corresponding component of G -bundles of degree d . There is a notion of semistability for G -bundles defined by Ramanathan [37] which generalizes semistability of vector bundles on C . The locus of semistable G -bundles in $\text{Bun}_G(C)$ is open, and corresponds to an open substack $\text{Bun}_G^d(C)^{\text{ss}} \subset \text{Bun}_G^d(C)$. The stack $\text{Bun}_G^d(C)^{\text{ss}}$ admits a projective good moduli space over k [38, 39], which we denote by $M_G^d(C)$.

THEOREM 4.3. — *Let C_1, C_2 be two smooth projective curves of the same genus over an algebraically closed field k of characteristic 0. Let G be a connected reductive group over k , and fix $d \in \pi_1(G)$. Then, the moduli spaces $M_G^d(C_1)$ and $M_G^d(C_2)$ of semistable G -bundles on C_1 and C_2 satisfy Property 4.1. If $k = \mathbb{C}$, then $M_G^d(C_1)$ and $M_G^d(C_2)$ satisfy Property 4.2.*

Proof. — Let g denote the genus of the curves C_1 and C_2 . The stack $\mathcal{M}_g$ of smooth curves of genus g is a smooth, separated and irreducible Deligne–Mumford stack of finite type over k [17]. Let $U \rightarrow \mathcal{M}_g$ be an étale cover by a quasi-compact scheme U . Since $\mathcal{M}_g$ is connected, we may assume that the points of $\mathcal{M}_g$ corresponding to C_1 and C_2 lie in the image of a connected component S of U .

Let $C \rightarrow S$ be the family of curves given by the map $S \rightarrow \mathcal{M}_g$. There is an algebraic stack $\mathrm{Bun}_G(C/S) \rightarrow S$ parametrizing G -bundles on the fibers of $C \rightarrow S$; this stack is locally of finite type and has affine relative diagonal over S (by [22, Theorem 1.3], with $S = A$, $Z = C \times A$ and $X = BG$). This stack is known to be smooth (see the proof of [24, Proposition 1], which applies over a general base scheme S). There is an open and closed substack $\mathrm{Bun}_G^d(C/S) \subset \mathrm{Bun}_G(C/S)$ parametrizing G -bundles that have degree d in the sense of [25]. It is known that the subfunctor $\mathrm{Bun}_G^d(C/S)^{\mathrm{ss}} \subset \mathrm{Bun}_G^d(C/S)$ parametrizing semistable G -bundles of degree d is represented by an open substack (see for example [20] for openness of semistability in the setting of families). It follows as a special case of [29, Theorem 1.1] that $\mathrm{Bun}_G^d(C/S)^{\mathrm{ss}}$ admits a locally projective good moduli space $f: M_G^d(C/S) \rightarrow S$ (alternatively, the GIT constructions of Schmitt [42] can be modified to apply to over such base S using Seshadri’s relative GIT [44]).

Let $s_1, s_2 \in S(k)$ be two points such that the fibers of $C \rightarrow S$ over s_1 and s_2 are isomorphic to C_1 and C_2 , respectively. Since the formation of the good moduli space of $\mathrm{Bun}_G(C/S)_d^{\mathrm{ss}}$ commutes with base-change on the base S [1, Proposition 4.7(i)], the fibers of f over s_1 and s_2 are isomorphic to $M_G^d(C_1)$ and $M_G^d(C_2)$, respectively. By applying Corollary 3.10 to f , we get that $M_G^d(C_1)$ and $M_G^d(C_2)$ satisfy Property 4.1.

If $k = \mathbb{C}$ then, by Theorem 3.15, $M_G^d(C_1)$ and $M_G^d(C_2)$ satisfy Property 4.2. $\square$

Remark 4.4. — It follows as in Proposition 3.13 that the moduli spaces of semistable G -bundles on any two curves of the same genus are homeomorphic. This can also be seen directly as a consequence of the Narasimhan–Seshadri correspondence for G -bundles (see e.g. [6, Theorem 1.0.3(3)]).

4.2. Moduli of sheaves on surfaces with ω -negative polarization

In this subsection, we show that the (intersection) cohomology and the homeomorphism type of certain moduli spaces of sheaves on smooth surfaces vary nicely in families. Our key assumption will be that the polarization used to define semistability of sheaves satisfies the following property.

DEFINITION 4.5. — *Let X be a smooth projective surface over a field K equipped with an ample line bundle $\mathcal{L}$. We say that $\mathcal{L}$ is an ω -negative polarization if $\mathcal{L} \cdot \omega_X < 0$, where $\omega_X = \det(\Omega_{X/K}^1)$ denotes the canonical bundle of X . In this case, we say that the pair $(X, \mathcal{L})$ is an ω -negatively polarized surface.*

The main motivation for this definition is that it ensures that $\mathrm{Ext}_X^2(\mathcal{F}, \mathcal{F}) = 0$ for any Gieseker $\mathcal{L}$ -semistable sheaf of dimension 2 or 1. See the proof of Proposition 4.9.

Example 4.6. — The following are examples of ω -negatively polarized surfaces.

- (a) Any smooth del Pezzo surface X admits the ω -negative polarization ω_X^{-1} .
- (b) All iterated blowups of $\mathbb{P}^2$ at finitely many points admit ω -negative polarizations.
- (c) Any ruled smooth surface admits an ω -negative polarization.

Remark 4.7. — The Enriques–Kodaira classification of surfaces implies that a smooth surface over k admits a negative polarization if and only if its Kodaira dimension is $-\infty$. Therefore the examples (b) and (c) above are exhaustive.

DEFINITION 4.8. — *Let S be a Noetherian scheme over k . A family of ω -negatively polarized surfaces consists of a pair $(X \rightarrow S, \mathcal{L})$ where $X \rightarrow S$ is a smooth proper family equipped with a relatively ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ such that for all $s \in S$ the fiber $(X_s, \mathcal{L}_s)$ is an ω -negatively polarized surface.*

Let $(X \rightarrow S, \mathcal{L})$ be a family of ω -negatively polarized surfaces over a Noetherian k -scheme S , and fix a polynomial $P \in \mathbb{Q}[n]$. We denote by $\mathrm{Coh}_{X/S}^{P, \mathrm{ss}}$ the stack of Gieseker $\mathcal{L}$ -semistable pure sheaves on X with Hilbert polynomial P . Recall that a coherent sheaf $\mathcal{F}$ on a variety over a field is called pure if for all coherent subsheaves $0 \neq \mathcal{E} \subset \mathcal{F}$ we have $\dim(\mathrm{Supp}(\mathcal{E})) = \dim(\mathrm{Supp}(\mathcal{F}))$.

To be more precise, for any S -scheme $T \rightarrow S$ the groupoid $\mathrm{Coh}_{X/S}^{P, \mathrm{ss}}$ consists of T -flat families of finitely presented quasicoherent $\mathcal{O}_{X_T}$ -modules $\mathcal{F}$ on X_T such that for all geometric points $\bar{t} \rightarrow T$ the base-change $\mathcal{F}|_{X_{\bar{t}}}$ is a Gieseker $\mathcal{L}_{X_{\bar{t}}}$ -semistable pure sheaves with Hilbert polynomial P . The stack $\mathrm{Coh}_{X/S}^{P, \mathrm{ss}} \rightarrow S$ has affine relative diagonal and is of finite type over S , as it is an open substack of the stack of coherent sheaves which satisfies both properties [46, Tag 08KA]. If the field k is of characteristic 0, then $\mathrm{Coh}_{X/S}^{P, \mathrm{ss}}$

admits a good moduli space $M_{X/S}^P$ which is locally projective over S ([45, Section 1], [30, Theorem 1.1], [23, Theorem 5.2]).

Note that, if the Hilbert polynomial P has degree 2, then $\mathrm{Coh}_{X/S}^{P,ss}$ parametrizes torsion-free sheaves.

PROPOSITION 4.9. — *Let $(X \rightarrow S, \mathcal{L})$ be a family of ω -negatively polarized surfaces over a Noetherian k -scheme S . Suppose that the Hilbert polynomial P has degree 2. Then the stack $\mathrm{Coh}_{X/S}^{P,ss}$ is smooth over S .*

Proof. — By [46, Tag 0DP0 and Tag 02HT] applied to the finite type morphism $\mathrm{Coh}_{X/S}^{P,ss} \rightarrow S$, it suffices to show the infinitesimal lifting criterion for square-zero thickenings for small extensions $A \rightarrow B$ of local Artin k -algebras over S . We denote by κ the common residue field of A and B , and we set I to be the ideal of B in A . Choose a morphism $\mathrm{Spec}(B) \rightarrow \mathrm{Coh}_{X/S}^{P,ss}$ corresponding to a B -flat family of sheaves $\mathcal{F}$ on the base-change X_B . We denote by $\mathcal{F}_\kappa$ the restriction to X_κ . By [46, Tag 08VW], there is an obstruction $\mathrm{ob} \in \mathrm{Ext}_{X_\kappa}^2(\mathcal{F}_\kappa, \mathcal{F}_\kappa) \otimes_\kappa I$ such that $\mathrm{ob} = 0$ if and only if a lift $\mathrm{Spec}(A) \rightarrow \mathrm{Coh}_{X/S}^{P,ss}$ exists. We conclude the proof by showing that $\mathrm{Ext}_{X_\kappa}^2(\mathcal{F}_\kappa, \mathcal{F}_\kappa) = 0$. By Serre duality for the smooth surface X_κ , this is isomorphic to $\mathrm{Hom}_{X_\kappa}(\mathcal{F}_\kappa, \mathcal{F}_\kappa \otimes \omega_{X_\kappa})$. By assumption $\mathcal{F}_\kappa$ is a $\mathcal{L}_\kappa$ -Gieseker semistable sheaf on X_κ , and therefore the same holds for the twist $\mathcal{F}_\kappa \otimes \omega_{X_\kappa}$. Since $\mathcal{L}_\kappa \cdot \omega_{X_\kappa} < 0$, it follows that the slope of $\mathcal{F}_\kappa \otimes \omega_{X_\kappa}$ is strictly smaller than the slope of $\mathcal{F}_\kappa$. Since both are Gieseker semistable, it follows that all homomorphisms from $\mathcal{F}_\kappa$ to $\mathcal{F}_\kappa \otimes \omega_{X_\kappa}$ are zero by [26, Lemma 1.3.3]. $\square$

THEOREM 4.10. — *Let $(X \rightarrow S, \mathcal{L})$ be a family of ω -negatively polarized surfaces over a connected finite type k -scheme S , where k is an algebraically closed field of characteristic 0. Fix a Hilbert polynomial P of degree 2. For any two k -points $s, t \in S(k)$, let $M_{X_s}^P$ (resp. $M_{X_t}^P$) denote the corresponding moduli space of Gieseker $\mathcal{L}_s$ -semistable torsion-free sheaves on X_s (resp. Gieseker $\mathcal{L}_t$ -semistable sheaves on X_t) [45, Section 1]. Then $M_{X_s}^P$ and $M_{X_t}^P$ satisfy Property 4.1. If $k = \mathbb{C}$, then $M_{X_s}^P$ and $M_{X_t}^P$ satisfy Property 4.2.*

Proof. — By smoothness of the stack $\mathrm{Coh}_{X/S}^{P,ss}$, proven in Proposition 4.9, its good moduli space $M_{X/S}^P \rightarrow S$ is locally isotrivial in the sense of Definition 3.1 (Corollary 2.4). The fibers of $M_{X/S}^P \rightarrow S$ over s and t are $M_{X_s}^P$ and $M_{X_t}^P$ respectively, since the formation of good moduli spaces commutes with base change from S [1, Proposition 4.7(i)]. The result follows from Corollary 3.10 and, in the case $k = \mathbb{C}$, from Theorem 3.15. $\square$

By the smoothness of $X \rightarrow S$, there is a determinant morphism $\det: \mathrm{Coh}_{X/S}^{P, \mathrm{ss}} \rightarrow \mathrm{Pic}_{X/S}$ to the relative Picard scheme $\mathrm{Pic}_{X/S}$ [26, Section 2.1]. The obstructions to lift sheaves $\mathcal{F}_\kappa$ under this morphism belong to the trace-zero Ext group $\mathrm{Ext}_0^2(\mathcal{F}_\kappa, \mathcal{F}_\kappa)$, and so in particular they are 0 by the proof of Proposition 4.9. It follows that $\det: \mathrm{Coh}_{X/S}^{P, \mathrm{ss}} \rightarrow \mathrm{Pic}_{X/S}$ is smooth. By the universal property of good moduli spaces, there is an induced proper morphism $\overline{\det}: M_{X/S}^P \rightarrow \mathrm{Pic}_{X/S}$. For a point $p \in \mathrm{Pic}_{X/S}(k)$ corresponding to a point $s \in S(k)$ and a line bundle L on X_s , the fiber $\overline{\det}^{-1}(p)$ is the moduli space $M_{X_s, L}^P$ of sheaves on X_s with Hilbert polynomial P and fixed determinant L .

PROPOSITION 4.11. — *In the situation of Theorem 4.10, fix line bundles L_1 and L_2 on X_s and X_t respectively that are algebraically equivalent in the sense that the points of $\mathrm{Pic}_{X/S}$ that they lie in the same connected component. Then the moduli spaces M_{X_s, L_1}^P and M_{X_t, L_2}^P of sheaves with fixed determinant satisfy Property 4.1. If $k = \mathbb{C}$, then M_{X_s, L_1}^P and M_{X_t, L_2}^P satisfy Property 4.2.*

Proof. — The map $\overline{\det}: M_{X/S}^P \rightarrow \mathrm{Pic}_{X/S}$ is locally isotrivial by Corollary 2.4. Therefore we may apply Corollary 3.10 and, in the case $k = \mathbb{C}$, Theorem 3.15, to conclude the result. $\square$

Le Potier Morphism. Let $(X \rightarrow S, \mathcal{L})$ be a smooth family of ω -negatively polarized surfaces, and fix a Hilbert polynomial P of degree 1. There is the Le Potier morphism $h: \mathrm{Coh}_{X/S}^{P, \mathrm{ss}} \rightarrow \mathrm{Hilb}_{X/S}^1$ originally defined by Le Potier [33], where $\mathrm{Hilb}^1(X/S)$ denotes the (relative) Hilbert scheme of subschemes of dimension 1 on the fibers of $X \rightarrow S$. For any S -scheme T , the morphism h sends a T -flat family of pure sheaves of dimension 1 in $\mathrm{Coh}_{X/S}^{P, \mathrm{ss}}(T)$ to its zero Fitting support, which in this case can be checked to be a relative Cartier divisor on $X_T \rightarrow T$ in $\mathrm{Hilb}_{X/S}^1(T)$. By the universal property of good moduli spaces, this induces the proper Le Potier morphism $\bar{h}: M_{X/S}^P \rightarrow \mathrm{Hilb}_{X/S}^1$. For every geometric point $s \rightarrow S$, the base change yields the proper Le Potier morphism $\bar{h}_s: M_{X_s}^P \rightarrow \mathrm{Hilb}_{X_s}^1$, which induces the perverse filtration on the intersection cohomology group (resp. cohomology group) $IH^\bullet(M_{X_s}^P, \overline{\mathbb{Q}}_\ell)$ (resp. $H^\bullet(M_{X_s}^P, \overline{\mathbb{Q}}_\ell)$).

PROPOSITION 4.12. — *Let $(X \rightarrow S, \mathcal{L})$ be a smooth family of ω -negatively polarized over a connected finite type k -scheme S , where k is an algebraically closed field of characteristic 0. Fix a Hilbert polynomial P of degree 1. Then, for any two points $s, t \in S(k)$, the corresponding moduli spaces $M_{X_s}^P$ and $M_{X_t}^P$ satisfy Property 4.1, where in addition it can*

be ensured that the isomorphisms $IC^\bullet(M_{X_s}^P, \overline{\mathbb{Q}}_\ell) \cong IC^\bullet(M_{X_t}^P, \overline{\mathbb{Q}}_\ell)$ and $H^\bullet(M_{X_s}^P, \overline{\mathbb{Q}}_\ell) \cong H^\bullet(M_{X_t}^P, \overline{\mathbb{Q}}_\ell)$ are filtered isomorphisms with respect to the perverse filtrations induced by the corresponding Le Potier morphisms $\bar{h}_s$ and $\bar{h}_t$.

If $k = \mathbb{C}$, then $M_{X_s}^P$ and $M_{X_t}^P$ satisfy Property 4.2 and the corresponding isomorphisms $IC^\bullet(M_{X_s}^P) \cong IC^\bullet(M_{X_t}^P)$ and $H^\bullet(M_{X_s}^P) \cong H^\bullet(M_{X_t}^P)$ can be arranged to be filtered isomorphisms with respect to the perverse filtrations induced by the corresponding Le Potier morphisms.

Proof. — By a similar reduction as in Corollary 3.10, we can assume without loss of generality that S is a smooth curve. Then, using the triviality of the vanishing cycle functor proven in Section 3, the proof of [11, Theorem 3.2.1(i)] applies verbatim to show that the corresponding specialization isomorphisms from the (intersection) cohomology of any closed S -fiber to the (intersection) cohomology of the geometric generic fiber are compatible with the perverse filtrations: while the proof in [11] is in the context of the complex topology, the argument found there is formal and also applies in the étale setting. $\square$

BIBLIOGRAPHY

- [1] J. ALPER, “Good moduli spaces for Artin stacks”, *Ann. Inst. Fourier* **63** (2013), no. 6, p. 2349-2402.
- [2] J. ALPER, J. HALL & D. RYDH, “A Luna étale slice theorem for algebraic stacks”, *Ann. Math. (2)* **191** (2020), no. 3, p. 675-738.
- [3] ———, “The étale local structure of algebraic stacks”, 2023, <https://arxiv.org/abs/1912.06162v3>.
- [4] J. ALPER, D. HALPERN-LEISTNER & J. HEINLOTH, “Existence of moduli spaces for algebraic stacks”, *Invent. Math.* **234** (2023), no. 3, p. 949-1038.
- [5] J. ALPER & F. JANDA, “Structure results for torus fixed loci”, 2024, <https://arxiv.org/abs/2402.10823>.
- [6] V. BALAJI & C. S. SESHADRI, “Moduli of parahoric $\mathcal{G}$ -torsors on a compact Riemann surface”, *J. Algebr. Geom.* **24** (2015), no. 1, p. 1-49.
- [7] A. A. BEĬLINSON, J. BERNSTEIN, P. DELIGNE & O. GABBER, “Faisceaux pervers”, in *Analysis and topology on singular spaces, I (Luminy, 1981)*, Astérisque, vol. 100, Société Mathématique de France, 1982, p. 5-171.
- [8] B. BHATT & P. SCHOLZE, “The pro-étale topology for schemes”, in *De la géométrie algébrique aux formes automorphes (I)* (J.-B. Bost, P. Boyer, A. Genestier, L. Laforgue, S. Lysenko, S. Morel & B. C. Ngô, eds.), Astérisque, vol. 369, Société Mathématique de France, 2015, p. 99-201.
- [9] M. A. DE CATALDO, “Perverse Leray filtration and specialisation with applications to the Hitchin morphism”, *Math. Proc. Camb. Philos. Soc.* **172** (2022), no. 2, p. 443-487.
- [10] M. A. DE CATALDO, A. F. HERRERO & A. IBÁÑEZ NÚÑEZ, “Relative étale slices and cohomology of moduli spaces”, 2023, <https://arxiv.org/abs/2307.00350>.

- [11] M. A. DE CATALDO & D. MAULIK, “The perverse filtration for the Hitchin fibration is locally constant”, *Pure Appl. Math. Q.* **16** (2020), no. 5, p. 1441-1464.
- [12] M. A. DE CATALDO & L. MIGLIORINI, “The projectors of the decomposition theorem are motivated”, *Math. Res. Lett.* **22** (2015), no. 4, p. 1061-1088.
- [13] M. A. DE CATALDO & S. ZHANG, “Projective completion of moduli of t -connections on curves in positive and mixed characteristic”, *Adv. Math.* **401** (2022), article no. 108329 (43 pages).
- [14] P. DELIGNE, “Théorème de Lefschetz et critères de dégénérescence de suites spectrales”, *Publ. Math., Inst. Hautes Étud. Sci.* **35** (1968), p. 259-278.
- [15] ———, *Cohomologie étale*, Lecture Notes in Mathematics, vol. 569, Springer, 1977, Séminaire de géométrie algébrique du Bois-Marie SGA 4 $\frac{1}{2}$.
- [16] P. DELIGNE & N. KATZ, *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, vol. 340, Springer, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7, II).
- [17] P. DELIGNE & D. MUMFORD, “The irreducibility of the space of curves of given genus”, *Publ. Math., Inst. Hautes Étud. Sci.* **36** (1969), p. 75-109.
- [18] T. Ekedahl, “On the adic formalism”, in *The Grothendieck Festschrift, Vol. II*, Progress in Mathematics, vol. 87, Birkhäuser, 1990, p. 197-218.
- [19] M. GORESKY & R. MACPHERSON, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 14, Springer, 1988.
- [20] S. GURJAR & N. NITSURE, “Schematic Harder–Narasimhan stratification for families of principal bundles in higher dimensions”, *Math. Z.* **289** (2018), no. 3-4, p. 1121-1142.
- [21] J. HALL & D. RYDH, “Perfect complexes on algebraic stacks”, *Compos. Math.* **153** (2017), no. 11, p. 2318-2367.
- [22] ———, “Coherent Tannaka duality and algebraicity of Hom-stacks”, *Algebra Number Theory* **13** (2019), no. 7, p. 1633-1675.
- [23] D. HALPERN-LEISTNER, A. F. HERRERO & T. JONES, “Moduli spaces of sheaves via affine Grassmannians”, *J. Reine Angew. Math.* **809** (2024), p. 159-215.
- [24] J. HEINLOTH, “Uniformization of $\mathcal{G}$ -bundles”, *Math. Ann.* **347** (2010), no. 3, p. 499-528.
- [25] N. HOFFMANN, “On moduli stacks of G -bundles over a curve”, in *Affine flag manifolds and principal bundles*, Trends in Mathematics, Springer, 2010, p. 155-163.
- [26] D. HUYBRECHTS & M. LEHN, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, 2010.
- [27] L. ILLUSIE, “Autour du théorème de monodromie locale”, in *Périodes p -adiques (Bures-sur-Yvette, 1988)*, Astérisque, vol. 223, Société Mathématique de France, 1994, p. 9-57.
- [28] ———, “Perversité et variation”, *Manuscr. Math.* **112** (2003), no. 3, p. 271-295.
- [29] A. LANGER, “Moduli spaces of principal bundles on singular varieties”, *Kyoto J. Math.* **53** (2013), no. 1, p. 3-23.
- [30] ———, “Semistable modules over Lie algebroids in positive characteristic”, *Doc. Math.* **19** (2014), p. 509-540.
- [31] Y. LASZLO & M. OLSSON, “The six operations for sheaves on Artin stacks. II. Adic coefficients”, *Publ. Math., Inst. Hautes Étud. Sci.* **107** (2008), p. 169-210.
- [32] ———, “Perverse t -structure on Artin stacks”, *Math. Z.* **261** (2009), no. 4, p. 737-748.
- [33] J. LE POTIER, “Faisceaux semi-stables de dimension 1 sur le plan projectif”, *Rev. Roum. Math. Pures Appl.* **38** (1993), no. 7-8, p. 635-678.

- [34] Q. LU & W. ZHENG, “Duality and nearby cycles over general bases”, *Duke Math. J.* **168** (2019), no. 16, p. 3135-3213.
- [35] D. MUMFORD, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Tata Institute of Fundamental Research and Hindustan Book Agency, 2008.
- [36] J. P. PRIDHAM, “Presenting higher stacks as simplicial schemes”, *Adv. Math.* **238** (2013), p. 184-245.
- [37] A. RAMANATHAN, “Stable principal bundles on a compact Riemann surface”, *Math. Ann.* **213** (1975), p. 129-152.
- [38] ———, “Moduli for principal bundles over algebraic curves. I”, *Proc. Indian Acad. Sci., Math. Sci.* **106** (1996), no. 3, p. 301-328.
- [39] ———, “Moduli for principal bundles over algebraic curves. II”, *Proc. Indian Acad. Sci., Math. Sci.* **106** (1996), no. 4, p. 421-449.
- [40] M. SAITO, “Modules de Hodge polarisables”, *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 6, p. 849-995.
- [41] ———, “Mixed Hodge modules”, *Publ. Res. Inst. Math. Sci.* **26** (1990), no. 2, p. 221-333.
- [42] A. H. W. SCHMITT, *Geometric invariant theory and decorated principal bundles*, Zürich Lectures in Advanced Mathematics, European Mathematical Society, 2008.
- [43] C. SCHNELL, “An overview of Morihiro Saito’s theory of mixed Hodge modules”, in *Representation theory, automorphic forms & complex geometry*, International Press, 2019, p. 27-80.
- [44] C. S. SESHADRI, “Geometric reductivity over arbitrary base”, *Adv. Math.* **26** (1977), no. 3, p. 225-274.
- [45] C. T. SIMPSON, “Moduli of representations of the fundamental group of a smooth projective variety. I”, *Publ. Math., Inst. Hautes Étud. Sci.* **79** (1994), p. 47-129.
- [46] THE STACKS PROJECT AUTHORS, “Stacks Project”, <http://stacks.math.columbia.edu>.
- [47] S. SUN, “Decomposition theorem for perverse sheaves on Artin stacks over finite fields”, *Duke Math. J.* **161** (2012), no. 12, p. 2297-2310.
- [48] B. TEISSIER, “Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney”, in *Algebraic geometry (La Rábida, 1981)*, Lecture Notes in Mathematics, vol. 961, Springer, 1982, p. 314-491.

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