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E-POLYNOMIALS OF GENERIC $\mathrm{GL}_n \rtimes \langle\sigma\rangle$-CHARACTER VARIETIES: UNBRANCHED CASE

by Cheng SHU

ABSTRACT. — For any unbranched double covering of a compact Riemann surface, we study the associated character varieties that are unitary in the global sense, which we call $\mathrm{GL}_n \rtimes \langle\sigma\rangle$-character varieties. We introduce $k > 0$ punctures on the surface, restrict the monodromies around the punctures to generic semi-simple conjugacy classes in GL_n, and compute the E-polynomials of these character varieties using the character table of $\mathrm{GL}_n(q)$. The result is expressed as the inner product of certain symmetric functions. We are then led to a conjectural formula for the mixed Hodge polynomial, which is built out of (modified) Macdonald polynomials, their self-pairings, and self-pairings of wreath Macdonald polynomials.

RÉSUMÉ. — Pour tout revêtement non ramifié de degré deux d'une surface de Riemann compacte, nous étudions les variétés de caractère associées qui sont unitaires dans le sens global, et nous les appellerons variétés de $\mathrm{GL}_n \rtimes \langle\sigma\rangle$-caractère. Nous enlevons k points de la surface donnée, exigeons que les monodromies autour des points enlevés appartiennent aux classes de conjugaison semi-simples génériques dans GL_n, et calculons les E-polynômes de ces variétés de caractère en utilisant la table des caractères de $\mathrm{GL}_n(q)$. Le résultat est exprimé comme le produit scalaire de certaines fonctions symétriques. Nous proposons une formule conjecturale pour le polynôme de Hodge mixte, qui est construit à partir des polynômes de Macdonald, de leurs produits scalaires, et des produits scalaires des polynômes de Macdonald en couronne.

1. Introduction

In this article, we will call Macdonald polynomials what are usually called *modified* Macdonald polynomials in the literature, since only this version of Macdonald polynomial appears.

Keywords: character varieties, mixed Hodge polynomial, symmetric functions.
2020 *Mathematics Subject Classification:* 14M35, 20C33, 05E05.

1.1. Background

Given a compact Riemann surface and a reductive algebraic group G, one can define three moduli spaces: the moduli space of Higgs G-bundles, the moduli space of flat G-connections and the G-character variety. The cohomologies of these spaces are identified via the nonabelian Hodge theory and the Riemann–Hilbert correspondence. Early study of these cohomology groups focused on the computation of Betti numbers. The method was to exploit the $\mathbb{C}^*$-action on the moduli of Higgs bundles (see for example [5, 9]). However, such a computation is only manageable when G has small rank. In general, counting points over finite fields proved to be a much more efficient way to compute the cohomology.

The project of counting points of character varieties over finite fields was initiated by Hausel and Rodriguez-Villegas in [8]. For character varieties, the counting does not produce the Poincaré polynomial. Instead, one obtains the *E-polynomial* of the character variety over complex numbers, according to a theorem of Katz [8, Appendix]. The E-polynomial has as coefficients the alternating sum of the dimensions of the graded pieces of the weight filtration. The computation relies on our knowledge of the character table of the finite group of Lie type $\mathrm{GL}_n(q)$. With the E-polynomial obtained, in a subsequent work joint with Letellier, they proposed a conjectural formula for the mixed Hodge polynomial. This polynomial encodes the dimension of each graded piece of the weight filtration in each cohomological degree. Remarkably, the formula is dominated by (modified) Macdonald polynomials and their self-pairings. Recently, by counting Higgs bundles over finite fields, Schiffmann and Mellit have obtained a closed formula for the Poincaré polynomial of the moduli space of (parabolic) Higgs bundles with $G = \mathrm{GL}_n$ and arbitrary n, thus proving the conjecture of Hausel–Letellier–Rodriguez-Villegas at the level of Poincaré polynomials (see [15, 16, 17]). Some other related conjectures have also been proved by Mellit (see [13, 14]).

1.2. $\mathrm{GL}_n\langle\sigma\rangle$-character varieties

In this article, we study character varieties that are unitary in the global sense. Let Σ' be a compact Riemann surface of genus g, and let $p' : \widetilde{\Sigma}' \to \Sigma'$ be an unbranched double covering. We will assume $g > 0$ throughout, so that such a covering exists. We introduce $k > 0$ punctures on Σ', resulting in the non-compact surface Σ. Denote by $p \colon \widetilde{\Sigma} \to \Sigma$ the restriction of p'

to Σ. Let $\mathcal{C} = (C_j)_{1\leqslant j\leqslant k}$ be a tuple of semi-simple conjugacy classes in GL_n. Finally, let σ be an order 2 exterior automorphism of GL_n (see Section 2.1 for the definition of σ), and denote by $\mathrm{GL}_n\langle\sigma\rangle$ the semi-direct product $\mathrm{GL}_n \rtimes \langle\sigma\rangle$. Our $\mathrm{GL}_n\langle\sigma\rangle$-character variety is defined by

$$\mathrm{Ch}_{\mathcal{C}}(\Sigma) := \left\{ (A_i,B_i)_i(X_j)_j \in \mathrm{GL}_n^{2g} \times \prod_{j=1}^{k} C_j \;\middle|\; A_1\sigma(B_1)A_1^{-1}B_1^{-1}\prod_{i=2}^{g}[A_i,B_i]\prod_{j=1}^{k}X_j = 1 \right\} /\!\!/ \mathrm{GL}_n,$$

where the bracket means the commutator. Note that $A_1\sigma(B_1)A_1^{-1}B_1^{-1}$ is just $[A_1\sigma, B_1]$. Also note that the defining equation is different from the branched case [20, Section 1.3]. This is the moduli space of the homomorphisms ρ that fit into the following commutative diagram:

$$\begin{tikzcd} \pi_1(\Sigma) \arrow[rr, "\rho"] \arrow[dr, "q_1"'] & & \mathrm{GL}_n\langle\sigma\rangle \arrow[dl, "q_2"] \\ & \mathrm{Gal}(\widetilde{\Sigma}/\Sigma) & \end{tikzcd}$$

where $\mathrm{Gal}(\widetilde{\Sigma}/\Sigma) \cong \mathbb{Z}/2\mathbb{Z}$ is the group of covering transformations, q_1 is the quotient by $\pi_1(\widetilde{\Sigma})$, and q_2 is the quotient by the identity component. The monodromy of ρ at the punctures must lie in the corresponding conjugacy classes $(C_j)_{1\leqslant j\leqslant k}$.

The character variety $\mathrm{Ch}_{\mathcal{C}}(\Sigma)$, via the nonabelian Hodge theory, is diffeomorphic to the moduli space of (parabolic) unitary Higgs bundles, that is, torsors under the unitary group scheme over Σ equipped with a Higgs field. The cohomology of this kind of moduli spaces is interesting in representation theory. See for example the work of Laumon and Ngô [10].

In a previous article [20], the author studied the case of branched coverings. Using the character table of $\mathrm{GL}_n(q)\langle\sigma\rangle$, a formula for the E-polynomial of the character variety was obtained in loc. cit. Surprisingly, the formula seems to suggest that the mixed Hodge polynomial is dominated by Macdonald polynomials and wreath Macdonald polynomials (see below). Roughly speaking, a wreath Macdonald polynomial appears precisely when the homomorphism ρ enters the twisted component $\mathrm{GL}_n\,\sigma$. In the current situation, a generator of $\pi_1(\Sigma)$ associated to a genus is mapped into $\mathrm{GL}_n\,\sigma$. We will see that this generator should give rise to a self-pairing of wreath Macdonald polynomial.

1.3. Macdonald polynomials and wreath Macdonald polynomials

We write $\mathbf{Sym}[\mathbf{z}]$ for the ring of symmetric functions in the variables $\mathbf{z} = (z_1, z_2, \dots)$ over the base field $\mathbb{Q}(z, w)$, the function field in z and w. The ring $\mathbf{Sym}[\mathbf{z}]$ is equipped with an inner product $\langle -, - \rangle$, called the Hall inner product. It can be deformed into another inner product $\langle -, - \rangle_{z,w}$, called the Macdonald inner product. We denote by $\mathcal{P}$ the set of all partitions, including the empty one. For any $\lambda \in \mathcal{P}$, denote by $H_\lambda(\mathbf{z}; z, w)$ the corresponding Macdonald polynomial. Denote by

$$N_\lambda(z,w) = \big\langle H_\lambda(\mathbf{z}; z, w), H_\lambda(\mathbf{z}; z, w) \big\rangle_{z,w}$$

the self-pairing of a Macdonald polynomial under the Macdonald inner product. It has an explicit expression (see [6, Corollary 5.4.8]):

$$N_\lambda(z,w) = \prod_{x \in \lambda} (z^{a(x)+1} - w^{l(x)})(z^{a(x)} - w^{l(x)+1}), \tag{1.1}$$

where x runs over the boxes in the Young diagram of λ, and $a(x)$ and $l(x)$ denote the arm-length and leg-length, respectively. We will also need a deformation of it:

$$N_\lambda(u,z,w) = \prod_{x \in \lambda} (z^{a(x)+1} - uw^{l(x)})(z^{a(x)} - u^{-1}w^{l(x)+1}). \tag{1.2}$$

Wreath Macdonald polynomials were introduced by Haiman in [6], and their existence was proved by Bezrukavnikov and Finkelberg in [1]. These are a family of (tensor) symmetric functions depending on an integer $r > 1$ and an r-core. An r-core is simply a partition λ such that there is no box $x \in \lambda$ with $h(x) \equiv 0 \mod r$, where

$$h(x) = a(x) + l(x) + 1$$

denotes the *hook length* of x. In this article, we set $r = 2$. The wreath Macdonald polynomials with a given 2-core are parametrised by $\mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$, and the set of all wreath Macdonald polynomials, with varying 2-cores, is in bijection with $\mathcal{P}$. For any $\lambda \in \mathcal{P}$, write

$$\widetilde{N}_\lambda(z,w) = \prod_{\substack{x \in \lambda \\ h(x) \equiv 0 \bmod 2}} (z^{a(x)+1} - w^{l(x)})(z^{a(x)} - w^{l(x)+1}), \tag{1.3}$$

and

$$\widetilde{N}_\lambda(u,z,w) = \prod_{\substack{x \in \lambda \\ h(x) \equiv 0 \bmod 2}} (z^{a(x)+1} - uw^{l(x)})(z^{a(x)} - u^{-1}w^{l(x)+1}). \tag{1.4}$$

Up to a "wreath-∇" term (see [20, Section 8.2.2]) that plays no role in what follows, these are the self-pairing and the deformed self-pairing of the wreath Macdonald polynomial corresponding to λ.

1.4. The generating series

We define two generating series, which only depend on g and k:

$$\Omega_{\bullet}(z,w) \coloneqq \sum_{\lambda\in\mathcal{P}} \frac{N_\lambda(zw,z^2,w^2)^{g-1}\widetilde{N}_\lambda(zw,z^2,w^2)}{\widetilde{N}_\lambda(z^2,w^2)} \prod_{j=1}^{k} H_\lambda(\mathbf{z}_j;z^2,w^2),$$

$$\Omega_{*}(z,w) \coloneqq \sum_{\lambda\in\mathcal{P}} \frac{N_\lambda(zw,z^2,w^2)^{2g-1}}{N_\lambda(z^2,w^2)} \prod_{j=1}^{k} H_\lambda(\mathbf{z}_j;z^2,w^2)^2,$$

where $\{\mathbf{z}_1,\dots,\mathbf{z}_k\}$ are independent variables.

Remark 1.1. — According to [20], the "wreath-∇" term of the self-pairing of a wreath Macdonald polynomial should not be deformed. Therefore in $\Omega_{\bullet}(z,w)$, these terms cancel out. This is the reason why we omit them from the definition of $\widetilde{N}_\lambda(z,w)$.

The reader can observe that the series $\Omega_{\bullet}(z,w)$ is roughly what we get from the series $\Omega_0(z,w)$ and $\Omega_1(z,w)$ defined in [20] by pretending that the number of ramification points is equal to 0. The essential difference is that there is no restriction on the 2-core in $\Omega_{\bullet}(z,w)$, and this is an obstruction to a uniform formula that works in both the branched case and the unbranched case. The origin of this difference lies in the representation theory of $\mathrm{GL}_n(q)\langle\sigma\rangle$. As is explained in [20], if a unipotent character of $\mathrm{GL}_n(q)$ corresponds to a partition with 2-core larger than (1), then its extension to $\mathrm{GL}_n(q)\sigma$ must vanish on every semi-simple conjugacy class. In the unbranched case, the conjugacy classes are contained in $\mathrm{GL}_n(q)$ and thus impose no restriction on the 2-core.

The series $\Omega_{*}(z,w)$ resembles the series defined by Hausel–Letellier–Rodriguez-Villegas for a Riemann surface of genus $2g-1$, but the Macdonald polynomial is squared.

1.5. The conjectural mixed Hodge polynomial

For any tuple of semi-simple conjugacy classes $\mathcal{C}=(C_j)_{1\leqslant j\leqslant k}$, let $\boldsymbol{\mu}=(\mu_j)_{1\leqslant j\leqslant k}$ be the tuple of partitions of n such that each μ_j is defined by the multiplicities of the eigenvalues of C_j. Let $\mathrm{Ch}_{\mathcal{C}}$ be the $\mathrm{GL}_n\langle\sigma\rangle$-character variety defined by $\mathcal{C}$. Denote by d the dimension of $\mathrm{Ch}_{\mathcal{C}}$, which only depends on $\boldsymbol{\mu}$, g and n. For any $\lambda \in \mathcal{P}$, denote by $h_\lambda(\mathbf{z})$ the corresponding complete symmetric function. The summand in Ω_* corresponding to the empty partition is equal to 1; therefore, we can apply the formal expansion

$$\frac{1}{1+x} = \sum_{m\geqslant 0}(-1)^m x^m$$

to $1+x=\Omega_*$. Define the rational functions in z and w:

$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) := \left\langle \frac{\Omega_\bullet(z,w)^2}{\Omega_*(z,w)}, \prod_{j=1}^{k} h_{\mu_j}(\mathbf{z}_j) \right\rangle. \tag{1.5}$$

The definition resembles the branched case, but does not depend on the parity of n.

Conjecture 1.2. — *Let $\mathcal{C}=(C_j)_{1\leqslant j\leqslant k}$ be a strongly generic tuple of semi-simple conjugacy classes in GL_n. Then:*

(i) *the rational function $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$ is a polynomial; it has degree d in each variable, and each monomial in it has even degree; moreover, $\mathbb{H}_{\boldsymbol{\mu}}(-z,w)$ has non-negative integer coefficients;*

(ii) *the mixed Hodge polynomial $H_{\mathrm{c}}(\mathrm{Ch}_{\mathcal{C}};x,y,t)$ is a polynomial in $q:=xy$ and t;*

(iii) *the mixed Hodge polynomial is given by the following formula:*

$$H_{\mathrm{c}}(\mathrm{Ch}_{\mathcal{C}};q,t) = (t\sqrt{q})^d\,\mathbb{H}_{\boldsymbol{\mu}}\left(-t\sqrt{q},\frac{1}{\sqrt{q}}\right);$$

in particular, it only depends on $\boldsymbol{\mu}$ and not on the generic eigenvalues of $\mathcal{C}$.

Remark 1.3. — Although $\mathcal{C}$ is a tuple of conjugacy classes of $\mathrm{GL}_n(q)$, the definition of genericity is very different from that in [7] (see Section 4.2).

The above conjectural formula, combined with some fundamental symmetries in Macdonald polynomials and wreath Macdonald polynomials, implies the following.

Conjecture 1.4 (Curious Poincaré duality). — *We have*

$$H_{\mathrm{c}}\left(\mathrm{Ch}_{\mathcal{C}};\frac{1}{qt^2},t\right) = (qt)^{-d}H_{\mathrm{c}}(\mathrm{Ch}_{\mathcal{C}};q,t).$$

1.6. Main theorem and evidences of the conjecture

Our main theorem is a formula for the E-polynomial.

THEOREM 1.5. — *The $t = -1$ specialisation of part* (iii) *of Conjecture 1.2 holds:*

$$H_{\mathrm{c}}(\mathrm{Ch}_{\mathcal{C}}; q, -1) = (\sqrt{q})^d \mathbb{H}_{\mathbf{B}}\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right).$$

Then the Curious Poincaré duality specialises to the following symmetry.

THEOREM 1.6. — *The $t = -1$ specialisation of Conjecture 1.4 holds:*

$$E(\mathrm{Ch}_{\mathcal{C}}; q) = q^d E(\mathrm{Ch}_{\mathcal{C}}; q^{-1}).$$

Our computation of the E-polynomial follows the line of [20]. However, there are major differences. On the one hand, in view of the defining equation of the character variety (see Section 1.2), the computation will only use the characters of $\mathrm{GL}_n(q)$, not those of $\mathrm{GL}_n(q)\langle\sigma\rangle$. The consequence is that partitions are allowed to have arbitrary 2-cores, as we have seen. On the other hand, due to the presence of the exterior automorphism σ, only those irreducible characters of $\mathrm{GL}_n(q)$ that are σ-stable have nontrivial contributions to the E-polynomial. This results in the expression (1.5) which looks more like the series in [20] instead of the one in [7]. Therefore, we will use a mixture of techniques from [7, 20].

The E-polynomial is the main evidence of our conjecture. However, as we can observe from the definition of $\Omega_{\bullet}(z, w)$, if we make the specialisations $z = \sqrt{q}$ and $w = \sqrt{q}^{-1}$, i.e. pass to the E-polynomial, then the wreath terms $\widetilde{N}_{\lambda}$ become invisible. This is one reason for which the conjectural formula in the unbranched case is formulated after the branched case. In [20], it is the representation theory of $\mathrm{GL}_n(q)\langle\sigma\rangle$ that suggests the possible connection to wreath Macdonald polynomials. In the current circumstance, the wreath terms can only be seen when we explicitly compute the mixed Hodge polynomials for $n = 1$ and $n = 2$ (and arbitrary $g > 0$). These computations are given at the end of the article. Without our previous experience with the branched case, we would have no idea how to explain these unexpected terms.

To test our conjectural formula[1] we have computed $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ when $n = 3$, $g = 1, 2, 3$, and:

$\boldsymbol{k} = 1$: $\mathcal{C} = (C_1)$ with C_1 regular;

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(1) The author uses MATLAB.

$k = 2$: $\mathcal{C} = (C_1, C_2)$ with C_1 and C_2 regular;
$k = 2$: $\mathcal{C} = (C_1, C_2)$ with C_1 regular and C_2 conjugate to $\mathrm{diag}(a, a, b)$, $a \neq b$.

There is no geometric result in the literature that allows us to compute the mixed Hodge polynomial of these character varieties when $n = 3$. However, our computation shows that part (i) of Conjecture 1.2 is true in every case we have tested. Since part (i) of the conjecture is a sufficient condition for the formula in part (iii) to be a polynomial in q and t with non-negative integer coefficients, this indicates a geometric origin of $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$.

Let us give one example. We put $n = 3$, $g = 1$, $k = 1$, and take one regular semi-simple conjugacy class. The complete symmetric function $h_1(\mathbf{z})^3$ corresponds to this conjugacy class. Unwinding the definitions, we have

$$\begin{aligned}
\frac{\Omega_\bullet(z,w)^2}{\Omega_*(z,w)} = {} & \frac{\widetilde{N}_{(2)}(zw,z^2,w^2)}{\widetilde{N}_{(2)}(z^2,w^2)} H_{(1)}H_{(2)} + \frac{\widetilde{N}_{(1^2)}(zw,z^2,w^2)}{\widetilde{N}_{(1^2)}(z^2,w^2)} H_{(1)}H_{(1^2)} \\
& + \frac{\widetilde{N}_{(3)}(zw,z^2,w^2)}{\widetilde{N}_{(3)}(z^2,w^2)} H_{(3)} + \frac{\widetilde{N}_{(1^3)}(zw,z^2,w^2)}{\widetilde{N}_{(1^3)}(z^2,w^2)} H_{(1^3)} + H_{(21)} \\
& - \frac{N_{(1)}(zw,z^2,w^2)}{N_{(1)}(z^2,w^2)} H_{(1)}^3 + \text{terms not of degree } 3,
\end{aligned}$$

where we have omitted the variables of the Macdonald polynomials. Note that the partition (21) is a 2-core, and so $\widetilde{N}_{(21)}(z, w) = 1$. Consequently, $H_{(21)}$ has coefficient 1. For the same reason, we do not see the contribution of partition (1) in the coefficients of $H_{(1)}H_{(2)}$ and $H_{(1)}H_{(1^2)}$. Taking the Hall inner product with $h_1(\mathbf{z})^3$ gives

$$\begin{aligned}
\mathbb{H}_{\boldsymbol{\mu}}(z,w) = {} & \frac{(z^3-w)^2}{(z^4-1)(z^2-w^2)}(3+3z^2) + \frac{(z-w^3)^2}{(1-w^4)(z^2-w^2)}(3+3w^2) \\
& + \frac{(z^3-w)^2}{(z^4-1)(z^2-w^2)}\big(1+2z^2(z^2+1)+z^6\big) \\
& + \frac{(z-w^3)^2}{(1-w^4)(z^2-w^2)}\big(1+2w^2(w^2+1)+w^6\big) \\
& + \big(1+2(z^2+w^2)+z^2w^2\big) - 6\frac{(z-w)^2}{(z^2-1)(1-w^2)} \\
= {} & w^6 + w^4 z^2 + 2\,w^4 - 2\,w^3 z + w^2 z^4 + 3\,w^2 z^2 + 8\,w^2 \\
& - 2\,w\,z^3 - 4\,w\,z + z^6 + 2\,z^4 + 8\,z^2 + 5.
\end{aligned}$$

Organisation of the article

As in [7, 20], there are three key ingredients in computing the E-polynomial, and they are collected in Sections 2 and 3:

(1) the Frobenius formula (2.3) that reduces the point-counting problem to the evaluation of σ-stable irreducible characters of $\mathrm{GL}_n(q)$;
(2) a theorem of Lusztig and Srinivasan that expresses an irreducible character of $\mathrm{GL}_n(q)$ as a linear combination of Deligne–Lusztig characters $R^G_{T_w}\theta_{T_w}$ (3.2);
(3) the character formula (3.1) which reduces the computation of $R^G_{T_w}\theta_{T_w}$ to the Green function and the linear characters θ_{T_w}.

The Green functions can be realised as transition matrices between symmetric functions (2.9). The linear characters are computed in Section 5. In Section 4, we give the definition of our character variety, as well as generic conjugacy classes, which is a key assumption in our main theorem. The computation of E-polynomial is achieved in Section 6, using results from previous sections. In the last section, we check that our conjectural formula is consistent with some known cohomological results when $n = 1$ and $n = 2$.

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2. Preliminaries

2.1. General notations

Let G be an algebraic group, and let X, Y and Z be subvarieties of G. We will write $N_X(Y) = \{x \in X \mid xYx^{-1} = Y\}$. If $g \in G$, we denote by $C_G(g) = \{h \in G \mid hgh^{-1} = g\}$ the centraliser of g in G. The identity component of an algebraic group G is denoted by G°. If ϕ is an automorphism or endomorphism of a set X, we denote by X^ϕ the subset of the fixed points of ϕ. If G is a finite group, then $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G.

For any $n \in \mathbb{Z}_{>0}$, we denote by $\mathfrak{S}_n$ the symmetric group and by $\mathfrak{W}_n$ the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the factors.

If σ is an order 2 automorphism of GL_n, then we can define a semi-direct product $\mathrm{GL}_n \rtimes \langle\sigma\rangle$. The σ that we will use is defined as follows. Denote by J_n the matrix

$$(J_n)_{ij} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

If n is even, define the diagonal matrix $t_0 = \operatorname{diag}(a_1, \ldots, a_n)$ with $a_i = 1$ if $i \leqslant N$ and $a_i = -1$ otherwise. Write $J'_n = t_0 J_n$. Write

$$\mathcal{J}_n := \begin{cases} J'_n & \text{if } n \text{ is even,} \\ J_n & \text{if } n \text{ is odd.} \end{cases} \tag{2.1}$$

Define $\sigma \in \operatorname{Aut} G$ as sending g to $\mathcal{J}_n g^{-t} \mathcal{J}_n^{-1}$, where g^{-t} is the transpose-inverse of g. We have $(G^\sigma)^\circ \cong \mathrm{Sp}_{2N}$ if $n = 2N$ and $(G^\sigma)^\circ \cong \mathrm{SO}_{2N+1}$ if $n = 2N+1$.

2.2. Algebraic groups defined over a finite field

2.2.1.

We will denote by $p > 2$ a prime number and q a power of p. We denote by $\Bbbk$ an algebraic closure of $\mathbb{F}_q$. An algebraic group G over $\Bbbk$ is defined over $\mathbb{F}_q$ if there is an algebraic group G_0 over $\mathbb{F}_q$ such that $G_0 \otimes_{\mathbb{F}_q} \Bbbk \cong G$. If this is the case, we denote by F the geometric Frobenius endomorphism of G. We denote by G^F the finite subgroup of F-fixed points. We may also denote it by $G(q)$. If $X \subset G$ is a subvariety, we say that X is F-stable if $F(X) = X$.

2.2.2.

Let G be a connected reductive group defined over $\mathbb{F}_q$, and let $T \subset G$ be an F-stable maximal torus. Denote by W the Weyl group of G defined by T. The G^F-conjugacy classes of F-stable maximal tori are parametrised by the *F-conjugacy* classes of W. Two elements v and w of W are F-conjugate if there is some $x \in W$ such that $xvF(x)^{-1} = w$. This bijection is given as follows. Let $w \in W$ and choose $\dot{w} \in N_G(T)$ representing w. By the

Lang–Steinberg theorem, there exists $g \in G$ such that $g^{-1}F(g) = \dot{w}$. Then gTg^{-1} is an F-stable maximal torus whose G^F-conjugacy class corresponds to w. Conversely, any F-stable maximal torus can be written as gTg^{-1} for some $g \in G$. Since $g^{-1}F(g)$ normalises T, it defines an element of W whose F-conjugacy class corresponds to the G^F-conjugacy class of gTg^{-1}.

Fix an F-stable Levi subgroup $L_0 \subset G$. The G^F-conjugacy classes of the G-conjugates of L_0 are in bijection with the F-conjugacy classes of $N_G(L_0)/L_0$. This bijection is completely analogous to the torus case above. Let $w \in N_G(L_0)/L_0$, choose $\dot{w} \in N_G(L_0)$ representing it, and let $g \in G$ be such that $g^{-1}F(g) = \dot{w}$. Then $L := gL_0g^{-1}$ is also an F-stable Levi subgroup. Define $F_w := \operatorname{ad}\dot{w} \circ F$. It is another Frobenius endomorphism on G, and L_0 is F_w-stable. Now $\operatorname{ad} g$ is an isomorphism between L_0 and L. We have a commutative diagram:

$$\begin{tikzcd} L_0 \arrow[r, "\operatorname{ad} g"] \arrow[d, "F_w"'] & L \arrow[d, "F"] \\ L_0 \arrow[r, "\operatorname{ad} g"'] & L. \end{tikzcd} \tag{2.2}$$

In particular, $L_0^{F_w} \cong L^F$.

2.3. Frobenius formula

2.3.1.

Let G be a finite group and let σ be an automorphism of G of order 2. Then σ induces an action on $\operatorname{Irr}(G)$ by composition. Denote by $\operatorname{Irr}(G)^\sigma$ the subset of σ-fixed elements.

Proposition 2.1. — *Let $\mathcal{C} = (C_j)_{1\leqslant j\leqslant k}$ be an arbitrary k-tuple of conjugacy classes in G. Then we have the following counting formula:*

$$\#\left\{(A_i,B_i)_i(X_j)_j \in G^{2g}\times\prod_{j=1}^k C_j \;\middle|\; A_1\sigma(B_1)A_1^{-1}B_1^{-1}\prod_{i=2}^g[A_i,B_i]\prod_{j=1}^k X_j = 1\right\} = |G|\sum_{\chi\in\operatorname{Irr}(G)^\sigma}\left(\frac{|G|}{\chi(1)}\right)^{2g-2}\prod_{i=1}^k\frac{|C_i|\chi(C_i)}{\chi(1)}. \tag{2.3}$$

Proof. — We will follow the strategy in [7, 20]. The left-hand side of (2.3) turns out to be the value at $1 \in G$ of the convolution of a collection of functions on G. The computation of this value can then be reduced to the Fourier transforms of individual functions in the convolution. Recall that if f_1 and f_2 are complex-valued functions on G that are invariant under conjugation by G, then their convolution product is defined by

$$(f_1 * f_2)(x) = \sum_{yz=x} f_1(y) f_2(z),$$

which is also invariant under conjugation. Define such a function n^1 by

$$n^1(x) = \#\big\{(A,B) \in G^2 \bigm| ABA^{-1}B^{-1} = x\big\},$$

and another function n' by

$$n'(x) = \#\big\{(A,B) \in G^2 \bigm| A\sigma(B)A^{-1}B^{-1} = x\big\}.$$

Denote by n^{g-1} the convolution product of $g-1$ copies of n^1. Denote by 1_{C_i} the characteristic function of the class C_i. Then

$$\begin{aligned}\#\Bigg\{(A_i,B_i)_i(X_j)_j \in G^{2g} \times \prod_{j=1}^k C_j \;\Bigg|\; & \\ A_1\sigma(B_1)A_1^{-1}B_1^{-1}\prod_{i=2}^{g}[A_i,B_i]\prod_{j=1}^{k} X_j = 1\Bigg\}& \\ = (n' * n^{g-1} * 1_{C_1} * \cdots * 1_{C_k})(1).&\end{aligned}$$

By the compatibility between Fourier transform and convolution (see [20, (2.7.2.6) and (2.7.2.7)]), we have

$$\begin{aligned}(2.4)\quad & (n' * n^{g-1} * 1_{C_1} * \cdots * 1_{C_k})(1) \\ &= \frac{1}{|G|}\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^2\mathcal{F}(n')(\chi)\big(\mathcal{F}(n^1)(\chi)\big)^{g-1}\prod_{i=1}^{k}\frac{|C_i|\chi(C_i)}{\chi(1)},\end{aligned}$$

where for any function f on G and any character χ,

$$\mathcal{F}(f)(\chi) \coloneqq \sum_{h\in G}\frac{f(h)\chi(h)}{\chi(1)}.$$

According to [7, Lemma 3.1.3], we have

$$\mathcal{F}(n^1)(\chi) = \left(\frac{|G|}{\chi(1)}\right)^2.$$

It remains to compute $\mathcal{F}(n')(\chi)$.

Let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible representation with character χ. We have

$$\sum_{h\in G} \frac{n'(h)\chi(h)}{\chi(1)} = \sum_{(a,b)\in G^2} \frac{\chi\big(a\sigma(b)a^{-1}b^{-1}\big)}{\chi(1)}$$
$$= \frac{1}{\chi(1)} \operatorname{Tr} \sum_{(a,b)\in G^2} \rho\big(a\sigma(b)a^{-1}b^{-1}\big).$$

Note that $\sum_a \rho\big(a\sigma(b)a^{-1}\big)$ is an endomorphism of V that commutes with $\rho(h)$ for any $h \in G$. By Schur's lemma, it must be a scalar endomorphism. We deduce that

$$\sum_a \rho\big(a\sigma(b)a^{-1}\big) = \frac{|G|\cdot\chi\circ\sigma(b)}{\chi(1)} \operatorname{Id}_V .$$

Using the orthogonality of irreducible characters, we compute

$$\frac{1}{\chi(1)} \operatorname{Tr} \sum_{(a,b)\in G^2} \rho\big(a\sigma(b)a^{-1}b^{-1}\big) = \frac{1}{\chi(1)} \sum_{b\in G} \frac{|G|\cdot\chi\circ\sigma(b)}{\chi(1)} \chi(b^{-1})$$
$$= \begin{cases} 0 & \text{if } \chi\circ\sigma \neq \chi, \\ \left(\frac{|G|}{\chi(1)}\right)^2 & \text{otherwise.} \end{cases}$$

We conclude that in (2.4), only the σ-stable irreducible characters contribute to the sum. $\square$

2.4. Symmetric functions

Now the letter q denotes an indeterminate.

2.4.1.

Let $\mathbf{z} = (z_1, z_2, \dots)$ be an infinite series of variables. The ring of symmetric functions over a field $\mathbb{K}$ is denoted by $\mathbf{Sym}_{\mathbb{K}}[\mathbf{z}]$. In this article, $\mathbb{K}$ can be $\mathbb{Q}$, $\mathbb{Q}(q)$, or $\mathbb{Q}(q,t)$, i.e. functions fields in variables q, t over rational numbers $\mathbb{Q}$, and in these cases, we will write $\mathbf{Sym}_q[\mathbf{z}]$ and $\mathbf{Sym}_{q,t}[\mathbf{z}]$. We will omit the subscript if there is no confusion with the base ring. For any partition $\lambda \in \mathcal{P}$, the corresponding Schur symmetric functions, monomial symmetric functions, complete symmetric functions and the power sum symmetric functions are denoted by $s_\lambda(\mathbf{z})$, $m_\lambda(\mathbf{z})$, $h_\lambda(\mathbf{z})$ and $p_\lambda(\mathbf{z})$ respectively.

For any partition $\lambda = (1^{m_1}, 2^{m_2}, \dots)$, define $z_\lambda = \prod_i i^{m_i} m_i!$. We have

$$p_\lambda(\mathbf{z}) = \sum_{\tau \in \mathcal{P}} \chi_\lambda^\tau s_\tau(\mathbf{z}),$$
$$s_\lambda(\mathbf{z}) = \sum_{\tau} z_\tau^{-1} \chi_\tau^\lambda p_\tau(\mathbf{z}),$$

where χ_λ^τ is the value of the irreducible character of $\mathfrak{S}_n$ (with $n = |\tau| = |\lambda|$) corresponding to τ at a conjugacy class corresponding to λ.

2.4.2.

There are two families of Hall–Littlewood symmetric functions defined in [12], and they are denoted by $P_\lambda(\mathbf{z}; q)$ and $Q_\lambda(\mathbf{z}; q)$, for $\lambda \in \mathcal{P}$. The function $b_\lambda(q)$ is the difference between them:

$$b_\lambda(q) P_\lambda(\mathbf{z}; q) = Q_\lambda(\mathbf{z}; q).$$

By [12, (2.6) and (2.7)], we have

$$\frac{1}{\left|\mathrm{GL}_n(q)\right|} = q^{-2n((1^n))-n} b_{(1^n)}(q^{-1})^{-1}. \tag{2.5}$$

We have

$$s_\lambda(\mathbf{z}) = \sum_{\tau} K_{\lambda,\tau}(q) P_\tau(\mathbf{z}; q), \tag{2.6}$$

for some polynomials $K_{\lambda,\tau}(q)$ in q and they are called Kostka–Foulkes polynomials. The modified Kostka–Foulkes polynomials are defined by

$$\widetilde{K}_{\lambda,\tau}(q) = q^{n(\tau)} K_{\lambda,\tau}(q^{-1}). \tag{2.7}$$

The modified Hall–Littlewood function is defined by

$$\widetilde{H}_\lambda(\mathbf{z}; q) = \sum_{\tau} \widetilde{K}_{\tau,\lambda}(q) s_\tau(\mathbf{z}). \tag{2.8}$$

Finally, the Green polynomial is defined by

$$\mathcal{Q}_\lambda^\tau(q) = \sum_{\nu} \chi_\lambda^\nu \widetilde{K}_{\nu\tau}(q). \tag{2.9}$$

Note that $\left(\mathcal{Q}_\lambda^\tau(q)\right)_{\tau\lambda}$ is the transition matrix between the modified Hall–Littlewood functions and the power sum symmetric functions.

2.4.3.

The ring $\mathbf{Sym}_{\mathbb{K}}[\mathbf{z}]$ has a natural λ-ring structure $\{p_n\}_{n\in\mathbb{Z}_{>0}}$ with p_n sending $p_1(\mathbf{z})$ to $p_n(\mathbf{z})$ and sending q and t to their n-th powers. Let A be a λ-ring containing $\mathbb{K}$ as a λ-subring. Given any element x of A, there is a unique λ-ring homomorphism $\varphi_x\colon \mathbf{Sym}_{\mathbb{K}}[\mathbf{z}] \to A$ sending $p_1(\mathbf{z})$ to x. For any $F \in \mathbf{Sym}_{\mathbb{K}}[\mathbf{z}]$, its image under this map is denoted by $F[x]$. Concretely, we can write F as a polynomial $f\big(p_n(\mathbf{z}) \,\big|\, n \in \mathbb{Z}_{>0}\big)$, we have $F[x] = f\big(p_n(x) \,\big|\, n \in \mathbb{Z}_{>0}\big)$. This is the plethystic substitution for F.

2.4.4.

The Hall inner product on $\mathbf{Sym}[\mathbf{z}]$ is defined by

$$\big\langle s_\lambda(\mathbf{z}), s_\mu(\mathbf{z})\big\rangle = \delta_{\lambda,\mu},$$

for any partitions λ and μ. We have

$$\big\langle p_\lambda(\mathbf{z}), p_\mu(\mathbf{z})\big\rangle = z_\lambda \delta_{\lambda,\mu}.$$

The (q,t)-inner product on $\mathbf{Sym}_{q,t}[\mathbf{z}]$ is defined by

$$\big\langle F[\mathbf{z}], G[\mathbf{z}]\big\rangle_{q,t} \coloneqq \big\langle F[\mathbf{z}], G[(q-1)(1-t)\,\mathbf{z}]\big\rangle$$

for any F, $G \in \mathbf{Sym}_{q,t}[\mathbf{z}]$.

3. σ-stable irreducible characters of $\mathrm{GL}_n(q)$

In this section, we recall the character formula for Deligne–Lusztig inductions, the construction of σ-stable irreducible characters of $\mathrm{GL}_n(q)$, and introduce some combinatorial data called *types*. Types will be used in our explicit computation of the E-polynomial in Section 6.

3.1. Deligne–Lusztig inductions

3.1.1.

Let H be a connected reductive group defined over $\mathbb{F}_q$ and denote by F the geometric Frobenius endomorphism. If $L \subset H$ is an F-stable Levi subgroup, we denote by R_L^H the Deligne–Lusztig induction, which is a $\overline{\mathbb{Q}}_\ell$-linear map from the set of L^F-invariant functions on L^F to the set of

H^F-invariant functions on H^F. We fix an F-stable maximal torus $T \subset H$, and denote by W_H the Weyl group of H defined by T. We will assume that T is split so that F acts trivially on W_H. For any $w \in W_H$, we choose an F-stable maximal torus T_w corresponding to the conjugacy class of w. Let $\mathbf{1}$ be the trivial character of T_w^F. The Green function $Q^H_{T_w}(u)$ is defined as the restriction of $R^H_{T_w}\mathbf{1}$ to H^F_u, the subset of unipotent elements. These functions only depend on the F-conjugacy class of w.

Proposition 3.1 ([4, Proposition 12.2]). — *Let $s \in H^F$ be a semi-simple element and $u \in H^F$ a unipotent element that commutes with s. Let θ be a $\overline{\mathbb{Q}}_\ell$-valued function on T_w^F. We have*

$$R^H_{T_w}\theta(us) = \frac{1}{\left|C_H(s)^{\circ F}\right|} \sum_{\{h \in H^F \mid hsh^{-1} \in T_w\}} Q^{C_H(s)^\circ}_{h^{-1}T_w h}(u)\theta(hsh^{-1}). \tag{3.1}$$

3.1.2.

Fix a semi-simple element $s \in H^F$. For any $\tau \in W_H$, put

$$A_\tau = \left\{h \in H \;\middle|\; hsh^{-1} \in T_\tau\right\}, \qquad A_\tau^F = A_\tau \cap H^F.$$

The set A_τ^F is crucial for the computation of Deligne–Lusztig characters. If A_τ^F is non-empty, we may assume that $s \in T_\tau^F$, since $R^H_{T_\tau}\theta$ is invariant under conjugation by H^F. Put $L := C_H(s)^\circ$. For any $h \in A_\tau^F$, $h^{-1}T_\tau h$ is an F-stable maximal torus of L. In particular, T_τ is an F-stable maximal torus of L. Let $W_L := W_L(T_\tau)$ be the Weyl group of L defined by T_τ. The L^F-conjugacy classes of the F-stable maximal tori of L are parametrised by the F-conjugacy classes of W_L. There is a natural map from A_τ^F to the set of F-conjugacy classes of W_L, sending $h \in A_\tau^F$ to the class of $h^{-1}T_\tau h$. Denote by B_τ the image of this map. Let $\nu \in W_L$ represent an F-conjugacy class of W_L and denote by $A_{\tau,\nu}^F$ the inverse image in A_τ^F of this F-conjugacy class.

Let $g_\tau \in H$ be such that $T_\tau = g_\tau T g_\tau^{-1}$. Write $s_0 = g_\tau^{-1}sg_\tau$ and $L_0 = C_H(s_0)$. Let W_{L_0} be the Weyl group of L_0 defined by T. The F-conjugacy classes of W_L are in natural bijection with the τ-conjugacy classes of W_{L_0}. Two elements v and w are τ-conjugate if there is some $x \in W_L$ such that $v = xw\tau x^{-1}\tau^{-1}$. Therefore, we can identify B_τ with a set of τ-conjugacy classes of W_{L_0}.

Proposition 3.2. — *We have*

$$|A_{\tau,\nu}^F| = |L^F| z_\tau z_\nu^{-1},$$

where z_τ is the cardinality of the centraliser of τ in W_H, and z_ν is the cardinality of the stabiliser of ν under the τ-twisted conjugation by W_{L_0}.

Proof. — This is [7, Equation (4.3.3)], but we can give a direct proof (without assuming $H = \mathrm{GL}_n$). We begin by showing that $A_\tau = N_H(T_\tau)\cdot L$. Let $h \in A_\tau$, then $h^{-1}T_\tau h \subset L$. There exists $l \in L$ such that $lh^{-1}T_\tau hl^{-1} = T_\tau$; that is, $h = nl$ for some $n \in N_H(T_\tau)$. Conversely, any product nl with $n \in N_H(T_\tau)$ and $l \in L$ lies in A_τ. We have the following bijections:

$$A_\tau^F/L^F \cong \big(N_H(T_\tau)/N_L(T_\tau)\big)^F \cong \big(N_H(T)/N_{L_0}(T)\big)^{F_\tau} \cong (W_H/W_{L_0})^\tau.$$

The first bijection is induced by the identifications $A_\tau = N_H(T_\tau)\cdot L$ and $(A_\tau/L)^F \cong A_\tau^F/L^F$. The second is induced by the conjugation by g_τ^{-1}, while the Frobenius endomorphism is twisted according to the diagram (2.2). More precisely, $F_\tau = \operatorname{ad}\dot\tau \circ F$, where $\dot\tau := g_\tau^{-1}F(g_\tau)$ represents $\tau \in W_H$. The last bijection is obtained by taking the quotient by T. Since T is assumed to be split, the action of F on W_H is trivial, and we are left with the conjugation action of τ. Suppose that hL^F corresponds to wW_{L_0} under these bijections. Since wW_{L_0} is preserved under the conjugation by τ, we have $w^{-1}\tau w\tau^{-1} \in W_{L_0}$. We claim that the following two statements are equivalent:

(i) the L^F-conjugacy class of the maximal torus $h^{-1}T_\tau h$ corresponds to the τ-conjugacy class of $\nu \in W_{L_0}$;

(ii) $w^{-1}\tau w\tau^{-1}$ lies in the τ-conjugacy class of ν.

Since $h^{-1}T_\tau h = l^{-1}T_\tau l$, the L^F-conjugacy class of this torus corresponds to the F-conjugacy class of $lF(l)^{-1}$. But h is fixed by F, and so $lF(l)^{-1} = n^{-1}F(n)$. Taking the conjugation by g_τ^{-1} and the quotient by T sends n to w and $F(n)$ to $\tau w\tau^{-1}$. The claim is proved. It remains to count the number of cosets wW_{L_0} such that $w^{-1}\tau w\tau^{-1}$ lies in the τ-conjugacy class of ν. Equivalently, we count the cardinality of

$$\mathbb{W}(\tau,\nu) := \big\{w \in W_H \;\big|\; w^{-1}\tau w\tau^{-1} \text{ lies in the } \tau\text{-conjugacy class of } \nu\big\}$$

and then divide by $|W_{L_0}|$. If w_1 and w_2 are such that $w_1^{-1}\tau w_1\tau^{-1} = w_2^{-1}\tau w_2\tau^{-1}$, then $w_2w_1^{-1}$ commutes with τ. Conversely, if $w_1 \in \mathbb{W}(\tau,\nu)$ and $c \in C_{W_H}(\tau)$, then cw_1 satisfies $(cw_1)^{-1}\tau cw_1\tau^{-1} = w_1^{-1}\tau w_1\tau^{-1}$. Therefore, we have

$$\big|\mathbb{W}(\tau,\nu)\big| = \big|C_{W_H}(\tau)\big|\cdot\frac{|W_{L_0}|}{z_\nu} = z_\tau z_\nu^{-1}|W_{L_0}|.$$

It follows that $|A_{\tau,\nu}^F| = |L^F|z_\tau z_\nu^{-1}$. $\square$

3.2. σ-stable irreducible characters

Write $G = \mathrm{GL}_n$. Denote by $T \subset G$ the maximal torus consisting of diagonal matrices, and by $B \subset G$ the Borel subgroup consisting of upper triangular matrices. Denote by Δ the set of simple roots of G defined by (T,B).

3.2.1.

Let $M\subset G$ be an F-stable Levi subgroup and let $T_M\subset M$ be an F-stable maximal torus. Let W_M be the Weyl group of M defined by T_M. For any $w\in W_M$, we choose an F-stable maximal torus $T_w\subset M$ whose M^F-conjugacy class corresponds to the F-conjugacy class of w. Let θ be a regular linear character of M^F (see [11, Section 3.1, (a) and (b)]). Since F preserves T_M, it acts on W_M. Let $\varphi\in\mathrm{Irr}(W_M)^F$. It extends to a character $\widetilde{\varphi}\in\mathrm{Irr}\big(W_M\rtimes\langle F\rangle\big)$. Such an extension is not unique. Write

$$R^G_\varphi\theta := |W_M|^{-1}\sum_{w\in W_M}\widetilde{\varphi}(wF)R^G_{T_w}\theta_{T_w}, \tag{3.2}$$

where θ_{T_w} is the restriction of θ to T^F_w. By [11, Theorem 3.2], for some choice of $\widetilde{\varphi}$, the function $\epsilon_G\epsilon_M R^G_\varphi\theta$ is an irreducible character of $\mathrm{GL}_n(q)$.

3.2.2.

For any subset $I\subset\Delta$, denote by L_I the corresponding standard Levi subgroup of G with respect to (T,B). If L_I is σ-stable, then we may choose an isomorphism

$$L_I\cong\mathrm{GL}_{n_0}\times\prod_i(\mathrm{GL}_{n_i}\times\mathrm{GL}_{n_i}).$$

If we write the elements of L_I as block diagonal matrices, then L_I is of the form

$$\begin{pmatrix}\ddots &&&&&&\\ & \boxed{\mathrm{GL}_{n_i}} &&&&&\\ && \ddots &&&&\\ &&& \boxed{\mathrm{GL}_{n_0}} &&&\\ &&&& \ddots &&\\ &&&&& \boxed{\mathrm{GL}_{n_i}} &\\ &&&&&& \ddots\end{pmatrix}.$$

Write $W_I := N_G(L_I)/L_I$. Since σ leaves L_I stable, it induces an automorphism of W_I. As is explained in [20, Section 3.2.2], there is an isomorphism

$$W_{(G^\sigma)^\circ}\big((L_I^\sigma)^\circ\big) \xrightarrow{\sim} W_I^\sigma.$$

Therefore, for any $w \in W_I^\sigma$, we may choose $\dot{w} \in (G^\sigma)^\circ$ representing w. If $g \in (G^\sigma)^\circ$ is such that $g^{-1}F(g) = \dot{w}$, then define $L_{I,w} := gL_Ig^{-1}$. It is an F-stable and σ-stable Levi subgroup of G. If M is an F-stable and σ-stable Levi factor of a σ-stable parabolic subgroup, then M is necessarily G^F-conjugate to an $L_{I,w}$ for some I and w.

3.2.3.

Let $M = L_{I,w}$ for some I and w as above. Write M as $M_0 \times M_1$ with $M_0 \cong \mathrm{GL}_{n_0}$ and $M_1 \cong \prod_i(\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i})$, where n_0 and the n_i's are as in the previous paragraph. Let $T_M \subset M$ be an F-stable and σ-stable maximal torus. We can write $T_M = T_0 \times T_1$ with $T_0 \subset M_0$ and $T_1 \subset M_1$. We may assume that $T_0 \subset M_0$ is a split maximal torus. Denote by W_0 (resp. W_1) the Weyl group of M_0 (resp. M_1) defined by T_0 (resp. T_1).

Let n_+ and n_- be some non-negative integers such that $n_+ + n_- = n_0$. Let $M_{00} \subset M_0$ be a Levi subgroup containing T_0 such that $M_{00}^F \cong \mathrm{GL}_{n_+}(q) \times \mathrm{GL}_{n_-}(q)$. Its Weyl group is isomorphic to $\mathfrak{S}_{n_+} \times \mathfrak{S}_{n_-}$.

Let θ_1 be a linear character of M_1^F that is σ-stable. Then $\theta := \mathbf{1}\boxtimes\eta\boxtimes\theta_1$ is a linear character of $M_{00} \times M_1$, where $\mathbf{1}$ is the trivial character of $\mathrm{GL}_{n_+}(q)$ and η is the order 2 irreducible character of $\mathbb{F}_q^*$ and is regarded as a character of $\mathrm{GL}_{n_-}(q)$ via the determinant map. The actions of F and σ preserve M_0 and M_1, and thus they induce actions on W_0 and W_1. Let $\varphi_1 \in \mathrm{Irr}(W_1)^F \cap \mathrm{Irr}(W_1)^\sigma$ and let $(\varphi_+, \varphi_-) \in \mathrm{Irr}(\mathfrak{S}_{n_+} \times \mathfrak{S}_{n_-})$. Then $\varphi := (\varphi_+, \varphi_-, \varphi_1)$ is an irreducible character of the Weyl group of $M_{00} \times M_1$.

For any connected reductive algebraic group H defined over $\mathbb{F}_q$, define $\epsilon_H := (-1)^{rk_H}$ where rk_H is the $\mathbb{F}_q$-rank of H. Note that $M_{00} \times M_1$ and M have the same $\mathbb{F}_q$-rank.

Theorem 3.3 ([19, Propositions 5.2.2 and 5.2.3]). — *Suppose that θ is a regular linear character of $M_{00}^F \times M_1^F$. Then for some choice of the extension $\widetilde{\varphi}$, the function $\epsilon_G\epsilon_M R_\varphi^G\theta$ is a σ-stable irreducible character of* $\mathrm{GL}_n(q)$. *Moreover, every σ-stable irreducible character of* $\mathrm{GL}_n(q)$ *is of this form.*

If the M_1-component of M is trivial, then $\epsilon_G\epsilon_M R_\varphi^G\theta$ is called a *quadratic-unipotent character*. Note that if we further require $n_- = 0$, then it is a unipotent character in the usual sense.

3.2.4.

Let $M = L_{I,w}$ with $w = 1$, i.e. a standard Levi subgroup. Then $\epsilon_G = \epsilon_M$ and $M_1^F \cong \prod_i \big(\mathrm{GL}_{n_i}(q) \times \mathrm{GL}_{n_i}(q)\big)$ and F acts trivially on $W_1 \cong \prod_i (\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_i})$. A σ-stable linear character of M_1^F is of the form $(\zeta_i, \zeta_i^{-1})_i$, where ζ_i is a linear character of $\mathrm{GL}_{n_i}(q)$ for each i. The linear character θ in Section 3.2.3 is regular if and only if $\zeta_i \neq \zeta_j^{\pm}$ whenever $i \neq j$ and $\zeta_i^2 \neq \mathbf{1}$ for all i. The set of σ-stable regular linear characters of M_1^F is denoted by $\mathrm{Irr}^{\sigma}_{\mathrm{reg}}(M_1^F)$.

3.3. Types

Types are some combinatorial data that are used to describe σ-stable irreducible characters associated to standard Levi subgroups.

3.3.1.

Denote by $\mathfrak{T}$ the set of data of the form

$$\boldsymbol{\omega} = \omega_+ \omega_- (\omega_i)_i,$$

where ω_+ and ω_- are partitions and $(\omega_i)_i$ is an unordered sequence of nontrivial partitions. An element of $\mathfrak{T}$ is called a type. For any type $\boldsymbol{\omega}$, we denote by $\boldsymbol{\omega}_* = (\omega_i)_i$ the part without the components ω_+ and ω_-, so we may write $\boldsymbol{\omega} = \omega_+\omega_-\boldsymbol{\omega}_*$. The sequence $\boldsymbol{\omega}_*$ is equivalent to a sequence $(m_\lambda)_{\lambda \in \mathcal{P}}$, where $\mathcal{P}$ is the set of partitions and m_λ is the multiplicity of λ in $\boldsymbol{\omega}_*$. The size of a type is defined by $|\boldsymbol{\omega}| := |\omega_+| + |\omega_-| + \sum_i |\omega_i|$. For any $a \in \mathbb{Z}_{\geqslant 0}$, we denote by $\mathfrak{T}(a)$ the subset of types of size a. If we require $\boldsymbol{\omega}_*$ to be an ordered sequence of partitions, then we call $\boldsymbol{\omega}$ an *ordered type*. We denote by $\widetilde{\mathfrak{T}}$ the set of ordered types. The subsets of $\widetilde{\mathfrak{T}}$ and $\mathfrak{T}$ consisting of elements with $\omega_+ = \omega_- = \varnothing$ are denoted by $\widetilde{\mathfrak{T}}_*$ and $\mathfrak{T}_*$.

Denote by $l(\boldsymbol{\omega}_*)$ the length of $\boldsymbol{\omega}_*$. Write $\boldsymbol{\omega}_* = (m_\lambda)_{\lambda \in \mathcal{P}}$ and define

$$N(\boldsymbol{\omega}_*) = \prod_\lambda m_\lambda!, \tag{3.3}$$

and

$$K(\boldsymbol{\omega}_*) = (-1)^{l(\boldsymbol{\omega}_*)} l(\boldsymbol{\omega}_*)!. \tag{3.4}$$

Define $\{\boldsymbol{\omega}_*\} := (2m_\lambda)_{\lambda \in \mathcal{P}}$ and $\{\boldsymbol{\omega}\} := \omega_+\omega_-\{\boldsymbol{\omega}_*\}$. Define a map $[-] \colon \mathfrak{T} \to \mathcal{P}$ by requiring that the parts of $[\boldsymbol{\omega}]$ are the union of the parts of the partitions ω_+, ω_- and ω_i's.

3.3.2.

Denote by $\widetilde{\mathfrak{T}}_s$ the subset of $\widetilde{\mathfrak{T}}$ consisting of elements of the form

$$\boldsymbol{\omega} = (1^{n_+})(1^{n_-})((1^{n_i}))_i.$$

We define its dual type by $\boldsymbol{\omega}^* = (n_+)(n_-)((n_i))_i$. Elements of $\widetilde{\mathfrak{T}}_s$ can also be represented by a sequence of integers: $\boldsymbol{\omega} = n_+n_-(n_i)_i$ if $\boldsymbol{\omega} = (1^{n_+})(1^{n_-})((1^{n_i}))_i$. There is a surjective map from $\widetilde{\mathfrak{T}}$ to $\widetilde{\mathfrak{T}}_s$ sending $\boldsymbol{\omega} = \omega_+\omega_-(\omega_i)_i$ to $|\omega_+||\omega_-|(|\omega_i|)_i$. For $\boldsymbol{\omega} \in \widetilde{\mathfrak{T}}_s$, we denote by $\widetilde{\mathfrak{T}}(\boldsymbol{\omega})$ the inverse image of $\boldsymbol{\omega}$ under this map. For $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \widetilde{\mathfrak{T}}$, we write $\boldsymbol{\alpha} \approx \boldsymbol{\beta}$, if they lie in the same $\widetilde{\mathfrak{T}}(\boldsymbol{\omega})$.

Let $s \in \mathrm{GL}_n$ be a semi-simple element. Suppose that the multiplicities of its eigenvalues are given by a sequence of integers $(n_i)_i$. Then we say the conjugacy class of s is of type $\boldsymbol{\omega} = \boldsymbol{\omega}_* = ((1^{n_i}))_i$.

3.3.3.

For any $\chi \in \mathrm{Irr}\big(\mathrm{GL}_n(q)\big)^\sigma$, there exists some F-stable and σ-stable Levi subgroup M such that $\chi = \epsilon_G\epsilon_M R_\varphi^G\theta$ as in Theorem 3.3. This character is induced from a smaller Levi subgroup $M_{00} \times M_1$. Denote by $\mathrm{Irr}_{\mathrm{st}}^\sigma \subset \mathrm{Irr}\big(\mathrm{GL}_n(q)\big)^\sigma$ the subset of characters such that M can be chosen to be a standard Levi subgroup. Define a map

$$\pi\colon \mathrm{Irr}_{\mathrm{st}}^\sigma \longrightarrow \mathfrak{T} \tag{3.5}$$

as follows. Let $\varphi = (\varphi_+, \varphi_-, \varphi_1)$ as in Section 3.2.3, and let ω_+ and ω_- be the partitions corresponding to the characters φ_+ and φ_-. Since the Weyl group of M_1 is isomorphic to $\prod_i(\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_i})$, φ_1 gives a sequence of partitions $\boldsymbol{\omega}''_*$. Each partition in $\boldsymbol{\omega}''_*$ has even multiplicity. Let $\boldsymbol{\omega}'_*$ be such that $\{\boldsymbol{\omega}'_*\} = \boldsymbol{\omega}''_*$. Then we define $\pi(\chi) \coloneqq \omega_+\omega_-\boldsymbol{\omega}'_*$. For any $\boldsymbol{\omega} \in \mathfrak{T}$, define

$$\mathrm{Irr}_{\boldsymbol{\omega}}^\sigma \coloneqq \pi^{-1}(\boldsymbol{\omega}). \tag{3.6}$$

It is empty unless $|\{\boldsymbol{\omega}\}| = n$. Its elements are called σ-stable irreducible characters of type $\boldsymbol{\omega}$.

For all σ-stable characters of the same type, the Levi subgroup M in Theorem 3.3 can be chosen to be the same, and the characters φ are the same by definition. Therefore, there is a surjective map

$$\mathrm{Irr}_{\mathrm{reg}}^\sigma(M_1^F) \longrightarrow \mathrm{Irr}_{\boldsymbol{\omega}}^\sigma. \tag{3.7}$$

The cardinality of the fibre is equal to $2^{l(\boldsymbol{\omega}_*)}N(\boldsymbol{\omega}_*)$. This can be explained as follows (see also [20, Section 3.5.2]). Write $\theta = (\zeta_i, \zeta_i^{-1})_i \in \mathrm{Irr}_{\mathrm{reg}}^\sigma(M_1^F)$

as in Section 3.2.4. Replacing ζ_i by ζ_i^{-1} does not change the resulting σ-stable irreducible character of $\mathrm{GL}_n(q)$, hence a factor 2 for each i. If the partition λ appears m_λ times in $\boldsymbol{\omega}_*$, then an arbitrary permutation of the corresponding factors of θ does not change the resulting σ-stable irreducible character of $\mathrm{GL}_n(q)$, hence the factor $N(\boldsymbol{\omega}_*)$.

3.3.4.

Let L be a σ-stable Levi subgroup of GL_n containing a split maximal torus T. Denote by W_L the Weyl group of L defined by T. By choosing an isomorphism $L \cong \mathrm{GL}_{n_+} \times \mathrm{GL}_{n_-} \times \prod_i \mathrm{GL}_{n_i}$, we can parametrise the conjugacy classes of W_L by $\widetilde{\mathfrak{T}}(\boldsymbol{\omega})$, with $\boldsymbol{\omega} = (1^{n_+})(1^{n_-})((1^{n_i}))_i$. (If L is the centraliser of a semi-simple element, then we make the convention that $n_+ = n_- = 0$.) Then the map $[-]\colon \mathfrak{T} \to \mathcal{P}$ is simply the map between conjugacy classes induced by the inclusion $W_L \hookrightarrow W$ where $W \cong \mathfrak{S}_n$ is the Weyl group of GL_n defined by T.

Suppose that $\boldsymbol{\omega}$ is the type of an element $s \in T^F$. Let B_τ be the set defined in Section 3.1.2.

Lemma 3.4. — *Let $\boldsymbol{\tau}$ be the partition corresponding to the conjugacy class of τ. Then*

$$B_\tau = \left\{\boldsymbol{\nu} \in \widetilde{\mathfrak{T}}(\boldsymbol{\omega}) \;\middle|\; [\boldsymbol{\nu}] = \boldsymbol{\tau}\right\}.$$

Proof. — Unwind the definitions. □

4. $\mathrm{GL}_n\langle\sigma\rangle$-character varieties

In this section, we give the definition of $\mathrm{GL}_n\langle\sigma\rangle$-character varieties and the definition of generic conjugacy classes. We also explain how to pass to finite fields.

4.1. Definition of $\mathrm{GL}_n\langle\sigma\rangle$-character varieties

4.1.1.

Let $p'\colon \widetilde{\Sigma}' \to \Sigma'$ be an unbranched double covering of compact Riemann surfaces, and let $\mathcal{R} = \{x_j\}_{1\leqslant j\leqslant k} \subset \Sigma'$ be a finite set. Put $\Sigma = \Sigma' \setminus \mathcal{R}$ and $\widetilde{\Sigma} = p'^{-1}(\Sigma)$ and denote by $p\colon \widetilde{\Sigma} \to \Sigma$ the restriction of p'. Fix the base points of $\widetilde{\Sigma}$ and Σ so that we have an injective homomorphism of fundamental groups $\pi_1(\widetilde{\Sigma}) \to \pi_1(\Sigma)$. Let $\operatorname{Rep}(\Sigma)$ be the space of homomorphisms ρ that make the following diagram commute:

$$\begin{tikzcd} \pi_1(\Sigma) \arrow[rr, "\rho"] \arrow[dr, "q_1"'] & & \mathrm{GL}_n\langle\sigma\rangle \arrow[dl, "q_2"] \\ & \mathrm{Gal}(\widetilde{\Sigma}/\Sigma) \cong \boldsymbol{\mu}_2 & \end{tikzcd}$$

where q_1 is the quotient by $\pi_1(\widetilde{\Sigma})$ and q_2 is the quotient by GL_n. Our $\mathrm{GL}_n\langle\sigma\rangle$-character variety $\operatorname{Ch}(\Sigma)$ is defined as the categorical quotient of $\operatorname{Rep}(\Sigma)$ for the conjugation action of GL_n on $\mathrm{GL}_n\langle\sigma\rangle$.

4.1.2.

Denote by g the genus of Σ'. We may choose the generators α_i, β_i, $1 \leqslant i \leqslant g$, and γ_j, $1 \leqslant j \leqslant k$, of $\pi_1(\Sigma)$ that satisfy the relation

$$\prod_{i=1}^{g}[\alpha_i,\beta_i]\prod_{j=1}^{k}\gamma_j = 1, \tag{4.1}$$

and the condition: the generators β_1, α_i, β_i, $2 \leqslant i \leqslant g$, and γ_j, $1 \leqslant j \leqslant k$, are the images of some elements of $\pi_1(\widetilde{\Sigma})$, while α_1 lies in $\pi_1(\Sigma) \setminus \pi_1(\widetilde{\Sigma})$. Let $\mathcal{C} = (C_j)_{1\leqslant j\leqslant k}$ be a k-tuple of semi-simple conjugacy classes contained in GL_n. We define the subvariety $\operatorname{Rep}_{\mathcal{C}}(\Sigma) \subset \operatorname{Rep}(\Sigma)$ as consisting of $\rho \in \operatorname{Rep}(\Sigma)$ such that $\rho(\gamma_j) \in C_j$ for all j. Then we define $\operatorname{Ch}_{\mathcal{C}}(\Sigma) := \operatorname{Rep}_{\mathcal{C}}(\Sigma)/\!\!/\mathrm{GL}_n$. The defining equation of $\operatorname{Rep}_{\mathcal{C}}(\Sigma)$ is then

$$\operatorname{Rep}_{\mathcal{C}}(\Sigma) = \left\{ (A_i,B_i)_i(X_j)_j \in \mathrm{GL}_n^{2g} \times \prod_{j=1}^{k} C_j \;\middle|\; A_1\sigma(B_1)A_1^{-1}B_1^{-1}\prod_{i=2}^{g}[A_i,B_i]\prod_{j=1}^{k}X_j = 1 \right\}. \tag{4.2}$$

4.2. Generic conjugacy classes

4.2.1.

Let $\mathcal{C} = (C_j)_{1\leqslant j\leqslant k}$ be a tuple of semi-simple conjugacy classes in GL_n. For each j, let $t_j \in T$ be a representative of C_j, and write $t_j = (a_{j,1},\dots,a_{j,n})$

with each $a_{j,\gamma} \in \mathbb{C}^*$. Write $\Lambda = \{1, \dots, n\}$. For any j and any subset $\mathbf{A} \subset \Lambda$, write $[\mathbf{A}]_j = \prod_{\gamma \in \mathbf{A}} a_{j,\gamma}$. Following [21, Section 4.4], we say that $\mathcal{C}$ is generic if for any $1 \leqslant M \leqslant [n/2]$ and any $2k$-tuple $(\mathbf{A}_1, \dots, \mathbf{A}_k, \mathbf{B}_1, \dots, \mathbf{B}_k)$ of subsets of Λ such that

- $|\mathbf{A}_1| = \cdots = |\mathbf{A}_k| = |\mathbf{B}_1| = \cdots = |\mathbf{B}_k| = M$;
- $\mathbf{A}_j \cap \mathbf{B}_j = \varnothing$, for all j,

we have

$$[\mathbf{A}_1]_1 \cdots [\mathbf{A}_k]_k [\mathbf{B}_1]_1^{-1} \cdots [\mathbf{B}_k]_k^{-1} \neq 1. \tag{4.3}$$

We say that $\mathcal{C}$ is *strongly generic* if for any M and $(\mathbf{A}_1, \dots, \mathbf{A}_k, \mathbf{B}_1, \dots, \mathbf{B}_k)$ as above, we have

$$[\mathbf{A}_1]_1 \cdots [\mathbf{A}_k]_k [\mathbf{B}_1]_1^{-1} \cdots [\mathbf{B}_k]_k^{-1} \neq \pm 1. \tag{4.4}$$

Remark 4.1. — In [7], one may take a central conjugacy class that is also generic. But in our situation, this is not possible. See [21, Remark 3].

In fact, according to the above definition, in order for the tuple $\mathcal{C}$ to be generic, it is necessary that there exists a regular conjugacy class among the C_j's.

4.2.2.

We can also define generic conjugacy classes in the finite group $\mathrm{GL}_n(q)\langle\sigma\rangle$. Let T be the maximal torus consisting of diagonal matrices. Let $\mathcal{C} = (C_j)_{1 \leqslant j \leqslant k}$ be a tuple of semi-simple conjugacy classes in $\mathrm{GL}_n(q)$, such that each C_j has a representative $t_j \in T^F$. Note that not all semi-simple conjugacy classes of $\mathrm{GL}_n(q)$ have representatives of this form. For each j, let $(a_{j,1}, \dots, a_{j,n}) \in (\mathbb{F}_q^*)^n$ be the eigenvalues of t_j. With the same notations as in the previous paragraph, we say that $\mathcal{C}$ is generic (resp. strongly generic) if for any $1 \leqslant M \leqslant [n/2]$, $(\mathbf{A}_1, \dots, \mathbf{A}_k, \mathbf{B}_1, \dots, \mathbf{B}_k)$, we have

$$[\mathbf{A}_1]_1 \cdots [\mathbf{A}_k]_k [\mathbf{B}_1]_1^{-1} \cdots [\mathbf{B}_k]_k^{-1} \neq 1 \quad (\text{resp. } \pm 1). \tag{4.5}$$

Lemma 4.2. — *Let $(C_j)_{1 \leqslant j \leqslant k}$ be a generic tuple of conjugacy classes, with eigenvalues given by $(a_{j,1}, \dots, a_{j,n})$. Let M and M' be positive integers such that $M + M' \leqslant [n/2]$. Let*

$$(\mathbf{A}_1, \dots, \mathbf{A}_k, \mathbf{B}_1, \dots, \mathbf{B}_k) \qquad \text{and} \qquad (\mathbf{A}'_1, \dots, \mathbf{A}'_k, \mathbf{B}'_1, \dots, \mathbf{B}'_k)$$

be two $2k$-tuples of subsets of Λ such that

- $|\mathbf{A}_1| = \cdots = |\mathbf{A}_k| = |\mathbf{B}_1| = \cdots = |\mathbf{B}_k| = M$;
- $|\mathbf{A}'_1| = \cdots = |\mathbf{A}'_k| = |\mathbf{B}'_1| = \cdots = |\mathbf{B}'_k| = M'$;
- *the sets $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{A}'_j$ and $\mathbf{B}'_j$ are mutually disjoint, for all j.*

Then,

$$[\mathbf{A}_1]_1 \cdots [\mathbf{A}_k]_k [\mathbf{B}_1]_1^{-1} \cdots [\mathbf{B}_k]_k^{-1} \neq \left([\mathbf{A}'_1]_1 \cdots [\mathbf{A}'_k]_k [\mathbf{B}'_1]_1^{-1} \cdots [\mathbf{B}'_k]_k^{-1}\right)^{\pm 1}.$$

Proof. — Suppose

$$[\mathbf{A}_1]_1 \cdots [\mathbf{A}_k]_k [\mathbf{B}_1]_1^{-1} \cdots [\mathbf{B}_k]_k^{-1} \neq \left([\mathbf{A}'_1]_1 \cdots [\mathbf{A}'_k]_k [\mathbf{B}'_1]_1^{-1} \cdots [\mathbf{B}'_k]_k^{-1}\right)^{\epsilon},$$

with $\epsilon = 1$. For each j, put $\widetilde{\mathbf{A}}_j = \mathbf{A}_j \sqcup \mathbf{B}'_j$ and $\widetilde{\mathbf{B}}_j = \mathbf{B}_j \sqcup \mathbf{A}'_j$. Then,

$$[\widetilde{\mathbf{A}}_1]_1 \cdots [\widetilde{\mathbf{A}}_k]_k [\widetilde{\mathbf{B}}_1]_1^{-1} \cdots [\widetilde{\mathbf{B}}_k]_k^{-1} = 1,$$

which is a contradiction. The case $\epsilon = -1$ is similar. □

4.3. The R-model

As in [20], we define the character variety over a particular base ring so that when passing to finite fields the conjugacy classes remain generic.

4.3.1.

The generic conjugacy classes $(C_j)_j$ are represented by some n-tuples of complex numbers

$$(a_1^j, \dots, a_1^j, \dots, a_{l_j}^j, \dots, a_{l_j}^j),$$

satisfying

(i) $a_r^j \neq a_s^j$ for any j and any $r \neq s$;
(ii) $[\mathbf{A}_1]_1 \cdots [\mathbf{A}_k]_k [\mathbf{B}_1]_1^{-1} \cdots [\mathbf{B}_k]_k^{-1} \neq 1$, for any $1 \leqslant M \leqslant [n/2]$, any $2k$-tuple $(\mathbf{A}_1, \dots, \mathbf{A}_k, \mathbf{B}_1, \dots, \mathbf{B}_k)$ of subsets of Λ such that:
   - $|\mathbf{A}_1| = \cdots = |\mathbf{A}_k| = |\mathbf{B}_1| = \cdots = |\mathbf{B}_k| = M$;
   - $\mathbf{A}_j \cap \mathbf{B}_j = \varnothing$, for all j.

For each j and each $1 \leqslant r \leqslant l_j$, choose $b_r^j \in \mathbb{C}$ such that $(b_r^j)^2 = a_r^j$. Denote by R_0 the subring of $\mathbb{C}$ generated by $\left\{(b_r^j)^{\pm 1} \,\middle|\, \text{all } r, j\right\}$. Let $S \subset R_0$ be the multiplicative subset generated by

(i) $a_r^j - a_s^j$ for any j and $r \neq s$;
(ii) $[\mathbf{A}_1]_1 \cdots [\mathbf{A}_k]_k [\mathbf{B}_1]_1^{-1} \cdots [\mathbf{B}_k]_k^{-1} - 1$, for any M and $(\mathbf{A}_1, \dots, \mathbf{A}_k, \mathbf{B}_1, \dots, \mathbf{B}_k)$ as above.

Define the ring of generic eigenvalues as $R := S^{-1}R_0$. We will see below that the character variety is defined over R.

Denote by $\mu_j = (m_{j,r})_{1 \leqslant r \leqslant l_j} \in \mathfrak{T}_s$ the type of C_j, so that $m_{j,r}$ is the multiplicity of a_r^j.

4.3.2.

Let $\mathcal{A}_0$ be the polynomial ring over R with $n^2(2g+k)$ indeterminates thought of as the entries of some $n\times n$ matrices $A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k$, with $\det A_i$, $\det B_i$, $\det X_j$, $1\leqslant i\leqslant g$, $1\leqslant j\leqslant k$, inverted. Let $I_0\subset\mathcal{A}_0$ be the ideal generated by

(i) the entries of $A_1\sigma(B_1)A_1^{-1}B_1^{-1}[A_2,B_2]\cdots[A_g,B_g]X_1\cdots X_k-\mathrm{Id}$ (note that σ is defined over $\mathbb{Z}$);
(ii) for all $1\leqslant j\leqslant k$, the entries of $\prod_{r=1}^{l_j}(X_j-a_r^j\,\mathrm{Id})$;
(iii) for all $1\leqslant j\leqslant k$, the entries of the coefficients of the following polynomial in an auxiliary variable t:

$$\det(t\,\mathrm{Id}-X_j)-\prod_{r}^{l_j}(t-a_r^j)^{m_{j,r}}. \tag{4.6}$$

Define $\mathcal{A}\coloneqq\mathcal{A}_0/\sqrt{I_0}$. Then $\mathbf{Rep}_{\mathcal{C}}\coloneqq\operatorname{Spec}\mathcal{A}$ is the R-model of $\mathrm{Rep}_{\mathcal{C}}$. Let GL_n act on B_1, A_i, B_i, $2\leqslant i\leqslant g$, and X_j, $1\leqslant j\leqslant k$ by conjugation, and act on A_1, by the σ-twisted conjugation. Then $\mathbf{Ch}_{\mathcal{C}}\coloneqq\operatorname{Spec}\mathcal{A}^{\mathrm{GL}_n(R)}$ is the R-model of $\mathrm{Ch}_{\mathcal{C}}$, since taking invariants commutes with flat base change (see [18, Section I.2 Lemma 2]).

4.3.3.

Let $\phi\colon R\to\mathbb{F}_q$ be any ring homomorphism. For each $1\leqslant j\leqslant k$, denote by C_j^{ϕ} the subvariety of GL_n over $\mathbb{F}_q$ defined by Section 4.3.2(ii)–(iii). Denote by $\mathrm{Rep}_{\mathcal{C}}^{\phi}$ (resp. $\mathrm{Ch}_{\mathcal{C}}^{\phi}$) the variety over $\mathbb{F}_q$ obtained by base change from $\mathbf{Rep}_{\mathcal{C}}$ (resp. $\mathbf{Ch}_{\mathcal{C}}$). We have

$$\mathrm{Rep}_{\mathcal{C}}^{\phi}(\mathbb{F}_q)=\left\{(A_i,B_i)(X_j)\in\mathrm{GL}_n(q)^{2g}\times\prod_{j=1}^{k}C_j^{\phi}(\mathbb{F}_q)\;\middle|\; A_1\sigma(B_1)A_1^{-1}B_1^{-1}\prod_{i=2}^{g}[A_i,B_i]\prod_{j=1}^{k}X_j=1\right\}, \tag{4.7}$$

where $C_j^{\phi}(\mathbb{F}_q)$ is a conjugacy class in $\mathrm{GL}_n(q)$ with eigenvalues $\phi(a_r^j)$, $1\leqslant r\leqslant l_j$. By the definition of the base ring R, the tuple of conjugacy classes $\mathcal{C}^{\phi}$ is also generic.

Remark 4.3. — Our construction of R guarantees that $\phi(a_r^j)\in(\mathbb{F}_q^*)^2$.

Proposition 4.4. — *We have,*

$$\left|\mathrm{Ch}^{\phi}_{\mathcal{C}}(\mathbb{F}_q)\right| = \frac{1}{\left|\mathrm{GL}_n(\mathbb{F}_q)\right|}\left|\mathrm{Rep}^{\phi}_{\mathcal{C}}(\mathbb{F}_q)\right|. \tag{4.8}$$

Proof. — The same as [20, Proposition 4.3.3]. □

5. Two lemmas

In this section, we denote by T the maximal torus consisting of diagonal matrices and denote by W the Weyl group of GL_n defined by T.

5.1. A constraint on types

5.1.1.

We show that only a particular subset of the set of σ-stable irreducible characters of $\mathrm{GL}_n(q)$ contributes to the E-polynomial.

Lemma 5.1. — *Let $(C_1,\ldots,C_k)$ be a tuple of semi-simple conjugacy classes in $\mathrm{GL}_n(q)$ such that each C_j has a representative $s_j \in T^F$. Let M be a σ-stable and F-stable Levi factor of a σ-stable parabolic subgroup of $\mathrm{GL}_n(\Bbbk)$. Suppose that $(C_j)_j$ is strongly generic and that for all j, we have $C_j \cap M \neq \varnothing$. Then M is $\mathrm{GL}_n(q)$-conjugate to a σ-stable standard Levi subgroup.*

Proof. — We may assume that $M = L_{I,w}$ for some σ-stable I and some $w \in W_I^{\sigma}$ (see Section 3.2.2). There exists $\dot{w} \in (G^{\sigma})^{\circ}$ representing w and $g \in (G^{\sigma})^{\circ}$ such that $g^{-1}F(g) = \dot{w}$. Then $L_{I,w} = gL_Ig^{-1}$.

For each j, let $t_j \in C_j \cap M$, then we have the following:

- $g^{-1}t_jg$ lies in L_I;
- there exists $l_j \in L_I$ such that $l_jg^{-1}t_jgl_j^{-1}$ lies in T;
- there exists $w_j \in W$ such that

$$l_jg^{-1}t_jgl_j^{-1} = w_js_jw_j^{-1}.$$

For any connected reductive algebraic group H, denote by

$$\mathbf{D}_H \colon H \longrightarrow H/[H,H] \cong Z_H^{\circ}/\big(Z_H^{\circ} \cap [H,H]\big)$$

the natural surjection. We have

$$\mathbf{D}_{L_I}\big((l_jg^{-1}t_jgl_j^{-1})\sigma(l_jg^{-1}t_jgl_j^{-1})\big) = \mathbf{D}_{L_I}(g^{-1}t_j\sigma(t_j)g).$$

We deduce that

$$\mathbf{D}_{L_I}\left(\prod_{j=1}^{2k} g^{-1}t_j\sigma(t_j)g\right) = \mathbf{D}_{L_I}\left(\prod_{j=1}^{2k}(w_js_jw_j^{-1})\sigma(w_js_jw_j^{-1})\right), \tag{5.1}$$

and we denote by l this element of $Z_{L_I}^{\circ}/\big(Z_{L_I}^{\circ}\cap[L_I,L_I]\big)$.

We may choose an isomorphism $L_I\cong \mathrm{GL}_{n_0}\times\prod_i(\mathrm{GL}_{n_i}\times\mathrm{GL}_{n_i})$, so that

$$Z_{L_I}^{\circ}/\big(Z_{L_I}^{\circ}\cap[L_I,L_I]\big)\cong \Bbbk^*\times\prod_i(\Bbbk^*\times\Bbbk^*).$$

The action of σ on the factor $\Bbbk^*\times\Bbbk^*$ corresponding to each i sends (x,y) to (y^{-1},x^{-1}), and on the first factor sends x to x^{-1}. From the right-hand side of (5.1), we see that l is a fixed point of σ. We can then write $l=\big(l_0,(l_i,l_i^{-1})\big)$ under the above isomorphism.

In view of the right-hand side of (5.1), each l_i is of the form

$$[\mathbf{A}_1]_1\cdots[\mathbf{A}_k]_k[\mathbf{B}_1]_1^{-1}\cdots[\mathbf{B}_k]_k^{-1}$$

(see Section 4.2.1 for the notations). We can then apply Lemma 4.2 and conclude that $l_i\neq l_j^{\pm1}$ whenever $i\neq j$. Moreover, all l_i lie in $\mathbb{F}_q$. However, the left-hand side of (5.1) shows that l is an F_w-stable element (see (2.2)). According to [20, Section 3.2.2], W_I^{σ} is of the form $\prod_r\mathfrak{W}_{N'_r}$ for some positive integers N'_r, so w consists of some signed cycles. It is easy to see that if there is some signed cycle in w that has size larger than 1, then $l_i=l_j$ for some $i\neq j$, which is a contradiction. Finally, if there is some negative cycle of size 1, then the corresponding factor l_i satisfies $l_i=l_i^q=l_i^{-1}$ and so must be equal to ±1. This is impossible due to the strongly generic condition (4.4). We conclude that $w=1$. $\square$

Let $\chi\in\mathrm{Irr}\big(\mathrm{GL}_n(q)\big)^{\sigma}$ and let $\mathcal{C}$ be as in the above lemma. Suppose that $\chi\notin\mathrm{Irr}_{\mathrm{st}}^{\sigma}$, and that it is obtained from a Levi subgroup M as in Section 3.2.3. By the above lemma, there exists some C_j such that $C_j\cap M=\varnothing$. The definition of $R_{\varphi}^{G}\theta$ and (3.1) imply that χ must vanish on C_j.

5.2. Sum of linear characters

5.2.1.

Let $M=L_I$ be a σ-stable standard Levi subgroup. As in Section 3.2.3, we write $M=M_0\times M_1$, where $M_0=\mathrm{GL}_{n_0}$ and $M_1=\prod_{i=1}^{l}(\mathrm{GL}_{n_i}\times\mathrm{GL}_{n_i})$ for some integers l, n_0, and n_i. Let n_+ and n_- be some non-negative integers such that $n_++n_-=n_0$, and let $M_{00}\cong\mathrm{GL}_{n_+}\times\mathrm{GL}_{n_-}$ be a Levi subgroup of M_0 containing $M_0 \cap T$. Denote by W_{00} (resp. W_1) the Weyl group of M_{00} (resp. M_1) defined by T. For each $j \in \{1, \dots k\}$, let

$$w_j \in W_{00} \times W_1 \subset W_M(T),$$

and choose an F-stable maximal torus $T_{w_j} \subset M_{00} \times M_1$ corresponding to the conjugacy class of w_j. Recall that $\mathrm{Irr}^{\sigma}_{\mathrm{reg}}(M_1^F)$ is the set of σ-stable regular linear characters of M_1^F defined in Section 3.2.4. An element $\theta \in \mathrm{Irr}^{\sigma}_{\mathrm{reg}}(M_1^F)$ determines a linear character $\mathbf{1} \boxtimes \eta \boxtimes \theta$ of $M_{00}^F \times M_1^F$ as in Section 3.2.3. For each j, we denote by θ_j the restriction of this linear character to $T_{w_j}^F$.

Let $\mathcal{C} = (C_1, \dots, C_k)$ be a tuple of semi-simple conjugacy classes in $\mathrm{GL}_n(q)$ with representatives $s_j \in T^F$. For each j, let $h_j \in \mathrm{GL}_n(q)$ be such that $h_j s_j h_j^{-1} \in T_{w_j}$.

Lemma 5.2. — *Suppose that $\mathcal{C}$ is strongly generic. We have:*

$$\sum_{\theta \in \mathrm{Irr}^{\sigma}_{\mathrm{reg}}(M_1^F)} \prod_{j=1}^{k} \theta_j(h_j s_j h_j^{-1}) = (-2)^l l!. \tag{5.2}$$

Proof. — We first give an expression for $\theta_j(h_j s_j h_j^{-1})$. Let $g_j \in \mathrm{GL}_n$ be such that $g_j T g_j^{-1} = T_{w_j}$. Write $x = g_j^{-1} h_j$. Then $x s_j x^{-1} \in T$. We deduce that $x = nl$ with n normalising T and l centralising s_j. Write

$$n s_j n^{-1} = \mathrm{diag}(a_{j,1}, \dots, a_{j,n}) \in T^F.$$

Note that this element also lies in $T^{F_{w_j}}$ since $x s_j x^{-1}$ lies in $T^{F_{w_j}}$. We write w_j as a permutation $\prod_{i \in \Lambda_j} c_{I_{j,i}}$ of $\{1, \dots, n\}$, where $I_{j,i}$'s are disjoint subsets of $\{1, \dots, n\}$ indexed by Λ_j, and $c_{I_{j,i}}$ is a cyclic permutation of $I_{j,i}$. The character $\theta_j \circ \mathrm{ad}\, g_j$ of $T^{F_{w_j}}$ can then be written as $(\alpha_{j,i})_{i \in \Lambda_j}$, where each $\alpha_{j,i}$ is a character of $\mathbb{F}_q^*$. With these notations, we have

$$\theta_j(h_j s_j h_j^{-1}) = \prod_{i \in \Lambda_j} \left(\alpha_{j,i} \left(\prod_{\gamma \in I_{j,i}} a_{j,\gamma} \right) \right).$$

(by Remark 4.3, the component η of θ has no contribution to this value).

Let θ be represented by a tuple $(\zeta_1, \dots, \zeta_l)$ of distinct characters of $\mathbb{F}_q^*$, where l is the number of factors $\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i}$ in M_1. For each $r \in \{1, \dots, l\}$ and each j, put

$$I_{j,r}^{+} := \bigsqcup_{\{i \in \Lambda_j \mid \alpha_{j,i} = \zeta_r\}} I_{j,i}, \qquad I_{j,r}^{-} := \bigsqcup_{\{i \in \Lambda_j \mid \alpha_{j,i} = \zeta_r^{-1}\}} I_{j,i},$$

and for each r,

$$a_r \coloneqq \prod_{j=1}^{k}\left(\prod_{\gamma\in I_{j,r}^{+}} a_{j,\gamma} \prod_{\gamma\in I_{j,r}^{-}} a_{j,\gamma}^{-1}\right).$$

Then,

$$\prod_{j=1}^{k}\prod_{i\in\Lambda_j}\alpha_{j,i}\left(\prod_{\gamma\in I_{j,i}} a_{j,\gamma}\right)=\prod_{r=1}^{l}\zeta_r(a_r).$$

Therefore,

$$\sum_{\theta\in\mathrm{Irr}^{\sigma}_{\mathrm{reg}}(M_1^F)}\prod_{j=1}^{k}\theta_j(h_js_jh_j^{-1})=\sum_{\substack{(\zeta_1,\ldots,\zeta_l)\\ \text{regular}}}\prod_{r=1}^{l}\zeta_r(a_r).$$

Remark 4.3 says that each a_r is in fact a square in $\mathbb{F}_q^*$. Therefore we can apply [20, Lemma 5.2.2, Equations (5.2.1.2) and (5.2.2.1)] and obtain

$$\sum_{\substack{(\zeta_1,\ldots,\zeta_l)\\ \text{regular}}}\prod_{r=1}^{l}\zeta_r(a_r)=\sum_{P_1\prec P_0}\mu(P_1,P_0)(-1)^{l(P_1)}2^l=(-2)^l l!.$$

□

6. Computation of E-polynomials

The goal of this section is to prove Theorem 6.4.

6.1. Symmetric functions associated to types

6.1.1.

For any $\boldsymbol{\omega}=\omega_+\omega_-(\omega_i)_i\in\mathfrak{T}$, define the associated Schur symmetric function by

$$s_{\boldsymbol{\omega}}(\mathbf{z})\coloneqq s_{\omega_+}(\mathbf{z})s_{\omega_-}(\mathbf{z})\prod_i s_{\omega_i}(\mathbf{z}).$$

Then $p_{\boldsymbol{\omega}}(\mathbf{z})$, $m_{\boldsymbol{\omega}}(\mathbf{z})$, $h_{\boldsymbol{\omega}}(\mathbf{z})$, $P_{\boldsymbol{\omega}}(\mathbf{z})$ and $\widetilde{H}_{\boldsymbol{\omega}}(\mathbf{z})$ can be defined similarly. Define $z_{\boldsymbol{\omega}}\coloneqq z_{\omega_+}z_{\omega_-}\prod_i z_{\omega_i}$ and $n(\boldsymbol{\omega})\coloneqq n(\omega_+)+n(\omega_-)+\sum_i n(\omega_i)$. For any $\boldsymbol{\alpha}$, $\boldsymbol{\beta}\in\widetilde{\mathfrak{T}}$, if $\boldsymbol{\alpha}\approx\boldsymbol{\beta}$, we define $\chi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\coloneqq\chi^{\alpha_+}_{\beta_+}\chi^{\alpha_-}_{\beta_-}\prod_i\chi^{\alpha_i}_{\beta_i}$; otherwise we put $\chi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}=0$. If $\boldsymbol{\alpha}\approx\boldsymbol{\beta}$, we define

$$K_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q)\coloneqq K_{\beta_+,\alpha_+}(q)K_{\beta_-,\alpha_-}(q)\prod_i K_{\beta_i,\alpha_i}(q);$$

otherwise, we define $K_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q)=0$. Similarly, we can define $\widetilde{K}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q)$, so we have $\widetilde{K}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q)=q^{n(\boldsymbol{\alpha})}K_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q^{-1})$.

Lemma 6.1. — *Let $\boldsymbol{\mu}\in\mathfrak{T}_s$. Then the following identity holds:*

$$h_{\boldsymbol{\mu}^*}\left[\frac{\mathbf{z}}{1-q}\right]=(-1)^{|\boldsymbol{\mu}|}q^{-n(\boldsymbol{\mu})-|\boldsymbol{\mu}|}b_{\boldsymbol{\mu}}(q^{-1})^{-1}\widetilde{H}_{\boldsymbol{\mu}}(\mathbf{z};q).$$

Proof. — See [7, Lemma 2.3.6]. □

Lemma 6.2. — *Let $\boldsymbol{\alpha}\in\widetilde{\mathfrak{T}}$ and $\boldsymbol{\beta}\in\widetilde{\mathfrak{T}}_s$. Then*

$$\left\langle s_{\boldsymbol{\alpha}}(\mathbf{x}),\widetilde{H}_{\boldsymbol{\beta}}(\mathbf{x},q)\right\rangle=\sum_{\boldsymbol{\tau}\in\widetilde{\mathfrak{T}}}\frac{z_{[\boldsymbol{\tau}]}\chi^{\boldsymbol{\alpha}}_{\boldsymbol{\tau}}}{z_{\boldsymbol{\tau}}}\sum_{\substack{\boldsymbol{\nu}\in\widetilde{\mathfrak{T}}\\ [\boldsymbol{\nu}]=[\boldsymbol{\tau}]}}\frac{\mathcal{Q}^{\boldsymbol{\beta}}_{\boldsymbol{\nu}}(q)}{z_{\boldsymbol{\nu}}}.$$

Proof. — See [7, Lemma 2.3.5]. □

6.2. A combinatorial formula for irreducible characters

For simplicity, we will write $\mathfrak{S}_+=\mathfrak{S}_{n_+}$ and $\mathfrak{S}_-=\mathfrak{S}_{n_-}$.

6.2.1.

Let χ be a σ-stable irreducible character of $\mathrm{GL}_n(q)$. According to Section 5.1.1, we may assume that χ is induced from some σ-stable standard Levi subgroup $M=M_0\times M_1$, and we can write χ as a linear combination of Deligne–Lusztig characters:

$$\chi=|\mathfrak{S}_+\times\mathfrak{S}_-\times W_1|^{-1}\sum_{\substack{w=(w_+,w_-,w_1)\\ \in\mathfrak{S}_+\times\mathfrak{S}_-\times W_1}}\varphi_+(w_+)\varphi_-(w_-)\varphi(w_1)R^G_{T_w}\theta_w. \tag{6.1}$$

Suppose that χ is of type $\boldsymbol{\alpha}$. Fixing an isomorphism $W_1\cong\prod_i(\mathfrak{S}_{n_i}\times\mathfrak{S}_{n_i})$, the irreducible character $(\varphi_+,\varphi_-,\varphi)$ of $\mathfrak{S}_+\times\mathfrak{S}_-\times W_1$ corresponds to a unique ordered type, the unordered version of which is exactly $\{\boldsymbol{\alpha}\}$. In this subsection, $\boldsymbol{\alpha}$ will be regarded as an element of $\widetilde{\mathfrak{T}}$. The sum over w only depends on its conjugacy class, which we denote by $\mathcal{O}(w)$. Note that the conjugacy classes of $\mathfrak{S}_+\times\mathfrak{S}_-\times W_1$ are parametrised by a subset of $\widetilde{\mathfrak{T}}$. The type of $\mathcal{O}(w)$ will be denoted by $\boldsymbol{\tau}$. We have

$$\frac{\left|\mathcal{O}(w)\right|}{|\mathfrak{S}_+\times\mathfrak{S}_-\times W_1|}=\frac{1}{z_{\boldsymbol{\tau}}},\qquad \varphi_+(w_+)\varphi_-(w_-)\varphi(w_1)=\chi^{\{\boldsymbol{\alpha}\}}_{\boldsymbol{\tau}}. \tag{6.2}$$

We will therefore replace the summation $\sum_w(-)$ by $\sum_{\mathcal{O}(w)}\left|\mathcal{O}(w)\right|\cdot(-)$. However, for each $\mathcal{O}(w)$, we will choose a w representing it so that T_w and θ_w have definite meanings.

If $s\in T^F$ is a semi-simple element, then the character formula (3.1) reads

$$R^G_{T_w}\theta_w(s)=\frac{1}{\left|C_G(s)^F\right|}\sum_{\{h\in G^F\mid hsh^{-1}\in T_w\}}Q^{C_G(s)}_{h^{-1}T_wh}(1)\theta_w(hsh^{-1}). \tag{6.3}$$

Denote by $\boldsymbol{\mu}\in\widetilde{\mathfrak{T}}_s$ the type of the conjugacy class of s. Since T is split, the $C_G(s)^F$-conjugacy classes of the F-stable maximal tori of $C_G(s)$ are parametrised by the conjugacy classes of $W_{C_G(s)}(T)$, which are in bijection with a subset of $\widetilde{\mathfrak{T}}$. If $h^{-1}T_wh$ is of type $\boldsymbol{\nu}\in\widetilde{\mathfrak{T}}$ (with $\nu_+=\nu_-=\varnothing$), it is known that $\mathcal{Q}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(q)=Q^{C_G(s)}_{h^{-1}T_wh}(1)$, where the left-hand side is the Green polynomial defined by symmetric functions.

Let A^F_τ and B_τ be as in Section 3.1.2. Recall the notation $z_{\boldsymbol{\nu}}$ therein. Combining (6.1) and (6.3) gives

$$\begin{aligned}\chi(s)&=\sum_{\boldsymbol{\tau}}\sum_{h\in A^F_\tau}\frac{1}{\left|C_G(s)^F\right|}\cdot\frac{\chi^{\{\boldsymbol{\alpha}\}}_{\boldsymbol{\tau}}Q^{C_G(s)}_{h^{-1}T_wh}(1)}{z_{\boldsymbol{\tau}}}\theta_w(hsh^{-1})\\ &\overset{①}{=}\sum_{\boldsymbol{\tau}}\sum_{\{\boldsymbol{\nu}\in B_\tau\}}\frac{1}{\left|C_G(s)^F\right|}\cdot\frac{\chi^{\{\boldsymbol{\alpha}\}}_{\boldsymbol{\tau}}\mathcal{Q}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(q)}{z_{\boldsymbol{\tau}}}\sum_{h\in A^F_{\boldsymbol{\tau},\boldsymbol{\nu}}}\theta_w(hsh^{-1}).\end{aligned} \tag{6.4}$$

Equality ① uses the surjective map $A^F_\tau\to B_\tau$ described in Section 3.1.2.

6.2.2.

Now we can work in the context of Section 5.2.1. For each j, let $\boldsymbol{\mu}_j$ be the type of the semi-simple conjugacy class C_j.

Lemma 6.3. — *We have*

$$\sum_{\chi\in\mathrm{Irr}^\sigma_{\boldsymbol{\omega}}}\prod_{j=1}^k\chi(C_j)=\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}\prod_{j=1}^k\left\langle s_{\{\boldsymbol{\omega}\}}(\mathbf{z}_j),\widetilde{H}_{\boldsymbol{\mu}_j}(\mathbf{x}_j,q)\right\rangle,$$

where for each j, $\mathbf{z}_j$ is an independent family of variables.

Proof. — We calculate

$$
\begin{aligned}
&\sum_{\chi\in\mathrm{Irr}^{\sigma}_{\boldsymbol{\omega}}}\prod_{j=1}^{k}\chi(C_j)\\
&\overset{①}{=}\frac{1}{2^{l(\boldsymbol{\omega}_*)}N(\boldsymbol{\omega}_*)}\sum_{\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_k}\sum_{\substack{(\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_k)\\ \in\prod_j B_{\tau_j}}}\prod_{j=1}^{k}\frac{1}{\left|C_G(s)^F\right|}\cdot\frac{\chi^{\{\boldsymbol{\omega}\}}_{\boldsymbol{\tau}_j}\mathcal{Q}^{\boldsymbol{\mu}_j}_{\boldsymbol{\nu}_j}(q)}{z_{\boldsymbol{\tau}_j}}\\
&\qquad\cdot\left(\sum_{(h_1,\ldots,h_k)\in\prod_j A^F_{\boldsymbol{\tau}_j,\boldsymbol{\nu}_j}}\;\sum_{\theta\in\mathrm{Irr}^{\sigma}_{\mathrm{reg}}(M_1^F)}\prod_{j=1}^{k}\theta_j(h_js_jh_j^{-1})\right)\\
&\overset{②}{=}\frac{1}{2^{l(\boldsymbol{\omega}_*)}N(\boldsymbol{\omega}_*)}\sum_{\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_k}\sum_{\substack{(\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_k)\\ \in\prod_j B_{\tau_j}}}\prod_{j=1}^{k}\frac{1}{\left|C_G(s)^F\right|}\cdot\frac{\chi^{\{\boldsymbol{\omega}\}}_{\boldsymbol{\tau}_j}\mathcal{Q}^{\boldsymbol{\mu}_j}_{\boldsymbol{\nu}_j}(q)}{z_{\boldsymbol{\tau}_j}}\\
&\qquad\cdot\left(\sum_{(h_1,\ldots,h_k)\in\prod_j A^F_{\boldsymbol{\tau}_j,\boldsymbol{\nu}_j}}(-1)^{l(\boldsymbol{\omega}_*)}2^{l(\boldsymbol{\omega}_*)}l(\boldsymbol{\omega}_*)!\right)\\
&\overset{③}{=}\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}\prod_{j=1}^{k}\sum_{\boldsymbol{\tau}_j}\frac{z_{[\boldsymbol{\tau}_j]}\chi^{\{\boldsymbol{\omega}\}}_{\boldsymbol{\tau}_j}}{z_{\boldsymbol{\tau}_j}}\sum_{\{\boldsymbol{\nu}_j\mid[\boldsymbol{\nu}_j]=[\boldsymbol{\tau}_j]\}}\frac{\mathcal{Q}^{\boldsymbol{\mu}_j}_{\boldsymbol{\nu}_j}(q)}{z_{\boldsymbol{\nu}_j}}\\
&\overset{④}{=}\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}\prod_{j=1}^{k}\left\langle s_{\{\boldsymbol{\omega}\}}(\mathbf{z}_j),\widetilde{H}_{\boldsymbol{\mu}_j}(\mathbf{z}_j,q)\right\rangle,
\end{aligned}
$$

In equality ①, we have applied (6.4) to each $\chi(C_j)$. The factor $2^{l(\boldsymbol{\omega}_*)}N(\boldsymbol{\omega}_*)$ comes from the surjective map (3.7). In equality ②, we have used Lemma 5.2. We have ③ because the summation over h_j is independent of h_j, which produces $\left|C_G(s)^F\right|z_{[\boldsymbol{\tau}_j]}/z_{\boldsymbol{\nu}_j}$ by Proposition 3.2. Finally, equality ④ uses Lemma 6.2. □

6.3. Some notations

6.3.1.

For any partition λ, the Hook polynomial $H_\lambda(q)$ is defined by:

$$H_\lambda(q):=\prod_{x\in\lambda}(1-q^{h(x)}),\tag{6.5}$$

where x runs over the boxes in the Young diagram of λ, and $h(x)$ is the hook length of x. If we denote by λ^* the dual partition of λ, then

$$\sum_{x\in\lambda} h(x) = |\lambda| + n(\lambda) + n(\lambda^*). \tag{6.6}$$

For any $\boldsymbol{\omega} = \omega_+\omega_-(\omega_i) \in \mathfrak{T}$, put $H_{\boldsymbol{\omega}}(q) \coloneqq H_{\omega_+}(q)H_{\omega_-}(q)\prod_i H_{\omega_i}(q)$. Then for $\chi \in \mathrm{Irr}^{\sigma}_{\boldsymbol{\omega}}$, we have (see [12, Chapter IV, Section 6 (6.7)]):

$$\frac{\left|\mathrm{GL}_n(q)\right|}{\chi(1)} = (-1)^n q^{\frac{1}{2}n(n-1)-n(\{\boldsymbol{\omega}\})} H_{\{\boldsymbol{\omega}\}}(q). \tag{6.7}$$

Recall the notation $\{\boldsymbol{\omega}\}$ in Section 3.3.1.

6.3.2.

According to [21, Theorem 4.6], the dimension of the character variety is given by:

$$\begin{aligned} d &\coloneqq (2g-2)\dim G + \sum_{j=1}^{k} \dim C_j \\ &= (2g-2)\dim G + \sum_{j=1}^{k}\big(\dim G - \dim C_G(s_j)\big) \\ &= (2g-2)n^2 + kn^2 - \sum_{j=1}^{k}\big(2n(\boldsymbol{\mu}_j) + n\big) \\ &= n^2(2g+k-2) - kn - \sum_{j=1}^{k} 2n(\boldsymbol{\mu}_j). \end{aligned}$$

Note that $n((1^n))$ is equal to the number of positive roots in GL_n.

6.4. The formula for E-polynomials

6.4.1.

Define the infinite series:

$$\Omega_\bullet(q) \coloneqq \sum_{\lambda\in\mathcal{P}} q^{|\lambda|(1-g)}\big(q^{-n(\lambda)}H_\lambda(q)\big)^{2g+k-2}\prod_{j=1}^{k} s_\lambda\Big[\frac{\mathbf{z}_j}{1-q}\Big],$$

$$\Omega_*(q) \coloneqq \sum_{\lambda\in\mathcal{P}} q^{|\lambda|(2-2g)}\big(q^{-n(\lambda)}H_\lambda(q)\big)^{4g+2k-4}\prod_{j=1}^{k} s_\lambda\Big[\frac{\mathbf{z}_j}{1-q}\Big]^2,$$

Recall that for each $1 \leqslant j \leqslant k$, we denote by $\boldsymbol{\mu}_j$ the type of the semi-simple conjugacy class C_j.

Theorem 6.4. — *Suppose that $\mathcal{C}$ is generic. Then with the notations above, we have*

$$\left|\mathrm{Ch}_{\mathcal{C}}(\mathbb{F}_q)\right| = q^{\frac{1}{2}d}\left\langle \frac{\Omega_\bullet(q)^2}{\Omega_*(q)}, \prod_{j=1}^{k} h_{\boldsymbol{\mu}_j^*}(\mathbf{z}_j)\right\rangle. \tag{6.8}$$

Proof. — Proposition 4.4, equation (4.7) and Proposition 2.1 now give

$$\begin{aligned}\left|\mathrm{Ch}_{\mathcal{C}}(\mathbb{F}_q)\right| &= \sum_{\chi\in\mathrm{Irr}(G^F)^\sigma}\left(\frac{|G^F|}{\chi(1)}\right)^{2g-2}\prod_{j=1}^{k}\frac{|C_j|\chi(C_j)}{\chi(1)}\\ &= \sum_{\substack{\boldsymbol{\omega}\in\mathfrak{T}\\ |\{\boldsymbol{\omega}\}|=n}}\frac{|G^F|^{2g-2}\prod_{j=1}^{k}|C_j|}{\chi(1)^{2g+k-2}}\sum_{\chi\in\mathrm{Irr}^\sigma_{\boldsymbol{\omega}}}\prod_{j=1}^{k}\chi(C_j).\end{aligned} \tag{6.9}$$

In the second equality, we have used the fact that irreducible characters of the same type have the same degree (see (6.7)). This formula holds for any generic tuple of conjugacy classes.

We calculate

$$\begin{aligned}&\left|\mathrm{Ch}_{\mathcal{C}}(\mathbb{F}_q)\right|\\ &= \sum_{\substack{\boldsymbol{\omega}\in\mathfrak{T}\\ |\{\boldsymbol{\omega}\}|=n}}\frac{|G^F|^{2g-2}\prod_{j=1}^{k}|C_j|}{\chi(1)^{2g+k-2}}\sum_{\chi\in\mathrm{Irr}^\sigma_{\boldsymbol{\omega}}}\prod_{j=1}^{k}\chi(C_j)\\ &\overset{①}{=} \sum_{\boldsymbol{\omega}\in\mathfrak{T}}\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}\frac{|G^F|^{2g-2}\prod_{j=1}^{k}|C_j|}{\chi(1)^{2g+k-2}}\prod_{j=1}^{k}\left\langle s_{\{\boldsymbol{\omega}\}}(\mathbf{z}_j), \widetilde{H}_{\boldsymbol{\mu}_j}(\mathbf{z}_j,q)\right\rangle\\ &= \sum_{\boldsymbol{\omega}\in\mathfrak{T}}\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}\left(\frac{|G^F|}{\chi(1)}\right)^{2g+k-2}\prod_{j=1}^{k}\frac{|C_j|}{|G^F|}\prod_{j=1}^{k}\left\langle s_{\{\boldsymbol{\omega}\}}(\mathbf{z}_j), \widetilde{H}_{\boldsymbol{\mu}_j}(\mathbf{z}_j,q)\right\rangle\\ &\overset{②}{=} \sum_{\boldsymbol{\omega}\in\mathfrak{T}}\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}\left((-1)^n H_{\{\boldsymbol{\omega}\}}(q)q^{\frac{1}{2}n(n-1)-n(\{\boldsymbol{\omega}\})}\right)^{2g+k-2}\\ &\qquad\cdot\prod_{j=1}^{k}\left\langle s_{\{\boldsymbol{\omega}\}}(\mathbf{z}_j), \frac{|C_j|}{|G^F|}\widetilde{H}_{\boldsymbol{\mu}_j}(\mathbf{z}_j,q)\right\rangle\end{aligned} \tag{6.10}$$

$$
\begin{aligned}
&= (-1)^{kn} q^{\frac{1}{2}\left(n^2(2g+k-2)-kn\right)} \\
&\cdot \left\langle \sum_{\boldsymbol{\omega}\in\mathfrak{T}} \frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)} q^{(1-g)|\{\boldsymbol{\omega}\}|} \left(H_{\{\boldsymbol{\omega}\}}(q) q^{-n(\{\boldsymbol{\omega}\})}\right)^{2g+k-2} \prod_{j=1}^{k} s_{\{\boldsymbol{\omega}\}}(\mathbf{z}_j), \right. \\
&\qquad\qquad \left. \prod_{j=1}^{k} \frac{|C_j|}{|G^F|} \widetilde{H}_{\boldsymbol{\mu}_j}(\mathbf{z}_j, q) \right\rangle.
\end{aligned}
$$

In ① the sum can be taken over the entire $\mathfrak{T}$ because if $\boldsymbol{\omega}$ was not of size N then the inner product of symmetric functions would vanish. We have also used Lemma 6.3 in this equality. Equality ② uses (6.7).

Equation (2.5) shows that

$$
\frac{|C_j|}{|G^F|} = q^{-2n(\boldsymbol{\mu}_j)-|\boldsymbol{\mu}_j|} b_{\boldsymbol{\mu}_j}(q^{-1})^{-1}. \tag{6.11}
$$

Combined with Lemma 6.1, this shows

$$
q^{-n(\boldsymbol{\mu}_j)} h_{\boldsymbol{\mu}_j^*}\left[\frac{\mathbf{z}_j}{1-q}\right] = (-1)^{|\boldsymbol{\mu}_j|} \frac{|C_j|}{|G^F|} \widetilde{H}_{\boldsymbol{\mu}_j}(\mathbf{z}_j; q). \tag{6.12}
$$

For any symmetric functions $u(\mathbf{z})$ and $v(\mathbf{z})$, we have

$$
\left\langle u\left[\frac{\mathbf{z}}{1-q}\right], v(\mathbf{z}) \right\rangle = \left\langle u(\mathbf{z}), v\left[\frac{\mathbf{z}}{1-q}\right] \right\rangle.
$$

This can be checked on the basis of power sums. We can then move $1/(1-q)$ to the left-hand side of the inner product.

Rewrite the left-hand side of the inner product (6.10) as follows:

$$
\begin{aligned}
&\sum_{\boldsymbol{\omega}\in\mathfrak{T}} \frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)} q^{(1-g)|\{\boldsymbol{\omega}\}|} \left(H_{\{\boldsymbol{\omega}\}}(q) q^{-n(\{\boldsymbol{\omega}\})}\right)^{2g+k-2} \prod_{j=1}^{k} s_{\{\boldsymbol{\omega}\}}\left[\frac{\mathbf{z}_j}{1-q}\right] \\
&= \left(\sum_{\omega_+\in\mathcal{P}} q^{(1-g)|\omega_+|} \left(H_{\omega_+}(q) q^{-n(\omega_+)}\right)^{2g+k-2} \prod_{j=1}^{k} s_{\omega_+}\left[\frac{\mathbf{z}_j}{1-q}\right] \right) \\
&\quad \cdot \left(\sum_{\omega_-\in\mathcal{P}} q^{(1-g)|\omega_-|} \left(H_{\omega_-}(q) q^{-n(\omega_-)}\right)^{2g+k-2} \prod_{j=1}^{k} s_{\omega_-}\left[\frac{\mathbf{z}_j}{1-q}\right] \right) \\
&\quad \cdot \left(\sum_{\boldsymbol{\omega}_*} \frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)} q^{(2-2g)|\boldsymbol{\omega}_*|} \left(H_{\boldsymbol{\omega}_*}(q)^2 q^{-2n(\boldsymbol{\omega}_*)}\right)^{2g+k-2} \prod_{j=1}^{k} s_{\boldsymbol{\omega}_*}\left[\frac{\mathbf{z}_j}{1-q}\right]^2 \right).
\end{aligned}
$$

By definition

$$
\begin{aligned}
&\frac{1}{\Omega_*(q)}\\
&\quad=\sum_{m\geqslant 0}(-1)^m\sum_{\substack{\boldsymbol{\omega}_*=(m_\lambda)_\lambda\in\mathfrak{T}_*\\ l(\boldsymbol{\omega}_*)=m}}\frac{m!}{\prod_\lambda m_\lambda!}q^{(2-2g)|\boldsymbol{\omega}_*|}\left(H_{\boldsymbol{\omega}_*}(q)^2q^{-2n(\boldsymbol{\omega}_*)}\right)^{2g+k-2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot\prod_{j=1}^k s_{\boldsymbol{\omega}_*}\left[\frac{\mathbf{z}_j}{1-q}\right]^2\\
&\quad=\sum_{\boldsymbol{\omega}_*\in\mathfrak{T}_*}\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}q^{(2-2g)|\boldsymbol{\omega}_*|}\left(H_{\boldsymbol{\omega}_*}(q)^2q^{-2n(\boldsymbol{\omega}_*)}\right)^{2g+k-2}\prod_{j=1}^k s_{\boldsymbol{\omega}_*}\left[\frac{\mathbf{z}_j}{1-q}\right]^2,
\end{aligned}
$$

whence

$$
\sum_{\boldsymbol{\omega}\in\mathfrak{T}}\frac{K(\boldsymbol{\omega}_*)}{N(\boldsymbol{\omega}_*)}q^{(1-g)|\{\boldsymbol{\omega}\}|}\left(H_{\{\boldsymbol{\omega}\}}(q)q^{-n(\{\boldsymbol{\omega}\})}\right)^{2g+k-2}\prod_{j=1}^k s_{\{\boldsymbol{\omega}\}}\left[\frac{\mathbf{z}_j}{1-q}\right]^2
=\frac{\Omega_\bullet(q)^2}{\Omega_*(q)}.
$$

Therefore,

$$
\left|\mathrm{Ch}_{\mathcal{C}}(\mathbb{F}_q)\right|=q^{\frac{1}{2}\left(n^2(2g+k-2)-kn\right)-\sum_{j=1}^k n(\boldsymbol{\mu}_j)}\cdot\left\langle\frac{\Omega_\bullet(q)^2}{\Omega_*(q)},\prod_{j=1}^k h_{\boldsymbol{\mu}_j^*}(\mathbf{z}_j)\right\rangle.
$$

Finally, we use Section 6.3.2. □

Corollary 6.5.

(i) *The $t=-1$ specialisation of part* (iii) *of Conjecture 1.2 holds.*
(ii) *The $t=-1$ specialisation of Conjecture 1.4 holds.*

Proof. — Specialising $\Omega_\bullet(z,w)$ and $\Omega_*(z,w)$ to $z=\sqrt{q}$, $w=\sqrt{q}^{-1}$, we get

$$
\Omega_\bullet(\sqrt{q},\sqrt{q}^{-1})=\sum_{\lambda\in\mathcal{P}}N_\lambda(q,q^{-1})^{g-1}\prod_{j=1}^k H_\lambda(\mathbf{z}_j;q,q^{-1}),
$$

$$
\Omega_*(\sqrt{q},\sqrt{q}^{-1})=\sum_{\lambda\in\mathcal{P}}N_\lambda(q,q^{-1})^{2g-2}\prod_{j=1}^k H_\lambda(\mathbf{z}_j;q,q^{-1})^2.
$$

Then use [20, Lemmas 8.1.3 and 8.1.4] and [6, Proposition 3.3.2]. This proves (i).

By [20, Remark 8.1.2], we have

$$
\Omega_\bullet(z,w)=\Omega_\bullet(w,z),\qquad \Omega_*(z,w)=\Omega_*(w,z),
$$

and so

$$\mathbb{H}_{\boldsymbol{\mu}}(z,w)=\mathbb{H}_{\boldsymbol{\mu}}(w,z),\qquad \mathbb{H}_{\boldsymbol{\mu}}(z,w)=\mathbb{H}_{\boldsymbol{\mu}}(-z,-w).$$

We deduce from this that part (iii) of Conjecture 1.2 implies Conjecture 1.4. Therefore the $t=-1$ specialisation holds. □

7. Mixed Hodge polynomial

7.1. Evidences of the conjecture

We check that our conjectural formula for the mixed Hodge polynomial is consistent with known cohomological results when $n=1$ and $n=2$. For simplicity, we assume $k=1$.

7.1.1.

We first consider the case $n=1$. The definition of the character variety reads

$$\mathrm{Ch}_{\mathcal{C}}(\Sigma)=\left\{(A_i,B_i)_i\in \mathrm{GL}_1^{2g}\ \middle|\ B_1^{-2}X_1=1\right\}/\!\!/\,\mathrm{GL}_1,$$

for a given element X_1 of GL_1. The action of GL_1 on A_1 is given by

$$gA_1\sigma(g)^{-1}=g^2A_1,\quad \text{for } g\in\mathrm{GL}_1,$$

and its action on all other components is trivial. We see that there are two connected components corresponding to the two solutions of B_1, and each connected component is a $(2g-2)$-dimensional torus. The mixed Hodge polynomial is then $2(t+qt^2)^{2g-2}$.

The formula in Conjecture 1.2(iii) gives

$$\begin{aligned}2(t\sqrt{q})^{2g-2}N_{(1)}(zw,z^2,w^2)^{2g-2}\big|_{z=-t\sqrt{q},\,w=\sqrt{q}^{-1}}\\ =2(t\sqrt{q})^{2g-2}\left(t\sqrt{q}+\frac{1}{\sqrt{q}}\right)^{2g-2},\end{aligned}$$

as expected.

7.1.2.

Then we consider the case $n=2$. In this case, the centraliser of σ in GL_2 is just SL_2, and σ acts on the center by inversion. Now the character variety is defined as

$$\mathrm{Ch}_{\mathcal{C}}(\Sigma) := \left\{ (A_i,B_i)_i(X_1) \in \mathrm{GL}_2^{2g} \times C_1 \;\middle|\; A_1\sigma(B_1)A_1^{-1}B_1^{-1}\prod_{i=2}^{g}[A_i,B_i]X_1 = 1 \right\} /\!\!/ \mathrm{GL}_2,$$

where C_1 is the conjugacy class of $\mathrm{diag}(a,a^{-1})$ with $a^2 \neq 1$. For any $1 \leqslant i \leqslant g$, write $A_i = a_i\alpha_i$ and $B_i = b_i\beta_i$, with a_i, $b_i \in \mathrm{SL}_2$ and α_i, $\beta_i \in \mathbb{C}^*$, the center of GL_2. Thus we realise $\mathrm{Ch}_{\mathcal{C}}$ as a quotient of

$$M := \left\{ (a_i,b_i)_i(\alpha_i,\beta_i)_i(X_1) \in \mathrm{SL}_2^{2g} \times (\mathbb{C}^*)^{2g} \times C_1 \;\middle|\; \prod_{i=1}^{g}[a_i,b_i]\beta_1^{-2}X_1 = 1 \right\} /\!\!/ \mathrm{GL}_2,$$

where GL_2 acts on X_1, a_i and b_i, $1 \leqslant i \leqslant g$, by conjugation, acts trivially on β_1, α_i, β_i, $2 \leqslant i \leqslant g$, and acts on α_1 by

$$g\colon \alpha_1 \longmapsto \det(g)\alpha_1, \quad \text{for } g \in \mathrm{GL}_2.$$

The finite group $\Gamma := \{\pm \mathrm{Id}\}^{2g} \subset (\mathbb{C}^*)^{2g}$ acts on M in such a way that $\mathrm{Ch}_{\mathcal{C}} \cong M/\Gamma$. Define

$$M_{\pm} := \left\{ (a_i,b_i)_i(X_1) \in \mathrm{SL}_2^{2g} \times C_1 \;\middle|\; \prod_{i=1}^{g}[a_i,b_i]X_1 = \pm 1 \right\} /\!\!/ \mathrm{SL}_2,$$

where SL_2 acts by conjugation on each component. Since it is necessary that $\beta_1^{-2} = \pm \mathrm{Id}$, we have

$$M = M_+ \times \{\pm 1\} \times (\mathbb{C}^*)^{2g-2} \sqcup M_- \times \{\pm\sqrt{-1}\} \times (\mathbb{C}^*)^{2g-2},$$

where $\{\pm 1\}$ and $\{\pm\sqrt{-1}\}$ are the solutions of $\beta_1^2 = \pm \mathrm{Id}$. The induced Γ-action on $M_+ \times \{\pm 1\} \times (\mathbb{C}^*)^{2g-2}$ is as follows. Let $(e_i)_{1\leqslant i\leqslant 2g} \in \Gamma$, then e_2 identifies the two components:

$$e_2 : M_+ \times \{+1\} \times (\mathbb{C}^*)^{2g-2} \xrightarrow{\sim} M_+ \times \{-1\} \times (\mathbb{C}^*)^{2g-2},$$

while e_i, $i \neq 2$, acts in the usual manner. Denote by Γ_0 the subgroup of Γ consisting of elements whose e_2-component is trivial. We see that

$$\big(M_+ \times \{\pm 1\} \times (\mathbb{C}^*)^{2g-2}\big)/\Gamma \cong \big(M_+ \times (\mathbb{C}^*)^{2g-2}\big)/\Gamma_0.$$

The same argument applies to M_-. We see that

$$\mathrm{Ch}_{\mathcal{C}} \cong \big(M_+ \times (\mathbb{C}^*)^{2g-2}\big)/\Gamma_0 \sqcup \big(M_- \times (\mathbb{C}^*)^{2g-2}\big)/\Gamma_0.$$

Considering the cohomology with compact support, we have

$$H_{\mathrm{c}}\big(\big(M_+ \times (\mathbb{C}^*)^{2g-2}\big)/\Gamma_0\big) \cong H_{\mathrm{c}}(M_+)^{\Gamma_0} \otimes H_{\mathrm{c}}\big((\mathbb{C}^*)^{2g-2}\big),$$

since the action of Γ_0 on the cohomology of $(\mathbb{C}^*)^{2g-2}$ is trivial. Denote by κ the irreducible character of Γ that is nontrivial only on the e_2-component. Then

$$H_{\mathrm{c}}(M_+)^{\Gamma_0} = H_{\mathrm{c}}(M_+)_{\mathrm{st}} \oplus H_{\mathrm{c}}(M_+)_\kappa,$$

where $H_{\mathrm{c}}(M_+)_{\mathrm{st}} = H_{\mathrm{c}}(M_+)^\Gamma$ is the Γ-invariant subspace and $H_{\mathrm{c}}(M_+)_\kappa$ is the κ-isotypic component. This is also a direct sum of mixed Hodge structures. The mixed Hodge polynomial of $H_{\mathrm{c}}(M_+)_{\mathrm{st}}$ can be deduced from the mixed Hodge polynomial of the GL_2-character variety defined by the conjugacy class C_1. The mixed Hodge polynomial of $H_{\mathrm{c}}(M_+)_\kappa$ is equal to the mixed Hodge polynomial of

$$H_{\mathrm{c}}(M_+)_{\mathrm{var}} := \bigoplus_{\substack{\gamma\in\mathrm{Irr}(\Gamma)\\ \gamma\neq 1}} H_{\mathrm{c}}(M_+)_\gamma$$

divided by $(2^{2g}-1)$, since the mapping class group acts transitively on the set of nontrivial irreducible characters of Γ. We call $H_{\mathrm{c}}(M_+)_{\mathrm{var}}$ the variant part of the cohomology.

7.1.3.

According to [2, Section 4.4], the Poincaré polynomial of the SL_2-character variety with one regular semisimple conjugacy class is given by

$$\frac{(1+t^3)^{2g}+t^{4g-1}(1+t)^{2g}\big((2g-1)t-2g\big)}{(t^2-1)^2}+(2^{2g}-1)t^{4g-2}(t+1)^{2g-2}.$$

We deduce from this expression the Poincaré polynomial of the cohomology with compact support by Poincaré duality. The variant part is

$$(2^{2g}-1)t^{6g-4}(t+1)^{2g-2}. \tag{7.1}$$

On the other hand, using the character table of $\mathrm{SL}_2(q)$ in [4, Section 15.9], we compute the E-polynomial of this character variety. The result is

$$(q-1)^{2g-2}\big((2^{2g}-2)q^{2g-1}+q^{2g-1}(q+1)^{2g-1}+(q+1)^{2g-1}\big). \tag{7.2}$$

The stable part of the E-polynomial can be computed using the character table of $\mathrm{GL}_2(q)$ in [4, Section 15.9], and the result is

$$(q-1)^{2g-2}\big(-q^{2g-1}+q^{2g-1}(q+1)^{2g-1}+(q+1)^{2g-1}\big). \tag{7.3}$$

We deduce the variant part of the E-polynomial:

$$(2^{2g}-1)q^{2g-1}(q-1)^{2g-2}. \tag{7.4}$$

The variant part of the Poincaré polynomial and the variant part of the E-polynomial have an obvious common deformation:

$$(2^{2g-1}-1)t^{6g-4}q^{2g-1}(qt+1)^{2g-2}. \tag{7.5}$$

If we write $z=-t\sqrt{q}$ and $w=1/\sqrt{q}$, then this is just

$$(2^{2g-1}-1)(t\sqrt{q})^{6g-4}(z-w)^{2g-2},$$

with $6g-4$ equal to the dimension of the SL_2-character variety.

Remark 7.1. — In principle, we can use the method of [3, Section 4.4] to prove that each cohomological degree has only one weight, and then the variant part of the mixed Hodge polynomial is given by (7.5). Since our goal is to check that the conjectural formula is consistent with known results, we omit the proof.

The stable part of the mixed Hodge polynomial can be computed using the conjectural formula of Hausel–Letellier–Rodriguez-Villegas. Taking into account the variant part, we see that the mixed Hodge polynomial of $\mathrm{Ch}_{\mathcal{C}}$ is given by

$$\begin{aligned}2\frac{(z^3-w)^{2g}(z-w)^{2g-2}}{(z^4-1)(z^2-w^2)}(1+z^2)&\\ +2\frac{(z-w^3)^{2g}(z-w)^{2g-2}}{(z^2-w^2)(1-w^4)}(1+w^2)&\\ -2\frac{(z-w)^{4g-2}}{(z^2-1)(1-w^2)}+2(z-w)^{4g-4}&\end{aligned}$$

with $z=-t\sqrt{q}$ and $w=1/\sqrt{q}$, and multiplied by $(t\sqrt{q})^{6g-4}$. This is exactly the formula predicted by Conjecture 1.2.

BIBLIOGRAPHY

[1] R. Bezrukavnikov & M. Finkelberg, "Wreath Macdonald polynomials and the categorical McKay correspondence", *Camb. J. Math.* **2** (2014), no. 2, p. 163-190.

[2] H. U. Boden & K. Yokogawa, "Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves. I", *Int. J. Math.* **7** (1996), no. 5, p. 573-598.

[3] M. A. A. de Cataldo, T. Hausel & L. Migliorini, "Topology of Hitchin systems and Hodge theory of character varieties: the case A_1", *Ann. Math. (2)* **175** (2012), no. 3, p. 1329-1407.

[4] F. Digne & J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 21, Cambridge University Press, 1991.

[5] P. B. Gothen, “The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface”, *Int. J. Math.* **5** (1994), no. 6, p. 861-875.

[6] M. Haiman, “Combinatorics, symmetric functions, and Hilbert schemes”, in *Current developments in mathematics, 2002* (D. Jerison, G. Lusztig, B. Mazur, T. Mrowka, W. Schmid, R. Stanley & S.-T. Yau, eds.), International Press, 2003, p. 39-111.

[7] T. Hausel, E. Letellier & F. Rodriguez-Villegas, “Arithmetic harmonic analysis on character and quiver varieties”, *Duke Math. J.* **160** (2011), no. 2, p. 323-400.

[8] T. Hausel & F. Rodriguez-Villegas, “Mixed Hodge polynomials of character varieties”, *Invent. Math.* **174** (2008), no. 3, p. 555-624.

[9] N. J. Hitchin, “The self-duality equations on a Riemann surface”, *Proc. Lond. Math. Soc. (3)* **55** (1987), no. 1, p. 59-126.

[10] G. Laumon & B. C. Ngô, “Le lemme fondamental pour les groupes unitaires”, *Ann. Math. (2)* **168** (2008), no. 2, p. 477-573.

[11] G. Lusztig & B. Srinivasan, “The characters of the finite unitary groups”, *J. Algebra* **49** (1977), no. 1, p. 167-171.

[12] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, Clarendon Press, 1995.

[13] A. Mellit, “Integrality of Hausel–Letellier–Villegas kernels”, *Duke Math. J.* **167** (2018), no. 17, p. 3171-3205.

[14] ———, “Cell decompositions of character varieties”, 2019, `https://arxiv.org/abs/1905.10685`.

[15] ———, “Poincaré polynomials of character varieties, Macdonald polynomials and affine Springer fibers”, *Ann. Math. (2)* **192** (2020), no. 1, p. 165-228.

[16] ———, “Poincaré polynomials of moduli spaces of Higgs bundles and character varieties (no punctures)”, *Invent. Math.* **221** (2020), no. 1, p. 301-327.

[17] O. Schiffmann, “Indecomposable vector bundles and stable Higgs bundles over smooth projective curves”, *Ann. Math. (2)* **183** (2016), no. 1, p. 297-362.

[18] C. S. Seshadri, “Geometric reductivity over arbitrary base”, *Adv. Math.* **26** (1977), no. 3, p. 225-274.

[19] C. Shu, “The character table of $\mathrm{GL}_n(q) \rtimes \langle\sigma\rangle$”, *Adv. Math.* **403** (2022), article no. 108357 (100 pages).

[20] ———, “E-polynomials of generic $\mathrm{GL}_n \rtimes \langle\sigma\rangle$-character varieties: branched case”, *Forum Math. Sigma* **11** (2023), article no. e116 (72 pages).

[21] ———, “On character varieties with non-connected structure groups”, *J. Algebra* **631** (2023), p. 484-516.

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