Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Article à paraître, mis en ligne le 16 juillet 2026, 65 p.

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 16 juillet 2026.

GEVREY REGULARITY AND SUMMABILITY OF THE FORMAL SOLUTIONS IN TIME OF SOME INHOMOGENEOUS LINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

by Pascal REMY

ABSTRACT. — In this article, we investigate the Gevrey regularity and the summability of the formal solutions in time of some inhomogeneous linear systems of partial differential equations with analytic coefficients at the origin of $\mathbb{C}^2$. Given such a system, we first prove that the Gevrey regularity of its formal solutions follows a noteworthy dichotomy with respect to the s-Gevrey regularity of the inhomogeneity: for $s \geqslant s_0$ with s_0 a nonnegative rational number entirely determined by a convenient Newton polygon attached to the given system, the solutions inherit the Gevrey regularity of the inhomogeneity; for $s < s_0$, if any exists, the solutions keep the s_0-Gevrey regularity defined by the structure of the initial system. Then, assuming $s_0 > 0$ and some convenient additional assumptions on the system, we give a necessary and sufficient condition for the $1/s_0$-summability of the formal solutions in a given direction. Several examples illustrate these results.

RÉSUMÉ. — Dans cet article, nous nous intéressons à la régularité Gevrey et à la sommabilité des solutions formelles en temps de certains systèmes linéaires non homogènes d'équations aux dérivées partielles à coefficients analytiques à l'origine de $\mathbb{C}^2$. Etant donné un tel système, nous montrons tout d'abord que la régularité Gevrey de ses solutions formelles suit une remarquable dichotomie liée à la régularité Gevrey-s de son second membre: lorsque $s \geqslant s_0$, où s_0 est un nombre rationnel positif ou nul entièrement déterminé par le polygone de Newton associé au système, les solutions formelles héritent de cette régularité Gevrey-s; lorsque $s < s_0$, si une telle valeur existe, les solutions formelles conservent la régularité Gevrey-s_0 définie par la structure même du système. En supposant ensuite $s_0 > 0$ et en ajoutant quelques hypothèses supplémentaires sur le système, nous donnons une condition nécessaire et suffisante pour que ses solutions formelles soient $1/s_0$-sommables dans une direction donnée. Plusieurs exemples viennent illustrer ces différents résultats

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Keywords: Gevrey regularity, Summability, Inhomogeneous linear partial differential equation, Inhomogeneous linear system, Formal power series, Divergent power series, Newton polygon.
2020 *Mathematics Subject Classification:* 35C10, 35C20, 35G35, 40B05.

1. Introduction

1.1. Setting the problem

For several years, various works have been done on the divergent solutions of some classes of linear partial differential equations or integro-differential equations in two variables or more, allowing thus to formulate many results on Gevrey properties, summability or multisummability (see for instance [3, 5, 7, 8, 9, 10, 11, 21, 22, 23, 27, 28, 29, 30, 31, 32, 33, 34, 38, 39, 41, 42, 43, 56, 57, 58] and the references therein). In recent years, these results have also been extended to more general semilinear and nonlinear partial differential equations ([12, 13, 14, 15, 16, 17, 18, 24, 35, 37, 44, 45, 46, 47, 48, 49, 50, 52, 53, 54, 55, 60] and the reference therein).

In this article, we consider an inhomogeneous linear system of partial differential equations

$$\begin{cases} \partial_t^{\kappa} u - \displaystyle\sum_{i\in\mathcal{K}} \sum_{q\in Q_i} A_{i,q}(t,x)\partial_t^i\partial_x^q u = \widetilde{f}(t,x) \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j=0,\dots,\kappa-1 \end{cases} \tag{1.1}$$

in two variables $(t,x)\in\mathbb{C}^2$, where

- $\kappa \geqslant 1$ is a positive integer;
- $\mathcal{K}$ is a nonempty subset of $\{0,\dots,\kappa-1\}$;
- Q_i is a nonempty finite subset of $\mathbb{N}$ for all $i\in\mathcal{K}$ ($\mathbb{N}$ denotes the set of all the nonnegative integers) and at least one of the Q_i's is not reduced to $\{0\}$;
- the coefficients $A_{i,q}(t,x)\in\mathcal{M}_n(\mathcal{O}(D_{\rho_0}\times D_{\rho_1}))$ are nonzero matrices of size $n\times n$ with analytic coefficients on a polydisc $D_{\rho_0}\times D_{\rho_1}$ centered at the origin of $\mathbb{C}^2$ (D_ρ denotes the disc with center $0\in\mathbb{C}$ and radius $\rho>0$) for all $i\in\mathcal{K}$ and $q\in Q_i$;
- the inhomogeneity $\widetilde{f}(t,x)$ is a matrix of size $n\times 1$ whose terms are formal power series in t with analytic coefficients in D_{ρ_1} (we denote by $\widetilde{f}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$ which may be convergent, or not[(1)];
- the initial conditions $\varphi_j(x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1}))$ are matrices of size $n\times 1$ with analytic coefficients on D_{ρ_1} for all $j=0,\dots,\kappa-1$.

System (1.1) is fundamental in many physical, chemical, biological, and ecological problems. Examples below present some of these problems.

[(1)] We denote $\widetilde{f}$ with a tilde to emphasize the possible divergence of the series $\widetilde{f}$.

Example 1.1 (Acoustic equations). — Let us respectively denote by $v(t,x)$ and $p(t,x)$ the velocity and the pressure of a fluid with density $\rho(t,x)$ and compressibility coefficient $c(t,x)$. Then, under some convenient hypotheses, the vector

$$u = \begin{pmatrix} v \\ p \end{pmatrix}$$

is a solution of a linear system of the form

$$\partial_t u + \begin{pmatrix} 0 & a(t,x) \\ b(t,x) & 0 \end{pmatrix} \partial_x u = f(t,x) \tag{1.2}$$

with $a = 1/\rho$, $b = \rho c^2$ and $f(t,x)$ an analytic function at the origin of $\mathbb{C}^2$. Note that a system of the same type as System (1.2) also occurs when one searchs "planar" solutions of Maxwell's equations. In this case, the functions $a(t,x)$ and $b(t,x)$ are related to the dielectric permittivity ε_0, to the magnetic permeability μ_0 and to a characteristic $c_0(t,x)$ of a non-conducting medium where neither currents nor charges are present.

Example 1.2 (Quasistationary electric oscillations). — Let us now consider the propagation of quasistationary electric oscillations in cables with self-induction $L(t,x)$, ohmic resistance $R(t,x)$, capacitance $C(t,x)$ and creepage $A(t,x)$. Then, the vector

$$u = \begin{pmatrix} v \\ i \end{pmatrix}$$

with v the voltage and i the current is a solution of a linear system of the form

$$\partial_t u + \begin{pmatrix} 0 & c(t,x) \\ \ell(t,x) & 0 \end{pmatrix} \partial_x u + \begin{pmatrix} a(t,x) & 0 \\ 0 & r(t,x) \end{pmatrix} u = f(t,x) \tag{1.3}$$

where $c = 1/C$, $\ell = 1/L$, $a = A/C$, $b = R/L$, and where $f(t,x)$ is an analytic function at the origin of $\mathbb{C}^2$.

Example 1.3 (Reaction-diffusion systems). — Reaction-diffusion systems are mathematical models that correspond to several physical phenomena. The most common is the change in time and space of the concentration of one or more chemical substances: local chemical reactions in which the substances are transformed into each other, and diffusion which causes the substances to spread out over a surface in space. In their "simplest" form, reaction-diffusion systems read as

$$\partial_t u = A(t,x)\partial_x^2 u + B(t,x)u, \tag{1.4}$$

where $A(t,x) \in \mathcal{M}_n(\mathcal{O}(D_{\rho_0} \times D_{\rho_1}))$ is a nonzero $n \times n$-diagonal matrix of diffusion coefficients, and where $B(t,x) \in \mathcal{M}_n(\mathcal{O}(D_{\rho_0} \times D_{\rho_1}))$ accounts for all local reaction.

Reaction-diffusion systems are naturally applied in chemistry, but can also describe dynamical processes of non-chemical nature. Many examples can be found in biology, geology, physics and ecology: e.g., morphogenesis, ecological invasions, spread of epidemics, tumour growth, dynamics of fission waves, neutron diffusion theory, wound healing, and visual hallucinations.

The work presented in this article is a natural extension of the work [43] in which we only considered the case $n = 1$, that is the case of the classical inhomogeneous linear partial differential equations. Before stating our various results (see Theorems 2.5 and 3.11) describing the Gevrey regularity and making explicit a characterization of the summability of the formal series solutions in time of System (1.1), let us first start by recalling some known results about these ones.

Notation 1.4. — All along the article,

- we write any formal power series $\widetilde{g}(t,x) \in \mathcal{O}(D_{\rho_1})[[t]]$ in the form
$$\widetilde{g}(t,x) = \sum_{j \geqslant 0} g_{j,*}(x) \frac{t^j}{j!} = \sum_{\nu \geqslant 0} \widetilde{g}_{*,\nu}(t) \frac{x^\nu}{\nu!} = \sum_{j,\nu \geqslant 0} g_{j,\nu} \frac{t^j}{j!} \frac{x^\nu}{\nu!}$$
with $g_{j,*}(x) \in \mathcal{O}(D_{\rho_1})$, $\widetilde{g}_{*,\nu}(t) \in \mathbb{C}[[t]]$(2) and $g_{j,\nu} \in \mathbb{C}$ for all $j, \nu \geqslant 0$:
- for any $n \times n$-square matrix M, we denote by $M^{[k,\ell]}$ with $1 \leqslant k, \ell \leqslant n$ its term at row k and column ℓ;
- for any $n \times 1$-column matrix M, we denote by $M^{[k]}$ with $1 \leqslant k \leqslant n$ its term at row k.

1.2. Formal solutions and known results

First of all, we have the following.

Proposition 1.5. — *System* (1.1) *is formally well-posed*(3)*, that is admits a unique formal solution* $\widetilde{u}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$.

(2) As for the inhomogeneity $\widetilde{f}(t,x)$, we denote $\widetilde{g}_{*,\nu}(t)$ with a tilde to emphasize the possible divergence of these series.

(3) This notion, introduced by W. Balser and V. Kostov in [2], extends to formal power series solutions the classical notion of *well-posed problem* of J. Hadamard [6].

Proof. — Let us set

$$\mathcal{L}_k(A_{i,q}) = \Big\{\ell \in \{1,\dots,n\} \,:\, A_{i,q}^{[k,\ell]}(t,x) \not\equiv 0\Big\}$$

for all $k=1,\dots,n$, all $i\in\mathcal{K}$ and all $q\in Q_i$. Some of these sets may of course be empty but, given our assumption $A_{i,q}(t,x)\not\equiv 0$, there exists at least one $k\in\{1,\dots,n\}$ such that $\mathcal{L}_k(A_{i,q})\neq\emptyset$. For any $\ell\in\mathcal{L}_k(A_{i,q})$, if any exists, we write the term $A_{i,q}^{[k,\ell]}(t,x)$ in the form

$$A_{i,q}^{[k,\ell]}(t,x) = t^{v_{i,q}^{[k,\ell]}} a_{i,q}^{[k,\ell]}(t,x)$$

where

- $a_{i,q}^{[k,\ell]}(t,x)\in\mathcal{O}(D_{\rho_0}\times D_{\rho_1})$ is analytic on the polydisc $D_{\rho_0}\times D_{\rho_1}$ and satisfies $a_{i,q}^{[k,\ell]}(0,x)\not\equiv 0$;
- the valuation $v_{i,q}^{[k,\ell]}\geqslant 0$ of $A_{i,q}^{[k,\ell]}(t,x)$ with respect to t is a nonnegative integer.

With these notations, System (1.1) becomes the system

$$\text{(1.5)}\qquad \begin{cases} \partial_t^\kappa u^{[k]} - \displaystyle\sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell\in\mathcal{L}_k(A_{i,q})} t^{v_{i,q}^{[k,\ell]}} a_{i,q}^{[k,\ell]}(t,x)\partial_t^i\partial_x^q u^{[\ell]} = \widetilde{f}^{[k]}(t,x) \\ \partial_t^j u^{[k]}(t,x)_{|t=0} = \varphi_j^{[k]}(x), j=0,\dots,\kappa-1 \end{cases}$$

for all $k=1,\dots,n$, with the classical convention that the third sum is zero when $\mathcal{L}_k(A_{i,q})=\emptyset$.

Writing then the analytic coefficients $a_{i,q}^{[k,\ell]}(t,x)$ and the inhomogeneities $\widetilde{f}^{[k]}(t,x)$ in the form

$$a_{i,q}^{[k,\ell]}(t,x) = \sum_{j\geqslant 0} a_{i,q;j,*}^{[k,\ell]}(x)\frac{t^j}{j!} \quad\text{and}\quad \widetilde{f}^{[k]}(t,x) = \sum_{j\geqslant 0} f_{j,*}^{[k]}(x)\frac{t^j}{j!}$$

with $a_{i,q;j,*}^{[k,\ell]}(x), f_{j,*}^{[k]}(x)\in\mathcal{O}(D_{\rho_1})$ for all $j\geqslant 0$ and all i,q,k,ℓ, and looking for the n components $\widetilde{u}^{[k]}(t,x)$ of $\widetilde{u}(t,x)$ on the same type:

$$\widetilde{u}^{[k]}(t,x) = \sum_{j\geqslant 0} u_{j,*}^{[k]}(x)\frac{t^j}{j!}$$

with $u_{j,*}^{[k]}(x)\in\mathcal{O}(D_{\rho_1})$ for all $j\geqslant 0$ and all $k=1,\dots,n$, one easily checks that the coefficients $u_{j,*}^{[k]}(x)$ are uniquely determined for all $k=1,\dots,n$

and all $j \geqslant 0$ by the recurrence relation

$$u^{[k]}_{j+\kappa,*}(x) = f^{[k]}_{j,*}(x) + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell\in\mathcal{L}_k(A_{i,q})}\sum_{j_0+j_1=j-v^{[k,\ell]}_{i,q}} \frac{j!}{j_0!j_1!} a^{[k,\ell]}_{i,q;j_0,*}(x)\partial_x^q u^{[\ell]}_{j_1+i,*} \tag{1.6}$$

together with the initial conditions $u^{[k]}_{j,*}(x) = \varphi^{[k]}_j(x)$ for all $k = 1,\dots,n$ and all $j = 0,\dots,\kappa-1$, which ends the proof of Proposition 1.5. □

In [43, 45], the author proved for the inhomogeneous linear partial differential equation

$$\begin{cases} \partial_t^\kappa u - \displaystyle\sum_{i\in\mathcal{K}}\sum_{q\in Q_i} t^{v_{i,q}} a_{i,q}(t,x)\partial_t^i\partial_x^q u = \widetilde{f}(t,x) \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j = 0,\dots,\kappa-1, \end{cases} \tag{1.7}$$

that is for System (1.1) with $n = 1$, that the Gevrey regularity of the formal solution $\widetilde{u}(t,x)$ depends both on the Gevrey regularity of the inhomogeneity $\widetilde{f}(t,x)$ and on the structure of (1.7), that is on the linear operator

$$\partial_t^\kappa - \sum_{i\in\mathcal{K}}\sum_{q\in Q_i} t^{v_{i,q}} a_{i,q}(t,x)\partial_t^i\partial_x^q.$$

Proposition 1.6 ([43, 45]). — *Let s_0 be the nonnegative rational number defined by*

$$s_0 = \max\left(0, \max_{i\in\mathcal{K}}\left(\max_{q\in Q_i} \frac{q-\kappa+i}{v_{i,q}+\kappa-i}\right)\right).$$

Then,

1. *$\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ are simultaneously s-Gevrey for any $s \geqslant s_0$;*
2. *$\widetilde{u}(t,x)$ is generically s_0-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_0$.*

Thus, the Gevrey regularity of $\widetilde{u}(t,x)$ follows a noteworthy dichotomy, governed by the critical value s_0, between the Gevrey regularity of the inhomogeneity $\widetilde{f}(t,x)$ and the one induced by the very structure of (1.7).

Considering then this critical value s_0, and under some convenient assumptions, the author gave a necessary and sufficient condition under which the formal solution $\widetilde{u}(t,x)$ is k_0-summable, with $k_0 = 1/s_0$, in a given direction $\arg(t) = \theta$.

Proposition 1.7 ([43]). — *Let us denote by*

- *q_i the maximum of all the x-derivative orders $q \in Q_i$;*
- *$q^\star$ the maximum of all the q_i's for $i \in \mathcal{K}$,*

and let us assume the following conditions:

- $v_{i,q_i} = 0$ *for all* $i \in \mathcal{K}$*;*
- *there exists a unique* $i \in \mathcal{K}$*, denoted by* $i^\star$*, such that* $q^\star \in Q_i$*;*
- $q^\star + i^\star > \kappa$ *and* $a_{i^\star,q^\star}(0,0) \neq 0$*.*

Then, $s_0 = q^\star/(\kappa - i^\star) - 1$ *and the following assertions hold for any direction* $\arg(t) = \theta \in \mathbb{R}/2\pi\mathbb{Z}$ *issuing from* 0*:*

(1) $\widetilde{u}(t,x)$ *is* k_0*-summable, with* $k_0 = 1/s_0$*, in the direction* θ *if and only if the inhomogeneity* $\widetilde{f}(t,x)$ *and the formal power series* $\widetilde{u}_{*,\nu}(t) \in \mathbb{C}[\![t]\!]$ *for* $\nu = 0, \ldots, q^\star - 1$ *are* k_0*-summable in the direction* θ*.*

(2) *Moreover, the* k_0*-sum* $u(t,x)$*, if any exists, satisfies* (1.7) *in which* $\widetilde{f}(t,x)$ *is replaced by its* k_0*-sum* $f(t,x)$ *in the direction* θ*.*

In the present work, we propose to extend these two results to the general System (1.1). Section 2 is devoted to the study of the Gevrey regularity of its formal solution $\widetilde{u}(t,x)$. After briefly recalling the definition and some basic properties of the s-Gevrey formal power series in $\mathcal{O}(D_{\rho_1})[\![t]\!]$ which are needed, we introduce a convenient critical value $s_0 \geqslant 0$ associated with System (1.1) and we prove that $\widetilde{u}(t,x)$ and the inhomogeneity $\widetilde{f}(t,x)$ of System (1.1) are simultaneously s-Gevrey for any $s \geqslant s_0$, and that $\widetilde{u}(t,x)$ is generically s_0-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_0$ (Theorem 2.5). In the latter case, we give in particular an explicit example in which $\widetilde{u}(t,x)$ is s'-Gevrey for no $s' < s_0$, that is $\widetilde{u}(t,x)$ is exactly s_0-Gevrey. In Section 3, we investigate, under some convenient assumptions (see Hypothesis 3.5), the k_0-summability of $\widetilde{u}(t,x)$ with $k_0 = 1/s_0$, and we give a necessary and sufficient condition under which the formal solution $\widetilde{u}(t,x)$ is k_0-summable in a given direction $\arg(t) = \theta$ (Theorem 3.11). In Section 5, we present some technical results (Nagumo norms, matrices with analytic coefficients and generalized binomial coefficients) which are needed for the proofs of our two main results. This section can also be read independently of the rest of the article, so as not to burden the main proofs.

2. Gevrey regularity

All along the article, we consider t as the variable and x as a parameter. Thereby, to define the notion of *Gevrey formal power series in* $\mathcal{O}(D_{\rho_1})[\![t]\!]$, one extends the classical notion of *Gevrey formal power series in* $\mathbb{C}[\![t]\!]$ to families parametrized by x in requiring similar conditions, the estimates being however uniform with respect to x [51, Chapter 3]. Note that, doing so, any formal power series in $\mathcal{O}(D_{\rho_1})[\![t]\!]$ can also be seen as a formal power series in t with coefficients in a convenient Banach space defined as the space of functions that are holomorphic on a convenient disc D_r and continuous up to its boundary, equipped with the usual supremum norm. For a general study of the formal power series with coefficients in a Banach space, we refer for instance to [1].

Before stating our main result (see Theorem 2.5 below), let us first recall for the convenience of the reader some definitions and basic properties about the Gevrey formal power series in $\mathcal{O}(D_{\rho_1})[\![t]\!]$, which are needed in the sequel.

2.1. Gevrey formal power series

Definition 2.1. — *Let $s \geqslant 0$. A formal power series*

$$\widetilde{u}(t,x) = \sum_{j\geqslant 0} u_{j,*}(x)\frac{t^j}{j!} \in \mathcal{O}(D_{\rho_1})[\![t]\!]$$

is said to be Gevrey of order s *(in short,* s-Gevrey*) if there exist three positive constants $0 < r \leqslant \rho_1$, $C > 0$ and $K > 0$ such that the inequalities*

$$|u_{j,*}(x)| \leqslant CK^j\Gamma(1+(s+1)j)$$

hold for all $x \in D_r$ and all $j \geqslant 0$.

In other words, Definition 2.1 means that $\widetilde{u}(t,x)$ is s-Gevrey in t, uniformly in x on a neighborhood of $x = 0$ in $\mathbb{C}$.

Notation 2.2. — We denote by $\mathcal{O}(D_{\rho_1})[\![t]\!]_s$ the set of all the formal series in $\mathcal{O}(D_{\rho_1})[\![t]\!]$ which are s-Gevrey.

Observe that the set $\mathbb{C}\{t,x\}$ of germs of analytic functions at the origin of $\mathbb{C}^2$ coincides with the union $\bigcup_{\rho_1>0}\mathcal{O}(D_{\rho_1})[\![t]\!]_0$. In particular, any element of $\mathcal{O}(D_{\rho_1})[\![t]\!]_0$ is convergent and $\mathbb{C}\{t,x\} \cap \mathcal{O}(D_{\rho_1})[\![t]\!] = \mathcal{O}(D_{\rho_1})[\![t]\!]_0$.

Observe also that the sets $\mathcal{O}(D_{\rho_1})[\![t]\!]_s$ are filtered as follows

$$\mathcal{O}(D_{\rho_1})[\![t]\!]_0 \subset \mathcal{O}(D_{\rho_1})[\![t]\!]_s \subset \mathcal{O}(D_{\rho_1})[\![t]\!]_{s'} \subset \mathcal{O}(D_{\rho_1})[\![t]\!]$$

for all s and s' satisfying $0 < s < s' < +\infty$.

Following Proposition 2.3 specifies the algebraic structure of $\mathcal{O}(D_{\rho_1})[\![t]\!]_s$.

Proposition 2.3. — *Let $s \geqslant 0$. The set $\mathcal{O}(D_{\rho_1})[\![t]\!]_s$ endowed with the usual algebraic operations and the usual derivations ∂_t and ∂_x is a $\mathbb{C}$-differential algebra.*

Proof. — See [51, Proposition 3.4]. □

2.2. Main result and applications

Let us first extend Definition 2.1 to the elements of $\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$.

Definition 2.4. — *A $n \times 1$-column matrix $\widetilde{u}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$ is said to be s-*Gevrey *if its components $\widetilde{u}^{[k]}(t,x) \in \mathcal{O}(D_{\rho_1})[[t]]$ are s-Gevrey for all $k = 1,\dots,n$.*

We are now able to state the main result in view in this section.

Theorem 2.5 (Gevrey Regularity Theorem). — *For all $i \in \mathcal{K}$ and all $q \in Q_i$, let us denote by $v_{i,q}$ the nonnegative integer defined as the smallest valuation of all the nonzero terms of $A_{i,q}(t,x)$:*

$$v_{i,q} = \min_{k\in\{1,\dots,n\}}\left(\min_{\ell\in\mathcal{L}_k(A_{i,q})} v_{i,q}^{[k,\ell]}\right).$$

Let s_0 be the nonnegative rational number defined by

$$s_0 = \max\left(0, \max_{i\in\mathcal{K}}\left(\max_{q\in Q_i}\frac{q-\kappa+i}{v_{i,q}+\kappa-i}\right)\right).$$

Then,

(1) *$\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ are simultaneously s-Gevrey for any $s \geqslant s_0$;*
(2) *$\widetilde{u}(t,x)$ is generically s_0-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_0$.*

Remark 2.6. — As in Proposition 1.6, we observe again that the Gevrey regularity of $\widetilde{u}(t,x)$ follows a noteworthy dichotomy, governed by the critical value s_0, between the Gevrey regularity of the inhomogeneity $\widetilde{f}(t,x)$ and the one induced by the structure of System (1.1), that is by the linear operator

$$\Delta = \partial_t^\kappa - \sum_{i\in\mathcal{K}}\sum_{q\in Q_i} A_{i,q}(t,x)\partial_t^i\partial_x^q.$$

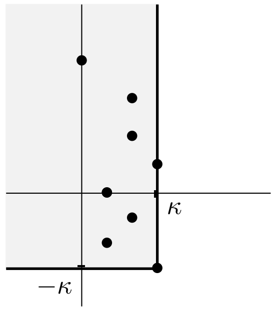
Remark 2.7. — Let us denote by $C(a,b)$ the domain

$$C(a,b) = \{(y_1,y_2)\in\mathbb{R}^2 \,:\, y_1 \leqslant a \text{ and } y_2 \geqslant b\}$$

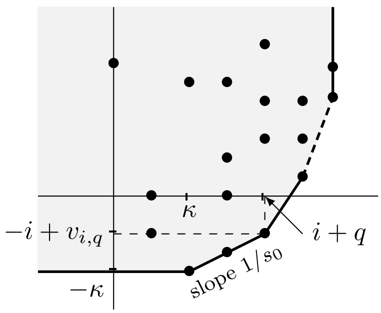
for any $a,b\in\mathbb{R}$. Let us also denote by $\mathcal{N}_t(\Delta)$ the convex hull of the domain

$$C(\kappa,-\kappa) \cup \bigcup_{i\in\mathcal{K}}\bigcup_{q\in Q_i} C(i+q,-i+v_{i,q}).$$

When $i+q \leqslant \kappa$ for all $i\in\mathcal{K}$ and all $q\in Q_i$, the domain $\mathcal{N}_t(\Delta)$ is reduced to the domain $C(\kappa,-\kappa)$ and has no positive slope (see Figure 2.1a). On the other hand, if $i+q > \kappa$ for some $i\in\mathcal{K}$ and $q\in Q_i$, the domain $\mathcal{N}_t(\Delta)$ has at least one positive slope (see Figure 2.1b) and the critical value s_0 is the inverse of the smallest of these slopes. By analogy with the linear partial differential equations, we refer the domain $\mathcal{N}_t(\Delta)$ as the *associated Newton polygon of System* (1.1) *at* $t = 0$.



(a) Case $i+q \leqslant \kappa$
for all $i \in \mathcal{K}$ and all $q \in Q_i$



(b) Case $i+q > \kappa$
for some $i \in \mathcal{K}$ and $q \in Q_i$

Figure 2.1. The domain $\mathcal{N}_t(\Delta)$

Before we move on to the proof of both parts of Theorem 2.5, let us first formulate two corollaries that deal with convergent inhomogeneity $\widetilde{f}(t,x)$ as well as some additional examples.

Corollary 2.8 (Necessary and sufficient condition for convergence). *Assume that* $i+q \leqslant \kappa$ *for all* $i \in \mathcal{K}$ *and all* $q \in Q_i$*, that is* $s_0 = 0$*. Then, the formal solution* $\widetilde{u}(t,x)$ *is convergent if and only if the inhomogeneity* $\widetilde{f}(t,x)$ *is convergent.*

Remark 2.9. — When $i+q \leqslant \kappa$ for all $i \in \mathcal{K}$ and all $q \in Q_i$, and $\widetilde{f}(t,x)$ is convergent, we find here the classical Cauchy–Kovalevskaya Theorem.

Corollary 2.10 (Maillet–Ramis type Theorem). — *Assume that the inhomogeneity* $\widetilde{f}(t,x)$ *is convergent. Then, the formal solution* $\widetilde{u}(t,x)$ *is either convergent or* s_0*-Gevrey for some positive rational number* $s_0 > 0$ *defined as the inverse of the smallest positive slope of the associated Newton polygon.*

As said at the beginning of the introduction, System (1.1) is fundamental in many physical, chemical, biological, and ecological problems. We present below two examples arising from these different problems.

Example 2.11. — Let us first consider the acoustic equations (1.2) and System (1.3) modelling the quasistationary electric oscillations. From Theorem 2.5, the value s_0 attached to these two systems is $s_0 = 0$. Then, since their inhomogeneities are convergent at the origin of $\mathbb{C}^2$, Corollary 2.8 tells us that their formal solutions are also convergent at the origin of $\mathbb{C}^2$. In particular, they define analytic solutions in a neighborhood of $0 \in \mathbb{C}^2$, whose explicit forms are given by relations (1.6).

Example 2.12. — Let us now consider the reaction-diffusion system (1.4). Since the derivative order in x is greater than the derivative order in t, System (1.4) is a non-Kovalevskian system and its formal solution $\widetilde{u}(t,x)$ is in general[(4)] divergent. More precisely, Theorem 2.5 tells us that $\widetilde{u}(t,x)$ is 1-Gevrey. Indeed, the value s_0 attached to System (1.4) is $s_0 = 1$ and the inhomogeneity of System (1.4) is zero.

Let us now move on to the proof of Theorem 2.5. We start with the first and most technical point.

2.3. Proof of the first point of Theorem 2.5

According to Definition 2.4, relations (1.5) and Proposition 2.3, it is clear that the following implication holds:

$$\begin{aligned}\widetilde{u}(t,x) \text{ is } s\text{-Gevrey} &\Longrightarrow \widetilde{u}^{[k]}(t,x) \in \mathcal{O}(D_{\rho_1})[\![t]\!]_s \text{ for all } k=1,\ldots,n\\ &\Longrightarrow \widetilde{f}^{[k]}(t,x) \in \mathcal{O}(D_{\rho_1})[\![t]\!]_s \text{ for all } k=1,\ldots,n\\ &\Longrightarrow \widetilde{f}(t,x) \text{ is } s\text{-Gevrey}.\end{aligned}$$

Reciprocally, let us fix $s \geqslant s_0$ and let us assume that the inhomogeneity $\widetilde{f}(t,x)$ is s-Gevrey. By definition (see Definitions 2.1 and 2.4), there exist three positive constants $0 < r < \rho_1$, $C > 0$ and $K > 0$ such that the following inequality holds for all $j \geqslant 0$, all $x \in D_r$ and all $k = 1,\ldots,n$:

$$\left|f_{j,*}^{[k]}(x)\right| \leqslant CK^j\Gamma(1+(s+1)j). \tag{2.1}$$

We must prove that the coefficients $u_{j,*}^{[k]}(x)$ of the formal series $\widetilde{u}^{[k]}(t,x)$ satisfy similar inequalities for all $k = 1,\ldots,n$. To do that, we shall proceed in a classical way based on the Nagumo norms and on a technique of majorant series, as already developed, for instance, in [3, 41, 43, 45]. However, as we shall see below, our calculations to determine a convenient majorant series appear to be much more complicated because we work now with systems and no longer with equations.

(4) We write "in general" since, for certain initial data $\varphi_j(x)$, $\widetilde{u}(t,x)$ can be convergent (it is for instance the case when all the φ_j are zero).

2.3.1. First step: some preliminary inequalities

From relations (1.6), we get, for all $j \geqslant 0$, all $x \in D_r$ and all $k = 1, \ldots, n$, the recurrence relations

$$\frac{u_{j+\kappa,*}^{[k]}(x)}{\Gamma(1+(s+1)(j+\kappa))} = \frac{f_{j,*}^{[k]}(x)}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell\in\mathcal{L}_k(A_{i,q})} \sum_{j_0+j_1=j-v_{i,q}^{[k,\ell]}} \frac{j!}{j_0!j_1!} \frac{a_{i,q;j_0,*}^{[k,\ell]}(x)\partial_x^q u_{j_1+i,*}^{[\ell]}}{\Gamma(1+(s+1)(j+\kappa))}$$

starting with the initial condition $u_{j,*}^{[k]}(x) = \varphi_j^{[k]}(x)$ for $j = 0, \ldots, \kappa-1$.

Let us now apply the Nagumo norms (see their definition and properties in Section 5.1) of indices $((j+\kappa)\sigma, r)$ with

$$\sigma = \max_{i\in\mathcal{K}} \left(\max_{q\in Q_i} q \right).$$

Notice that $\sigma \geqslant 1$ thanks to our assumption on the sets Q_i. From Property 1 of Proposition 5.2, we first receive for all $j \geqslant 0$ and all $k = 1, \ldots, n$ the following inequalities:

$$\frac{\left\|u_{j+\kappa,*}^{[k]}\right\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{\left\|f_{j,*}^{[k]}\right\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell\in\mathcal{L}_k(A_{i,q})} \sum_{j_0+j_1=j-v_{i,q}^{[k,\ell]}} \frac{j!}{j_0!j_1!} \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\partial_x^q u_{j_1+i,*}^{[\ell]}\right\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))}.$$

Writing then $(j+\kappa)\sigma$ in the form

$$(j+\kappa)\sigma = j_0\sigma + (j_1+i)\sigma + q + \sigma'_{i,q,k,\ell}$$

with

$$\sigma'_{i,q,k,\ell} = \left(\kappa - i + v_{i,q}^{[k,\ell]}\right)\sigma - q \geqslant 0$$

we finally get, from Properties 4-5 of Proposition 5.2, the following inequalities for all $j \geqslant 0$ and all $k = 1, \ldots, n$:

$$\frac{\left\|u_{j+\kappa,*}^{[k]}\right\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{\left\|f_{j,*}^{[k]}\right\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell\in\mathcal{L}_k(A_{i,q})} \sum_{j_0+j_1=j-v_{i,q}^{[k,\ell]}} C_{i,q,k,\ell,j} \frac{\left\|u_{j_1+i,*}^{[\ell]}\right\|_{(j_1+i)\sigma,r}}{\Gamma(1+(s+1)(j_1+i))}, \tag{2.2}$$

where $C_{i,q,k,\ell,j}$ is the positive real number defined for all $j \geqslant v_{i,q}^{[k,\ell]}$ by

$$C_{i,q,k,\ell,j} = \frac{\binom{j-v_{i,q}^{[k,\ell]}}{j_0}}{\binom{(s+1)(j-v_{i,q}^{[k,\ell]})}{(s+1)j_0}} \times \|1\|_{\sigma'_{i,q,k,\ell},r}(\sigma e)^q$$
$$\times \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\right\|_{j_0\sigma,r}}{\Gamma(1+(s+1)j_0)} \times \prod_{h=1}^{q}\left(j_1+i+\frac{h}{\sigma}\right)$$
$$\times \frac{j!\Gamma(1+(s+1)(j_1+i))\Gamma(1+(s+1)(j-v_{i,q}^{[k,\ell]}))}{(j-v_{i,q}^{[k,\ell]})!\Gamma(1+(s+1)j_1)\Gamma(1+(s+1)(j+\kappa))},$$

and where $\binom{j-v_{i,q}^{[k,\ell]}}{j_0}$ and $\binom{(s+1)(j-v_{i,q}^{[k,\ell]})}{(s+1)j_0}$ stand respectively for the binomial coefficient

$$\binom{j-v_{i,q}^{[k,\ell]}}{j_0} = \frac{(j-v_{i,q}^{[k,\ell]})!}{j_0!(j-v_{i,q}^{[k,\ell]}-j_0)!}$$

and for the generalized binomial coefficient

$$\binom{(s+1)(j-v_{i,q}^{[k,\ell]})}{(s+1)j_0} = \frac{\Gamma(1+(s+1)(j-v_{i,q}^{[k,\ell]}))}{\Gamma(1+(s+1)j_0)\Gamma(1+(s+1)(j-v_{i,q}^{[k,\ell]}-j_0))}.$$

2.3.2. Second step: bound of $C_{i,q,k,\ell,j}$

Let us first start with two technical lemmas.

Lemma 2.13. — *The following inequality holds for all $j \geqslant v_{i,q}^{[k,\ell]}$ and all $j_0 \in \{0,\dots,j-v_{i,q}^{[k,\ell]}\}$:*

$$\frac{\binom{j-v_{i,q}^{[k,\ell]}}{j_0}}{\binom{(s+1)(j-v_{i,q}^{[k,\ell]})}{(s+1)j_0}} \leqslant 1.$$

Proof. — Applying succesively the Vandermonde's Inequality (Proposition 5.5) and Proposition 5.7, we get

$$\binom{(s+1)(j-v_{i,q}^{[k,\ell]})}{(s+1)j_0} \geqslant \binom{s(j-v_{i,q}^{[k,\ell]})}{sj_0}\binom{j-v_{i,q}^{[k,\ell]}}{j_0} \geqslant \binom{j-v_{i,q}^{[k,\ell]}}{j_0}$$

for all $j \geqslant v_{i,q}^{[k,\ell]}$ and all $j_0 \in \{0,\dots,j-v_{i,q}^{[k,\ell]}\}$, and Lemma 2.13 follows. □

Lemma 2.14. — *There exist two positive constants $c_1, C_1 > 0$ such that the following estimate holds for all $j \geqslant 0$:*

$$c_1(j+1)^{s+1} \leqslant \frac{\Gamma(1+(s+1)(j+1))}{\Gamma(1+(s+1)j)} \leqslant C_1(j+1)^{s+1}.$$

Proof. — From the Stirling's Formula, we have the equivalence

$$\frac{\Gamma(1+(s+1)(j+1))}{\Gamma(1+(s+1)j)} \underset{j\to+\infty}{\sim} (s+1)^{s+1}(j+1)^{s+1}$$

which proves Lemma 2.14. $\square$

Let us now apply Lemmas 2.13 and 2.14: for all $j \geqslant v_{i,q}^{[k,\ell]}$ and all $j_0, j_1 \geqslant 0$ such that $j_0 + j_1 = j - v_{i,q}^{[k,\ell]}$, we have

$$\begin{aligned} C_{i,q,k,\ell,j} \leqslant \|1\|_{\sigma'_{i,q,k,\ell},r}(\sigma e)^q \times \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\right\|_{j_0\sigma,r}}{\Gamma(1+(s+1)j_0)} \times \prod_{h=1}^{q}\left(j_1+i+\frac{h}{\sigma}\right) \\ \times \frac{(j-v_{i,q}^{[k,\ell]}+1)\dots j \times C_1^i((j_1+1)\dots(j_1+i))^{s+1}}{c_1^{\kappa+v_{i,q}^{[k,\ell]}}((j-v_{i,q}^{[k,\ell]}+1)\dots(j+\kappa))^{s+1}}; \end{aligned}$$

hence,

$$\begin{aligned} C_{i,q,k,\ell,j} \leqslant \|1\|_{\sigma'_{i,q,k,\ell},r}(\sigma e)^q \frac{C_1^i}{c_1^{\kappa+v_{i,q}^{[k,\ell]}}} \times \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\right\|_{j_0\sigma,r}}{\Gamma(1+(s+1)j_0)} \\ \times \frac{(j-v_{i,q}^{[k,\ell]}+1)\dots j((j+1)\dots(j+i))^{s+1}(j+i+1)^q}{((j-v_{i,q}^{[k,\ell]}+1)\dots(j+\kappa))^{s+1}} \end{aligned}$$

since $j_1 \leqslant j$ and $h \leqslant q \leqslant \sigma$, and therefore

$$\begin{aligned} C_{i,q,k,\ell,j} &\leqslant \|1\|_{\sigma'_{i,q,k,\ell},r}(\sigma e)^q \frac{C_1^i}{c_1^{\kappa+v_{i,q}^{[k,\ell]}}} \times \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\right\|_{j_0\sigma,r}}{\Gamma(1+(s+1)j_0)} \\ &\quad \times \frac{(j-v_{i,q}^{[k,\ell]}+1)\dots j}{((j-v_{i,q}^{[k,\ell]}+1)\dots j)^{s+1}} \frac{(j+i+1)^q}{((j+i+1)\dots(j+\kappa))^{s+1}} \\ &\leqslant \|1\|_{\sigma'_{i,q,k,\ell},r}(\sigma e)^q \frac{C_1^i}{c_1^{\kappa+v_{i,q}^{[k,\ell]}}} \times \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\right\|_{j_0\sigma,r}}{\Gamma(1+(s+1)j_0)} \\ &\quad \times \frac{(j-v_{i,q}^{[k,\ell]}+1)\dots j}{((j-v_{i,q}^{[k,\ell]}+1)\dots j)^{s+1}} \frac{(j+i+1)^q}{(j+i+1)^{(s+1)(\kappa-i)}} \end{aligned}$$

since $\kappa > i$. Of course, we use here the classical convention that the product $(j - v_{i,q}^{[k,\ell]} + 1)\dots j$ is 1 as soon as $v_{i,q}^{[k,\ell]} = 0$. Observe also that all the calculations above remain valid when $i = 0$.

Lemma 2.15. — *Assume $v_{i,q}^{[k,\ell]} \geqslant 1$. Then, the following inequality holds for all $j \geqslant v_{i,q}^{[k,\ell]}$ and all $v \in \{0,\dots,v_{i,q}^{[k,\ell]}-1\}$:*

$$j - v \geqslant \frac{j+i+1}{2^{v+1}(i+1)}.$$

Proof. — We proceed by recursion on v.

For $v=0$, the hypotheses $j \geqslant v_{i,q}^{[k,\ell]} \geqslant 1$ and $i \geqslant 0$ implies

$$\frac{j+i+1}{2(i+1)} = \frac{j}{2(i+1)} + \frac{1}{2} \leqslant \frac{j}{2} + \frac{j}{2} = j;$$

hence, the inequality (and also Lemma 2.15 if $v_{i,q}^{[k,\ell]} = 1$).

Let us now assume $v_{i,q}^{[k,\ell]} \geqslant 2$ and let us suppose that the inequality holds for all $v' \in \{0,\dots,v\}$ for a certain $v \in \{0,\dots,v_{i,q}^{[k,\ell]}-2\}$. We must prove it for $v+1$. Applying our assumptions, we get

$$j-(v+1) = (j-1)-v \geqslant \frac{j+i}{2^{v+1}(i+1)}.$$

The sought inequality follows then from the relation

$$j+i \geqslant \frac{j+i+1}{2},$$

which ends the proof. □

Applying then Lemma 2.15, we deduce that

$$C_{i,q,k,\ell,j} \leqslant \|1\|_{\sigma'_{i,q,k,\ell},r} (\sigma e)^q \frac{C_1^i}{c_1^{\kappa+v_{i,q}^{[k,\ell]}}} \times \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\right\|_{j_0\sigma,r}}{\Gamma(1+(s+1)j_0)}$$
$$\times\, 2^{(s+1)\frac{v_{i,q}^{[k,\ell]}(v_{i,q}^{[k,\ell]}+1)}{2}} (i+1)^{(s+1)v_{i,q}^{[k,\ell]}} (j+i+1)^{q+v_{i,q}^{[k,\ell]}-(s+1)(v_{i,q}^{[k,\ell]}+\kappa-i)}$$

for all $j \geqslant v_{i,q}^{[k,\ell]}$ and all $j_0, j_1 \geqslant 0$ such that $j_0 + j_1 = j - v_{i,q}^{[k,\ell]}$. Observe here that all the inequalities on $C_{i,q,k,\ell,j}$ written above remain valid when $v_{i,q}^{[k,\ell]} = 0$.

Lemma 2.16. — *Let $s \geqslant s_0$. Then, the following inequality holds for all $i \in \mathcal{K}$, all $q \in Q_i$, all $k \in \{1,\dots,n\}$ and all $\ell \in \mathcal{L}_k(A_{i,q})$:*

$$(s+1)(v_{i,q}^{[k,\ell]} + \kappa - i) \geqslant q + v_{i,q}^{[k,\ell]}$$

Proof. — Let us first assume $q > \kappa - i$. Then, by definition of s_0 and $v_{i,q}$ (see Theorem 2.5), we easily get the inequality

$$s \geqslant s_0 \geqslant \frac{q-\kappa+i}{v_{i,q}+\kappa-i} \geqslant \frac{q-\kappa+i}{v_{i,q}^{[k,\ell]}+\kappa-i}.$$

When $q \leqslant \kappa - i$, we also have

$$s \geqslant s_0 \geqslant 0 \geqslant \frac{q-\kappa+i}{v_{i,q}^{[k,\ell]}+\kappa-i}.$$

Consequently, the inequality

$$s \geqslant \frac{q-\kappa+i}{v_{i,q}^{[k,\ell]}+\kappa-i} \tag{2.3}$$

is always true, and Lemma 2.16 follows by adding "+1" to the both sides of (2.3) and by multiplying by $v_{i,q}^{[k,\ell]}+\kappa-i>0$. □

Let us set

$$C_2 = \max_{\substack{i\in\mathcal{K}\\ q\in Q_i}}\left(\max_{\substack{k\in\{1,\ldots,n\}\\ \ell\in\mathcal{L}_k(A_{i,q})}}\left(\frac{2^{(s+1)\frac{v_{i,q}^{[k,\ell]}(v_{i,q}^{[k,\ell]}+1)}{2}}\left\|1\right\|_{\sigma'_{i,q,k,\ell},r}(\sigma e)^q C_1^i (i+1)^{(s+1)v_{i,q}^{[k,\ell]}}}{C_1^{\kappa+v_{i,q}^{[k,\ell]}}}\right)\right).$$

Then, Lemma 2.16 implies the final inequality

$$C_{i,q,k,\ell,j} \leqslant C_2 \frac{\left\|a_{i,q;j_0,*}^{[k,\ell]}\right\|_{j_0\sigma,r}}{\Gamma(1+(s+1)j_0)} \tag{2.4}$$

for all $j \geqslant v_{i,q}^{[k,\ell]}$ and all $j_0, j_1 \geqslant 0$ such that $j_0+j_1 = j - v_{i,q}^{[k,\ell]}$.

2.3.3. Third step: the majorant series method

We shall now bound the Nagumo norms $\left\|u_{j,*}^{[k]}\right\|_{j\sigma,r}$ for any $j \geqslant 0$ and any $k = 1,\ldots,n$ by using the classical majorant series method, which here consists of determining formal power series

$$v^{[k]}(X) = \sum_{j\geqslant 0} v_j^{[k]} X^j$$

which are convergent at the origin of $\mathbb{C}$, and whose the coefficients $v_j^{[k]}$ satisfy the following condition for all $j \geqslant 0$ and all $k = 1,\ldots,n$:

$$0 \leqslant \frac{\left\|u_{j,*}^{[k]}\right\|_{j\sigma,r}}{\Gamma(1+(s+1)j)} \leqslant v_j^{[k]}. \tag{2.5}$$

First of all, let us set

$$g_{j,s}^{[k]} = \frac{\left\|f_{j,*}^{[k]}\right\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))} \quad \text{and} \quad \alpha_{j,i,q,s}^{[k,\ell]} = \frac{\left\|a_{i,q;j,*}^{[k,\ell]}\right\|_{j\sigma,r}}{\Gamma(1+(s+1)j)}.$$

Lemma 2.17. — *There exist two positive constants $B', C' > 0$ such that the following inequalities hold for all $j \geqslant 0$, all $i \in \mathcal{K}$, all $q \in Q_i$, all $k \in \{1, \dots, n\}$ and all $\ell \in \mathcal{L}_k(A_{i,q})$:*

$$g_{j,s}^{[k]} \leqslant C'B'^j \quad \textit{and} \quad \alpha_{j,i,q,s}^{[k,\ell]} \leqslant C'B'^j.$$

Proof. — According to our assumption on the coefficients $f_{j,*}^{[k]}(x)$ (see inequality (2.1)), we first deduce from Properties 3-4 of Proposition 5.2 that the following inequality holds for all $j \geqslant 0$ and all $k = 1, \dots, n$:

$$0 \leqslant g_{j,s}^{[k]} \leqslant \frac{\left\|f_{j,*}^{[k]}\right\|_{0,r}\|1\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{CK^j\Gamma(1+(s+1)j)r^{(j+\kappa)\sigma}}{\Gamma(1+(s+1)(j+\kappa))}.$$

The sought inequality on $g_{j,s}^{[k]}$ follows then from the fact that the quotient

$$\frac{\Gamma(1+(s+1)j)}{\Gamma(1+(s+1)(j+\kappa))}$$

is bounded (see Lemma 2.14).

The inequality on $\alpha_{j,i,q,s}^{[k,\ell]}$, due to the analyticity on $D_{\rho_0} \times D_{\rho_1}$ of the function $a_{i,q}^{[k,\ell]}(t,x)$, is proved in a similar way and is left to the reader. □

We are now able to bound the Nagumo norms $\left\|u_{j,*}^{[k]}\right\|_{j\sigma,r}$. Recall that, due to inequalities (2.2) and (2.4), the following holds for all $j \geqslant 0$ and all $k = 1, \dots, n$:

$$\frac{\left\|u_{j+\kappa,*}^{[k]}\right\|_{(j+\kappa)\sigma,r}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant g_{j,s}^{[k]} + C_2 \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell\in\mathcal{L}_k(A_{i,q})} \sum_{j_0+j_1=j-v_{i,q}^{[k,\ell]}} \alpha_{j_0,i,q,s}^{[k,\ell]} \frac{\left\|u_{j_1+i,*}^{[\ell]}\right\|_{(j_1+i)\sigma,r}}{\Gamma(1+(s+1)(j_1+i))}, \tag{2.6}$$

where C_2 is a positive constant independent of i, q, k and ℓ.

Let us now introduce, for any $k = 1, \dots, n$, the formal power series

$$v^{[k]}(X) = \sum_{j\geqslant 0} v_j^{[k]} X^j,$$

the coefficients of which are recursively determined for all $j \geqslant 0$ by the relations

$$v_{j+\kappa}^{[k]} = g_{j,s}^{[k]} + C_2 \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell\in\mathcal{L}_k(A_{i,q})} \sum_{j_0+j_1=j-v_{i,q}^{[k,\ell]}+i} \alpha_{j_0,i,q,s}^{[k,\ell]} v_{j_1}^{[\ell]},$$

starting with the initial conditions

$$v_0^{[k]} = \left\|u_{0,*}^{[k]}\right\|_{0,r} = \left\|\varphi_0^{[k]}\right\|_{0,r}, \text{ and, for } j = 1, \ldots, \kappa - 1 \text{ (if } \kappa \geqslant 2\text{):}$$

$$\begin{aligned} v_j^{[k]} &= \frac{\left\|u_{j,*}^{[k]}\right\|_{j\sigma,r}}{\Gamma(1+(s+1)j)} + \sum_{(i,q,\ell)\in V_j^{[k]}} \sum_{j_0+j_1=j-\kappa-v_{i,q}^{[k,\ell]}+i} \alpha_{j_0,i,q,s}^{[k,\ell]} v_{j_1}^{[\ell]} \\ &= \frac{\left\|\varphi_j^{[k]}\right\|_{j\sigma,r}}{\Gamma(1+(s+1)j)} + \sum_{(i,q,\ell)\in V_j^{[k]}} \sum_{j_0+j_1=j-\kappa-v_{i,q}^{[k,\ell]}+i} \alpha_{j_0,i,q,s}^{[k,\ell]} v_{j_1}^{[\ell]} \end{aligned}$$

where

$$V_j^{[k]} = \{(i,q,\ell) \in \mathcal{K} \times Q_i \times \mathcal{L}_k(A_{i,q}); j - \kappa - v_{i,q}^{[k,\ell]} + i \geqslant 0\}.$$

Observe that the condition "$\kappa > i$ for all $i \in \mathcal{K}$" implies $j - \kappa - v_{i,q}^{[k,\ell]} + i < j$; hence the initial conditions on the $v_j^{[k]}$'s with $j = 1, \ldots, \kappa - 1$ make sense for all $k = 1, \ldots, n$. Observe also that the set $V_j^{[k]}$ may be empty.

Reasoning by induction on $j \geqslant 0$, one can easily check that the inequality (2.5) holds for all $j \geqslant 0$ and all $k = 1, \ldots, n$.

On the other hand, let us define the $n \times n$-matrix $\alpha(X) \in \mathcal{M}_n(\mathbb{C}[\![X]\!])$ by

$$\alpha(X) = C_2 \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \alpha_{i,q}(X)$$

with $\alpha_{i,q}(X) \in \mathcal{M}_n(\mathbb{C}[\![X]\!])$ such that

$$\alpha_{i,q}^{[k,\ell]}(X) = \begin{cases} X^{\kappa-i+v_{i,q}^{[k,\ell]}} \displaystyle\sum_{j\geqslant 0} \alpha_{j,i,q,s}^{[k,\ell]} X^j & \text{if } \ell \in \mathcal{L}_k(A_{i,q}) \\ 0 & \text{otherwise} \end{cases}$$

for all $k, \ell \in \{1, \ldots, n\}$. Let us also define the $n \times 1$-column matrix $h(X) \in \mathcal{M}_{n,1}(\mathbb{C}[\![X]\!])$ by

$$h^{[k]}(X) = \sum_{j=0}^{\kappa-1} \frac{\left\|\varphi_j^{[k]}\right\|_{j\sigma,r}}{\Gamma(1+(s+1)j)} X^j + X^\kappa \sum_{j\geqslant 0} g_{j,s}^{[k]} X^j$$

for all $k \in \{1, \ldots, n\}$. Note that, due to Lemma 2.17, the coefficients of $\alpha(X)$ and $h(X)$ are all analytic at the origin of $\mathbb{C}$.

Then, the $n \times 1$-column matrix

$$v(X) = \begin{pmatrix} v^{[1]}(X) \\ \vdots \\ v^{[n]}(X) \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{C}[\![X]\!])$$

is the unique solution of this type of the system

$$(I_n - \alpha(X))v(X) = h(X),$$

where I_n stands for the $n \times n$-identity matrix. Since $\alpha(0) = 0$, we deduce from Proposition 5.4 that the matrix $I_n - \alpha(X)$ is invertible and that its inverse is a $n \times n$-matrix with analytic coefficients at the origin of $\mathbb{C}$. Consequently, the coefficients $v^{[k]}(X) \in \mathbb{C}[\![X]\!]$ of $v(X)$ are also analytic at the origin of $\mathbb{C}$ for all $k = 1, \ldots, n$. This brings then us to the following result which allows to bound the Nagumo norms $\left\|u_{j,*}^{[k]}\right\|_{j\sigma,r}$ for all $j \geqslant 0$ and all $k = 1, \ldots, n$.

Lemma 2.18. — *There exist two positive constants $C'', K'' > 0$ such that the following inequaliy holds for all $j \geqslant 0$ and all $k = 1, \ldots, n$:*

$$\left\|u_{j,*}^{[k]}\right\|_{j\sigma,r} \leqslant C'' K''^j \Gamma(1 + (s+1)j).$$

We are now able to conclude the proof of the first point of Theorem 2.5.

2.3.4. Fourth step: conclusion

We must prove that the sup-norm of the $u_{j,*}^{[k]}(x)$ has estimates similar to the ones we arrived at for the norms $\left\|u_{j,*}^{[k]}\right\|_{j\sigma_s,r}$ (see Lemma 2.18). To this end, we proceed by shrinking the disc D_r. Let $0 < r' < r$. Then, for all $j \geqslant 0$, all $k = 1, \ldots, n$ and all $x \in D_{r'}$, we have

$$\left|u_{j,*}^{[k]}(x)\right| = \left|u_{j,*}^{[k]}(x) d_r(x)^{j\sigma} \frac{1}{d_r(x)^{j\sigma}}\right| \leqslant \frac{\left|u_{j,*}^{[k]}(x) d_r(x)^{j\sigma}\right|}{(r-r')^{j\sigma}} \leqslant \frac{\left\|u_{j,*}^{[k]}\right\|_{j\sigma,r}}{(r-r')^{j\sigma}}$$

and, consequently,

$$\left|u_{j,*}^{[k]}(x)\right| \leqslant C'' \left(\frac{K''}{(r-r')^{\sigma}}\right)^j \Gamma(1 + (s+1)j)$$

by applying Lemma 2.18. This proves that the formal series $\widetilde{u}^{[k]}(t,x)$ are s-Gevrey for all $k = 1, \ldots, n$, which ends the proof of the first point of Theorem 2.5 thanks to Definition 2.4.

2.4. Proof of the second point of Theorem 2.5

In this section, we assume $s_0 > 0$, that is there exists at least one $i \in \mathcal{K}$ and one $q \in Q_i$ such that $q + i > \kappa$, and we fix $s \in [0, s_0[$. According to Definition 2.4, the filtration of the s-Gevrey spaces $\mathcal{O}(D_{\rho_1})[\![t]\!]_s$ (see page 8) and the first point of Theorem 2.5, it is clear that we have the following implications:

$$\begin{aligned}\widetilde{f}(t,x) \text{ is } s\text{-Gevrey} &\Longrightarrow \widetilde{f}^{[k]}(t,x)\in\mathcal{O}(D_{\rho_1})[\![t]\!]_s \text{ for all } k=1,\dots,n\\ &\Longrightarrow \widetilde{f}^{[k]}(t,x)\in\mathcal{O}(D_{\rho_1})[\![t]\!]_{s_0} \text{ for all } k=1,\dots,n\\ &\Longrightarrow \widetilde{f}(t,x) \text{ is } s_0\text{-Gevrey}\\ &\Longrightarrow \widetilde{u}(t,x) \text{ is } s_0\text{-Gevrey.}\end{aligned}$$

Therefore, to conclude that we can not say better about the Gevrey order of $\widetilde{u}(t,x)$, that is $\widetilde{u}(t,x)$ is *generically* s_0-Gevrey, we need to find an example for which the formal solution $\widetilde{u}(t,x)$ of System (1.1) is s'-Gevrey for no $s'<s_0$.

When $n=1$, that is when System (1.1) is reduced to an inhomogeneous linear partial differential equation, such examples can be found, for instance, in [3, 41, 43, 45]. In the case $n\geqslant 2$, such an example is given by the following.

Proposition 2.19. — *Let us consider the system*

$$\begin{cases}\partial_t u-\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}\partial_x^2 u=\widetilde{f}(t,x)\\[2ex] u(0,x)=\dfrac{1}{1-x}\begin{pmatrix}1\\1\end{pmatrix}\in\mathcal{M}_{2,1}(\mathcal{O}(D_1))\end{cases}\tag{2.7}$$

with $a,b>0$, $\widetilde{f}(t,x)\in\mathcal{M}_{2,1}(\mathcal{O}(D_1)[\![t]\!]_s)$ and $\partial_x^q f_{j,*}^{[k]}(0)\geqslant 0$ for all $q\geqslant 0$, all $j\geqslant 0$ and all $k=1,2$. Then, the formal solution $\widetilde{u}(t,x)$ of System (2.7) *is exactly* 1*-Gevrey.*

Proof. — Since the critical value of System (2.7) is $s_0=1$, it results from the above calculations that it is sufficient to prove that $\widetilde{u}(t,x)$ is s'-Gevrey for no $s'<1$.

According to the general relations (1.6), the coefficients $u_{j,*}^{[1]}(x)$ and $u_{j,*}^{[2]}(x)$ of the components $\widetilde{u}^{[1]}(t,x)$ and $\widetilde{u}^{[2]}(t,x)$ of the formal solution $\widetilde{u}(t,x)$ of System (2.7) are recursively determined from

$$u_{0,*}^{[1]}(x)=u_{0,*}^{[2]}(x)=\frac{1}{1-x}$$

by the relations

$$\begin{cases}u_{j+1,*}^{[1]}(x)=f_{j,*}^{[1]}(x)+a\partial_x^2 u_{j,*}^{[2]}(x)\\ u_{j+1,*}^{[2]}(x)=f_{j,*}^{[2]}(x)+b\partial_x^2 u_{j,*}^{[1]}(x).\end{cases}$$

Reasoning then by recursion on $j \geqslant 0$, one easily checks that

$$\begin{cases} u_{j,*}^{[1]}(x) = a^{\lceil j/2 \rceil} b^{\lfloor j/2 \rfloor} \dfrac{(2j)!}{(1-x)^{2j+1}} + \mathrm{rem}_j^{[1]}(x) \\ u_{j,*}^{[2]}(x) = a^{\lfloor j/2 \rfloor} b^{\lceil j/2 \rceil} \dfrac{(2j)!}{(1-x)^{2j+1}} + \mathrm{rem}_j^{[2]}(x) \end{cases}$$

for all $j \geqslant 0$, where $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) stands for the ceil (resp. the floor) of $x \in \mathbb{R}$, and where $\mathrm{rem}_j^{[k]}(x)$, $k = 1, 2$, are linear combinations with nonnegative coefficients of terms of the form $\partial_x^q f_{\ell,*}^{[k']}(x)$ with $k' = 1, 2$, $\ell \in \{0, \dots, j-1\}$ and $q \in \{0, \dots, 2j\}$.

Consequently, using our assumptions on a, b and $\widetilde{f}(t,x)$, we deduce that

$$u_{j,*}^{[k]}(0) \geqslant m^j (2j)! \text{ with } m = \min(a,b)$$

for all $j \geqslant 0$ and all $k = 1, 2$.

Let us now assume that $\widetilde{u}(t,x)$ is s'-Gevrey for some $s' < 1$, that is $\widetilde{u}^{[1]}(t,x)$ and $\widetilde{u}^{[2]}(t,x)$ are both s'-Gevrey. Then, applying Definition 2.1 to the above inequalities, we get the relations

$$1 \leqslant CK^j \frac{\Gamma(1 + (s'+1)j)}{\Gamma(1+2j)}$$

for all $j \geqslant 0$ and some convenient positive constants C and K independent of j. Proposition 2.19 follows then, since such inequalities are impossible. Indeed, it results from the Stirling's Formula that

$$CK^j \frac{\Gamma(1 + (s'+1)j)}{\Gamma(1+2j)} \underset{j \to +\infty}{\sim} C \sqrt{\frac{s'+1}{2}} \left(\frac{K e^{1-s'} (s'+1)^{s'+1}}{4} \right)^j j^{(s'-1)j}$$

which approaches 0 when j tends to infinity. □

3. Summability

In this section, we assume that the critical value s_0 introduced in Theorem 2.5 is positive, and we are interested in the k_0-summability, with $k_0 = 1/s_0$, of the formal solution $\widetilde{u}(t,x)$ of System (1.1). More precisely, our aim is to make explicit a necessary and sufficient condition under which $\widetilde{u}(t,x)$ is k_0-summable in a given direction $\arg(t) = \theta$.

Before specifying the framework of our study (Hypothesis 3.5) and stating our main result (Theorem 3.11), let us start by recalling some notions about the k_0-summability of formal power series in $\mathcal{O}(D_{\rho_1})[\![t]\!]$.

3.1. k_0-summable formal power series

Still considering t as the variable and x as a parameter, one extends, in the similar way as the s-Gevrey formal series (see Definition 2.1), the classical notion of k_0-summability of formal series in $\mathbb{C}[\![t]\!]$ to the notion of k_0-summability of formal series in $\mathcal{O}(D_{\rho_1})[\![t]\!]$ in requiring similar conditions, the estimates being however uniform with respect to x.

Among the many equivalent definitions of k_0-summability in a given direction $\arg(t) = \theta$ at $t = 0$ [51] (see also [20, Section 5]), we choose in this article a generalization of Ramis' definition which states that a formal series $\widetilde{g}(t) \in \mathbb{C}[\![t]\!]$ is k_0-summable in direction θ if there exists a holomorphic function g which is $1/k_0$-Gevrey asymptotic to $\widetilde{g}$ in an open sector $\Sigma_{\theta,>\pi/k_0}$ bisected by θ and with opening larger than π/k_0 [40, Definition 3.1]. To express the $1/k_0$-Gevrey asymptotic, there also exist various equivalent ways. We choose here the one which sets conditions on the successive derivatives of g [51, Chapter 6] (see also [25, p. 171] or [40, Theorem 2.4] for instance).

Definition 3.1 (k_0-summability). — *Let $k_0 > 0$ and $s_0 = 1/k_0$. A formal series $\widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1})[\![t]\!]$ is said to be* k_0-summable in the direction $\arg(t) = \theta$ *if there exist a sector $\Sigma_{\theta,>\pi/k_0}$, a radius $0 < r_1 \leqslant \rho_1$ and a function $u(t,x)$ called* k_0-sum of $\widetilde{u}(t,x)$ in direction θ *such that*

(1) *u is defined and holomorphic on $\Sigma_{\theta,>\pi/k_0} \times D_{r_1}$;*
(2) *for any $x \in D_{r_1}$, the map $t \mapsto u(t,x)$ has $\widetilde{u}(t,x) = \sum_{j\geqslant 0} u_{j,*}(x)\frac{t^j}{j!}$ as its Taylor expansion at 0 on $\Sigma_{\theta,>\pi/k_0}$;*
(3) *for any proper[5] subsector $\Sigma \Subset \Sigma_{\theta,>\pi/k_0}$, there exist two positive constants $C, K > 0$ such that the following estimate holds for all $p \geqslant 0$ and all $(t,x) \in \Sigma \times D_{r_1}$:*

$$|\partial_t^p u(t,x)| \leqslant CK^p\Gamma(1+(s_0+1)p).$$

We denote by $\mathcal{O}(D_{\rho_1})\{t\}_{k_0;\theta}$ the subset of $\mathcal{O}(D_{\rho_1})[\![t]\!]$ consisting of all the k_0-summable formal series in the direction $\arg(t) = \theta$.

Notice that, for any fixed $x \in D_{r_1}$, the k_0-summability of $\widetilde{u}(t,x)$ coincides with the classical k_0-summability. Consequently, Watson's lemma [20, Theorem 5.1.3] implies the unicity of its k_0-sum, if any exists.

Notice also that the k_0-sum of a k_0-summable formal series $\widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1})\{t\}_{k_0;\theta}$ may be analytic with respect to x on a disc D_{r_1} smaller than the common disc D_{ρ_1} of analyticity of the coefficients $u_{j,*}(x)$ of $\widetilde{u}(t,x)$.

---

(5) A subsector Σ of a sector Σ' is said to be a *proper subsector*, and one denotes $\Sigma \Subset \Sigma'$, if its closure in $\mathbb{C}$ is contained in $\Sigma' \cup \{0\}$.

Denote by $\partial_t^{-1}\widetilde{u}$ (resp. $\partial_x^{-1}\widetilde{u}$) the anti-derivative of $\widetilde{u}$ with respect to t (resp. x) which vanishes at $t=0$ (resp. $x=0$). Following Proposition 3.2, which is proved for instance in [42, Proposition 2], specifies the algebraic structure of $\mathcal{O}(D_{\rho_1})\{t\}_{k_0;\theta}$.

Proposition 3.2. — *Let $k_0>0$ and $\theta\in\mathbb{R}/2\pi\mathbb{Z}$. Then, the set $\mathcal{O}(D_{\rho_1})\{t\}_{k_0;\theta}$ endowed with the usual algebraic operations and the usual derivations ∂_t and ∂_x is a $\mathbb{C}$-differential subalgebra of the Gevrey space $\mathcal{O}(D_{\rho_1})[\![t]\!]_{s_0}$. Moreover, it is stable under the anti-derivations ∂_t^{-1} and ∂_x^{-1}.*

With respect to t, the k_0-sum $u(t,x)$ of a k_0-summable series $\widetilde{u}(t,x)\in\mathcal{O}(D_{\rho_1})\{t\}_{k_0;\theta}$ is analytic on an open sector for which there is no control on the angular opening except that it must be larger than π/k_0 (hence, it contains a closed sector $\overline{\Sigma}_{\theta,\pi/k_0}$ bisected by θ and with opening π/k_0) and no control on the radius except that it must be positive. Thereby, the k_0-sum $u(t,x)$ is well-defined as a section of the sheaf of analytic functions in (t,x) on a germ of closed sector of opening π/k_0 (that is, a closed interval $\bar{I}_{\theta,\pi/k_0}$ of length π/k_0 on the circle S^1 of directions issuing from 0; see [26, 1.1] or [19, I.2]) times $\{0\}$ (in the plane $\mathbb{C}$ of the variable x). We denote by $\mathcal{O}_{\bar{I}_{\theta,\pi/k_0}\times\{0\}}$ the space of such sections.

Corollary 3.3. — *The operator of k_0-summation*

$$\begin{array}{rccc}\mathcal{S}_{k_0;\theta}: & \mathcal{O}(D_{\rho_1})\{t\}_{k_0;\theta} & \longrightarrow & \mathcal{O}_{\bar{I}_{\theta,\pi/k_0}\times\{0\}}\\ & \widetilde{u}(t,x) & \longmapsto & u(t,x)\end{array}$$

is a homomorphism of $\mathbb{C}$-differential algebras for the derivations ∂_t and ∂_x. Moreover, it commutes with the anti-derivations ∂_t^{-1} and ∂_x^{-1}.

The following proposition investigates the k_0-summability of the analytic functions at the origin $(0,0)\in\mathbb{C}^2$.

Proposition 3.4. — *Let $a(t,x)$ be an analytic function on a polydisc $D_{\rho_0}\times D_{\rho_1}$. Then, $a(t,x)\in\mathcal{O}(D_{\rho_1})\{t\}_{k_0;\theta}$ for any $k_0>0$ and any direction $\theta\in\mathbb{R}/2\pi\mathbb{Z}$.*

Proof. — Let us fix $k_0>0$ and put $s_0=1/k_0$. We also consider a direction $\theta\in\mathbb{R}/2\pi\mathbb{Z}$, and four radii $0<\rho_j''<\rho_j'<\rho_j$ with $j=0,1$. Let us first start by observing that $a(t,x)$ is its own Taylor series at $(0,0)$ on $D_{\rho_0}\times D_{\rho_1}$. On the other hand, we derive from the Cauchy Integral Formula

$$\partial_t^p a(t,x)=\frac{p!}{(2i\pi)^2}\int_{\substack{|t'-t|=\rho_0'-\rho_0''\\ |x'-x|=\rho_1'-\rho_1''}}\frac{a(t',x')}{(t'-t)^{p+1}(x'-x)}\mathrm{d}t'\mathrm{d}x'$$

the inequalities

$$|\partial_t^p a(t,x)| \leqslant \alpha\left(\frac{1}{\rho_0'-\rho_0''}\right)^p p!$$

for all $p \geqslant 0$ and all $(t,x) \in D_{\rho_0''} \times D_{\rho_1''}$, where α stands for the maximum of $|a(t,x)|$ on the closed polydisc $\overline{D}_{\rho_0'} \times \overline{D}_{\rho_1'}$ ($\overline{D}_\rho$ denotes the closed disc with center $0 \in \mathbb{C}$ and radius $\rho > 0$). Observing then that $p! = \Gamma(1+p) \leqslant \Gamma(1+(s_0+1)p)$ for all $p \geqslant 0$ (it is clear for $p=0$ and stems from the increase of the Gamma function on $[2,+\infty[$ for $p \geqslant 1$), we finally get the inequalities

$$|\partial_t^p a(t,x)| \leqslant \alpha\left(\frac{1}{\rho_0'-\rho_0''}\right)^p \Gamma(1+(s_0+1)p)$$

for all $p \geqslant 0$ and all $(t,x) \in D_{\rho_0''} \times D_{\rho_1''}$. The choice of an arbitrary sector $\Sigma_{\theta,>\pi/k_0} \subset D_{\rho_0''}$ ends the proof of Proposition 3.4. □

Let us now turn to the study of the k_0-summability of the formal solution $\widetilde{u}(t,x)$ of System (1.1).

3.2. Main result and applications

3.2.1. Notations and framework of the study

In the sequel of the article, we denote by

- $q^\star$ the maximum of all the x-derivative orders $q \in \bigcup_{i\in\mathcal{K}} Q_i$;
- $i^\star$ the maximum of all the t-derivative orders $i \in \mathcal{K}$ such that $q^\star \in Q_i$;
- q_i the maximum of all the x-derivative orders $q \in Q_i \backslash \{q^\star\}$ if any exists (that is $Q_i \neq \{q^\star\}$),

and by

- $\mathcal{K}_{q^\star} = \{i \in \mathcal{K} \,:\, q^\star \in Q_i\}$;
- $\mathcal{K}'_{q^\star} = \mathcal{K}_{q^\star} \backslash \{i^\star\}$;
- $Q'_i = Q_i \backslash \{q^\star\}$ (hence, $Q'_i = Q_i$ if $i \notin \mathcal{K}_{q^\star}$).

Thereby, $i^\star$ stands for the maximum of all the t-derivative orders $i \in \mathcal{K}_{q^\star}$ and q_i stands for the maximum of all the x-derivative orders $q \in Q'_i$. Observe that $\mathcal{K}'_{q^\star}$ and some Q'_i may be empty.

HYPOTHESIS 3.5. — *We assume the four following conditions:*

(1) $i^\star + q^\star > \kappa$;
(2) $A_{i^\star,q^\star}(0,0) \in \mathcal{M}_n(\mathbb{C})$ *is invertible (hence, in particular,* $v_{i^\star,q^\star} = 0$*);*
(3) $s_0 = \dfrac{q^\star - \kappa + i^\star}{\kappa - i^\star}$;

(4) $v_{i,q_i} = 0$ *for all* $i > i^\star$, *if some exists.*

Remark 3.6. — Since $i^\star < \kappa$, condition (1) implies $q^\star \geqslant 2$.

Remark 3.7. — Hypothesis 3.5 tells us that the associated Newton polygon $\mathcal{N}_t(\Delta)$ of System (1.1) is the convex hull

$$\mathrm{CH}(C(\kappa,-\kappa) \cup C(i^\star + q^\star, -i^\star))$$

of $C(\kappa,-\kappa) \cup C(i^\star + q^\star, -i^\star)$ (see Remark 2.7 for the definition of the domain $C(a,b)$). Indeed,

- when $i \leqslant i^\star$, the definition of $q^\star$ implies the inclusions
$$C(i+q, -i+v_{i,q}) \subset C(i^\star + q^\star, -i^\star)$$
for all $q \in Q_i$;
- when $i > i^\star$, we first have the inclusions
$$C(i+q, -i+v_{i,q}) \subset C(i+q_i, -i)$$
for all $q \in Q_i$, and then the inclusion
$$C(i+q_i,-i) \subset \mathrm{CH}(C(\kappa,-\kappa) \cup C(i^\star + q^\star, -i^\star)),$$
which is obvious when $i + q_i \leqslant \kappa$ (we have indeed $C(i+q_i,-i) \subset C(\kappa,-\kappa)$ by definition of these sets), and stems from condition (3) when $i + q_i > \kappa$ since we have the inequalities
$$0 < k_0 = \frac{1}{s_0} = \frac{\kappa - i^\star}{q^\star - \kappa + i^\star} \leqslant \frac{\kappa - i}{q_i - \kappa + i} = k_0',$$
where $k_0 = 1/s_0$ is the slope of the segment with end points $(\kappa,-\kappa)$ and $(i^\star + q^\star, -i^\star)$, and where k_0' is the slope of the segment with end points $(\kappa,-\kappa)$ and $(i+q_i,-i)$ (see Figure 3.1).

In particular, $\mathcal{N}_t(\Delta)$ has a unique positive slope, the latter being equal to

$$\frac{1}{s_0} = \frac{\kappa - i^\star}{q^\star - \kappa + i^\star}$$

and determining by the segment with end points $(\kappa,-\kappa)$ and $(i^\star + q^\star, -i^\star)$ as shown on Figure 3.1.

The following technical lemma, which is a direct consequence of Hypothesis 3.5, will play a key role in the proof of our main result (see Section 3.3).

Lemma 3.8. — *The following relations hold:*

(1) $(s_0+1)(\kappa - i^\star) = q^\star$;
(2) $(s_0+1)(i - i^\star) \leqslant q^\star - q$ *for all* $i \in \mathcal{K}$ *and all* $q \in Q_i'$.

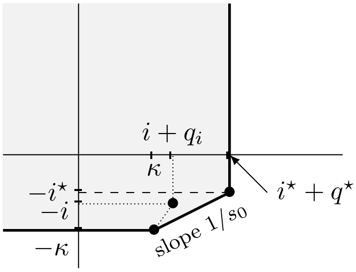


Figure 3.1. The Newton polygon $\mathcal{N}_t(\Delta)$ under Hypothesis 3.5

Proof. — The first point is straigthforward from condition (3) of Hypothesis 3.5.

Let us prove the second point. The inequality is obvious when $i \leqslant i^\star$ since $q^\star - q \geqslant 0$ for all $q \in Q'_i$. When $i > i^\star$, if any i exists, Remark 3.7 and Figure 3.1 imply the relation

$$\frac{i^\star + q^\star - i - q_i}{i - i^\star} \geqslant s_0.$$

Adding then "+1" to the both sides of this inequality and multiplying by $i - i^\star > 0$, we finally get

$$(s_0+1)(i-i^\star) \leqslant q^\star - q_i$$

and therefore

$$(s_0+1)(i-i^\star) \leqslant q^\star - q$$

since $q_i \geqslant q$ for all $q \in Q'_i$. This ends the proof of Lemma 3.8. □

Let us now turn to the formal solution $\widetilde{u}(t,x)$ of System (1.1).

3.2.2. A preliminary calculation

Let us write the components $\widetilde{u}^{[k]}(t,x)$ of the formal solution $\widetilde{u}(t,x)$, the components $\widetilde{f}^{[k]}(t,x)$ of the inhomogeneity $\widetilde{f}(t,x)$ and the coefficients $a_{i,q}^{[k,\ell]}(t,x)$ for $k \in \{1,\dots,n\}$, $i \in \mathcal{K}$, $q \in Q_i$ and $\ell \in \mathcal{L}_k(A_{i,q})$ in the form

$$\widetilde{u}^{[k]}(t,x) = \sum_{\nu\geqslant 0} \widetilde{u}^{[k]}_{*,\nu}(t)\frac{x^\nu}{\nu!}, \quad \widetilde{f}^{[k]}(t,x) = \sum_{\nu\geqslant 0} \widetilde{f}^{[k]}_{*,\nu}(t)\frac{x^\nu}{\nu!},$$
$$a_{i,q}^{[k,\ell]}(t,x) = \sum_{\nu\geqslant 0} a_{i,q;*,\nu}^{[k,\ell]}(t)\frac{x^\nu}{\nu!}$$

with $\widetilde{u}^{[k]}_{*,\nu}(t), \widetilde{f}^{[k]}_{*,\nu}(t) \in \mathbb{C}[[t]]$ and $a^{[k,\ell]}_{i,q;*,\nu}(t) \in \mathcal{O}(D_{\rho_0})$ for all $\nu \geqslant 0$. By identifying the terms in x^ν in (1.5), we first get the identities

$$\partial_t^\kappa \widetilde{u}^{[k]}_{*,\nu}(t) = \widetilde{f}^{[k]}_{*,\nu}(t) + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell\in\mathcal{L}_k(A_{i,q})}\sum_{\nu_0+\nu_1=\nu}\binom{\nu}{\nu_0} t^{v^{[k,\ell]}_{i,q}} a^{[k,\ell]}_{i,q;*,\nu_0}(t)\partial_t^i \widetilde{u}^{[\ell]}_{*,\nu_1+q}(t)$$

for all $k=1,\dots,n$ and all $\nu \geqslant 0$. Rewriting them in the form

$$\begin{aligned}\partial_t^\kappa \widetilde{u}^{[k]}_{*,\nu}(t) = \widetilde{f}^{[k]}_{*,\nu}(t) &+ \sum_{i\in\mathcal{K}_{q^\star}}\sum_{\ell\in\mathcal{L}_k(A_{i,q^\star})} t^{v^{[k,\ell]}_{i,q^\star}} a^{[k,\ell]}_{i,q^\star;*,0}(t)\partial_t^i \widetilde{u}^{[\ell]}_{*,\nu+q^\star}(t)\\ &+ \sum_{i\in\mathcal{K}_{q^\star}}\sum_{\ell\in\mathcal{L}_k(A_{i,q^\star})}\sum_{\substack{\nu_0+\nu_1=\nu\\ \nu_1\neq\nu}}\binom{\nu}{\nu_0,} t^{v^{[k,\ell]}_{i,q^\star}} a^{[k,\ell]}_{i,q;*,\nu_0}(t)\partial_t^i \widetilde{u}^{[\ell]}_{*,\nu_1+q^\star}(t)\\ &+ \sum_{i\in\mathcal{K}}\sum_{q\in Q'_i}\sum_{\ell\in\mathcal{L}_k(A_{i,q})}\sum_{\nu_0+\nu_1=\nu}\binom{\nu}{\nu_0} t^{v^{[k,\ell]}_{i,q}} a^{[k,\ell]}_{i,q;*,\nu_0}(t)\partial_t^i \widetilde{u}^{[\ell]}_{*,\nu_1+q}(t)\end{aligned}$$

and returning to a matrix notation we finally get the relations

$$\begin{aligned}\Delta_{q^\star}\widetilde{u}_{*,\nu+q^\star}(t) = \partial_t^\kappa \widetilde{u}_{*,\nu}(t) - \widetilde{f}_{*,\nu}(t)\\ - \sum_{i\in\mathcal{K}_{q^\star}}\sum_{\substack{\nu_0+\nu_1=\nu\\ \nu_1\neq\nu}}\binom{\nu}{\nu_0}\partial_x^{\nu_0}A_{i,q}(t,x)_{|x=0}\partial_t^i\widetilde{u}_{*,\nu_1+q^\star}(t)\\ - \sum_{i\in\mathcal{K}}\sum_{q\in Q'_i}\sum_{\nu_0+\nu_1=\nu}\binom{\nu}{\nu_0}\partial_x^{\nu_0}A_{i,q}(t,x)_{|x=0}\partial_t^i\widetilde{u}_{*,\nu_1+q}(t)\end{aligned}\tag{3.1}$$

where $\Delta_{q^\star}$ is the linear differential operator of order $i^\star$ defined by

$$\Delta_{q^\star} = \sum_{i\in\mathcal{K}_{q^\star}} A_{i,q^\star}(t,0)\partial_t^i.$$

Observe that all the indices $\nu_1+q^\star$ and ν_1+q of the terms $\widetilde{u}_{*,\nu_1+q^\star}(t)$ and $\widetilde{u}_{*,\nu_1+q}(t)$ in the right hand-side of (3.1) are $< \nu+q^\star$.

Lemma 3.9. — *The following identity holds for all $i \in \{0,\dots,\kappa-1\}$ and all $\nu \geqslant 0$:*

$$\partial_t^i\widetilde{u}_{*,\nu}(t)_{|t=0} = \partial_x^\nu\varphi_i(x)_{|x=0}.$$

Proof. — Let $k \in \{1,\dots,n\}$. Lemma 3.9 is straightforward from the identities

$$\partial_t^i\widetilde{u}^{[k]}_{*,\nu}(t) = \sum_{j\geqslant 0} u^{[k]}_{j+i,\nu}\frac{t^j}{j!}$$

and

$$\partial_x^\nu \varphi_i^{[k]}(x) = \partial_x^\nu u_{i,*}^{[k]}(x) = \sum_{m\geqslant 0} u_{i,m+\nu}^{[k]} \frac{x^m}{m!},$$

which provide the relation $\partial_t^i \widetilde{u}_{*,\nu}^{[k]}(t)_{|t=0} = u_{i,\nu}^{[k]} = \partial_x^\nu \varphi_i^{[k]}(x)_{|x=0}$. $\square$

Consequently, since condition (2) of Hypothesis 3.5 and Proposition 5.4 imply that $A_{i^\star,q^\star}^{-1}(t,0)$ is well-defined and with analytic coefficients at the origin of $\mathbb{C}$, we deduce from relations (3.1) that each coefficient $\widetilde{u}_{*,\nu+q^\star}(t)$ is uniquely determined from the inhomogeneity $\widetilde{f}(t,x)$, the initial conditions $\varphi_j(x)$ (Lemma 3.9 above implies indeed the relations $\partial_t^i \widetilde{u}_{*,\nu+q^\star}(t)_{|t=0} = \partial_x^{\nu+q^\star}\varphi_i(x)_{|x=0}$ for all $i = 0,\ldots,i^\star-1$, which allow to fully solve the differential equation (3.1)), and from the formal series $\widetilde{u}_{*,\nu'}(t)$ with $\nu' = 0,\ldots,q^\star-1$. In particular, the same applies to the formal solution $\widetilde{u}(t,x)$.

3.2.3. The summability theorem

Before stating our summability theorem, let us first extend Definition 3.1 to the elements of $\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$.

Definition 3.10. — *Let $k_0 > 0$.*

- *A $n\times 1$-column matrix $\widetilde{u}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$ is said to be k_0-summable in the direction $\arg(t) = \theta$ if its components $\widetilde{u}^{[k]}(t,x) \in \mathcal{O}(D_{\rho_1})[[t]]$ are k_0-summable in the direction θ for all $k = 1,\ldots,n$.*
- *The k_0-sum $u(t,x)$ of $\widetilde{u}(t,x)$, if any exists, is then the $n\times 1$-matrix whose the k^{th} component if the k_0-sum of $\widetilde{u}^{[k]}(t,x)$ for all $k = 1,\ldots,n$.*

According to the calculations above, we are now able to state the main result in view in this section.

Theorem 3.11 (Summability Theorem). — *Assume that Hypothesis 3.5 holds. Let $\arg(t) = \theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction issuing from 0. Let*

$$k_0 = \frac{1}{s_0} = \frac{\kappa - i^\star}{q^\star - \kappa + i^\star}.$$

Then,

(1) *The unique formal solution $\widetilde{u}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$ of System (1.1) is k_0-summable in the direction θ if and only if the inhomogeneity $\widetilde{f}(t,x)$ and the formal series $\partial_x^\nu \widetilde{u}(t,x)_{|x=0} = \widetilde{u}_{*,\nu}(t) \in \mathcal{M}_{n,1}(\mathbb{C}[[t]])$ for $\nu = 0,\ldots,q^\star-1$ are k_0-summable in the direction θ.*

(2) *Moreover, the* k_0*-sum* $u(t,x)$*, if any exists, satisfies System* (1.1) *in which* $\widetilde{f}(t,x)$ *is replaced by its* k_0*-sum* $f(t,x)$ *in the direction* θ*.*

Observe that Theorem 3.11 extends to System (1.1) the result already obtained by the author in [43] in the case $n=1$, and which is recalled in Proposition 1.7. Observe also that our present result significantly improves the one of [43] since we have here much weaker conditions on the valuations v_{i,q_i} and on the number of elements $i\in\mathcal{K}$ such that $q^\star\in Q_i$.

Before starting the proof of Theorem 3.11, let us illustrate it with two examples, the first one to clarify, for the convenience of the reader, the above observation; the second one dealing with the important example of the reaction-diffusion systems.

Example 3.12. — We restrict here to the case $n=1$.

- Let us first consider the equation

$$\begin{cases}\partial_t^2 u - a(t,x)\partial_t\partial_x^3 u - b(t,x)\partial_x u = \widetilde{f}(t,x)\\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j=0,1\end{cases}$$

  where the coefficients $a(t,x)$ and $b(t,x)$ are analytic on a polydisc $D_{\rho_0}\times D_{\rho_1}$, the initial data $\varphi_0(x)$ and $\varphi_1(x)$ are analytic on D_{ρ_1}, and where $\widetilde{f}(t,x)\in\mathcal{O}(D_{\rho_1})[[t]]$. We have $\kappa=2$, $\mathcal{K}=\{0,1\}$, $Q_0=\{1\}$ and $Q_1=\{3\}$; hence $(i^\star,q^\star)=(1,3)$. Assuming besides $a(0,0)\neq 0$, Hypothesis 3.5 holds. Consequently, $s_0=2$ and Theorem 3.11 tells us that its formal solution $\widetilde{u}(t,x)$ is 1/2-summable in the direction θ if and only if the inhomogeneity $\widetilde{f}(t,x)$ and the three formal power series $\widetilde{u}_{*,0}(t)$, $\widetilde{u}_{*,1}(t)$ and $\widetilde{u}_{*,2}(t)$ are 1/2-summable in the direction θ. Observe here that to apply Theorem 3.11, no assumption on the coefficient $b(t,x)$ is made, unlike in [43, Theorem 3] (see Proposition 1.7) where the condition $b(0,x)\not\equiv 0$ should also have been assumed.
- Let us now consider the equation

$$\begin{cases}\partial_t^3 u - a(t,x)\partial_t^2\partial_x u - b(t,x)\partial_t\partial_x^3 u - c(t,x)\partial_x^3 u = \widetilde{f}(t,x)\\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j=0,1,2\end{cases}$$

  where the coefficients $a(t,x)$, $b(t,x)$ and $c(t,x)$ are analytic on a polydisc $D_{\rho_0}\times D_{\rho_1}$, the initial data $\varphi_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ are analytic on D_{ρ_1}, and where $\widetilde{f}(t,x)\in\mathcal{O}(D_{\rho_1})[[t]]$. We have $\kappa=3$, $\mathcal{K}=\{0,1,2\}$, $Q_0=\{3\}$, $Q_1=\{3\}$ and $Q_2=\{1\}$; hence $(i^\star,q^\star)=(1,3)$ (here, $q^\star$ occurs twice in the sets Q_0 and Q_1). Assuming besides $a(0,x)\not\equiv 0$ and $b(0,0)\neq 0$, Hypothesis 3.5 holds.

Consequently, $s_0 = 1/2$ and Theorem 3.11 tells us that its formal solution $\widetilde{u}(t,x)$ is 2-summable in the direction θ if and only if the inhomogeneity $\widetilde{f}(t,x)$ and the three formal power series $\widetilde{u}_{*,0}(t)$, $\widetilde{u}_{*,1}(t)$ and $\widetilde{u}_{*,2}(t)$ are 2-summable in the direction θ. Observe here that this could not be obtained with the result of [43, Theorem 3] (see Proposition 1.7) since $q^\star$ could belong to only one set Q_i with $i \in \mathcal{K}$.

Example 3.13. — Let us consider the reaction-diffusion system (1.4). Applying first Theorem 2.5, we have shown in Section 2 that the value s_0 attached to System (1.4) is $s_0 = 1$ and, therefore, that the formal solution $\widetilde{u}(t,x)$ of System (1.4) is 1-Gevrey. Let us now assume that the diagonal coefficient $A^{[k,k]}(t,x)$ of the diffusion matrix $A(t,x)$ satisfy $A^{[k,k]}(0,0) \neq 0$ for all $k = 1,\dots,n$. Then, all the conditions required in Hypothesis 3.5 are satisfied and Theorem 3.11 above applies, providing thus a characterization of the 1-summability of $\widetilde{u}(t,x)$ in a given direction θ: $\widetilde{u}(t,x)$ is 1-summable in the direction θ if and only if its two formal coefficients $\widetilde{u}_{*,0}(t)$ and $\widetilde{u}_{*,1}(t)$ are 1-summable in the direction θ.

3.3. Proof of Theorem 3.11

Let us first start by observing that the necessary condition of the first point is straightforward from Definition 3.10 and Proposition 3.2, and that the second point stems obvious from Definition 3.10 and Corollary 3.3. Consequently, we are left to prove the sufficient condition of the first point. To this end, we fix from now on a direction θ and we suppose that the inhomogeneity $\widetilde{f}(t,x)$ and the formal series $\partial_x^\nu \widetilde{u}(t,x)_{|x=0} \in \mathcal{M}_{n,1}(\mathbb{C}[[t]])$ for $\nu = 0,\dots,q^\star - 1$ are all k_0-summable in the direction θ. To prove that the formal solution $\widetilde{u}(t,x)$ is also k_0-summable in this direction, we shall proceed through a fixed point method similar to the ones already used by W. Balser and M. Loday-Richaud in [3] and by the author in [41, 42, 43]. However, as we shall see below, the calculations are much more complicated because of the matrix coefficients and the fact that the set $\mathcal{K}'_{q^\star}$ may be nonempty.

3.3.1. An associated system and the fixed point procedure

Let us first define the element $\widetilde{v}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$ by

$$\widetilde{v}(t,x) = \sum_{j=0}^{i^\star-1} \left(\varphi_j(x) - \sum_{\nu=0}^{q^\star-1} \partial_x^\nu \varphi_j(x)_{|x=0} \right) \frac{t^j}{j!} + \sum_{\nu=0}^{q^\star-1} \widetilde{u}_{*,\nu}(t) \frac{x^\nu}{\nu!}.$$

Lemma 3.14. — *$\widetilde{v}(t,x)$ is k_0-summable in the direction θ.*

Proof. — Since the sum on j defines an analytic function on $\mathbb{C}\times D_{\rho_1}$, Lemma 3.14 is a direct consequence of Propositions 3.2 and 3.4 and our assumption on the formal series $\widetilde{u}_{*,\nu}(t)$ with $\nu=0,\dots,q^\star-1$. $\square$

Lemma 3.15. — *$\widetilde{u}^{[k]}(t,x)-\widetilde{v}^{[k]}(t,x)=O(t^{i^\star}x^{q^\star})$ for all $k=1,\dots,n$.*

Proof. — Let $k\in\{1,\dots,n\}$. Since

$$\varphi_j^{[k]}(x)=u_{j,*}^{[k]}(x)=\sum_{\nu\geqslant 0}u_{j,\nu}^{[k]}\frac{x^\nu}{\nu!}\quad\text{and}\quad \widetilde{u}_{*,\nu}^{[k]}(t)=\sum_{j\geqslant 0}u_{j,\nu}^{[k]}\frac{t^j}{j!},$$

the formal series $\widetilde{v}^{[k]}(t,x)$ can be rewritten in the form

$$\widetilde{v}^{[k]}(t,x)=\sum_{j=0}^{i^\star-1}\sum_{\nu\geqslant 0}u_{j,\nu}^{[k]}\frac{t^j}{j!}\frac{x^\nu}{\nu!}+\sum_{j\geqslant i^\star}\sum_{\nu=0}^{q^\star-1}u_{j,\nu}^{[k]}\frac{t^j}{j!}\frac{x^\nu}{\nu!}.$$

Hence,

$$\widetilde{u}^{[k]}(t,x)-\widetilde{v}^{[k]}(t,x)=\sum_{j\geqslant i^\star}\sum_{\nu\geqslant q^\star}u_{j,\nu}^{[k]}\frac{t^j}{j!}\frac{x^\nu}{\nu!}$$

and Lemma 3.15 follows. $\square$

Let us now introduce the element $\widetilde{w}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[\![t]\!])$ defined by the relation

$$\widetilde{u}(t,x)=\widetilde{v}(t,x)+\partial_t^{-i^\star}\partial_x^{-q^\star}\widetilde{w}(t,x).$$

With these notations, System (1.1) becomes

$$\begin{aligned}(3.2)\quad \partial_t^{\kappa-i^\star}\partial_x^{-q^\star}\widetilde{w}&-\sum_{i\in\mathcal{K}_{q^\star}}A_{i,q^\star}(t,x)\partial_t^{i-i^\star}\widetilde{w}\\ &-\sum_{i\in\mathcal{K}}\sum_{q\in Q_i'}A_{i,q}(t,x)\partial_t^{i-i^\star}\partial_x^{q-q^\star}\widetilde{w}\\ &=\widetilde{f}(t,x)-\partial_t^\kappa\widetilde{v}+\sum_{i\in\mathcal{K}}\sum_{q\in Q_i}A_{i,q}(t,x)\partial_t^i\partial_x^q\widetilde{v}.\end{aligned}$$

From condition (2) of Hypothesis 3.5 and Proposition 5.4, we know that the matrix $A_{i^\star,q^\star}(t,x)$ is invertible and that its inverse $A_{i^\star,q^\star}^{-1}(t,x)$ has analytic coefficients on $D_{\rho_0'}\times D_{\rho_1'}$ for some convenient radii $0<\rho_0'\leqslant\rho_0$ and $0<\rho_1'\leqslant\rho_1$. Consequently, denoting by

$$B_{i^\star,q^\star}(t,x)=A_{i^\star,q^\star}^{-1}(t,x)\in\mathcal{M}_n(\mathcal{O}(D_{\rho_0'}\times D_{\rho_1'}))$$

and by

$$B_{i,q}(t,x)=A_{i^\star,q^\star}^{-1}(t,x)A_{i,q}(t,x)\in\mathcal{M}_n(\mathcal{O}(D_{\rho_0'}\times D_{\rho_1'}))$$

for all $(i,q)\neq(i^\star,q^\star)$, we finally derive from (3.2) that $\widetilde{w}(t,x)$ is a solution of the system

$$D_{q^\star}w-H(t,x,w)=\widetilde{g}(t,x), \tag{3.3}$$

where $D_{q^\star}$ is the integral operator

$$D_{q^\star}=1+\sum_{i\in\mathcal{K}'_{q^\star}}B_{i,q^\star}(t,x)\partial_t^{i-i^\star},$$

where

$$H(t,x,w)=B_{i^\star,q^\star}(t,x)\partial_t^{\kappa-i^\star}\partial_x^{-q^\star}w-\sum_{i\in\mathcal{K}}\sum_{q\in Q'_i}B_{i,q}(t,x)\partial_t^{i-i^\star}\partial_x^{q-q^\star}w,$$

and where the inhomogeneity $\widetilde{g}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho'_1})[\![t]\!])$ is defined by

$$\widetilde{g}(t,x)=B_{i^\star,q^\star}(t,x)\left(\partial_t^\kappa\widetilde{v}-\sum_{i\in\mathcal{K}}\sum_{q\in Q_i}A_{i,q}(t,x)\partial_t^i\partial_x^q\widetilde{v}-\widetilde{f}(t,x)\right).$$

Lemma 3.16. — *$\widetilde{w}(t,x)$ is the unique formal solution of System* (3.3) *whose the components $\widetilde{w}^{[k]}(t,x)\in\mathbb{C}[\![t,x]\!]$ are formal power series in t and x for all $k=1,\dots,n$.*

Proof. — Let us first rewrite System (3.3) in the form

$$w^{[k]}+\sum_{i\in\mathcal{K}'_{q^\star}}\sum_{\ell=1}^{n}B_{i,q^\star}^{[k,\ell]}(t,x)\partial_t^{i-i^\star}w^{[\ell]}-\sum_{\ell=1}^{n}B_{i^\star,q^\star}^{[k,\ell]}(t,x)\partial_t^{\kappa-i^\star}\partial_x^{-q^\star}w^{[\ell]}$$
$$-\sum_{i\in\mathcal{K}}\sum_{q\in Q'_i}\sum_{\ell=1}^{n}B_{i,q}^{[k,\ell]}(t,x)\partial_t^{i-i^\star}\partial_x^{q-q^\star}w^{[\ell]}=\widetilde{g}^{[k]}(t,x)$$

for all $k=1,\dots,n$. Writing then the analytic coefficients $B_{i,q}^{[k,\ell]}(t,x)\in\mathcal{O}(D_{\rho'_0}\times D_{\rho'_1})$ and the inhomogeneities $\widetilde{g}^{[k]}(t,x)$ in the form

$$B_{i,q}^{[k,\ell]}(t,x)=\sum_{\nu\geqslant0}B_{i,q;*,\nu}^{[k,\ell]}(t)\frac{x^\nu}{\nu!}\quad\text{and}\quad\widetilde{g}^{[k]}(t,x)=\sum_{\nu\geqslant0}\widetilde{g}_{*,\nu}^{[k]}(t)\frac{x^\nu}{\nu!}$$

with $B_{i,q;*,\nu}^{[k,\ell]}(t)\in\mathcal{O}(D_{\rho'_0})$ and $\widetilde{g}_{*,\nu}^{[k]}(t)\in\mathbb{C}[\![t]\!]$ for all $\nu\geqslant0$ and all i,q,k,ℓ, and looking for the n components $\widetilde{w}^{[k]}(t,x)$ of a formal solution of System (3.3) on the same type:

$$\widetilde{w}^{[k]}(t,x)=\sum_{\nu\geqslant0}\widetilde{w}_{*,\nu}^{[k]}(t)\frac{x^\nu}{\nu!}$$

with $\widetilde{w}^{[k]}_{*,\nu}(t) \in \mathbb{C}[[t]]$ for all $\nu \geqslant 0$ and all $k = 1, \ldots, n$, one easily checks that the coefficients $\widetilde{w}^{[k]}_{*,\nu}(t)$ satisfy, for all $k = 1, \ldots, n$ and all $\nu \geqslant 0$, the recurrence relations

$$\begin{aligned}\widetilde{w}^{[k]}_{*,\nu} + \sum_{i\in\mathcal{K}'_{q^\star}} \sum_{\ell=1}^{n} \sum_{\nu_0+\nu_1=\nu} \binom{\nu}{\nu_0} B^{[k,\ell]}_{i,q^\star;*,\nu_0}(t)\partial_t^{i-i^\star}\widetilde{w}^{[\ell]}_{*,\nu_1} \\ - \sum_{\ell=1}^{n} \sum_{\nu_0+\nu_1=\nu} \binom{\nu}{\nu_0} B^{[k,\ell]}_{i^\star,q^\star;*,\nu_0}(t)\partial_t^{\kappa-i^\star}\widetilde{w}^{[\ell]}_{*,\nu_1-q^\star} \\ - \sum_{i\in\mathcal{K}} \sum_{q\in Q'_i} \sum_{\ell=1}^{n} \sum_{\nu_0+\nu_1=\nu} \binom{\nu}{\nu_0} B^{[k,\ell]}_{i,q;*,\nu_0}(t)\partial_t^{i-i^\star}\widetilde{w}^{[\ell]}_{*,\nu_1-q^\star+q} = \widetilde{g}^{[k]}_{*,\nu}(t)\end{aligned}$$

which can be rewritten in the form

$$\widetilde{w}^{[k]}_{*,\nu} + \sum_{i\in\mathcal{K}'_{q^\star}} \sum_{\ell=1}^{n} B^{[k,\ell]}_{i,q^\star;*,0}(t)\partial_t^{i-i^\star}\widetilde{w}^{[\ell]}_{*,\nu} = \widetilde{h}^{[k]}_{\nu}(t),$$

where $\widetilde{h}^{[k]}_{\nu}(t) \in \mathbb{C}[[t]]$ is a formal power series in t of the form

$$\widetilde{h}^{[k]}_{\nu}(t) = \widetilde{g}^{[k]}_{*,\nu}(t) + \mathrm{rem}^{[k]}_{\nu}(t),$$

the remainder term $\mathrm{rem}^{[k]}_{\nu}(t)$ being a linear combination with analytic coefficients in $D_{\rho'_0}$ of terms of the form $\partial_t^{\bullet}\widetilde{w}^{[\ell]}_{*,\nu'}$ with $\ell \in \{1, \ldots, n\}$ and $\nu' \in \{0, \ldots, \nu-1\}$.

Let us now fix $\nu \geqslant 0$ and let us set

$$\widetilde{w}^{[k]}_{*,\nu}(t) = \sum_{j\geqslant 0} \widetilde{w}^{[k]}_{j,\nu}\frac{t^j}{j!}, \quad \widetilde{h}^{[k]}_{\nu}(t) = \sum_{j\geqslant 0} \widetilde{h}^{[k]}_{j,\nu}\frac{t^j}{j!}$$
$$\text{and} \quad B^{[k,\ell]}_{i,q^\star;*,0}(t) = \sum_{j\geqslant 0} B^{[k,\ell]}_{i,q^\star;j,0}\frac{t^j}{j!}$$

with $\widetilde{w}^{[k]}_{j,\nu}, \widetilde{h}^{[k]}_{j,\nu}, B^{[k,\ell]}_{i,q^\star;j,0} \in \mathbb{C}$ for all $j \geqslant 0$ and all $i, q^\star, k, \ell$. Then, one can easily check this time that the coefficients $\widetilde{w}^{[k]}_{j,\nu}$ are uniquely determined for all $k = 1, \ldots, n$ and all $j \geqslant 0$ by the recurrence relations

$$\widetilde{w}^{[k]}_{j,\nu} + \sum_{i\in\mathcal{K}'_{q^\star}} \sum_{\ell=1}^{n} \sum_{j_0+j_1=j} \binom{j}{j_0} B^{[k,\ell]}_{i,q^\star;j_0,0}\widetilde{w}^{[\ell]}_{j_1-i^\star+i,\nu} = \widetilde{h}^{[k]}_{j,\nu}$$

since $i < i^\star$ for all $i \in \mathcal{K}'_{q^\star}$. Consequently, the coefficients $\widetilde{w}^{[k]}_{*,\nu}(t)$ are uniquely determined for all $k = 1, \ldots, n$ and all $\nu \geqslant 0$, and System (3.3) admits a unique formal solution in $\mathcal{M}_{n,1}(\mathbb{C}[[t,x]])$. Since $\widetilde{w}(t,x) \in$

$\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[[t]])$ is by construction a formal solution of this system, Lemma 3.16 follows. □

Let us now observe that Propositions 3.2 and 3.4, Lemma 3.14 and our assumption on $\widetilde{f}(t,x)$ imply that the inhomogeneity $\widetilde{g}(t,x)$ of System (3.3) is k_0-summable in the direction θ. Thereby, it is sufficient to prove that the formal series $\widetilde{w}(t,x)$ is also k_0-summable in the direction θ. To do that, we shall proceed as in [3, 41, 42, 43] by using a fixed point method as follows. Let us set

$$\widetilde{W}^{[k]}(t,x)=\sum_{\mu\geqslant 0}\widetilde{w}_{\mu}^{[k]}(t,x)$$

for all $k=1,\dots,n$, and let us choose the solution of System (3.3) recursively determined by the relations

$$\begin{cases} D_{q^\star}\widetilde{w}_0(t,x)=\widetilde{g}(t,x),\\ D_{q^\star}\widetilde{w}_{\mu+1}=H(t,x,\widetilde{w}_{\mu}) \quad \text{for all } \mu\geqslant 0\end{cases} \tag{3.4}$$

with

$$\widetilde{w}_{\mu}(t,x)=\begin{pmatrix}\widetilde{w}_{\mu}^{[1]}(t,x)\\ \vdots\\ \widetilde{w}_{\mu}^{[n]}(t,x)\end{pmatrix}\quad \text{for all } \mu\geqslant 0.$$

Lemma 3.17. — *Let $\widetilde{h}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[[t]])$. Then, the system*

$$D_q^\star y=\widetilde{h}(t,x) \tag{3.5}$$

admits a unique formal solution $\widetilde{y}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[[t]])$.

Proof. — Let us first rewrite System (3.5) in the form

$$y^{[k]}+\sum_{i\in\mathcal{K}'_{q^\star}}\sum_{\ell=1}^{n}B_{i,q^\star}^{[k,\ell]}(t,x)\partial_t^{i-i^\star}y^{[\ell]}=\widetilde{h}^{[k]}(t,x)$$

for all $k=1,\dots,n$. Writing then the analytic coefficients $B_{i,q^\star}^{[k,\ell]}(t,x)$ and the inhomogeneities $\widetilde{h}^{[k]}(t,x)$ in the form

$$B_{i,q^\star}^{[k,\ell]}(t,x)=\sum_{j\geqslant 0}B_{i,q^\star;j,*}^{[k,\ell]}(x)\frac{t^j}{j!}\quad\text{and}\quad \widetilde{h}^{[k]}(t,x)=\sum_{j\geqslant 0}h_{j,*}^{[k]}(x)\frac{t^j}{j!}$$

with $B_{i,q^\star;j,*}^{[k,\ell]}(x),h_{j,*}^{[k]}(x)\in\mathcal{O}(D_{\rho_1'})$ for all $j\geqslant 0$ and all i,k,ℓ, and looking for the n components $\widetilde{y}^{[k]}(t,x)$ of $\widetilde{y}(t,x)$ on the same type:

$$\widetilde{y}^{[k]}(t,x)=\sum_{j\geqslant 0}y_{j,*}^{[k]}(x)\frac{t^j}{j!}$$

with $y_{j,*}^{[k]}(x) \in \mathcal{O}(D_{\rho_1'})$ for all $j \geqslant 0$ and all $k = 1, \ldots, n$, one easily checks, due to the fact that $i^\star - i > 0$ for all $i \in \mathcal{K}'_{q^\star}$, that the coefficients $y_{j,*}^{[k]}(x)$ are uniquely determined for all $k = 1, \ldots, n$ and all $j \geqslant 0$ by the recurrence relation

$$y_{j,*}^{[k]}(x) = h_{j,*}^{[k]}(x) - \sum_{i \in \mathcal{K}'_{q^\star}} \sum_{\ell=1}^{n} \sum_{j_0+j_1=j} \binom{j}{j_0} B_{i,q^\star;j_0,*}^{[k,\ell]}(x) y_{j_1-i^\star+i,*}^{[\ell]}(x), \tag{3.6}$$

with the classical convention that the terms $y_{j_1-i^\star+i,*}^{[\ell]}$ are zero as soon as $j_1 < i^\star - i$; hence, Lemma 3.17. $\square$

Applying Lemma 3.17 to relations (3.4), we first deduce that all the $\widetilde{w}_\mu(t,x)$ are well-defined and in $\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[[t]])$. Moreover, according to the recurrence relations (3.6) and the definition of H (see page 32 and use the fact that $q^\star \geqslant 2$ (see Remark 3.6) and $q^\star - q \geqslant 1$ for all $q \in Q_i'$ and $i \in \mathcal{K}$), we observe that the components $\widetilde{w}_\mu^{[k]}(t,x)$ of $\widetilde{w}_\mu(t,x)$ are of order $O(x^\mu)$ in x for all $\mu \geqslant 0$ and all $k = 1, \ldots, n$. Thereby, for any $k \in \{1, \ldots n\}$, $\widetilde{W}^{[k]}(t,x)$ itself makes sense as a formal power series in t and x and, consequently, $\widetilde{W}^{[k]}(t,x) = \widetilde{w}^{[k]}(t,x)$ by unicity (see Lemma 3.16).

In Sections 3.3.3–3.3.5 below, we shall show that the formal series $\widetilde{w}_\mu^{[k]}(t,x)$ are all k_0-summable in the direction θ, and that their k_0-sums $w_\mu^{[k]}(t,x)$ satisfy inequalities of the form

$$|\partial_t^p w_\mu^{[k]}(t,x)| \leqslant C' K'^p \Gamma(1 + (s_0+1)p)(c|x|)^\mu \quad , C', K', c > 0$$

for all $k = 1, \ldots, n$, all $p, \mu \geqslant 0$ and all (t,x) in a convenient domain $\Sigma \times D_{r_1}$ independent of k, p and μ (see Proposition 3.28). These will then allow us to prove that the series

$$\sum_{\mu \geqslant 0} w_\mu^{[k]}(t,x)$$

defines, for all $k = 1, \ldots, n$ a holomorphic function, which is the k_0-sum of $\widetilde{w}^{[k]}(t,x)$ (see Section 3.3.6).

Before proving all this, let us start by looking at an auxiliary system that will allow us to define the functions $w_\mu^{[k]}(t,x)$ from the recursive relations (3.4).

3.3.2. An auxiliary system

Statement of the result. In this section, we fix a positive integer $I \geqslant 1$ and we consider the system of integral equations

$$z - \sum_{i=1}^{I} b_i(t,x) \partial_t^{-i} z = \widetilde{h}(t,x) \tag{3.7}$$

of unknown

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix},$$

where the coefficients $b_i(t,x)$ are $n\times n$-matrices with analytic entries on the polydisc $D_{\rho'_0}\times D_{\rho'_1}$, and where the inhomogeneity $\widetilde{h}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho'_1})[[t]])$ is k_0-summable in a direction θ. Using Definition 3.10 and Proposition 3.4, we assume besides that the coefficients $b_i(t,x)$ and the k_0-sum $h(t,x)$ of $\widetilde{h}(t,x)$ in the direction θ satisfy the following property: there exist a radius $0<r_1\leqslant\rho'_1$, a sector Σ bisected by θ, opening larger than π/k_0 and radius less 1, four constants $c\geqslant 0$, $A,A'>0$ and $B\geqslant 1$, and a polynomial $P(X)\in\mathbb{R}^+[X]$ with nonnegative coefficients such that

$$\begin{cases} |\partial_t^p b_i^{[k,\ell]}(t,x)|\leqslant AB^p\Gamma(1+(s_0+1)p) \\ |\partial_t^p h^{[k]}(t,x)|\leqslant A'B^p\Gamma(1+(s_0+1)(p+c))P(|x|) \end{cases} \tag{3.8}$$

for all $k,\ell\in\{1,\dots,n\}$, all $i=1,\dots,I$, all $p\geqslant 0$ and all $(t,x)\in\Sigma\times D_{r_1}$.

Adapting the proof of Lemma 3.17, it is clear that System (3.7) admits a unique formal solution $\widetilde{z}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho'_1})[[t]])$. The aim of this section is to prove the following.

Proposition 3.18. — *Let C_{s_0} be the positive real number defined by*

$$C_{s_0}=s'_0(2+\Gamma(s_0s'_0)) \quad \textit{with} \quad s'_0=\begin{cases} 1 & \textit{if } s_0\geqslant 1 \\ \left\lfloor \dfrac{1}{s_0}\right\rfloor+1 & \textit{if } s_0<1, \end{cases}$$

where the notation $\lfloor a\rfloor$ refers to the floor of $a\in\mathbb{R}$. Let Σ' be a proper subsector of Σ bisected by θ and opening larger than π/k_0[6]*. Then, the formal solution $\widetilde{z}(t,x)$ is k_0-summable in the direction θ and its k_0-sum $z(t,x)$ exists on $\Sigma'\times D_{r_1}$. Moreover,*

(1) *The following inequalities hold for all $k\in\{1,\dots,n\}$, all $p\geqslant 0$ and all $(t,x)\in\Sigma'\times D_{r_1}$:*

$$|\partial_t^p z^{[k]}(t,x)|\leqslant e^{nIA(I+C_{s_0})}A'B^p\Gamma(1+(s_0+1)(p+c))P(|x|). \tag{3.9}$$

(2) *The sum $z(t,x)$ satisfies the system of integral equations*

$$z-\sum_{i=1}^{I} b_i(t,x)\partial_t^{-i}z=h(t,x).$$

[6] Such a choice is already possible by definition of a proper subsector, see Footnote 5

Observe that the constant $e^{nIA(I+C_{s_0})}$ which occurs in (3.9) only depends on the homogenous part of the equation (3.7): number n of components, number I of coefficients and constant A defined by the coefficients $b_i(t,x)$. In particular, it is not depend on the bounds defined by the k_0-sum of the inhomogeneity $\widetilde{h}(t,x)$. Observe however that the latter are found as they are in the bounds (3.9).

Proof of Proposition 3.18. When all the coefficients $b_i(t,x)$ are zero, Proposition 3.18 is straightforward from the second inequality of (3.8) since System (3.7) is reduced to $z=\widetilde{h}$ and $e^{nIA(I+C_{s_0})}\geqslant 1$. When some of the coefficients $b_i(t,x)$ are nonzero, we proceed as in the proof of Theorem 3.11 throughout a fixed point method(7).

Let us set

$$\widetilde{Z}^{[k]}(t,x)=\sum_{\mu\geqslant 0}\widetilde{z}_\mu^{[k]}(t,x)$$

for all $k=1,\dots,n$, and let us choose the solution of System (3.7) recursively determined for all $\mu\geqslant 0$ by the relations

$$\widetilde{z}_{\mu+1}(t,x)=\sum_{i=1}^{I}b_i(t,x)\partial_t^{-i}\widetilde{z}_\mu \quad ,\widetilde{z}_\mu(t,x)=\begin{pmatrix}\widetilde{z}_\mu^{[1]}(t,x)\\ \vdots\\ \widetilde{z}_\mu^{[n]}(t,x)\end{pmatrix} \tag{3.10}$$

together with the initial condition $\widetilde{z}_0=\widetilde{h}$. Observe that $\widetilde{z}_\mu(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[[t]])$ for all $\mu\geqslant 0$. Observe also that the components $\widetilde{z}_\mu^{[k]}(t,x)$ of $\widetilde{z}_\mu(t,x)$ are of order $O(t^\mu)$ for all $\mu\geqslant 0$. Thereby, for any $k\in\{1,\dots,n\}$, $\widetilde{Z}^{[k]}(t,x)$ itself makes sense as a formal power series in t and x and, consequently, $\widetilde{Z}^{[k]}(t,x)=\widetilde{z}^{[k]}(t,x)$ by unicity.

Let us now denote by $z_0(t,x)$ the k_0-sum of $\widetilde{z}_0=\widetilde{h}$ in the direction θ and, for all $\mu>0$, let $z_\mu(t,x)$ be determined by the relations (3.10) in which all the $\widetilde{z}_\mu$ are replaced by z_μ. By construction, the components $z_\mu^{[k]}(t,x)$ of $z_\mu(t,x)$ are all defined and holomorphic on $\Sigma\times D_{r_1}$. Lemma 3.19 below provides some bounds on the successive derivatives of the functions $z_\mu^{[k]}(t,x)$.

(7) This method remains valid when all the $b_i(t,x)$'s are zero, but is, of course, much more complicated than the one given just before.

LEMMA 3.19. — *The inequalities*

$$|\partial_t^p z_\mu^{[k]}(t,x)| \leqslant A'B^p\Gamma(1+(s_0+1)(p+c))P(|x|) \times n^\mu \sum_{j=0}^{\min(\mu,p)} \frac{(IAC_{s_0})^j}{j!}\frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!} \tag{3.11}$$

hold for all $k\in\{1,\dots,n\}$, *all* $\mu,p\geqslant 0$ *and all* $(t,x)\in\Sigma\times D_{r_1}$.

Proof. — The proof proceeds by recursion on μ. The case $\mu=0$ is straightforward from the second inequality of (3.8) since we have $z_0=h$. Let us now suppose that the inequality (3.11) holds for a certain $\mu\geqslant 0$, and let us prove it for $z_{\mu+1}^{[k]}$ for any $k\in\{1,\dots,n\}$.

Applying the Leibniz Formula, we get

$$\partial_t^p z_{\mu+1}^{[k]}(t,x) = \sum_{i=1}^{I}\sum_{\ell=1}^{n} S_{p,\mu,k,\ell,i}(t,x)$$

for all $k\in\{1,\dots,n\}$, all $p\geqslant 0$ and all $(t,x)\in\Sigma\times D_{r_1}$, where the terms $S_{p,\mu,k,\ell,i}(t,x)$ are defined by

$$S_{p,\mu,k,\ell,i}(t,x) = \sum_{p_0+p_1=p}\binom{p}{p_0}\partial_t^{p_0}b_i^{[k,\ell]}(t,x)\partial_t^{p_1-i}z_\mu^{[\ell]}(t,x).$$

Lemma 3.20 just below provides then the sought inequality, which completes the proof of Lemma 3.19. □

LEMMA 3.20. — *Let* $p,\mu\geqslant 0$, $k,\ell\in\{1,\dots,n\}$ *and* $i\in\{1,\dots,I\}$. *Then,*

$$|S_{p,\mu,k,\ell,i}(t,x)| \leqslant A'B^p\Gamma(1+(s_0+1)(p+c))P(|x|)\sigma_{p,\mu}(t) \tag{3.12}$$

for all $(t,x)\in\Sigma\times D_{r_1}$, *with*

$$\sigma_{p,\mu}(t) = \frac{n^\mu}{I}\sum_{j=0}^{\min(\mu+1,p)}\frac{(IAC_{s_0})^j}{j!}\frac{(I^2A|t|)^{\mu+1-j}}{(\mu+1-j)!}.$$

Proof.

First case: $0\leqslant p<i$. — All the derivative orders p_1-i which occur in the definition of $S_{p,\mu,k,i}(t,x)$ are negative. Thereby, using the inequalities

$$|z_\mu^{[k]}(t,x)| \leqslant A'\Gamma(1+(s_0+1)c)P(|x|)\times n^\mu\frac{(I^2A|t|)^\mu}{\mu!} \tag{3.13}$$

for all $k=1,\dots,n$, the first inequality of (3.8), and the two inequalities $B\geqslant 1$ and $|t|\leqslant 1$, we get the relation

$$|S_{p,\mu,k,\ell,i}(t,x)| \leqslant A'B^p\Gamma(1+(s_0+1)(p+c))P(|x|)S'_{p,\mu}(t)$$

with

$$S'_{p,\mu}(t) = \sum_{p_0+p_1=p} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)c)}{\Gamma(1+(s_0+1)(p+c))} (nI^2)^\mu \frac{(A|t|)^{\mu+1}}{(\mu+1)!}.$$

Inequality (3.12) follows then from the technical Lemma 3.21 below which implies

$$\begin{aligned} S'_{p,\mu}(t) \leqslant (p+1)(nI^2)^\mu \frac{(A|t|)^{\mu+1}}{(\mu+1)!} \leqslant n^\mu I^{2\mu+1} \frac{(A|t|)^{\mu+1}}{(\mu+1)!} &= \frac{n^\mu}{I} \frac{(I^2A|t|)^{\mu+1}}{(\mu+1)!} \\ &\leqslant \sigma_{p,\mu}(t). \end{aligned}$$

Second case: $i \leqslant p < \mu + i$. — Let us write $S_{p,\mu,k,\ell,i}(t,x)$ in the form

$$\begin{aligned} S_{p,\mu,k,\ell,i}(t,x) = \sum_{\substack{p_0+p_1=p \\ p_1 \leqslant i-1}} \binom{p}{p_0} \partial_t^{p_0} b_i(t,x) \partial_t^{p_1-i} z_\mu^{[\ell]}(t,x) \\ + \sum_{\substack{p_0+p_1=p \\ p_1 \geqslant i}} \binom{p}{p_0} \partial_t^{p_0} b_i(t,x) \partial_t^{p_1-i} z_\mu^{[\ell]}(t,x). \end{aligned}$$

Observe that all the derivative orders $p_1 - i$ are negative in the first sum and satisfy $0 \leqslant p_1 - i \leqslant p - i < \mu$ in the second sum. Thereby, using the inequalities

$$\begin{aligned} |\partial_t^{p_1-i} z_\mu^{[\ell]}(t,x)| \leqslant A' B^{p_1-i} \Gamma(1+(s_0+1)(p_1-i+c)) P(|x|) \\ \times n^\mu \sum_{j=0}^{p_1-i} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!} \end{aligned}$$

for all $p_1 \in \{i, \ldots, p\}$ and, as in the first case, the inequalities (3.8), (3.13), $B \geqslant 1$ and $|t| \leqslant 1$, we get the relation

$$|S_{p,\mu,k,\ell,i}(t,x)| \leqslant A' B^p \Gamma(1+(s_0+1)(p+c)) P(|x|) S'_{p,\mu,i}(t)$$

with $S'_{p,\mu,i}(t) = S'_{p,\mu,i;1}(t) + S'_{p,\mu,i;2}(t)$, where

$$S'_{p,\mu,i;1}(t) = \sum_{\substack{p_0+p_1=p \\ p_1 \leqslant i-1}} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)c)}{\Gamma(1+(s_0+1)(p+c))} (nI^2)^\mu \frac{(A|t|)^{\mu+1}}{(\mu+1)!},$$

and where

$$\begin{aligned} S'_{p,\mu,i;2}(t) = \sum_{\substack{p_0+p_1=p \\ p_1 \geqslant i}} \sum_{j=0}^{p_1-i} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p_1-i+c))}{\Gamma(1+(s_0+1)(p+c))} \\ \times A n^\mu \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}. \end{aligned}$$

As in the first case, the technical Lemma 3.21 allows to bound the first sum $S'_{p,\mu,i;1}(t)$ of $S'_{p,\mu,i}(t)$:

$$S'_{p,\mu,i;1}(t) \leqslant i(nI^2)^{\mu}\frac{(A|t|)^{\mu+1}}{(\mu+1)!} \leqslant n^{\mu}I^{2\mu+1}\frac{(A|t|)^{\mu+1}}{(\mu+1)!} = \frac{n^{\mu}}{I}\frac{(I^2A|t|)^{\mu+1}}{(\mu+1)!}.$$

To bound the second sum $S'_{p,\mu,i;2}(t)$ of $S'_{p,\mu,i}(t)$, we use the second technical Lemma 3.22 below and we get:

$$\begin{aligned}
&S'_{p,\mu,i;2}(t)\\
&= \sum_{p_0=0}^{p-i}\sum_{j=0}^{p-i-p_0}\binom{p}{p_0}\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad \times An^{\mu}\frac{(IAC_{s_0})^j}{j!}\frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&= \sum_{j=0}^{p-i}\sum_{p_0=0}^{p-i-j}\binom{p}{p_0}\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad \times An^{\mu}\frac{(IAC_{s_0})^j}{j!}\frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&= \sum_{j=1}^{p-i+1}\sum_{p_0=0}^{p-i+1-j}\binom{p}{p_0}\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad \times An^{\mu}\frac{(IAC_{s_0})^{j-1}}{(j-1)!}\frac{(I^2A|t|)^{\mu+1-j}}{(\mu+1-j)!}\\
&\leqslant n^{\mu}\sum_{j=1}^{p-i+1}\frac{AC_{s_0}}{j}\frac{(IAC_{s_0})^{j-1}}{(j-1)!}\frac{(I^2A|t|)^{\mu+1-j}}{(\mu+1-j)!}\\
&= \frac{n^{\mu}}{I}\sum_{j=1}^{p-i+1}\frac{(IAC_{s_0})^{j}}{j!}\frac{(I^2A|t|)^{\mu+1-j}}{(\mu+1-j)!}.
\end{aligned}$$

Thereby, the term $S'_{p,\mu,i}(t)$ is bounded by

$$S'_{p,\mu,i}(t) \leqslant \frac{n^{\mu}}{I}\sum_{j=0}^{p-i+1}\frac{(IAC_s)^j}{j!}\frac{(I^2A|t|)^{\mu+1-j}}{(\mu+1-j)!} \leqslant \sigma_{p,\mu}(t)$$

since $p-i+1 \leqslant \min(\mu+1,p)$, which proves the inequality (3.12).

Third case: $p \geqslant \mu + i$. — We write now $S_{p,\mu,k,\ell,i}(t,x)$ in the form

$$S_{p,\mu,k,\ell,i}(t,x) = \sum_{\substack{p_0+p_1=p\\ p_1\leqslant i-1}} \binom{p}{p_0} \partial_t^{p_0} b_i(t,x) \partial_t^{p_1-i} z_\mu^{[\ell]}(t,x) + \sum_{\substack{p_0+p_1=p\\ i\leqslant p_1\leqslant \mu+i-1}} \binom{p}{p_0} \partial_t^{p_0} b_i(t,x) \partial_t^{p_1-i} z_\mu^{[\ell]}(t,x) + \sum_{\substack{p_0+p_1=p\\ p_1\geqslant \mu+i}} \binom{p}{p_0} \partial_t^{p_0} b_i(t,x) \partial_t^{p_1-i} z_\mu^{[\ell]}(t,x).$$

Here, the derivative orders $p_1 - i$ are negative in the first sum, satisfy $0 \leqslant p_1 - i < \mu$ in the second sum, and satisfy $p_1 - i \geqslant \mu$ in the third sum. Thereby, using the inequalities

$$|\partial_t^{p_1-i} z_\mu^{[\ell]}(t,x)| \leqslant A' B^{p_1-i} \Gamma(1+(s_0+1)(p_1-i+c)) P(|x|) \times n^\mu \sum_{j=0}^{p_1-i} \frac{(IAC_s)^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}$$

for all $p_1 \in \{i, \dots, \mu+i-1\}$, the inequalities

$$|\partial_t^{p_1-i} z_\mu^{[\ell]}(t,x)| \leqslant A' B^{p_1-i} \Gamma(1+(s_0+1)(p_1-i+c)) P(|x|) \times n^\mu \sum_{j=0}^{\mu} \frac{(IAC_s)^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}$$

for all $p_1 \in \{\mu+i, \dots, p\}$, and, as in the two previous cases, the inequalities (3.8), (3.13), $B \geqslant 1$ and $|t| \leqslant 1$, we get the relation

$$|S_{p,\mu,k,\ell,i}(t,x)| \leqslant A' B^\ell \Gamma(1+(s_0+1)(p+c)) P(|x|) S'_{p,\mu,i}(t)$$

with $S'_{p,\mu,i}(t) = S'_{p,\mu,i;1}(t) + S'_{p,\mu,i;2}(t)$, where

$$S'_{p,\mu,i;1}(t) = \sum_{\substack{p_0+p_1=p\\ p_1\leqslant i-1}} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)c)}{\Gamma(1+(s_0+1)(p+c))} (nI^2)^\mu \frac{(A|t|)^{\mu+1}}{(\mu+1)!},$$

and where

$$
\begin{aligned}
&S'_{p,\mu,i;2}(t)\\
&= \sum_{\substack{p_0+p_1=p\\ i\leqslant p_1\leqslant \mu+i-1}} \sum_{j=0}^{p_1-i} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p_1-i+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad\qquad \times An^{\mu} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&\qquad + \sum_{\substack{p_0+p_1=p\\ p_1\geqslant \mu+i}} \sum_{j=0}^{\mu} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p_1-i+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad\qquad \times An^{\mu} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&= \sum_{p_0=p-\mu-i+1}^{p-i} \sum_{j=0}^{p-i-p_0} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad\qquad \times An^{\mu} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&\qquad + \sum_{p_0=0}^{p-\mu-i} \sum_{j=0}^{\mu} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad\qquad \times An^{\mu} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&= \sum_{j=0}^{\mu} \sum_{p_0=p-\mu-i+1}^{p-i-j} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad\qquad \times An^{\mu} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&\qquad + \sum_{j=0}^{\mu} \sum_{p_0=0}^{p-\mu-i} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad\qquad \times An^{\mu} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}\\
&= \sum_{j=0}^{\mu} \sum_{p_0=0}^{p-i-j} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad\qquad \times An^{\mu} \frac{(IAC_{s_0})^j}{j!} \frac{(I^2A|t|)^{\mu-j}}{(\mu-j)!}
\end{aligned}
$$

$$= \sum_{j=1}^{\mu+1} \sum_{p_0=0}^{p-i+1-j} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))} \times An^{\mu} \frac{(IAC_{s_0})^{j-1}}{(j-1)!} \frac{(I^2A|t|)^{\mu+1-j}}{(\mu+1-j)!}.$$

Applying then the two technical Lemmas 3.21 and 3.22 as in the previous case to bound the two sums of $S'_{p,\mu,i}(t)$, we finally get

$$S'_{p,\mu,i}(t) \leqslant \frac{n^{\mu}}{I} \sum_{j=0}^{\mu+1} \frac{(IAC_s)^j}{j!} \frac{(I^2A|t|)^{\mu+1-j}}{(\mu+1-j)!};$$

hence $S'_{p,\mu,i}(t) \leqslant \sigma_{p,\mu}(t)$ since $p \geqslant \mu+i \geqslant \mu+1$. This proves the inequality (3.12) and completes the proof of Lemma 3.20. □

Lemma 3.21. — *Let $p \geqslant 0$ and $p_0 \in \{0,\dots,p\}$. Then,*

$$\binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)c)}{\Gamma(1+(s_0+1)(p+c))} \leqslant 1.$$

Proof. — Lemma 3.21 is clear when $p=0$. To prove it in the case $p \geqslant 1$, let us first observe that the increase of the Gamma function on $[2,+\infty[$ and the relation $\Gamma(a) \leqslant 1 = \Gamma(2)$ for all $a \in [1,2]$ imply

$$\Gamma(a) \leqslant \Gamma(b) \quad \text{for all } a \geqslant 1 \text{ and all } b \geqslant \max(2,a), \tag{3.14}$$

which leads us to the inequality

$$\Gamma(1+(s_0+1)c) \leqslant \Gamma(1+(s_0+1)(p-p_0+c)).$$

Thereby, applying the Vandermonde's Inequality (Proposition 5.5) and the classical properties on the variations and the bounds of the generalized binomial coefficients (Propositions 5.6 and 5.7), we get

$$\begin{aligned} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)c)}{\Gamma(1+(s_0+1)(p+c))} &\leqslant \frac{\binom{p}{p_0}}{\binom{(s_0+1)(p+c)}{(s_0+1)p_0}} \\ &\leqslant \frac{\binom{p}{p_0}}{\binom{(s_0+1)p}{(s_0+1)p_0}} \\ &\leqslant \frac{1}{\binom{s_0p}{s_0p_0}} \\ &\leqslant 1, \end{aligned}$$

which ends the proof. □

Lemma 3.22. — *Let $i \geqslant 1$, $j \geqslant 1$ and $p \geqslant i+j-1$. Then,*

$$\sum_{p_0=0}^{p-i+1-j} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-i-p_0+c))}{\Gamma(1+(s_0+1)(p+c))} \leqslant \frac{C_{s_0}}{j}.$$

Proof. — Let us denote by $S_{i,j,p}$ the sum of the left hand-side and let us first start by proving the inequality

$$S_{i,j,p} \leqslant S_{1,j,p} \quad \text{for all } p \geqslant i+j-1.$$

The latter is clear when $i=1$. When $i \geqslant 2$, let us observe that the inequalities

$$\begin{cases} 1+(s_0+1)(p-1-p_0+c) \geqslant \max(2, 1+(s_0+1)(p-i-p_0+c)) \\ 1+(s_0+1)(p-i-p_0+c) \geqslant 1 \end{cases}$$

hold for all $p_0 = 0, \dots, p-i+1-j$. Thereby, applying the relation (3.14), we get

$$S_{i,j,p} \leqslant \sum_{p_0=0}^{p-i+1-j} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-1-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}$$

and we conclude by using the fact that $p-i+1-j < p-j$, that $p \geqslant i+j-1$ implies $p \geqslant j$, and that all the terms of the sum $S_{1,j,p}$ are positive.

Let us now prove, for all $p \geqslant j$, the inequality

(3.15)
$$\sum_{p_0=0}^{p-j} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-1-p_0+c))}{\Gamma(1+(s_0+1)(p+c))} \leqslant \frac{C_{s_0}}{j}.$$

When $p=j=1$, the left hand-side of (3.15) is reduced to the quotient

$$\frac{\Gamma(1+(s_0+1)c)}{\Gamma(1+(s_0+1)(1+c))}$$

which is less than 1. Indeed, the relation (3.14) applied to the inequalities

$$\begin{cases} 1+(s_0+1)(1+c) \geqslant \max(2, 1+(s_0+1)c) \\ 1+(s_0+1)c \geqslant 1 \end{cases}$$

implies $\Gamma(1+(s_0+1)(1+c)) \geqslant \Gamma(1+(s_0+1)c)$. The inequality (3.15) follows then from the relation $C_{s_0} \geqslant 1$.

Let us now suppose $(p,j) \neq (1,1)$. Applying the pawn formula and the recurrence relation $\Gamma(1+z) = z\Gamma(z)$, we first have

$$\begin{aligned}
&\binom{p}{p_0}\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-1-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\qquad = \frac{p}{p-p_0}\binom{p-1}{p_0}\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-1-p_0+c))}{(s_0+1)(p+c)\Gamma((s_0+1)(p+c))}\\
&\qquad \leqslant \frac{1}{j}\binom{p-1}{p_0}\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-1-p_0+c))}{\Gamma((s_0+1)(p+c))}
\end{aligned}$$

for all $p_0 = 0, \ldots, p-j$. Observe that these relations make sense since $j \geqslant 1$. Observe also that the condition $(p,j) \neq (1,1)$ implies $p \geqslant 2$; hence, the inequalities

$$(s_0+1)(p+c) \geqslant 1 + (s_0+1)(p-1+c) \geqslant 2.$$

Thereby, using the increase of the Gamma function on $[2;+\infty[$ and applying successively the classical properties on the variations of the generalized binomial coefficients (Proposition 5.6) and the Vandermonde's Inequality (Proposition 5.5) and then, we get

$$\begin{aligned}
\binom{p}{p_0}\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-1-p_0+c))}{\Gamma(1+(s_0+1)(p+c))} &\leqslant \frac{1}{j}\frac{\binom{p-1}{p_0}}{\binom{(s_0+1)(p-1+c)}{(s_0+1)p_0}}\\
&\leqslant \frac{1}{j}\frac{\binom{p-1}{p_0}}{\binom{(s_0+1)(p-1)}{(s_0+1)p_0}}\\
&\leqslant \frac{1}{j}\frac{1}{\binom{s_0(p-1)}{s_0p_0}},
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\sum_{p_0=0}^{p-j}\binom{p}{p_0}&\frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)(p-1-p_0+c))}{\Gamma(1+(s_0+1)(p+c))}\\
&\leqslant \frac{1}{j}\sum_{p_0=0}^{p-j}\frac{1}{\binom{s_0(p-1)}{s_0p_0}}\\
&\leqslant \frac{1}{j}\sum_{p_0=0}^{p-1}\frac{1}{\binom{s_0(p-1)}{s_0p_0}}
\end{aligned}$$

since all the terms are positive. The inequality (3.15) follows then from Proposition 5.8, which ends the proof of Lemma 3.22. □

To end the proof of Proposition 3.18, we shall now prove that the series $\sum_{\mu\geqslant 0} z_\mu^{[k]}(t,x)$ are convergent for all $k=1,\dots,n$ and that their respective sums $z^{[k]}(t,x)$ are the k_0-sum of $\widetilde{z}^{[k]}(t,x)$ in the direction θ. From Lemma 3.19, we derive the inequalities

$$\begin{aligned}(3.16)\quad &|\partial_t^p z_\mu^{[k]}(t,x)|\\ &\leqslant A'B^p\Gamma(1+(s_0+1)(p+c))P(|x|)\sum_{j=0}^{\min(\mu,p)}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}\\ &\leqslant A'B^p\Gamma(1+(s_0+1)(p+c))P(r_1)\sum_{j=0}^{\min(\mu,p)}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}\end{aligned}$$

for all $k\in\{1,\dots,n\}$, all $\mu,p\geqslant 0$ and all $(t,x)\in\Sigma\times D_{r_1}$. Indeed, the radius of Σ is less 1 and the coefficients of the polynomial $P(X)$ are all nonnegative.

Lemma 3.23. — *Let $\alpha_{\mu,p}$ denote the positive real numbers defined for all $\mu,p\geqslant 0$ by*

$$\alpha_{\mu,p}=\sum_{j=0}^{\min(\mu,p)}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}.$$

Let $p\geqslant 0$. Then, the series $\sum_{\mu\geqslant 0}\alpha_{\mu,p}$ is convergent and its sum satisfies the relations:

$$(3.17)\qquad \sum_{\mu=0}^{+\infty}\alpha_{\mu,p}=e^{nI^2A}\sum_{j=0}^{p}\frac{(nIAC_{s_0})^j}{j!}\leqslant e^{nIA(I+C_{s_0})}.$$

Proof. — For any $M>p$, we have

$$\begin{aligned}&\sum_{\mu=0}^{M}\alpha_{\mu,p}\\ &=\sum_{\mu=0}^{p}\sum_{j=0}^{\min(\mu,p)}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}+\sum_{\mu=p+1}^{M}\sum_{j=0}^{\min(\mu,p)}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}\\ &=\sum_{\mu=0}^{p}\sum_{j=0}^{\mu}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}+\sum_{\mu=p+1}^{M}\sum_{j=0}^{p}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}\\ &=\sum_{j=0}^{p}\sum_{\mu=j}^{p}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}+\sum_{j=0}^{p}\sum_{\mu=p+1}^{M}\frac{(nIAC_{s_0})^j}{j!}\frac{(nI^2A)^{\mu-j}}{(\mu-j)!}\end{aligned}$$

$$= \sum_{j=0}^{p} \sum_{\mu=0}^{p-j} \frac{(nIAC_{s_0})^j}{j!} \frac{(nI^2A)^\mu}{\mu!} + \sum_{j=0}^{p} \sum_{\mu=p+1-j}^{M-j} \frac{(nIAC_{s_0})^j}{j!} \frac{(nI^2A)^\mu}{\mu!}$$
$$= \sum_{j=0}^{p} \left(\frac{(nIAC_{s_0})^j}{j!} \sum_{\mu=0}^{M-j} \frac{(nI^2A)^\mu}{\mu!} \right).$$

The convergence and the first equality of (3.17) stems then from the identiy

$$\lim_{M\to+\infty} \sum_{\mu=0}^{M-j} \frac{(nI^2A)^\mu}{\mu!} = e^{nI^2A} \quad \text{for all } j \in \{0,\dots,p\}.$$

As for the second inequality of (3.17), it follows from the relation

$$\sum_{j=0}^{p} \frac{(nIAC_{s_0})^j}{j!} \leqslant \sum_{j=0}^{+\infty} \frac{(nIAC_{s_0})^j}{j!} = e^{nIAC_{s_0}},$$

which completes the proof. □

Applying Lemma 3.23, we deduce from the second inequality of (3.16) that the series $\sum_{\mu\geqslant 0} \partial_t^p z_\mu^{[k]}(t,x)$ are normally convergent on $\Sigma \times D_{r_1}$ for all $k \in \{1,\dots,n\}$ and all $p \geqslant 0$ and satisfy the inequalities

$$\begin{aligned}\sum_{\mu\geqslant 0} |\partial_t^p z_\mu^{[k]}(t,x)| &\leqslant A'P(r_1)e^{nIA(I+C_s)}B^p\Gamma(1+(s_0+1)(p+c)) \\ &\leqslant A'P(r_1)\Gamma(1+(s_0+1)c)e^{nIA(I+C_s)}B^p\Gamma(1+(s_0+1)p) \\ &\qquad \times \binom{(s_0+1)(p+c)}{(s_0+1)c}\end{aligned}$$

for all $(t,x) \in \Sigma \times D_{r_1}$. Denoting then by $\lceil s_0 \rceil$ (resp. $\lceil c \rceil$) the ceil of s_0 (resp. c), we deduce from the classical properties on the bounds of the generalized binomial coefficients (Proposition 5.7) that

$$\sum_{\mu\geqslant 0} |\partial_t^p z_\mu^{[k]}(t,x)| \leqslant A''B''^p\Gamma(1+(s_0+1)p)$$

for all $k \in \{1,\dots,n\}$, all $p \geqslant 0$ and all $(t,x) \in \Sigma \times D_{r_1}$, with

$$A'' = A'P(r_1)\Gamma(1+(s_0+1)c)e^{nIA(I+C_s)}4^{(\lceil s_0\rceil+1)\lceil c\rceil} \quad \text{and}$$
$$B'' = 4^{\lceil s_0\rceil+1}B.$$

In particular, choosing any proper subsector $\Sigma' \Subset \Sigma$ bisected by θ and opening larger than π/k_0, the sums $z^{[k]}(t,x)$ of the series $\sum_{\mu\geqslant 0} z_\mu^{[k]}(t,x)$ are all well-defined, holomorphic on $\Sigma' \times D_{r_1}$, and satifies the inequalities

$$|\partial_t^p z^{[k]}(t,x)| \leqslant A''B''^p\Gamma(1+(s_0+1)p) \tag{3.18}$$

for all $k \in \{1, \dots, n\}$, $p \geqslant 0$ and $(t,x) \in \Sigma' \times D_{r_1}$. Hence, Conditions 1 and 3 of Definition 3.1 hold.

To prove the second condition of Definition 3.1, we proceed as follows. The Removable Singularities Theorem implies the existence of $\lim_{\substack{t\to 0\\ t\in\Sigma'}} \partial_t^p z^{[k]}(t,x)$ for all $x \in D_{r_1}$ (use here that the inequalities (3.18) are actually valid on $\Sigma \times D_{r_1}$) and, thereby, the existence of the Taylor series of $z^{[k]}$ at 0 on Σ' for all $x \in D_{r_1}$ (see for instance [51, Chapter 4]; see also [25, Corollary 1.1.3.3] and [20, Proposition 1.1.11]). On the other hand, considering recurrence relations (3.10) with $z_\mu^{[k]}(t,x)$ and the k_0-sum $h(t,x)$ instead of $\widetilde{z}_\mu^{[k]}(t,x)$ and $\widetilde{h}(t,x)$, it is clear that

$$z(t,x) = \begin{pmatrix} z^{[1]}(t,x) \\ \vdots \\ z^{[n]}(t,x) \end{pmatrix}$$

satisfies System (3.7) with right-hand side $h(t,x)$ in place of $\widetilde{h}(t,x)$ and, consequently, so does its Taylor series. Then, since System (3.7) has a unique formal solution $\widetilde{z}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[\![t]\!])$, we then conclude that the Taylor expansions of each of its components $z^{[k]}(t,x)$ are the formal series $\widetilde{z}^{[k]}(t,x)$. Hence, Condition 2 of Definition 3.1 holds.

This proves the k_0-summability of $\widetilde{z}(t,x)$ in the direction θ and the existence of its k_0-sum on $\Sigma' \times D_{r_1}$, where Σ' stands for any proper subsector of Σ which is bisected by θ and opening larger than π/k_0. As for the two additional properties on $z(t,x)$, the first one stems from Lemma 3.23 and the first inequality of (3.16), and the second one is straightforward from Corollary 3.3.

This achieves the proof of Proposition 3.18 and we can now return to the study of the formal series $\widetilde{w}_\mu^{[k]}(t,x)$, $k = 1, \dots, n$, defined by the relations (3.4).

3.3.3. k_0-summability of $\widetilde{w}_\mu(t,x)$ and some estimates

Let us first recall that the coefficients $B_{i,q}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_0'} \times D_{\rho_1'}))$ are $n \times n$-matrices with analytic entries in $D_{\rho_0'} \times D_{\rho_1'}$ for some convenient radii $0 < \rho_0' \leqslant \rho_0$ and $0 < \rho_1' \leqslant \rho_1$, and that the inhomogeneity $\widetilde{g}(t,x) \in \mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[\![t]\!])$ is k_0-summable in the direction θ. Thereby, denoting its k_0-sum by $g(t,x)$, we derive from Definitions 3.1 and 3.10 that there exist a sector

$$\Sigma_{\theta,>\pi/k_0} = \left\{ t \in \mathbb{C}^* \,:\, |t| < r \text{ and } \theta - \frac{\pi}{2k_0} - \varepsilon < \arg(t) < \theta + \frac{\pi}{2k_0} + \varepsilon \right\}, \quad \varepsilon > 0$$

bisected by θ, opening larger than π/k_0 and radius $r \leqslant \min(1, \rho_0')$, a radius $r_1 \leqslant \min(1, \rho_1')$, and two positive constants $C > 0$ and $K \geqslant 1$ such that[8]

$$\begin{cases} |\partial_t^p B_{i,q}^{[k,\ell]}(t,x)| \leqslant CK^p \Gamma(1 + (s_0+1)p) \\ |\partial_t^p g^{[k]}(t,x)| \leqslant CK^p \Gamma(1 + (s_0+1)p) \end{cases} \tag{3.19}$$

for all $k, \ell \in \{1, \dots, n\}$, all $p \geqslant 0$, all $i \in \mathcal{K}$, all $q \in Q_i$, and all $(t,x) \in \Sigma_{\theta,>\pi/k_0} \times D_{r_1}$.

Let us now fix a sector

$$\Sigma'_{\theta,>\pi/k_0} = \left\{ t \in \mathbb{C}^* \,:\, |t| < r' \text{ and } \theta - \frac{\pi}{2k_0} - \varepsilon' < \arg(t) < \theta + \frac{\pi}{2k_0} + \varepsilon' \right\}$$

with $0 < r' < r$ and $0 < \varepsilon' < \varepsilon$, and let us define for all $\mu \geqslant 0$ the sectors

$$\Sigma_\mu = \left\{ t \in \mathbb{C}^* \,:\, |t| < r_\mu \text{ and } \theta - \frac{\pi}{2k_0} - \varepsilon_\mu < \arg(t) < \theta + \frac{\pi}{2k_0} + \varepsilon_\mu \right\}$$

with

$$r_\mu = r' + \frac{r - r'}{2(\mu+1)} \quad \text{and} \quad \varepsilon_\mu = \varepsilon' + \frac{\varepsilon - \varepsilon'}{2(\mu+1)}.$$

By construction, all the sectors $\Sigma'_{\theta,>\pi/k_0}$ and Σ_μ are bisected by θ, opening larger than π/k_0 and radius less 1. Moreover, they satisfy the inclusions

$$\Sigma'_{\theta,>\pi/k_0} \Subset \Sigma_{\mu+1} \Subset \Sigma_\mu \Subset \Sigma_{\theta,>\pi/k_0}$$

for all $\mu \geqslant 0$.

The goal of this section is to prove the following result.

Proposition 3.24. — *Let C_{s_0} be the positive real number defined by*

$$C_{s_0} = s_0'(2 + \Gamma(s_0 s_0')) \quad \textit{with} \quad s_0' = \begin{cases} 1 & \textit{if } s_0 \geqslant 1 \\ \left\lfloor \dfrac{1}{s_0} \right\rfloor + 1 & \textit{if } s_0 < 1, \end{cases}$$

where the notation $\lfloor a \rfloor$ refers to the floor of $a \in \mathbb{R}$.

---

[8] Observe that such choices on the radii r and r_1, and on the constant K are already possible due to the definition of the k_0-summability (see Definition 3.1).

Let $(P_\mu(x))_{\mu\geqslant 0}$ be the sequence of polynomials in $\mathbb{R}^+[x]$ recursively determined from $P_0(x)\equiv 1$ by the relations

$$P_{\mu+1}(x)=CC_{s_0}\partial_x^{-q^\star}P_\mu(x)+CC_{s_0}\sum_{i\in\mathcal{K}}\sum_{q\in Q_i'}\frac{(\mu q^\star+q^\star-q)!}{(\mu q^\star+q^\star)!}\partial_x^{q-q^\star}P_\mu(x). \tag{3.20}$$

Then, for all $\mu\geqslant 0$, the element $\widetilde{w}_\mu(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[\![t]\!])$ is k_0-summable in the direction θ and its k_0-sum $w_\mu(t,x)$ exists on $\Sigma_\mu\times D_{r_1}$. Moreover,

(1) *The following inequalities hold for all $k\in\{1,\dots,n\}$, all $p,\mu\geqslant 0$ and all $(t,x)\in\Sigma_\mu\times D_{r_1}$:*

$$|\partial_t^p w_\mu^{[k]}(t,x)|\leqslant n^\mu e^{n(\mu+1)i^\star C(i^\star+C_{s_0})}CK^{p+\mu(\kappa-i^\star)}\times\Gamma(1+(s_0+1)p+\mu q^\star)P_\mu(|x|). \tag{3.21}$$

(2) *The sums $w_\mu(t,x)$ satisfy the recurrence relations*

$$\begin{cases}D_{q^\star}w_0(t,x)=g(t,x),\\ D_{q^\star}w_{\mu+1}=H(t,x,w_\mu)\quad\textit{for all } \mu\geqslant 0\end{cases}. \tag{3.22}$$

Proof. — The proof proceeds by recursion on $\mu\geqslant 0$.

The case $\mu=0$ stems from Proposition 3.18 applied with $\widetilde{z}=\widetilde{w}_0$, $\widetilde{h}=\widetilde{g}$, $I=i^\star$ (we have indeed $i\in\mathcal{K}'_{q^\star}\Rightarrow i^\star-i\in\{1,\dots,i^\star\}$), $\Sigma=\Sigma_{\theta,>\pi/k_0}$, $\Sigma'=\Sigma_0$, $A=A'=C$, $B=K$, $c=0$ and $P=P_0$.

Let us now suppose that Proposition 3.24 holds for any element $\widetilde{w}_{\mu'}(t,x)$ with $\mu'\in\{0,\dots,\mu\}$ for a certain $\mu\geqslant 0$, and let us prove it for $\widetilde{w}_{\mu+1}(t,x)$.

By assumption, the element $\widetilde{H}(t,x,\widetilde{w}_\mu)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1'})[\![t]\!])$ is k_0-summable in the direction θ and its k_0-sum is the function $H(t,x,w_\mu)$ (Corollary 3.3) which is holomorphic on $\Sigma_\mu\times D_{r_1}$. Besides, denoting by $C_{p,\mu}$ the positive constant

$$C_{p,\mu}=n^{\mu+1}e^{n(\mu+1)i^\star C(i^\star+C_{s_0})}CK^{p+(\mu+1)(\kappa-i^\star)}\Gamma(1+(s_0+1)p+(\mu+1)q^\star),$$

the two technical Lemmas 3.25 and 3.26 below provide us the inequality

$$|\partial_t^p H^{[k]}(t,x,w_\mu)|\leqslant C_{p,\mu}P_{\mu+1}(|x|)$$

for all $k\in\{1,\dots,n\}$, all $p\geqslant 0$ and all $(t,x)\in\Sigma_\mu\times D_{r_1}$. The sought result on $\widetilde{w}_{\mu+1}(t,x)$ follows then by applying Proposition 3.18 with $\widetilde{z}=\widetilde{w}_{\mu+1}$, $\widetilde{h}=H(t,x,\widetilde{w}_\mu)$, $I=i^\star$, $\Sigma=\Sigma_\mu$, $\Sigma'=\Sigma_{\mu+1}$, $A=C$, $A'=n^{\mu+1}e^{n(\mu+1)i^\star C(i^\star+C_{s_0})}CK^{(\mu+1)(\kappa-i^\star)}$, $B=K$, $c=(\mu+1)(\kappa-i^\star)$ (recall that $(s_0+1)(\kappa-i^\star)=q^\star$, see Lemma 3.8 (1)) and $P=P_{\mu+1}$. This completes the proof of Proposition 3.24. □

Lemma 3.25. — *Let $\mu \geqslant 0$ and $\varphi_\mu(t,x)$ the holomorphic function defined on $\Sigma_\mu \times D_{r_1}$ by*

$$\varphi_\mu(t,x) = B_{i^\star,q^\star}(t,x)\partial_t^{\kappa-i^\star}\partial_x^{-q^\star} w_\mu(t,x).$$

Then, the following inequalities

$$|\partial_t^p \varphi_\mu^{[k]}(t,x)| \leqslant C_{p,\mu} \times CC_{s_0}\partial_x^{-q^\star} P_\mu(|x|)$$

hold for all $k \in \{1,\ldots,n\}$, all $p \geqslant 0$ and all $(t,x) \in \Sigma_\mu \times D_{r_1}$.

Proof. — Let $k \in \{1,\ldots,n\}$ and $p \geqslant 0$. From the Leibniz Formula, we first get

$$\partial_t^p \varphi_\mu^{[k]}(t,x) = \sum_{p_0+p_1=p}\sum_{\ell=1}^{n}\binom{p}{p_0}\partial_t^{p_0}B_{i^\star,q^\star}^{[k,\ell]}(t,x)\partial_t^{\kappa-i^\star+p_1}\partial_x^{-q^\star} w_\mu^{[\ell]}(t,x)$$

for all $(t,x) \in \Sigma_\mu \times D_{r_1}$. The inequalities (3.19) and (3.21) (observe that $\kappa - i^\star + p_1 \geqslant 0$) and the identity $(s+1)(\kappa - i^\star) = q^\star$ (Lemma 3.8(1)) provide us the relations

$$\begin{aligned} &|\partial_t^p \varphi_\mu^{[k]}(t,x)| \\ &\quad \leqslant \sum_{p_0+p_1=p}\sum_{\ell=1}^{n}\binom{p}{p_0} CK^{p_0}\Gamma(1+(s_0+1)p_0)n^\mu e^{n(\mu+1)i^\star C(i^\star+C_{s_0})} \\ &\qquad \times CK^{p_1+(\mu+1)(\kappa-i^\star)}\Gamma(1+(s_0+1)p_1+(\mu+1)q^\star)\partial_x^{-q^\star}P_\mu(|x|) \end{aligned}$$

for all $(t,x) \in \Sigma_\mu \times D_{r_1}$, which we rewrite in the form

$$|\partial_t^p \varphi_\mu^{[k]}(t,x)| \leqslant C_{p,\mu} \times C\partial_x^{-q^\star}P_\mu(|x|)\sum_{p_0=0}^{p}\frac{\binom{p}{p_0}}{\binom{(s_1+1)p+(\mu+1)q^\star}{(s_0+1)p_0}}.$$

Lemma 3.25 follows by applying successively the Vandermonde's Inequality (Proposition 5.5)

$$\begin{aligned} \binom{(s_0+1)p+(\mu+1)q^\star}{(s_0+1)p_0} &\geqslant \binom{s_0 p}{s_0 p_0}\binom{p}{p_0}\binom{(\mu+1)q^\star}{(\mu+1)q^\star} \\ &= \binom{s_0 p}{s_0 p_0}\binom{p}{p_0} \end{aligned}$$

and Proposition 5.8. □

Lemma 3.26. — *Let $i \in \mathcal{K}$, $q \in Q_i'$, $\mu \geqslant 0$ and $\chi_{i,q,\mu}(t,x)$ the holomorphic function defined on $\Sigma_\mu \times D_{r_1}$ by*

$$\chi_{i,q,\mu}(t,x) = B_{i,q}(t,x)\partial_t^{i-i^\star}\partial_x^{q-q^\star} w_\mu(t,x).$$

Then, the following inequalities

$$|\partial_t^p \chi_{i,q,\mu}^{[k]}(t,x)| \leqslant C_{p,\mu} \times CC_{s_0} \frac{(\mu q^\star + q^\star - q)!}{(\mu q^\star + q^\star)!} \partial_x^{q-q^\star} P_\mu(|x|)$$

hold for all $k \in \{1,\dots,n\}$, all $p \geqslant 0$ and all $(t,x) \in \Sigma_\mu \times D_{r_1}$.

Proof. — Let $k \in \{1,\dots,n\}$ and $p \geqslant 0$. From the Leibniz Formula, we first get

$$\partial_t^p \chi_{i,q,\mu}^{[k]}(t,x) = \sum_{p_0+p_1=p} \sum_{\ell=1}^{n} \binom{p}{p_0} \partial_t^{p_0} B_{i,q}^{[k,\ell]}(t,x) \partial_t^{i-i^\star+p_1} \partial_x^{q-q^\star} w_\mu^{[\ell]}(t,x)$$

for all $(t,x) \in \Sigma_\mu \times D_{r_1}$. Let us now observe that the inequality (3.21) provides us the relation

$$|\partial_t^{i-i^\star+p_1} w_\mu^{[\ell]}(t,x)| \leqslant n^\mu e^{n(\mu+1)i^\star C(i^\star+C_{s_0})} CK^{p_1+\mu(\kappa-i^\star)+i-i^\star} \times \Gamma(1+(s_0+1)(i-i^\star+p_1)+\mu q^\star) P_\mu(|x|)$$

for all $\ell \in \{1,\dots,n\}$ and all $(t,x) \in \Sigma_\mu \times D_{r_1}$ when $i - i^\star + p_1 \geqslant 0$, and the relation

$$|\partial_t^{i-i^\star+p_1} w_\mu^{[\ell]}(t,x)| \leqslant n^\mu e^{n(\mu+1)i^\star C(i^\star+C_{s_0})} CK^{\mu(\kappa-i^\star)} \Gamma(1+\mu q^\star) P_\mu(|x|)$$

for all $\ell \in \{1,\dots,n\}$ and all $(t,x) \in \Sigma_\mu \times D_{r_1}$ when $i - i^\star + p_1 < 0$ (use here the fact that $|t| < r_\mu \leqslant 1$). Using then the inequalities $K \geqslant 1$ and $i - i^\star < \kappa - i^\star$, the relation $(s_0+1)(i-i^\star) \leqslant q^\star - q$ (Lemma 3.8 (2)) and the inequality (3.14), we finally get

$$|\partial_t^{i-i^\star+p_1} w_\mu^{[\ell]}(t,x)| \leqslant n^\mu e^{n(\mu+1)i^\star C(i^\star+C_{s_0})} CK^{p_1+(\mu+1)(\kappa-i^\star)} \times \Gamma(1+(s_0+1)p_1+\mu q^\star+q^\star-q) P_\mu(|x|)$$

for all $\ell \in \{1,\dots,n\}$, all $p_1 \in \{0,\dots,p\}$ and all $(t,x) \in \Sigma_\mu \times D_{r_1}$. This brings next us to the relations

$$\begin{aligned}|\partial_t^p \chi_{i,q,\mu}^{[k]}(t,x)| \leqslant \sum_{p_0+p_1=p} \sum_{\ell=1}^{n} \binom{p}{p_0} & CK^{p_0} \Gamma(1+(s_0+1)p_0) \\ & \times n^\mu e^{n(\mu+1)i^\star C(i^\star+C_{s_0})} CK^{p_1+(\mu+1)(\kappa-i^\star)} \\ & \times \Gamma(1+(s_0+1)p_1+\mu q^\star+q^\star-q) \partial_x^{q-q^\star} P_\mu(|x|)\end{aligned}$$

for all $(t,x) \in \Sigma_\mu \times D_{r_1}$, which we first rewrite in the form

$$\begin{aligned}|\partial_t^p \chi_{i,q,\mu}^{[k]}(t,x)| &\leqslant C_{p,\mu} \times C \partial_x^{q-q^\star} P_\mu(|x|) \\ &\times \sum_{p_0+p_1=p} \binom{p}{p_0} \frac{\Gamma(1+(s_0+1)p_0)\Gamma(1+(s_0+1)p_1+\mu q^\star+q^\star-q)}{\Gamma(1+(s_0+1)p+(\mu+1)q^\star)},\end{aligned}$$

then, in the form

$$|\partial_t^p \chi_{i,q,\mu}^{[k]}(t,x)| \leqslant C_{p,\mu} \times C\partial_x^{q-q^\star} P_\mu(|x|) \\ \times \sum_{p_0=0}^{p} \frac{\binom{p}{p_0}}{q!\binom{(s_0+1)p+(\mu+1)q^\star}{(s_0+1)p_0,(s_0+1)(p-p_0)+(\mu+1)q^\star-q,q}}.$$

Lemma 3.26 follows by applying successively the Vandermonde's Inequality (Proposition 5.5)

$$\binom{(s_0+1)p+(\mu+1)q^\star}{(s_0+1)p_0,(s_0+1)(p-p_0)+(\mu+1)q^\star-q,q} \\ \geqslant \binom{s_0 p}{s_0 p_0}\binom{p}{p_0}\binom{(\mu+1)q^\star}{q}$$

and Proposition 5.8. ☐

3.3.4. Bounds of the polynomials P_μ

In this section, we are interested in the polynomials P_μ introduced in Proposition 3.24. Our goal is to prove the following.

Proposition 3.27. — *Let B be the positive real number defined by*

$$B = CC_{s_0}\left(q^{\star q^\star} + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i'} q^{\star q^\star - q}\right).$$

Then, the inequalities

$$0 \leqslant P_\mu(x) \leqslant \frac{(Bx)^\mu}{(\mu q^\star)!}$$

hold for all $\mu \geqslant 0$ and all $x\in[0,1]$.

Proof. — The left inequality is obvious since all the coefficients of the $P_\mu(x)$'s are nonnegative. The right inequality is proved by recursion on $\mu \geqslant 0$ as follows.

The case $\mu = 0$ is clear since $P_0(x) \equiv 1$. Let us now suppose that Proposition 3.27 holds for all the polynomials $P_{\mu'}(x)$ with $\mu' \in \{0,\ldots,\mu\}$ for a certain $\mu \geqslant 0$, and let us prove it for the polynomial $P_{\mu+1}(x)$.

Applying the recurrence relation (3.20), the inequalities $q^\star \geqslant 2$ and $q^\star - q \geqslant 1$ when $q \in Q_i'$, and the fact that $x\in[0,1]$, we first get

$$P_{\mu+1}(x) \leqslant B' \frac{B^\mu x^{\mu+1}}{((\mu+1)q^\star)!}$$

for all $x \in [0,1]$, where B' is the positive real number defined by

$$B' = CC_{s_0}\alpha_{\mu,0} + CC_{s_0}\sum_{i\in\mathcal{K}}\sum_{q\in Q_i'}\alpha_{\mu,q}$$

with

$$\begin{aligned}\alpha_{\mu,q} &= \frac{(\mu q^\star+1)\dots(\mu q^\star+q^\star-q)}{(\mu+1)\dots(\mu+q^\star-q)}\\ &= \frac{q^{\star q^\star-q}\left(\mu+\dfrac{1}{q^\star}\right)\dots\left(\mu+\dfrac{q^\star-q}{q^\star}\right)}{(\mu+1)\dots(\mu+q^\star-q)} \leqslant q^{\star q^\star-q}.\end{aligned}$$

Consequently, $B' \leqslant B$ and the sought inequality on $P_{\mu+1}(x)$ follows, which ends the proof of Proposition 3.27. □

We are now able to improve the bounds of the functions $\partial_t^p w_\mu^{[k]}(t,x)$ given in Proposition 3.24 and conclude the proof of Theorem 3.11.

3.3.5. Final estimates on $w_\mu(t,x)$

According to the definition of the sectors Σ_μ (see page 49), we deduce from Propositions 3.24 and 3.27 that all the functions $w_\mu^{[k]}(t,x)$ are holomorphic on $\Sigma'_{\theta,>\pi/k_0}\times D_{r_1}$ and satisfy the inequalities

$$|\partial_t^p w_\mu^{[k]}(t,x)| \leqslant n^\mu e^{n(\mu+1)i^\star C(i^\star+C_{s_0})}CK^{p+\mu(\kappa-i^\star)} \times\Gamma(1+(s_0+1)p+\mu q^\star)\frac{(B|x|)^\mu}{(\mu q^\star)!} \tag{3.23}$$

for all $k\in\{1,\dots,n\}$, all $p,\mu\geqslant 0$ and all $(t,x)\in\Sigma'_{\theta,>\pi/k_0}\times D_{r_1}$ (recall that the radius r_1 was chosen so that $r_1\leqslant 1$). The following result specifies the latter.

Proposition 3.28. — *Let us set $K_1 = 2^{q^\star}K$ and $c = ne^{ni^\star C(i^\star+C_{s_0})}\times 2^{q^\star}K^{\kappa-i^\star}B$. Then, the inequalities*

$$|\partial_t^p w_\mu^{[k]}(t,x)| \leqslant e^{ni^\star C(i^\star+C_{s_0})}CK_1^p\Gamma(1+(s_0+1)p)(c|x|)^\mu$$

hold for all $k\in\{1,\dots,n\}$, all $p,\mu\geqslant 0$ and all $(t,x)\in\Sigma'_{\theta,>\pi/k_0}\times D_{r_1}$.

Proof. — We first derive from the recurrence relation $\Gamma(1+z)=z\Gamma(z)$ applied $\mu q^\star$ times the identity

$$\Gamma(1+(s_0+1)p+\mu q^\star) = \Gamma(1+(s_0+1)p)\prod_{j=1}^{\mu q^\star}((s_0+1)p+j).$$

Next, applying the inequality $s_0+1\leqslant q^\star$ (use Lemma 3.8 (1) and the fact that $\kappa>i^\star$), we get the following relations:

$$\begin{aligned}\Gamma(1+(s_0+1)p+\mu q^\star)&\leqslant \Gamma(1+(s_0+1)p)\prod_{j=1}^{\mu q^\star}(pq^\star+j)\\&=\Gamma(1+(s_0+1)p)\binom{pq^\star+\mu q^\star}{\mu q^\star}(\mu q^\star)!\\&\leqslant 2^{pq^\star+\mu q^\star}(\mu q^\star)!\Gamma(1+(s_0+1)p).\end{aligned}$$

Proposition 3.28 stems then from the inequality (3.23). □

We are now able to complete the proof of Theorem 3.11.

3.3.6. Conclusion

Let us now choose a proper subsector $\Sigma''_{\theta,>\pi/k_0}$ of $\Sigma'_{\theta,>\pi/k_0}$ bisected by the direction θ and opening larger than π/k_0 (recall that such a choice is already possible by definition of a proper subsector, see Footnote 5). Let us also choose a radius $0<r'_1<\min(r_1,1/c)$ and let us set

$$C_1:=e^{ni^\star C(i^\star+C_{s_0})}C\sum_{\mu\geqslant 0}(cr'_1)^\mu\in\mathbb{R}^*_+.$$

Thanks to Proposition 3.28, the series $\sum_{\mu\geqslant 0}\partial_t^p w_\mu^{[k]}(t,x)$ are normally convergent on $\Sigma'_{\theta,>\pi/k_0}\times D_{r'_1}$ for all $k\in\{1,\dots,n\}$ and $p\geqslant 0$ and satisfy the inequalities

$$\sum_{\mu\geqslant 0}\left|\partial_t^p w_\mu^{[k]}(t,x)\right|\leqslant C_1K_1^p\Gamma(1+(s_0+1)p)$$

for all $(t,x)\in\Sigma'_{\theta,>\pi/k_0}\times D_{r'_1}$. In particular, the sums $w^{[k]}(t,x)$ of the series $\sum_{\mu\geqslant 0}w_\mu^{[k]}(t,x)$ are well-defined, holomorphic on $\Sigma'_{\theta,>\pi/k_0}\times D_{r'_1}$ and satisfies the inequalities

$$\left|\partial_t^p w^{[k]}(t,x)\right|\leqslant C_1K_1^p\Gamma(1+(s_0+1)p)$$

for all $k\in\{1,\dots,n\}$, all $p\geqslant 0$ and all $(t,x)\in\Sigma'_{\theta,>\pi/k_0}\times D_{r'_1}$. Hence, Conditions 1 and 3 of Definition 3.1 hold, since $\Sigma''_{\theta,>\pi/k_0}\Subset\Sigma'_{\theta,>\pi/k_0}$.

To prove the second condition of Definition 3.1, we proceed as follows. The Removable Singularities Theorem implies the existence of $\lim\limits_{\substack{t\to 0\\ t\in\Sigma''_{\theta,>\pi/k_0}}}\partial_t^p w^{[k]}(t,x)$ for all $x\in D_{r'_1}$ and, thereby, the existence of the Taylor series of $w^{[k]}$ at 0 on $\Sigma''_{\theta,>\pi/k_0}$ for all $x\in D_{r'_1}$ (see for instance [51, Chapter 4]; see also [25, Corollary 1.1.3.3] and [20, Proposition 1.1.11]). On the other hand, considering recurrence relations (3.22), it is clear that

$$w(t,x) = \begin{pmatrix} w^{[1]}(t,x) \\ \vdots \\ w^{[n]}(t,x) \end{pmatrix}$$

satisfies System (3.3) with right-hand side $g(t,x)$ in place of $\widetilde{g}(t,x)$ and, consequently, so does its Taylor series. Then, since System (3.3) has a unique formal solution $\widetilde{w}(t,x) \in \mathcal{M}_{n,1}(\mathbb{C}[\![t,x]\!])$ (see Lemma 3.16), we then conclude that the Taylor expansions of each of its components $w^{[k]}(t,x)$ are the formal series $\widetilde{w}^{[k]}(t,x)$. Hence, Condition (2) of Definition 3.1 holds.

This proves the k_0-summability of $\widetilde{w}(t,x)$ and achieves the proof of the sufficient condition of the first point of Theorem 3.11, which ends its full proof.

4. A more general system

In this section, we consider the following more general system in two variables $(t,x) \in \mathbb{C}^2$ defined by the n inhomogeneous linear partial differential equations

$$\begin{cases} \partial_t^{\kappa_k} u^{[k]} - \displaystyle\sum_{i \in \mathcal{K}_k} \sum_{q \in Q_{k,i}} \sum_{\ell \in L_{k,i,q}} t^{v_{i,q}^{[k,\ell]}} a_{i,q}^{[k,\ell]}(t,x) \partial_t^i \partial_x^q u^{[\ell]} = \widetilde{f}^{[k]}(t,x) \\ \partial_t^j u^{[k]}(t,x)_{|t=0} = \varphi_j^{[k]}(x), j = 0, \dots, \kappa_k - 1 \end{cases} \tag{4.1}$$

for all $k = 1, \dots, n$, where

- $\kappa_k \geqslant 1$ is a positive integer for all $k = 1, \dots, n$;
- $\mathcal{K}_k$ is a nonempty subset of $\{0, \dots, \kappa_k - 1\}$ for all $k = 1, \dots, n$;
- $Q_{k,i}$ is a nonempty finite subset of $\mathbb{N}$ for all $k = 1, \dots, n$ and all $i \in \mathcal{K}_k$, and at least one of the $Q_{k,i}$'s is not reduced to $\{0\}$;
- $L_{k,i,q}$ is a possibly empty subset of $\{1, \dots, n\}$ for all $k = 1, \dots, n$, all $i \in \mathcal{K}_k$ and all $q \in Q_{k,i}$, and at least one of the $L_{k,i,q}$'s is nonempty;
- the valuation $v_{i,q}^{[k,\ell]} \geqslant 0$ are nonnegative integers for all $k = 1, \dots, n$, all $i \in \mathcal{K}_k$, all $q \in Q_{k,i}$ and all $\ell \in L_{k,i,q}$;
- the coefficients $a_{i,q}^{[k,\ell]}(t,x)$ are analytic functions on a polydisc $D_{\rho_0} \times D_{\rho_1}$ and satisfy $a_{i,q}^{[k,\ell]}(0,x) \not\equiv 0$ for all $k = 1, \dots, n$, all $i \in \mathcal{K}_k$, all $q \in Q_{k,i}$ and all $\ell \in L_{k,i,q}$;
- the inhomogeneities $\widetilde{f}^{[k]}(t,x)$ are formal power series in t with analytic coefficients in D_{ρ_1} for all $k = 1, \dots, n$;

- the initial conditions $\varphi_j^{[k]}(x)$ are analytic functions on D_{ρ_1} for all $k=1,\dots,n$ and all $j=0,\dots,\kappa_k-1$.

As for System (1.1), this system admits a unique formal solution $\widetilde{u}(t,x)\in\mathcal{M}_{n,1}(\mathcal{O}(D_{\rho_1})[\![t]\!])$: reasoning as in the proof of Proposition 1.5 (see page 4), one easily checks that its coefficients $u_{j,*}^{[k]}(x)$ are uniquely determined for all $k=1,\dots,n$ and all $j\geqslant 0$ by the recurrence relation

$$u_{j+\kappa_k,*}^{[k]}(x)=f_{j,*}^{[k]}(x)\\ +\sum_{i\in\mathcal{K}_k}\sum_{q\in Q_{k,i}}\sum_{\ell\in L_{k,i,q}}\sum_{j_0+j_1=j-v_{i,q}^{[k,\ell]}}\frac{j!}{j_0!j_1!}a_{i,q;j_0,*}^{[k,\ell]}(x)\partial_x^q u_{j_1+i,*}^{[\ell]}$$

together with the initial conditions $u_{j,*}^{[k]}(x)=\varphi_j^{[k]}(x)$ for all $k=1,\dots,n$ and all $j=0,\dots,\kappa_k-1$.

Let us denote by κ the maximum of $\kappa_1,\dots,\kappa_n$. The above relations can be used to explicitly determine all the functions $u_{j,*}^{[k]}$ for $k=1,\dots,n$ and $j=0,\dots,\kappa-1$. Therefore, deriving $\kappa-\kappa_k$ times with respect to t the k^{th} equation of System (4.1) for any $k=1,\dots,n$, we deduce that the formal solution $\widetilde{u}(t,x)$ of System (4.1) is also the formal solution of a system of the type (1.1). Consequently, Theorems 2.5 and 3.11 apply (if Hypothesis 3.5 is valid for the latter), providing thus the Gevrey regularity and a characterization of the k_0-summability of $\widetilde{u}(t,x)$.

Example 4.1. — Let us first consider the system

$$\begin{cases}\partial_t^2u^{[1]}=a\partial_x^4u^{[1]}+b\partial_tu^{[2]}\\ \partial_tu^{[2]}=c\partial_xu^{[2]}+du^{[1]}\\ u^{[1]}(0,x)=\varphi_0^{[1]}(x),\ \partial_tu^{[1]}(t,x)_{|t=0}=\varphi_1^{[1]}(x)\\ u^{[2]}(0,x)=\varphi_0^{[2]}(x)\end{cases}\tag{4.2}$$

where the coefficients a, b, c and d are nonzero complex numbers, and where the initial data $\varphi_0^{[1]}(x)$, $\varphi_1^{[1]}(x)$ and $\varphi_0^{[2]}(x)$ are analytic functions on a common disc D_{ρ_1}. From calculations above, System (4.2) admits a unique formal solution $\widetilde{u}(t,x)\in\mathcal{M}_{2,1}(\mathcal{O}(D_{\rho_1})[\![t]\!])$. More precisely, its coefficients $u_{j,*}^{[1]}(x)$ and $u_{j,*}^{[2]}(x)$ are uniquely determined for all $j\geqslant 0$ by the recurrence relations

$$\begin{cases}u_{j+2,*}^{[1]}(x)=a\partial_x^4u_{j,*}^{[1]}(x)+bu_{j+1,*}^{[2]}(x)\\ u_{j+1,*}^{[2]}(x)=c\partial_xu_{j,*}^{[2]}(x)+du_{j,*}^{[1]}(x)\end{cases}$$

together with the initial conditions $u_{0,*}^{[1]}(x)=\varphi_0^{[1]}(x)$, $u_{1,*}^{[1]}(x)=\varphi_1^{[1]}(x)$ and $u_{0,*}^{[2]}(x)=\varphi_0^{[2]}(x)$. In particular, these relations show us that $u_{1,*}^{[2]}(x)$ is equal to the analytic function $\varphi_1^{[2]}(x) \in \mathcal{O}(D_{\rho_1})$ defined by

$$\varphi_1^{[2]}(x) = c\partial_x\varphi_0^{[2]}(x) + d\varphi_0^{[1]}(x).$$

Consequently, deriving once with respect to t the second equation of System (4.2), we deduce that the formal solution $\widetilde{u}(t,x)$ is also the formal solution in $\mathcal{M}_{2,1}(\mathcal{O}(D_{\rho_1})[\![t]\!])$ of the system

$$\partial_t^2 u = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}\partial_t\partial_x u + \begin{pmatrix} 0 & b \\ d & 0 \end{pmatrix}\partial_t u + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\partial_x^4 u \tag{4.3}$$

with the initial data $u(0,x) = \varphi_0(x)$ and $\partial_t u(t,x)_{|t=0} = \varphi_1(x)$.

Applying first Theorem 2.5, we deduce that the value s_0 attached to System (4.3) is $s_0 = 1$, and thus that $\widetilde{u}(t,x)$ is 1-Gevrey. On the other hand, Theorem 3.11 can be not applied since condition (2) of Hypothesis 3.5 fails.

Example 4.2. — Let us now consider the system

$$\begin{cases} \partial_t^2 u^{[1]} = a\partial_t\partial_x^3 u^{[2]} \\ \partial_t u^{[2]} = c\partial_x^3 u^{[1]} + du^{[2]} \\ u^{[1]}(0,x) = \varphi_0^{[1]}(x),\ \partial_t u^{[1]}(t,x)_{|t=0} = \varphi_1^{[1]}(x) \\ u^{[2]}(0,x) = \varphi_0^{[2]}(x) \end{cases} \tag{4.4}$$

where the coefficients a, c and d are nonzero complex numbers, and where the initial data $\varphi_0^{[1]}(x)$, $\varphi_1^{[1]}(x)$ and $\varphi_0^{[2]}(x)$ are analytic functions on a common disc D_{ρ_1}. As in the previous example, one can easily check that the formal solution $\widetilde{u}(t,x) \in \mathcal{M}_{2,1}(\mathcal{O}(D_{\rho_1})[\![t]\!])$ of System (4.4) is also the formal solution in $\mathcal{M}_{2,1}(\mathcal{O}(D_{\rho_1})[\![t]\!])$ of the system

$$\partial_t^2 u = \begin{pmatrix} 0 & a \\ c & 0 \end{pmatrix}\partial_t\partial_x^3 u + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\partial_t u \tag{4.5}$$

together with the initial data $u(0,x) = \varphi_0(x)$ and $\partial_t u(t,x)_{|t=0} = \varphi_1(x)$ with $\varphi_1^{[2]}(x) = c\partial_x^3\varphi_0^{[1]}(x) + d\varphi_0^{[2]}(x)$.

Here, the value s_0 attached to System (4.5) is given by the pair $(i,q) = (1,3)$ and is equal to $s_0 = 1$. Consequently, Theorem 2.5 tells us that $\widetilde{u}(t,x)$ is 1-Gevrey. Moreover, since the matrix

$$\begin{pmatrix} 0 & a \\ c & 0 \end{pmatrix}$$

is invertible, all the conditions required in Hypothesis 3.5 are satisfied and Theorem 3.11 applies, providing thus a characterization of the 1-summability of $\widetilde{u}(t,x)$ in a given direction θ: $\widetilde{u}(t,x)$ is 1-summable in the direction θ if and only if its three formal coefficients $\widetilde{u}_{*,0}(t)$, $\widetilde{u}_{*,1}(t)$ and $\widetilde{u}_{*,2}(t)$ are 1-summable in the direction θ.

5. Appendices

5.1. Nagumo norms

In this section, we recall the definition of the Nagumo norms [4, 36, 59] and some of their properties.

Definition 5.1. — *Let $f \in \mathcal{O}(D_{\rho_1})$, $p \geqslant 0$ and $0 < r < \rho_1$ be. Then, the* Nagumo norm $\|f\|_{p,r}$ with indices (p, r) of f *is defined by*

$$\|f\|_{p,r} := \sup_{|x|\leqslant r} |f(x)d_r(x)^p|,$$

where $d_r(x)$ denotes the Euclidian distance $d_r(x) = r - |x|$.

Following Proposition 5.2 gives us some properties of the Nagumo norms.

Proposition 5.2. — *Let $f, g \in \mathcal{O}(D_{\rho_1})$, $p, p' \geqslant 0$ and $0 < r < \rho_1$ be. Then,*

(1) $\|\cdot\|_{p,r}$ *is a norm on $\mathcal{O}(D_{\rho_1})$.*
(2) $\pm f(x) \leqslant \|f\|_{p,r} d_r(x)^{-p}$ *for all $|x| < r$.*
(3) $\|f\|_{0,r} = \sup_{|x|\leqslant r} \big|f(x)\big|$ *is the usual sup-norm on the disc D_r.*
(4) $\|fg\|_{p+p',r} \leqslant \|f\|_{p,r} \|g\|_{p',r}$.
(5) $\|\partial_x f\|_{p+1,r} \leqslant e(p+1)\|f\|_{p,r}$.

Proof. — See [51, Proposition 12.2] □

Remark 5.3. — Inequalities (4)–(5) of Proposition 5.2 are the most important properties. Observe besides that the same index r occurs on their both sides, allowing thus to get estimates for the product fg in terms of f and g, and for the derivative $\partial_x f$ in terms of f without having to shrink the disc D_r.

5.2. Matrix with analytic coefficients

Proposition 5.4. — *Let $p \geqslant 1$ be and $A(X) \in \mathcal{M}_n(\mathbb{C}\{X\})$ with $X = (X_1, \ldots, X_p)$ a $n\times n$-matrix with analytic coefficients at the origin $(0, \ldots, 0)$ of $\mathbb{C}^p$. Assume that the matrix $A(0) \in \mathcal{M}_n(\mathbb{C})$ is invertible.*

Then, $A(X)$ is invertible in $\mathcal{M}_n(\mathbb{C}\{X\})$, that is its inverse has analytic coefficients at the origin of $\mathbb{C}^p$ too.

Proof. — Let us denote by $d(X) = \det(A(X))$ the determinant of $A(X)$. By assumption, $d(X) \in \mathbb{C}\{X\}$ is analytic at the origin of $\mathbb{C}^p$ and $d(0) \neq 0$. Consequently, the inverse $1/d(X)$ of $d(X)$ is well-defined and analytic on a convenient polydisc $D_{r_1} \times \cdots \times D_{r_p}$ centered at the origin of $\mathbb{C}^p$. From this and from the theoretical relation

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where $\operatorname{adj}(A)$ stands for the adjugate of A, that is the transpose of the cofactor matrix of A, we deduce that the matrix $A(X)$ is invertible and that its inverse has analytic coefficients on $D_{r_1} \times \cdots \times D_{r_p}$. This ends the proof. □

5.3. Generalized binomial and multinomial coefficients

In combinatorial analysis, the binomial coefficients $\binom{n}{m}$ and the multinomial coefficients $\binom{n}{n_1,\ldots,n_q}$ are defined for any nonnegative integers $0 \leqslant m \leqslant n$ and any tuples $(n, n_1, \ldots, n_q)$ of nonnegative integers satisfying $q \geqslant 2$ and $n_1 + \cdots + n_q = n$ by the relations

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \quad \text{and} \quad \binom{n}{n_1, \ldots, n_q} = \frac{n!}{n_1! \ldots n_q!}.$$

They respectively denote the number of ways of choosing m objects from a collection of n distinct objects without regard to order, and the number of ways of putting $n = n_1 + \cdots + n_q$ different objects into q different boxes with n_i in the i-th box for all $i = 1, \ldots, q$.

Using the fact that $n! = \Gamma(1+n)$ for any integer $n \geqslant 0$, one can easily extend the definitions of these coefficients to the case where their terms are no longer necessarily integers by setting

$$\binom{a}{b} = \frac{\Gamma(1+a)}{\Gamma(1+b)\Gamma(1+a-b)} \tag{5.1}$$

for any nonnegative real numbers $0 \leqslant b \leqslant a$ and

$$\binom{a}{a_1, \ldots, a_q} = \frac{\Gamma(1+a)}{\Gamma(1+a_1) \ldots \Gamma(1+a_q)} = \frac{\Gamma(1+a)}{\prod_{i=1}^q \Gamma(1+a_i)} \tag{5.2}$$

for any tuples $(a, a_1, \ldots, a_q)$ of nonnegative real numbers satisfying $q \geqslant 2$ and $a_1 + \cdots + a_q = a$. Observe that all these coefficients are positive.

The four propositions below, which are proved for instance in [51, Chapter 13], extend to the generalized binomial coefficients (5.1) and the generalized multinomial coefficients (5.2) some well-known results in combinatorial analysis.

Proposition 5.5 (Vandermonde's Inequality [51, Proposition 13.2]).

(1) *(Binomial case) Let* $0 \leqslant b \leqslant a$ *be two nonnegative real numbers and* $0 \leqslant m \leqslant n$ *two nonnegative integers. Then,*

$$\binom{a+n}{b+m} \geqslant \binom{a}{b}\binom{n}{m}.$$

(2) *(Multinomial case) Let* $q \geqslant 2$ *be an integer,* $(a, a_1, \ldots, a_q)$ *a tuple of nonnegative real numbers and* $(n, n_1, \ldots, n_q)$ *a tuple of nonnegative integers such that* $a_1 + \cdots + a_q = a$ *and* $n_1 + \cdots + n_q = n$*. Then,*

$$\binom{a+n}{a_1+n_1, \ldots, a_q+n_q} \geqslant \binom{a}{a_1, \ldots, a_q}\binom{n}{n_1, \ldots, n_q}.$$

Proposition 5.6 (Variations of the binomial coefficients [51, Prop. 13.4]).

(1) *Let* $b \geqslant 0$*. Then, the function* $\mathrm{Bin}_b : a \in [b, +\infty[\mapsto \binom{a}{b}$ *is increasing on* $[b, +\infty[$*.*

(2) *Let* $a > 0$*. Then, the function* $\mathrm{bin}_a : b \in [0, a] \mapsto \binom{a}{b}$ *is increasing on* $\left[0, \frac{a}{2}\right]$ *and decreasing on* $\left[\frac{a}{2}, a\right]$*. In particular,* $\mathrm{bin}_a(a/2)$ *is the maximum of* bin_a *on* $[0, a]$*.*

Proposition 5.7 (Bounds of the binomial coefficients [51, Proposition 13.5]). — *Let* $0 \leqslant b \leqslant a$ *and* A *a nonnegative integer greater than* $a/2$*. Then,*

$$1 \leqslant \binom{a}{b} \leqslant 4^A.$$

Proposition 5.8 (Sum of the inverses of binomial and multinomial coefficients [51, Proposition 13.6]). — *Let* $s > 0$ *be and let us set* $C_s = s'(2 + \Gamma(ss'))$*, where* $s' \geqslant 1$ *is the positive integer defined by*

$$s' = \begin{cases} 1 & \text{if } s \geqslant 1 \\ \left\lfloor \dfrac{1}{s} \right\rfloor + 1 & \text{if } s < 1, \end{cases}$$

where $\lfloor a \rfloor$ *stands for the floor of* $a \in \mathbb{R}$*.*

(1) *(Binomial case) The following inequality holds for all integers* $n \geqslant 0$*:*

$$\sum_{m=0}^{n} \frac{1}{\binom{sn}{sm}} \leqslant C_s.$$

(2) *(Multinomial case) The following inequality holds for all integers* $q \geqslant 2$ *and* $n \geqslant 0$*:*

$$\sum_{n_1+\cdots+n_q=n} \frac{1}{\binom{sn}{sn_1, \ldots, sn_q}} \leqslant C_s^{q-1}.$$

Acknowledgements

The author want to express its gratitude to the referee of the work for the suggestions and comments which helped to improve the manuscript.

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Manuscrit reçu le 17 juillet 2024,
révisé le 25 janvier 2025,
accepté le 17 février 2025.

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