Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Daniel Barrera Salazar & Santiago Molina

Triple product p-adic L-functions for Shimura curves over totally real number fields

Article à paraître, mis en ligne le 16 juillet 2026, 76 p.

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 16 juillet 2026.

TRIPLE PRODUCT p-ADIC L-FUNCTIONS FOR SHIMURA CURVES OVER TOTALLY REAL NUMBER FIELDS

by Daniel BARRERA SALAZAR & Santiago MOLINA (*)

Abstract. — Let F be a totally real number field. Using a recent geometric approach developed by Andreatta and Iovita we construct several variables p-adic families of finite slope quaternionic automorphic forms over F. It is achieved by interpolating the modular sheaves defined over some auxiliary unitary Shimura curves.

Secondly, we attach p-adic L-functions to triples of ordinary p-adic families of quaternionic automorphic eigenforms. This is done by relating trilinear periods to some trilinear products over unitary Shimura curves which can be interpolated adapting the work of Liu–Zhang–Zhang to our families.

Résumé. — Soit F un corps de nombres totalement réel. En utilisant une approche géométrique récemment développé par Andreatta et Iovita, nous construisons des familles p-adiques à plusieurs variables de formes automorphes quaternioniques de pente finie sur F. Cela est réalisé en interpolant les faisceaux modulaires définis sur certaines courbes de Shimura unitaires auxiliaires.

Deuxièmement, nous attachons des fonctions L p-adiques à des triplets de familles ordinaires p-adiques de formes automorphes quaternioniques propres. Cela est effectué en reliant des périodes trilinéaires à certains produits trilinéaires sur des courbes de Shimura unitaires, lesquels peuvent être interpolés en adaptant les travaux de Liu–Zhang–Zhang à nos familles.

Keywords: Triple product p-adic L-functions, Eigenvarieties, families of automorphic forms, unitary Shimura curves.
2020 *Mathematics Subject Classification:* 11S40, 11F67, 11R42, 11F85, 14G35.
(*) The authors are supported in part by DGICYT Grant MTM2015-63829-P. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 682152). D.B. was also funded by ANID FONDECYT grants 11201025 and 1241702. S. M. is also funded by Ministerio de Ciencia e innovación, project PID2022-137605NB-I00, and this paper is part of the R&D+i project PID2021-124613OB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, EU.

1. Introduction

1.1. Arithmetic, p-adic L-functions and p-adic families

Several recent works with important arithmetic applications use in a crucial way p-adic L-functions (see for example [5, 14, 23, 37]). In contrast with its complex counterpart, the theory of p-adic L-functions is far from being well established. Thus, depending on the context, their construction is performed with the available technology. This work is devoted to the construction of the so-called *triple product p-adic L-functions*.

During the nineties, Kato obtained deep results on the Birch and Swinnerton–Dyer conjecture in rank 0 for twists of elliptic curves over $\mathbb{Q}$ by Dirichlet characters. More recently, Bertolini–Darmon–Rotger in [5] and Darmon–Rotger in [14] developed analogous methods to treat twists by certain Artin representations of dimensions 2 and 4. In [6, 29, 30, 32], these methods were extended in different directions, as for example bounding certain Selmer groups and treating finite slope settings. We could say that in these situations a prominent role is played by the *unbalanced* p-adic L-function attached to a triple of p-adic families of modular forms. Such p-adic L-functions were constructed in [13, 16, 20, 21] in the ordinary case and in [1] for Coleman Families. On the other hand, *balanced* triple p-adic L-functions have been constructed in [21] and in [17].

This work grew up of the aim to generalize the methods mentioned above to totally real number fields. In the present paper, we furnish the p-adic L-functions that come into play in the ordinary setting. Our main results are:

(i) the construction of several-variables p-adic families of finite slope quaternionic automorphic forms over totally real number fields;
(ii) the construction of triple product p-adic L-functions for ordinary families obtained in (i).

To achieve (i), we use a geometrical approach (developed in [1]) that employs the theory of overconvergent modular forms in the required generality. For (ii) we work in a hybrid situation: unbalanced in a single component of the weight and balanced in the rest of the components. Our strategy combines the two approaches used in [1] and [17], and crucially adapts ideas and constructions performed in [31] to our situation. Moreover, based on the present work and inspired by the calculations in [1], in the more technical work [35] the construction in (ii) was generalized to the finite slope setting.

1.2. Main results

Let F be a totally real number field of degree $d = [F : \mathbb{Q}]$. We denote by Σ_F the set of real embeddings and fix $\tau_0 \in \Sigma_F$. Let $p > 2$ be a prime number and let $\mathfrak{p}_0$ be the prime over p associated to τ_0 under a fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. *We suppose F is unramified at p and the inertia degree of $\mathfrak{p}_0$ over p is 1.*

Let B be a quaternion algebra over F that splits at τ_0 and at every prime over p, and it is ramified at any $\tau \in \Sigma_F \setminus \{\tau_0\}$. As already mentioned, one of our goals is the construction of p-adic families of automorphic forms on $(B \otimes \mathbb{A})^\times$ using geometrical tools. One of the main obstructions to performing this task directly on Shimura curves of B is the lack of an adequate moduli problem. To remedy this issue we work with $D = B \otimes_F E$ for some CM extension E of F on which each prime of F over p splits. Let X be the unitary Shimura curve attached to D of level prime to $p \cdot \mathrm{disc}(B)$ (see Section 4.1), and let $\pi : A \to X$ be its universal abelian variety (of dimension $4d$). For each $\underline{k} \in \mathbb{N}[\Sigma_F]$, a precise piece of the sheaf of invariant differentials of A produces a modular sheaf $\omega^{\underline{k}}$ that gives rise to *modular forms for D*.

Previous works devoted to developing the theory of p-adic families for unitary Shimura curves ([8, 15, 25]) construct essentially 1-dimensional p-adic families. In fact, those works only treat the p-adic variation of powers of the invertible subsheaf of ω attached to τ_0. Nevertheless, for applications, it would be better to have more variation. Thus, our first task is to provide a more general theory of p-adic families in this context. The complexity of performing this was reflected, for example, in the fact that the rank of $\omega^{\underline{k}}$ grows with $\underline{k}$. Nowadays, we have enough technology to perform this task.

The *weight space for D* is the d-dimensional adic space, denoted $\mathcal{W}$, attached to the complete group algebra $\mathbb{Z}_p[\![(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times]\!]$. The *weight space for B* is the $d+1$-dimensional adic space attached to $\mathbb{Z}_p[\![(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times \mathbb{Z}_p^\times]\!]$, and it is denoted by $\mathcal{W}^G$ in the text. For each $n > 0$ we consider certain open subspaces $\mathcal{W}_n \subset \mathcal{W}$ and $\mathcal{W}_n^G \subset \mathcal{W}^G$ (see Section 6.2 and Section 7.1).

Let $\mathcal{X}$ be the adic analytic space attached to X. For $r > 0$, we denote by $\mathcal{X}_r$ the strict neighborhood of the $\mathfrak{p}_0$-ordinary locus of $\mathcal{X}$ where the universal abelian variety has a $\mathfrak{p}_0$-canonical subgroup of order $\leqslant r$. We have (see Section 6.3, Section 7.5, and Proposition 7.7):

Theorem 1.1. — *Let $n \leqslant r$.*

(i) There exist a sheaf of Banach modules $\mathcal{F}_n$ over $\mathcal{X}_r \times \mathcal{W}_n$ such that, for each classical weight $\underline{k} \in \mathbb{N}[\Sigma_F]$, the map $(\mathrm{id}, \underline{k}) : \mathcal{X}_r \to \mathcal{X}_r \times \mathcal{W}_n$ induces a natural embedding $\omega^{\underline{k}}|_{\mathcal{X}_r} \subset (\mathrm{id}, \underline{k})^*(\mathcal{F}_n)$ of Banach sheaves over $\mathcal{X}_r$.

(ii) There exists an adic space $\mathcal{E}_n$, equidimensional of dimension $d+1$, endowed with a locally free map $w : \mathcal{E}_n \to \mathcal{W}_n^G$ without torsion, which parametrizes systems of Hecke eigenvalues appearing in the space of automorphic forms for B.

The construction of the sheaves $\mathcal{F}_n$ is carried out using a slight modification of the machinery of *formal vector bundles with marked sections* introduced in [1]. We then exploit the description of automorphic forms for B in terms of modular forms for D. By using the sheaves $\mathcal{F}_n$ (more precisely, their duals), we produce the module of *p-adic families of locally analytic overconvergent automorphic forms on* $(B \otimes \mathbb{A}_F)^\times$. Such a module is projective and the usual Hecke operator U_p acts compactly on it. Using the theory developed in [3, Appendix B], we obtain the eigenvariety $\mathcal{E}_n$.

Now we explain our result on triple product p-adic L-functions. We say that a triplet $(\underline{k}_1, \underline{k}_2, \underline{k}_3) \in \mathbb{Z}[\Sigma_F]^3$ is *unbalanced at* τ_0 *with dominant weight* $\underline{k}_3$ if $k_{1,\tau} + k_{2,\tau} + k_{3,\tau}$ is even for every $\tau \in \Sigma_F$, $k_{3,\tau_0} \geqslant k_{1,\tau_0} + k_{2,\tau_0}$, and $(k_{1,\tau}, k_{2,\tau}, k_{3,\tau})$ is balanced for each $\tau \neq \tau_0$ (see Definition 3.4). The *interpolation region* for our p-adic L-functions is the set S_3 of triplets $((\underline{k}_1, \nu_1), (\underline{k}_2, \nu_2), (\underline{k}_3, \nu_3)) \in (\mathbb{Z}[\Sigma_F] \times \mathbb{Z})^3$ such that: (i) $(\underline{k}_1, \underline{k}_2, \underline{k}_3)$ is unbalanced at τ_0 with dominant weight $\underline{k}_3$; (ii) $k_{\tau,i} > 0$, for every $i = 1, 2, 3$ and $\tau \in \Sigma_F$; (iii) $\nu_3 = \nu_1 + \nu_2$.

Let μ_1, μ_2, and μ_3 be three ordinary eigenfamilies of automorphic forms on $(B \otimes \mathbb{A}_F)^\times$. We denote by Λ_1, Λ_2 and Λ_3 the rings over which these eigenfamilies are defined, respectively. Let x, y, z be a triplet of classical points corresponding to $((\underline{k}_1, \nu_1), (\underline{k}_2, \nu_2), (\underline{k}_3, \nu_3)) \in S_3$. We denote by π_x the automorphic representation of $(B \otimes \mathbb{A}_F)^\times$ generated by the specialization of μ_1 at x, and Π_x the corresponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Moreover, we denote by $\alpha_x^{\mathfrak{p}}$ and $\beta_x^{\mathfrak{p}}$ the roots of the corresponding Hecke polynomial at $\mathfrak{p}$. In the same way we obtain π_y, Π_y, $\alpha_y^{\mathfrak{p}}$, $\beta_y^{\mathfrak{p}}$, π_z, Π_z, $\alpha_z^{\mathfrak{p}}$, $\beta_z^{\mathfrak{p}}$. Let μ_z° be a newform of π_z. Then we have the following result (for further details, see Lemma 10.3 and Theorem 10.5):

Theorem 1.2. — *There exists $\mathcal{L}_p(\mu_1, \mu_2, \mu_3) \in \Lambda_1 \widehat{\otimes} \Lambda_2 \widehat{\otimes} \mathrm{Frac}(\Lambda_3)$ such that, for each classical point (x, y, z) corresponding to a triplet $((\underline{k}_1, \nu_1), (\underline{k}_2, \nu_2), (\underline{k}_3, \nu_3)) \in S_3$, we have:*

$$\mathcal{L}_p(\mu_1, \mu_2, \mu_3)(x, y, z) = K \cdot \left(\prod_{\mathfrak{p} \mid p} \frac{\mathcal{E}_{\mathfrak{p}}(x, y, z)}{\mathcal{E}_{\mathfrak{p},1}(z)} \right) \cdot \frac{L\left(\frac{1-\nu_1-\nu_2-\nu_3}{2}, \Pi_x \otimes \Pi_y \otimes \Pi_z\right)^{\frac{1}{2}}}{\langle \mu_z^\circ, \mu_z^\circ \rangle},$$

where K is a non-zero constant depending of (x,y,z),

$$\begin{aligned}&\mathcal{E}_{\mathfrak{p}}(x,y,z)\\ &=\begin{cases}\left(1-\frac{\beta_x^{\mathfrak{p}}\beta_y^{\mathfrak{p}}\alpha_z^{\mathfrak{p}}}{\varpi_{\mathfrak{p}}^{\underline{m}_{\mathfrak{p}}+\underline{2}}}\right)\left(1-\frac{\alpha_x^{\mathfrak{p}}\beta_y^{\mathfrak{p}}\beta_z^{\mathfrak{p}}}{\varpi_{\mathfrak{p}}^{\underline{m}_{\mathfrak{p}}+\underline{2}}}\right)\left(1-\frac{\beta_x^{\mathfrak{p}}\alpha_y^{\mathfrak{p}}\beta_z^{\mathfrak{p}}}{\varpi_{\mathfrak{p}}^{\underline{m}_{\mathfrak{p}}+\underline{2}}}\right)\left(1-\frac{\beta_x^{\mathfrak{p}}\beta_y^{\mathfrak{p}}\beta_z^{\mathfrak{p}}}{\varpi_{\mathfrak{p}}^{\underline{m}_{\mathfrak{p}}+\underline{2}}}\right), & \mathfrak{p}\neq\mathfrak{p}_0\\ \left(1-\frac{\alpha_x^{\mathfrak{p}_0}\alpha_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right)\left(1-\frac{\alpha_x^{\mathfrak{p}_0}\beta_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right)\left(1-\frac{\beta_x^{\mathfrak{p}_0}\alpha_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right)\left(1-\frac{\beta_x^{\mathfrak{p}_0}\beta_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right), & \mathfrak{p}=\mathfrak{p}_0\end{cases}\end{aligned}$$

$$\mathcal{E}_{\mathfrak{p},1}(z):=\begin{cases}(1-(\beta_z^{\mathfrak{p}})^2\varpi_{\mathfrak{p}}^{-\underline{k}_{3,\mathfrak{p}}-\underline{2}})\cdot(1-(\beta_z^{\mathfrak{p}})^2\varpi_{\mathfrak{p}}^{-\underline{k}_{3,\mathfrak{p}}-\underline{1}}), & \mathfrak{p}\neq\mathfrak{p}_0,\\ (1-(\beta_z^{\mathfrak{p}_0})^2p^{-k_{3,\tau_0}})\cdot(1-(\beta_z^{\mathfrak{p}_0})^2p^{1-k_{3,\tau_0}}), & \mathfrak{p}=\mathfrak{p}_0,\end{cases}$$

$m_0=\frac{k_{1,\tau_0}+k_{2,\tau_0}+k_{3,\tau_0}}{2}\geqslant 0$, $\underline{m}_{\mathfrak{p}}=\frac{\underline{k}_{1,\mathfrak{p}}+\underline{k}_{2,\mathfrak{p}}+\underline{k}_{3,\mathfrak{p}}}{2}=\left(\frac{k_{1,\tau}+k_{2,\tau}+k_{3,\tau}}{2}\right)_{\tau\sim\mathfrak{p}}$, *and $\tau\sim\mathfrak{p}$ means that the real embedding τ corresponds to an embeddings $F_{\mathfrak{p}}\hookrightarrow\mathbb{C}_p$ via ι_p.*

Our construction begins with a triplet of families of p-stabilized newforms of quaternionic automorphic representations. Note that, in our setting, there is not a perfect duality between automorphic forms and Hecke algebras. Consequently, our p-adic L-function is not canonical as an element of the Hecke algebra alone. Instead, it is canonical as an element of the Hecke algebra tensored with the module of isomorphisms between the dual of the Hecke algebra and a module of automorphic forms.

For the precise form of the constant K, refer to the end of the proof of Theorem 10.5. The starting point of our construction is a result by Harris–Kudla and Ichino, which relates the central value $L(\frac{1-\nu_1-\nu_2-\nu_3}{2},\Pi_x\otimes\Pi_y\otimes\Pi_z)$ to certain trilinear period integrals defined in terms of automorphic forms of π_1, π_2 and π_3. These trilinear periods can be geometrically interpreted as trilinear products of sections of modular sheaves over unitary Shimura curves (see Proposition 3.10 and Theorem 4.9). By adapting the p-adic interpolation of the integral powers of the Gauss–Manin connexion from [31] to our context, and drawing inspiration from the ideas in [1] and [17], we perform p-adic interpolation of the trilinear periods over the unitary Shimura curves. This is sufficient for the construction of the p-adic L-functions.

1.3. On our condition for weights

The weight space under consideration in this work is the $3(d+1)$-dimensional space $\mathcal{W}^G\times\mathcal{W}^G\times\mathcal{W}^G$. Moreover, the subset $S_3\subset\mathcal{W}^G\times\mathcal{W}^G\times\mathcal{W}^G$ of classical weights where our p-adic L-function has interpolation properties corresponds to triplets of weights that are unbalanced at τ_0 with dominant weight in the third coordinate, and balanced at the other $\tau \neq \tau_0$. Observe that τ_0 is the only place at infinity where B is splits.

In general, one could consider other sets of classical points of $\mathcal{W}^G \times \mathcal{W}^G \times \mathcal{W}^G$ with other conditions of balance/unbalance on the triplet of weights. Depending on the sign of the functional equation, one expects the existence of either p-adic L-functions or families of Galois cohomology classes with good interpolation properties on the given sets. Furthermore, if two of these sets of classical weights differ at a single place by its condition of balance/unbalance, then one would expect a reciprocity law relating these objects. This would generalize of the classical setting (see [13] and [14]) and is analogous to the Siegel case (see [33] and a ongoing work by Andreatta, Bertolini, Seveso, and Venerucci).

In this line of ideas, an interesting set to consider is the one in which the classical weights are *balanced everywhere*. In fact, in the ongoing work [4] with Cauchi and Rotger, we construct p-adic families of Galois cohomology classes that interpolate diagonal cohomology classes in such a set. A main goal of [4] is to prove a reciprocity law relating this family of Galois cohomology classes to the p-adic L-function constructed in the present work. From this, we expect arithmetic applications such as p-adic versions of Kato's theorem for elliptic curves over totally real number fields.

Our choice of the set S_3 is motivated by the fact that, since B will split at a single place at infinity, the quaternionic Shimura varieties involved in our construction are curves, and the geometry becomes substantially simpler. For other choices of the set of classical weights, one would have to work with higher-dimensional quaternionic Shimura varieties, where generalizing Andreatta–Iovita's ideas may be more involved, and the associated Galois representations do not fit exactly with the ones we are interested in. Nevertheless, we believe that future projects in this direction will be very interesting. Indeed, in the recent work [28], Kazi considers sets of weights that are unbalanced everywhere for $B = \mathrm{M}_2$ (Hilbert case) to construct twisted triple p-adic L-functions.

Acknowledgements

We would like to thank Fabrizio Andreatta, Yiwen Ding, Adrian Iovita, Michael Harris and Victor Rotger for helpful comments and conversations.

Part A. Background

2. Basic notations

2.1. Setting

Let F be a totally real field, $d = [F : \mathbb{Q}]$, $\mathcal{O}_F$ its ring of integers, and Σ_F the set of its real embeddings. Let $\mathbb{A}$, $\mathbb{A}_f$, $\mathbb{A}_F$, and $\mathbb{A}_{F,f}$ be the ring of adeles and finite adeles of $\mathbb{Q}$ and F, respectively, and let $|\cdot| : \mathbb{A}_F^\times/F^\times \to \mathbb{R}$ be the norm.

In all this paper, we fix an embedding $\tau_0 \in \Sigma_F$. We denote by $\underline{1} \in \mathbb{Z}[\Sigma_F]$ the element with each coordinate equal to 1. For $x \in F^\times$ and $\underline{k} \in \mathbb{Z}[\Sigma_F]$, we put $x^{\underline{k}} = \prod_{\tau\in\Sigma_F} \tau(x)^{k_\tau}$.

We fix an odd prime p, denote by Σ_p the set of embeddings of F in $\overline{\mathbb{Q}}_p$ and fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. For a prime $\mathfrak{p} \mid p$, let $F_\mathfrak{p}$ be the completion of F at $\mathfrak{p}$, $\Sigma_\mathfrak{p}$ the set of its embeddings in $\mathbb{C}_p$, $\mathcal{O}_\mathfrak{p}$ its ring of integers, $\kappa_\mathfrak{p}$ its residue field, $q_\mathfrak{p} = \sharp\kappa_\mathfrak{p}$ and $e_\mathfrak{p}$ the ramification index. We fix uniformizers $\varpi_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$. Using the embedding ι_p, we identify Σ_F with $\Sigma_p := \bigcup_{\mathfrak{p}|p} \Sigma_\mathfrak{p}$ naturally. Moreover, we will use the notation $\mathcal{O} := \mathcal{O}_F \otimes \mathbb{Z}_p$, which naturally decomposes as $\mathcal{O} = \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}$.

In all this text, we denote by $\mathfrak{p}_0$ the prime corresponding to τ_0 through ι_p, and we put $F_0 := F_{\mathfrak{p}_0}$, $\mathcal{O}_0 := \mathcal{O}_{\mathfrak{p}_0}$, and $\Sigma_0 := \Sigma_{\mathfrak{p}_0}$. Moreover, we will suppose that $[F_0 : \mathbb{Q}_p] = 1$ *and F is unramified at p.* We denote $\mathcal{O}^{\tau_0} := \prod_{\mathfrak{p}\neq\mathfrak{p}_0} \mathcal{O}_\mathfrak{p}$. Thus, we have the following decomposition $\mathcal{O} = \mathcal{O}_0 \times \mathcal{O}^{\tau_0} = \mathbb{Z}_p \times \mathcal{O}^{\tau_0}$.

We fix a quaternion algebra B over F that splits at τ_0 and at each $\mathfrak{p} \mid p$, and it is ramified at each $\tau \in \Sigma_F \setminus \{\tau_0\}$. For $\tau \in \Sigma_F$, we put $B_\tau := B \otimes_{F,\tau} \mathbb{R}$, fix an identification $\iota_{\tau_0} : B_{\tau_0} \xrightarrow{\cong} \mathrm{M}_2(\mathbb{R})$, and let $B_{\tau_0}^+ \subset B_{\tau_0}^\times$ be the elements of positive norm. For each $\tau \in \Sigma_F \smallsetminus \{\tau_0\}$, we fix an isomorphism $B \otimes_\tau \mathbb{C} \cong \mathrm{M}_2(\mathbb{C})$ and denote by $\iota_\tau : B_\tau^\times \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ the embedding obtained. We denote by $\mathrm{disc}(B)$ the discriminant of B.

Let $G := \mathrm{Res}_\mathbb{Q}^F B^\times$ and we denote by $G(\mathbb{Q})^+$ the subgroup of elements of $G(\mathbb{Q})$ such that its image in B_{τ_0} is contained in $B_{\tau_0}^+$. Let $\det : G \to \mathrm{Res}_\mathbb{Q}^F \mathbb{G}_{m,F}$ be the reduced norm, and let $G^* = G \times_{\mathrm{Res}_\mathbb{Q}^F \mathbb{G}_{m,F}} \mathbb{G}_{m,\mathbb{Q}} \subseteq G$.

Fix $\lambda \in \mathbb{Q}$ such that $\lambda < 0$ and p splits in $\mathbb{Q}(\sqrt{\lambda})$. Let $E := F(\sqrt{\lambda})$ and write $z \mapsto \bar{z}$ for the nontrivial automorphism of E/F. For $\tau \in \Sigma_F$, let $\tilde{\tau} : E \to \mathbb{C}$ be the embedding over τ such that $\tilde{\tau}(\sqrt{\lambda}) = \sqrt{\lambda}$.

Let $D := B \otimes_F E$ be a quaternion algebra over E, and let us consider the involution on D defined by $l = b \otimes z \mapsto \bar{l} := \bar{b} \otimes \bar{z}$, where $\bar{b}$ is the canonical involution of B. Let $\delta \in D^\times$ such that $\bar{\delta} = -\delta$(1) and define a

(1) Our δ corresponds to the product $\alpha\delta$ with the notation of [11]

new involution on D by $l \mapsto l^* := \delta^{-1}\bar{l}\delta$. Denote V the underlying $\mathbb{Q}$-vector space of D endowed with the natural left action of D. We have a symplectic bilinear form on V:

$$\Theta : V \times V \longrightarrow \mathbb{Q}, \quad (v,w) \longmapsto \mathrm{Tr}_{E/\mathbb{Q}}(\mathrm{Tr}_{D/E}(v\delta w^*)).$$

Let $G_D := \mathrm{Res}_{\mathbb{Q}}^{E} D^\times$ and let G' be the reductive group over $\mathbb{Q}$ such that, for each $\mathbb{Q}$-algebra R,

$$G'(R) = \{\mathrm{D-linear\ symplectic\ similitudes\ of\ } (V \otimes_{\mathbb{Q}} R, \Theta \otimes_{\mathbb{Q}} R)\}.$$

Denote $T_E := \mathrm{Res}_{\mathbb{Q}}^{E}\mathbb{G}_{m.E}/\mathrm{Res}_{\mathbb{Q}}^{F}\mathbb{G}_{m,F}$. Then we have the exact sequence

$$0 \longrightarrow G^* \longrightarrow G' \xrightarrow{\pi_T} T_E \longrightarrow 0. \tag{2.1}$$

In fact, if $d \in G' \subseteq D^\times$ and $\Theta(vd,wd) = \mu\Theta(v,w)$ with $\mu \in \mathbb{Q}$, then

$$\begin{aligned}\mathrm{Tr}_{E/\mathbb{Q}}(\mathrm{Tr}_{D/E}(vd\delta d^* w^*)) &= \mathrm{Tr}_{E/\mathbb{Q}}(d\bar{d}\mathrm{Tr}_{D/E}(v\delta w^*)) \\ &= \mathrm{Tr}_{E/\mathbb{Q}}(\mu \mathrm{Tr}_{D/E}(v\delta w^*)),\end{aligned}$$

thus, $\mu = d\bar{d}$. Hence, $d = be \in B^\times E^\times \subset D^\times$ with $\det(b)e\bar{e} = \mu$. We put $\pi_T(d) = \pi_T(be) = [e]$.

Remark 2.1. — If there exists an embedding $\varphi : E \hookrightarrow B$, then $D \simeq \mathrm{M}_2(E)$ and T_E acts on G^* by conjugation of $\varphi(e)$. In this situation, we have an isomorphism $G' \xrightarrow{\simeq} G^* \rtimes T_E$ given by $be \mapsto (b\varphi(e), e)$.

Let $A \subset B$ be an $\mathcal{O}_F$-order. Let $\psi : \mathbb{Q}(\sqrt{\lambda}) \to D$ given by $z \mapsto 1 \otimes z$ and fix a $\mathcal{O}_E$-algebra R such that $A \otimes_{\mathbb{Z}} R = \mathrm{M}_2(R) \otimes_{\mathbb{Z}} \mathcal{O}_F$. For any R-module M endowed with a linear action of $A \otimes_{\mathcal{O}_F} \mathcal{O}_E$, we put $M^\pm := \{v \in M : \psi(\sqrt{\lambda}) * v = \pm\sqrt{\lambda} \cdot v\}$. Then $A \otimes_{\mathbb{Z}} R = \mathrm{M}_2(R) \otimes_{\mathbb{Z}} \mathcal{O}_F$ acts on these modules, hence, we write $M^{\pm,1} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) M^\pm$ and $M^{\pm,2} := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) M^\pm$. Both $R \otimes_{\mathbb{Z}} \mathcal{O}_F$-modules are isomorphic through the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We have the following decomposition, that is an equality if $\mathrm{disc}(\mathbb{Q}(\sqrt{\lambda})) \in R^\times$,

$$M \supseteq M^+ \oplus M^- = M^{+,1} \oplus M^{-,1} \oplus M^{+,2} \oplus M^{-,2}.$$

2.2. Levels

Fix $\mathcal{O}_B \subset B$ a maximal order and let $\mathcal{O}_D := \mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_E$. Using this choices we extend the groups G_D and G from $\mathbb{Q}$ to $\mathbb{Z}$. Remark that $\mathcal{O}_D$ might not be maximal. In fact, since $\mathrm{disc}(D)$ is an ideal of $\mathcal{O}_F$ dividing $\mathrm{disc}(B)$, if $\mathfrak{q}$ is an ideal of F over an odd prime, inert in E, and such that $\mathfrak{q} \mid \mathrm{disc}(B)\mathrm{disc}(D)^{-1}$, then $(\mathcal{O}_D)_{\mathfrak{q}}$ is an Eichler order of level $\mathfrak{q}$. Nevertheless, $\mathcal{O}_D$ is maximal at every prime not dividing $\mathrm{disc}(B)\mathrm{disc}(D)^{-1}$.

If $\mathfrak{n}$ is an ideal of F prime to $\mathrm{disc}(B)$, we consider the open compact subgroup of $G_D(\widehat{\mathbb{Z}})$:

$$K_1^D(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_D(\widehat{\mathbb{Z}}) : \ c \equiv d-1 \equiv 0 \bmod \mathfrak{n}\mathcal{O}_E \right\}.$$

Moreover, we put $K_1^B(\mathfrak{n}) := K_1^D(\mathfrak{n}) \cap G(\widehat{\mathbb{Z}})$, $K_1^{'}(\mathfrak{n}) := K_1^D(\mathfrak{n}) \cap G'(\mathbb{A}_f)$ and $K_{1,1}^B(\mathfrak{n}) := K_1^D(\mathfrak{n}) \cap G^*(\mathbb{A}_f)$. Since $\pi_T(K_1^{'}(\mathfrak{n})) = (\widehat{\mathcal{O}}_F + \mathfrak{n}\widehat{\mathcal{O}}_E)^\times / \widehat{\mathcal{O}}_F^\times \subseteq T_E(\widehat{\mathbb{Z}})$, then from (2.1) we have $K_1^{'}(\mathfrak{n})/K_{1,1}^B(\mathfrak{n}) \cong (\widehat{\mathcal{O}}_F + \mathfrak{n}\widehat{\mathcal{O}}_E)^\times / \widehat{\mathcal{O}}_F^\times$. Put $\mathrm{Pic}(E/F, \mathfrak{n}) := T_E(\mathbb{A}_f)/[(\widehat{\mathcal{O}}_F + \mathfrak{n}\widehat{\mathcal{O}}_E)^\times / \widehat{\mathcal{O}}_F^\times] T_E(\mathbb{Q})$, and for each $t \in \mathrm{Pic}(E/F,\mathfrak{n})$ we fix a representative $b_t t \in G(\mathbb{A}_f)(E \otimes \mathbb{A}_{F,f})^\times = G'(\mathbb{A}_f)$ under π_T. We denote $\Gamma_{1,1}^t(\mathfrak{n}) := G^*(\mathbb{Q})_+ \cap b_t K_{1,1}^B(\mathfrak{n}) b_t^{-1}$ and $\Gamma_1^t(\mathfrak{n}) := G(\mathbb{Q})_+ \cap b_t K_1^B(\mathfrak{n}) b_t^{-1}$. We can define analogously $K_0^D(\mathfrak{n})$, $K_0^B(\mathfrak{n})$, $K_0'(\mathfrak{n})$, $K_{0,1}^B(\mathfrak{n})$, $\Gamma_{0,1}^t(\mathfrak{n})$ and $\Gamma_0^t(\mathfrak{n})$. For any $\mathfrak{q}$ coprime to $\mathfrak{n}$, write also $K_1'(\mathfrak{n},\mathfrak{q}) := K_1'(\mathfrak{n}) \cap K_0'(\mathfrak{q})$, and similarly for $\Gamma_{1,1}^t(\mathfrak{n},\mathfrak{q})$ and $\Gamma_1^t(\mathfrak{n},\mathfrak{q})$. Moreover, for each $[\mathfrak{c}] \in \mathrm{Pic}(\mathcal{O}_F) \simeq \mathbb{A}_{F,f}^\times / F_+^\times \widehat{\mathcal{O}}_F^\times$ we fix $b_\mathfrak{c} \in G(\mathbb{A}_f^p)$ such that $\det(b_\mathfrak{c}) = \mathfrak{c}$. We put $\Gamma_1^\mathfrak{c}(\mathfrak{n}) := G(\mathbb{Q})_+ \cap b_\mathfrak{c} K_1^B(\mathfrak{n}) b_\mathfrak{c}^{-1}$ and $\Gamma_{1,1}^\mathfrak{c}(\mathfrak{n}) := G^*(\mathbb{Q})_+ \cap b_\mathfrak{c} K_{1,1}^B(\mathfrak{n}) b_\mathfrak{c}^{-1}$.

3. Automorphic forms for G

In this section, we recall some facts about quaternionic automorphic forms over F. Moreover, we introduce some algebraic and analytic operations (triple products) and we recall Ichino's formula which relates them to central L-values. One of the main goals of this work is to p-adically deform these algebraic operations.

3.1. Quaternionic Automorphic Forms

We start by introducing some notations about local representations. Let $k > 1$ and ν be integers such that $k \equiv \nu \pmod 2$.

Let $\mathcal{D}(k,\nu)$ be the $(\mathcal{G},K)$-module of discrete series of $B_{\tau_0}^\times$ of weight k and central character $a \mapsto a^\nu$. It is the sub-representation of $C^\infty(B_{\tau_0}^{\times,+},\mathbb{C})$ generated by the holomorphic element $f_k\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (ad-bc)^{\frac{\nu+k}{2}}(ci+d)^{-k}$. Write also $f_{-k} := \overline{f}_k \in \mathcal{D}(k,\nu)$.

Let R, L be the Shimura–Mass operators defined in [9, Proposition 2.2.5], then f_k is also characterized by the relations

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} f_{\pm k} = \mathrm{e}^{\pm kit} f_{\pm k}, \quad a f_{\pm k} = a^\nu f_{\pm k}, \quad L f_k = 0 = R f_{-k}. \tag{3.1}$$

Let $\mathcal{P}(k,\nu)=\mathrm{Sym}^k(\mathbb{C}^2)\otimes\mathbb{C}[\frac{\nu-k}{2}]$ be the space of homogeneous polynomials of degree k endowed with the following action of $\mathrm{GL}_2(\mathbb{C})$: $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{C})$ and $P(x,y)\in\mathcal{P}(k,\nu)$ then

$$\gamma P(x,y):=\det(\gamma)^{(\nu-k)/2}P(ax+cy,bx+dy).$$

For each $\tau\in\Sigma_F\smallsetminus\{\tau_0\}$ we write $\mathcal{P}_\tau(k,\nu)$ to denote the $\mathbb{C}$-vector space $\mathcal{P}(k,\nu)$ endowed with the action of $B_\tau^\times$ given through the embedding ι_τ. Remark that we have an isomorphism:

$$\mathcal{P}_\tau(k,-\nu)^\vee\simeq\mathcal{P}_\tau(k,\nu),\quad \mu\longmapsto P_\mu(X,Y)=\mu\left(\begin{vmatrix} X & Y\\ x & y\end{vmatrix}^k\right). \tag{3.2}$$

We denote by $\mathcal{A}(\mathbb{C})$ the $\mathbb{C}$-vector space of functions $f:G(\mathbb{A})\to\mathbb{C}$ such that:

(i) There exists an open compact subgroup $U\subseteq G(\mathbb{A}_f)$ such that $f(gU)=f(g)$, for $g\in G(\mathbb{A})$.
(ii) $f|_{G(\mathbb{R})}\in C^\infty(G(\mathbb{R}),\mathbb{C})$.
(iii) f is K-finite where K is the maximal open compact of $G(\mathbb{R})$ i.e. its right translates by K span a finite-dimensional vector space.
(iv) f is $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the center of the universal enveloping algebra of the Lie algebra of $G(\mathbb{R})$.

Write ρ for the action of $G(\mathbb{A})$ given by right translations, then $(\mathcal{A}(\mathbb{C}),\rho)$ defines a smooth $G(\mathbb{A})$-representation. Moreover, $\mathcal{A}(\mathbb{C})$ is also equipped with a $G(\mathbb{Q})$-action: if $h\in G(\mathbb{Q})$, $g\in G(\mathbb{A})$, $f\in\mathcal{A}(\mathbb{C})$ we put $(h\cdot f)(g)=f(h^{-1}g)$.

Let $\Xi:=\{\underline{k}\in\mathbb{N}[\Sigma_F]\,|\,k_{\tau_0}>0,\ k_\tau\equiv k_{\tau'}\pmod 2\text{ for all }\tau,\tau'\}$, and for each $\nu\in\mathbb{Z}$ we put $\Xi_\nu=\{\underline{k}\in\Xi\,|\,k_{\tau_0}\equiv\nu\pmod 2\}$. If $\nu\in\mathbb{Z}$ and $\underline{k}\in\Xi_\nu$ we put $\mathcal{P}^{\tau_0}(\underline{k},\nu):=\bigotimes_{\tau\in\Sigma_F\smallsetminus\{\tau_0\}}\mathcal{P}_\tau(k_\tau,\nu)$ and $\mathcal{D}(\underline{k},\nu):=\mathcal{D}(k_{\tau_0},\nu)\otimes\mathcal{P}^{\tau_0}(\underline{k},\nu)$. It is a $G(\mathbb{R})$-representation (understanding that it is a $(\mathcal{G},K)$-module at τ_0) and the action of the center $(F\otimes\mathbb{R})^\times$ is given by the character $a\mapsto a^{\nu\underline{1}}$. For a given weight $(\underline{k},\nu)$, the space

$$\mathcal{A}(\underline{k},\nu):=\mathrm{Hom}_{G(\mathbb{R})}(\mathcal{D}(\underline{k},\nu),\mathcal{A}(\mathbb{C}))$$

is endowed with natural $G(\mathbb{Q})$ and $G(\mathbb{A}_f)$-actions, that commute with each other.

Definition 3.1. — *The elements of $H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k},\nu))$ are called* automorphic forms of weight $(\underline{k},\nu)$.

Remark 3.2. — We fix $\phi \in H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k},\nu))$, we identify $B_{\tau_0}^+/\mathrm{SO}(2)F_0^\times$ with the Poincaré upper half plane $\mathfrak{H}$ and define the holomorphic function

$$f_\phi : \mathfrak{H} \times G(\mathbb{A}_f) \longrightarrow \bigotimes_{\tau\in\Sigma_F\setminus\{\tau_0\}} \mathcal{P}_\tau(k_\tau,\nu)^\vee,$$

by $f_\phi(z,g_f)(P) := \phi(f_{k_{\tau_0}} \otimes P)(g_{\tau_0}g_f)f_{k_{\tau_0}}(g_{\tau_0})^{-1}$, where $z = g_{\tau_0}i \in \mathfrak{H}$ for some $g_{\tau_0} \in B_{\tau_0}^\times$, $g_f \in G(\mathbb{A}_f)$ and $P \in \bigotimes_{\tau\neq\tau_0} \mathcal{P}_\tau(k_\tau,\nu)$. If $\gamma \in G(\mathbb{Q})^+$ then:

$$f_\phi(\gamma z, \gamma g_f) = \det\gamma^{\frac{-\nu-k_{\tau_0}}{2}}(cz+d)^{k_{\tau_0}}\gamma\left(f_\phi(z,g_f)\right), \qquad \iota_{\tau_0}\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Since $\mathcal{D}(k_{\tau_0},\nu)$ is generated by $f_{k_{\tau_0}}$, to provide f_ϕ is equivalent to providing ϕ.

Definition 3.3. — *Let $\mathfrak{n}$ be an ideal of F prime to* $\mathrm{disc}(B)$ *and let $\chi : \mathbb{A}_F^\times/F^\times \to \overline{\mathbb{Q}}^\times$ be a finite character. Let $M_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n}),\chi)$ be the space of $\phi \in H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k},\nu))^{K_1^B(\mathfrak{n})}$, such that $\phi(f)(ag) = \chi(a)|a|^\nu\phi(f)(g)$ for $a \in \mathbb{A}_F^\times$, $g \in G(\mathbb{A})$, and $f \in \mathcal{D}(\underline{k},\nu)$.*

3.2. Archimedean trilinear products

Firstly we treat the local setting. Let $k_1,k_2,k_3 \in \mathbb{N}_{>0}$ and $\nu_1,\nu_2,\nu_3 = \nu_1+\nu_2 \in \mathbb{Z}$ such that $k_i \equiv \nu_i \pmod 2$ for $i = 1,2,3$. We consider the following two cases:

(1) *Unbalanced:* Suppose that $k_3 \geqslant k_1+k_2$ and $m := (k_1+k_2+k_3)/2 \in \mathbb{Z}$. We write $m_3 := \frac{-k_1-k_2+k_3}{2} \geqslant 0$, and consider the map $t_{\tau_0} : \mathcal{D}(k_3,\nu_1+\nu_2) \to \mathcal{D}(k_1,\nu_1)\otimes\mathcal{D}(k_2,\nu_2)$ given by:

$$t_{\tau_0}(f_{k_3}) = \sum_{j=0}^{m_3}(-1)^j\binom{m_3}{j}\binom{m-2}{k_1+j-1}R^j(f_{k_1})\otimes R^{m_3-j}(f_{k_2}). \tag{3.3}$$

This map is well defined since $t_{\tau_0}(f_{k_3})$ also satisfies the relations (3.1).

(2) *Balanced:* Suppose that, for $i = 1,2,3$, we have $2k_i \leqslant k_1+k_2+k_3$ and $m := (k_1+k_2+k_3)/2 \in \mathbb{Z}$. We write $m_i := m-k_i \geqslant 0$, for $i = 1,2,3$. For each $\tau \in \Sigma_F \setminus \{\tau_0\}$, we write $t_\tau^\vee : \mathcal{P}_\tau(k_1,\nu_1)^\vee \otimes \mathcal{P}_\tau(k_2,\nu_2)^\vee \to \mathcal{P}_\tau(k_3,-\nu_1-\nu_2)$ for the map given by:

$$t_\tau^\vee(\mu_1\otimes\mu_2)(x,y) = \mu_1\left(\mu_2\left(\begin{vmatrix} x & y \\ x_2 & y_2\end{vmatrix}^{m_1}\begin{vmatrix} x & y \\ x_1 & y_1\end{vmatrix}^{m_2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix}^{m_3}\right)\right);$$

where $\mu_1 \in \mathcal{P}_\tau(k_1,\nu_1)^\vee$ and $\mu_2 \in \mathcal{P}_\tau(k_2,\nu_2)^\vee$. Using the identification (3.2) we obtain a $B_\tau^\times$-equivariant morphism

$$t_\tau : \mathcal{P}_\tau(k_3,\nu_1+\nu_2) \longrightarrow \mathcal{P}_\tau(k_1,\nu_1)\otimes\mathcal{P}_\tau(k_2,\nu_2). \tag{3.4}$$

Now we consider the global setting. Let $\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \Xi$.

Definition 3.4. — $\underline{k}_1,\underline{k}_2,\underline{k}_3$ *are called* unbalanced at τ_0 with dominant weight $\underline{k}_3$ *if* $k_{1,\tau}+k_{2,\tau}+k_{3,\tau}$ *is even, for* $\tau\in\Sigma_F$, $k_{3,\tau_0}\geqslant k_{1,\tau_0}+k_{2,\tau_0}$, *and* $2k_{i,\tau}\leqslant k_{1,\tau}+k_{2,\tau}+k_{3,\tau}$, *for* $\tau\neq\tau_0$ *and* $i=1,2,3$.

Assume that $\underline{k}_1\in\Xi_{\nu_1}$, $\underline{k}_2\in\Xi_{\nu_2}$ and $\underline{k}_3\in\Xi_{\nu_1+\nu_2}$ are unbalanced at τ_0 with dominant weight $\underline{k}_3$. The products (3.3) and (3.4) provide a morphism of $G(\mathbb{R})$-representations

$$t_\infty : \mathcal{D}(\underline{k}_3,\nu_1+\nu_2)\longrightarrow\mathcal{D}(\underline{k}_1,\nu_1)\otimes\mathcal{D}(\underline{k}_2,\nu_2)$$

Thus, we obtain a global and $G(\mathbb{Q})$-equivariant *trilinear product*:

$$t:\mathcal{A}(\underline{k}_1,\nu_1)\otimes\mathcal{A}(\underline{k}_2,\nu_2)\longrightarrow\mathcal{A}(\underline{k}_3,\nu_1+\nu_2);\quad t(\phi_1,\phi_2)(f):=\phi_1\phi_2(t_\infty(f)),$$

for $\phi_1\in\mathcal{A}(\underline{k}_1,\nu_1)$, $\phi_2\in\mathcal{A}(\underline{k}_2,\nu_2)$ and $f\in\mathcal{D}(\underline{k}_3,\nu_1+\nu_2)$. Here

$$\phi_1\phi_2\left(\sum_j f_j^1\otimes f_j^2\right):=\sum_j\phi_1(f_j^1)\phi_2(f_j^2)$$

if $f_j^i\in\mathcal{D}(\underline{k}_i,\nu_i)$, for $i=1,2$. From this we obtain a trilinear product between automorphic forms:

$$\begin{aligned} t: H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k}_1,\nu_1))\times H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k}_2,\nu_2))&\\ \longrightarrow H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k}_3,\nu_1+\nu_2)).&\end{aligned}\tag{3.5}$$

3.3. Test vectors and non-archimedean trilinear products

Let L be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_L$ its ring of integers, ϖ an uniformizer, κ its residue field and $q:=\#\kappa$. Let W be a spherical representation of $\mathrm{GL}_2(L)$ with central character ϵ. Suppose there is a hermitian inner product $\langle\cdot\,,\cdot\rangle : W\times W\to\mathbb{C}$ such that $\langle gv,gv'\rangle=|\epsilon(\det(g))|\cdot\langle v,v'\rangle$, for $v,v'\in W$ and $g\in\mathrm{GL}_2(L)$.

We write $K:=\mathrm{GL}_2(\mathcal{O}_L)$ and $K_0(\varpi^n):=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in K:\ \varpi^n\mid c\}$, for $n\in\mathbb{N}$. Fix $v_0\in W^K\setminus\{0\}$ and let $v_n:=\left(\begin{smallmatrix}1&0\\0&\varpi^n\end{smallmatrix}\right)v_0\in W^{K_0(\varpi^n)}$.

If T is the usual Hecke operator, then there exists $a\in L^\times$ such that $Tv_0=a\cdot v_0$. Let $U:W\to W$ be the operator given by $Uv:=q^{-1}(\sum_{j\in\kappa}g_jv)$, with $g_j=\left(\begin{smallmatrix}\varpi&j\\0&1\end{smallmatrix}\right)$ for $j\in\kappa$. Then $Uv_0=a\cdot v_0-q^{-1}v_1$.

Lemma 3.5. — *if*

$$\chi = |\epsilon(\varpi)|^{-1}\epsilon(\varpi),$$

then there exists $\varrho(X,Y) \in \mathbb{Q}(\chi)[X,Y]$ *such that*

$$\langle v_n, v_m\rangle = \varrho(a, |\epsilon(\varpi)|)\cdot\langle v_0, v_0\rangle,$$

for all n *and* m.

Proof. — Firstly remark that, for $n \geqslant m$, we have

$$\langle v_n, v_m\rangle = |\epsilon(\varpi)|^m\langle v_{n-m}, v_0\rangle.$$

If $j \in \kappa$, then we have $g_j = \left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)v_n = v_n$, we obtain:

$$\begin{aligned}\langle Uv_0, v_n\rangle = q^{-1}\sum_{j\in\kappa}\langle g_jv_0, v_n\rangle &= |\epsilon(\varpi)|\overline{\epsilon(\varpi)}^{-1}\langle v_0, v_{n+1}\rangle\\ &= |\epsilon(\varpi)|^{-1}\epsilon(\varpi)\langle v_0, v_{n+1}\rangle.\end{aligned}$$

In the same way, we obtain $\langle v_n, Uv_0\rangle = |\epsilon(\varpi)|\epsilon(\varpi)^{-1}\langle v_{n+1}, v_0\rangle$. Thus,

$$\begin{aligned}\langle v_1, v_0\rangle &= |\epsilon(\varpi)|^{-1}\epsilon(\varpi)\langle v_0, Uv_0\rangle\\ &= \bar{a}|\epsilon(\varpi)|^{-1}\epsilon(\varpi)\langle v_0, v_0\rangle - q^{-1}|\epsilon(\varpi)|^{-1}\epsilon(\varpi)\langle v_0, v_1\rangle\\ &= \bar{a}|\epsilon(\varpi)|^{-1}\epsilon(\varpi)\langle v_0, v_0\rangle - q^{-1}\langle Uv_0, v_0\rangle\\ &= (\bar{a}|\epsilon(\varpi)|^{-1}\epsilon(\varpi) - aq^{-1})\langle v_0, v_0\rangle + q^{-2}\langle v_1, v_0\rangle.\end{aligned}$$

Since $\bar{a}|\epsilon(\varpi)|^{-1}\epsilon(\varpi) = a$, we obtain:

$$\langle v_1, v_0\rangle = \frac{a}{1+q^{-1}}\langle v_0, v_0\rangle \tag{3.6}$$

$$\text{and}\quad \langle v_{n+2}, v_0\rangle = a\langle v_{n+1}, v_0\rangle - q^{-1}\epsilon(\varpi)\langle v_n, v_0\rangle.$$

Then the lemma follows directly from the calculations above. □

Now, for $i = 1,2,3$, let W_i be spaces as above with central characters ϵ_i, spherical vectors $v_0^i \in W_i^K$, and let a_i be the eigenvalues of the usual Hecke operator. We denote by α_i, β_i the roots of the Hecke polynomial $X^2 - a_iX + \epsilon_i(\varpi)q^{-1}$. We assume that we have a $\mathrm{GL}_2(L)$-invariant trilinear product $t : W_1 \otimes W_2 \to W_3$ (i.e. $t(gv_1, gv_2) = gt(v_1, v_2)$ for each $g \in \mathrm{GL}_2(L)$).

We consider the operators $V : W_i \to W_i$, defined by $Vv := g_0^{-1}v$, and $U^* : W_i \to W_i$, by $U^*v := \frac{1}{q}\sum_{j\in\kappa}\left(\begin{smallmatrix}1 & 0\\ j\varpi & \varpi\end{smallmatrix}\right)v$, where $g_0 = \left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)$. If $v, v' \in W_3$ then $\langle Uv, v'\rangle = \chi_3\cdot\langle v, U^*v'\rangle$ with $\chi_3 := \epsilon_3(\varpi)|\epsilon_3(\varpi)|^{-1}$. Moreover, if $v \in W_i$, then we write $v^{[p]} = (1 - VU)v$.

If $n \in \mathbb{N}$, we write as above $v_n^i := \left(\begin{smallmatrix}1 & 0\\ 0 & \varpi^n\end{smallmatrix}\right)v_0^i$. Moreover, let $v_{\alpha_i} := (1-\beta_iV)v_0^i = v_0^i - \beta_i\epsilon_i(\varpi)^{-1}v_1^i$ and $v_{\beta_i}^* := v_1^i - \alpha_iv_0^i$. A straightforward calculation shows that $U^*v_{\beta_i}^* = \beta_iv_{\beta_i}^*$ and $Uv_{\alpha_i} = \alpha_i\cdot v_{\alpha_i}$. Since $\bar{\alpha}_3\chi_3 = \beta_3$, we deduce that, whenever $\alpha_3\neq\beta_3$, the map $v\mapsto\langle v,v^*_{\beta_3}\rangle\langle v_{\alpha_3},v^*_{\beta_3}\rangle^{-1}$ provides the projection into the subspace of W_3 where U acts as α_3.

Proposition 3.6. — *Assume that $\alpha_3\neq\beta_3$, then we have that*

$$\frac{\langle t(v_{\alpha_1},v_{\alpha_2}),v^*_{\beta_3}\rangle}{\langle v_{\alpha_3},v^*_{\beta_3}\rangle} = \frac{(1-\beta_1\alpha_2\alpha_3^{-1})(1-\alpha_1\beta_2\alpha_3^{-1})(1-\beta_1\beta_2\alpha_3^{-1})}{(1-\alpha_1\beta_1\alpha_2\beta_2\alpha_3^{-2})(1-\beta_3\alpha_3^{-1})}\cdot\frac{\langle t(v_0^1,v_0^2),v_0^3\rangle}{\langle v_0^3,v_0^3\rangle}$$

$$\frac{\langle t(v^{[p]}_{\alpha_1},v_{\alpha_2}),v^*_{\beta_3}\rangle}{\langle v_{\alpha_3},v^*_{\beta_3}\rangle} = \frac{(1-\beta_1\alpha_2\alpha_3^{-1})(1-\alpha_1\beta_2\alpha_3^{-1})(1-\beta_1\beta_2\alpha_3^{-1})(1-\alpha_1\alpha_2\alpha_3^{-1})}{(1-\alpha_1\beta_1\alpha_2\beta_2\alpha_3^{-2})(1-\beta_3\alpha_3^{-1})}\cdot\frac{\langle t(v_0^1,v_0^2),v_0^3\rangle}{\langle v_0^3,v_0^3\rangle}.$$

Proof. — Firstly observe that, if $v\in W_1^{K_0(\varpi)}$ and $v'\in W_2^{K_0(\varpi)}$ are test vectors, then we have:

$$\begin{aligned}
\langle t(Vv,Vv'),v^*_{\beta_3}\rangle &= \alpha_3^{-1}\chi_3\langle Vt(v,v'),U^*v^*_{\beta_3}\rangle\\
&= \alpha_3^{-1}\langle t(v,v'),v^*_{\beta_3}\rangle;\\
Ut(v^{[p]},Vv') &= \frac{1}{q}\sum_{j\in\kappa}t(g_jv^{[p]},g_jVv') = \frac{1}{q}\sum_{j\in\kappa}t(g_jv^{[p]},v')\\
&= t(Uv^{[p]},v')=0;\\
\langle t(v,Vv'),v^*_{\beta_3}\rangle &= \alpha_3^{-1}\langle U(t(v^{[p]},Vv')+t(VUv,Vv')),v^*_{\beta_3}\rangle\\
&= \alpha_3^{-1}\langle t(Uv,v'),v^*_{\beta_3}\rangle.
\end{aligned}$$

From these equations we obtain

$$\langle t(Vv,v'),v^*_{\beta_3}\rangle = \alpha_3^{-1}\langle t(v,Uv'),v^*_{\beta_3}\rangle. \tag{3.7}$$

Hence, the second equality of the proposition is a consequence of the first one as $\langle t(v^{[p]}_{\alpha_1},v_{\alpha_2}),v^*_{\beta_3}\rangle=\langle t(v_{\alpha_1},v_{\alpha_2}),v^*_{\beta_3}\rangle-\alpha_1\langle t(Vv_{\alpha_1},v_{\alpha_2}),v^*_{\beta_3}\rangle=(1-\alpha_1\alpha_3^{-1}\alpha_2)\langle t(v_{\alpha_1},v_{\alpha_2}),v^*_{\beta_3}\rangle$.

Secondly, we have

$$\begin{aligned}
\langle t(v_{\alpha_1},v_{\alpha_2}),v^*_{\beta_3}\rangle &= \langle t(v_0^1,v_{\alpha_2}),v^*_{\beta_3}\rangle-\beta_1\langle t(Vv_0^1,v_{\alpha_2}),v^*_{\beta_3}\rangle\\
&= (1-\beta_1\alpha_2\alpha_3^{-1})\langle t(v_0^1,v_{\alpha_2}),v^*_{\beta_3}\rangle\\
\langle t(v_0^1,v_{\alpha_2}),v^*_{\beta_3}\rangle &= (1-\beta_2\alpha_3^{-1}a_1)\langle t(v_0^1,v_0^2),v^*_{\beta_3}\rangle\\
&\quad + q^{-1}\epsilon_1(\varpi)\beta_2\alpha_3^{-1}\langle t(Vv_0^1,v_0^2),v^*_{\beta_3}\rangle
\end{aligned}$$

$$\begin{aligned}\langle t(Vv_0^1,v_0^2),v_{\beta_3}^*\rangle &= \alpha_3^{-1}a_2\langle t(v_0^1,v_0^2),v_{\beta_3}^*\rangle - \alpha_3^{-1}q^{-1}\epsilon_2(\varpi)\langle t(v_0^1,Vv_0^2),v_{\beta_3}^*\rangle\\ &= \alpha_3^{-1}\beta_2\langle t(v_0^1,v_0^2),v_{\beta_3}^*\rangle + \alpha_3^{-1}\alpha_2\langle t(v_0^1,v_{\alpha_2}),v_{\beta_3}^*\rangle,\end{aligned}$$

and therefore

$$\begin{aligned}&\langle t(v_{\alpha_1},v_{\alpha_2}),v_{\beta_3}^*\rangle\\ &\qquad= \frac{(1-\beta_1\alpha_2\alpha_3^{-1})(1-\alpha_1\beta_2\alpha_3^{-1})(1-\beta_1\beta_2\alpha_3^{-1})}{1-\alpha_1\beta_1\alpha_2\beta_2\alpha_3^{-2}}\langle t(v_0^1,v_0^2),v_{\beta_3}^*\rangle.\end{aligned}$$

Since $t(v_0^1,v_0^2)=Cv_0^3$ for some $C\in\mathbb{C}$, we compute using (3.6):

$$\begin{aligned}\langle t(v_0^1,v_0^2),v_{\beta_3}^*\rangle &= \langle t(v_0^1,v_0^2),v_1^3\rangle - \bar\alpha_3\langle t(v_0^1,v_0^2),v_0^3\rangle\\ &= \left(\frac{\bar a_3}{1+q^{-1}}-\bar\alpha_3\right)\langle t(v_0^1,v_0^2),v_0^3\rangle,\end{aligned}$$

and we similarly compute

$$\begin{aligned}\langle v_{\alpha_3},v_{\beta_3}^*\rangle &= \langle v_0^3,v_1^3\rangle - \beta_3\epsilon_3(\varpi)^{-1}\langle v_1^3,v_1^3\rangle - \bar\alpha_3\langle v_0^3,v_0^3\rangle + \bar\alpha_3\beta_3\epsilon_3(\varpi)\langle v_1^3,v_0^3\rangle\\ &= \left(\frac{\bar\beta_3-\bar\alpha_3+\bar\alpha_3q^{-1}(\bar\alpha_3\bar\beta_3^{-1}-1)}{1+q^{-1}}\right)\langle v_0^3,v_0^3\rangle,\end{aligned}$$

and the result follows. □

3.4. Ichino–Harris–Kudla formula and trilinear products

We recall a result by Harris–Kudla and Ichino which provides a formula describing the central critical value of triple product L-functions in terms of certain trilinear periods. Moreover, we relate those trilinear periods with the trilinear products introduced above.

Let Π_1,Π_2 and Π_3 be three irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Assume also that the corresponding central characters ε_i satisfy $\varepsilon_1\cdot\varepsilon_2=\varepsilon_3$. We denote by $L(s,\Pi_1\otimes\Pi_2\otimes\Pi_3)$ the complex triple product L-function attached to the tensor product $\Pi_1\otimes\Pi_2\otimes\Pi_3$.

Assume that there exist $\pi_1,\pi_2,\pi_3\subset H^0(G(\mathbb{Q}),\mathcal{A}(\mathbb{C}))$ irreducible automorphic representations of $G(\mathbb{A})$ associated to Π_1,Π_2 and Π_3, respectively, via the Jacquet–Langlands correspondence. Notice that $\varepsilon_3=\chi\circ|\cdot|^{\nu_3}$, for a finite character χ. This implies that $\nu_3=\nu_1+\nu_2$ and $|\varepsilon_3|^2=|\cdot|^{2\nu_3}$. Let $\widetilde\Pi_3$ be the contragradient representation of Π_3 and write $\Pi:=\Pi_1|\det|^{-\frac{\nu_1}{2}}\otimes\Pi_2|\det|^{-\frac{\nu_2}{2}}\otimes\widetilde\Pi_3|\det|^{\frac{\nu_3}{2}}$ for the corresponding unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A}_M)$, where $M=F\times F\times F$. Note that $\mathbb{A}_F^\times$ acts trivially when embedded diagonally in $\mathrm{GL}_2(\mathbb{A}_M)$. Moreover, $\pi:=\pi_1|\det|^{-\frac{\nu_1}{2}}\otimes\pi_2|\det|^{-\frac{\nu_2}{2}}\otimes\widetilde\pi_3|\det|^{\frac{\nu_3}{2}}$ is the Jacquet–Langlands lift of

Π. For $\varphi \in \pi$, $\widetilde{\varphi} \in \widetilde{\pi}$, where $\widetilde{\pi}$ is the contragradient representation, we consider the *trilinear period*:

$$I(\varphi \otimes \widetilde{\varphi}) := \int_{G(\mathbb{A})/G(\mathbb{Q})\mathbb{A}_F^\times} \int_{G(\mathbb{A})/G(\mathbb{Q})\mathbb{A}_F^\times} \varphi(g)\widetilde{\varphi}(g')\,\mathrm{d}g\,\mathrm{d}g',$$

here $\mathrm{d}g$ is the normalized Haar measure and, in the integral, we consider the natural diagonal embedding $\mathbb{A}_F \hookrightarrow \mathbb{A}_F \times \mathbb{A}_F \times \mathbb{A}_F$. The following is the main result of Harris–Kudla–Ichino in (see [18], [19], [22, Theorem 1.1, Remark 1.3]):

Theorem 3.7. — *For any $\varphi \in \pi$, $\widetilde{\varphi} \in \widetilde{\pi}$, we have*

$$\frac{I(\varphi \otimes \widetilde{\varphi})}{(\varphi, \widetilde{\varphi})} = \frac{1}{2^3} \cdot \zeta_F(2)^2 \cdot \frac{L(1/2, \Pi)}{L(1, \Pi, \mathrm{Ad})} \cdot \prod_v I_v(\varphi_v \otimes \widetilde{\varphi}_v),$$

where $I_v(\varphi_v \otimes \widetilde{\varphi}_v) := \xi_{F_v}(2)^2 \cdot \frac{L_v(1,\Pi_v,\mathrm{Ad})}{L_v(1/2,\Pi_v)} \cdot \int_{F_v^\times \backslash B_v^\times} \frac{(\pi_v(b_v)\varphi_v, \tilde{\varphi}_v)_v}{(\varphi_v, \tilde{\varphi}_v)_v}\,\mathrm{d}b_v$, for certain pairing $(\,\cdot\,,\cdot\,)$ between π and $\widetilde{\pi}$ compatible with local pairings $(\,\cdot\,,\cdot\,)_v$.

In order to interpret this result in terms of the trilinear products described in Section 3.2, we introduce some notations. Fix $i \in \{1, 2, 3\}$. We suppose that $\pi_{i,\infty} := \pi_i\mid_{G(\mathbb{R})} \simeq \mathcal{D}(\underline{k}_i, \nu_i)$, for $\underline{k}_i \in \Xi_{\nu_i}$, and

$$(\pi_i\mid_{G(\mathbb{A}_f)})^{K_1^B(\mathfrak{n}_i)} := ((\pi_i)_f)^{K_1^B(\mathfrak{n}_i)} \simeq \mathbb{C},$$

for some ideal $\mathfrak{n}_i$ prime to $\mathrm{disc}(B)$. Then, we can realize $(\pi_i)_f$ inside the space $H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k}_i, \nu_i))$, and we denote by $\phi_i^0 \in M_{(\underline{k}_i,\nu_i)}(\Gamma_1(\mathfrak{n}_i), \chi_i)$ a generator, for certain character χ_i. If $\phi \in H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k}_i, \nu_i))$, we define $\bar{\phi}, \phi^* \in H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k}_i, \nu_i))$ by:

$$\bar{\phi}(f)(g) := \overline{\phi(\bar{f})(g)},$$

$$\phi^*(f)(g) := \mathrm{sign}(f)^\nu \cdot \phi(f)\left(g\begin{pmatrix} 0 & -1 \\ \varpi_{\mathfrak{n}} & 0 \end{pmatrix}\right) \cdot \chi_i(\det(g))^{-1},$$

where $f \in \mathcal{D}(\underline{k}_i, \nu_i)$, $g \in G(\mathbb{A})$, $\varpi_{\mathfrak{n}} = \prod_{v\mid\mathfrak{n}} \varpi_v^{v_v(\mathfrak{n})} \in \mathbb{A}_F^\times$, ϖ_v is a uniformizer of the finite place v, and $\mathrm{sign}(f)$ is ± 1, if $f \in \mathbb{C}f_{\pm k_{i,\tau_0}} \otimes \mathcal{P}^{\tau_0}(\underline{k}_i, \nu_i)$. Observe that, if $\phi \in M_{(\underline{k}_i,\nu_i)}(\Gamma_1(\mathfrak{n}_i), \chi_i)$ then $\bar{\phi}, \phi^* \in M_{(\underline{k}_i,\nu_i)}(\Gamma_1(\mathfrak{n}_i), \chi_i^{-1})$. Moreover, the contragradient representation $\widetilde{\pi}_i^B$ is generated by $\overline{\phi_i^0}|\det|^{-\nu_i} \in M_{(\underline{k}_i,-\nu_i)}(\Gamma_1(\mathfrak{n}_i), \chi_i^{-1})$.

Lemma 3.8. — *There exists $c_i \in \mathbb{C}^\times$ such that $\overline{\phi_i^0} = c_i \cdot (\phi_i^0)^*$.*

Proof. — The result follows from the fact that both $\bar{\phi}_i^0$ and $(\phi_i^0)^*$ generate irreducible automorphic representations with the same Hecke eigenvalues at finite places not dividing $\mathfrak{n}_i\mathrm{disc}(B)$. ☐

Definition 3.9. — *Write $\underline{k}_3=(k_{3,\tau_0},\underline{k}_3^{\tau_0})$ and let*

$$\phi_1,\phi_2\in H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k}_3,\nu_3))$$

such that $\phi_i\mid_{\mathbb{A}_F^\times}=\varepsilon_3$. Put $\Upsilon^{\tau_0}:=\left|\begin{smallmatrix}X_1&Y_1\\X_2&Y_2\end{smallmatrix}\right|^{\underline{k}_3^{\tau_0}}$, write $[G]=G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{A}_F^\times$, and define the Hermitian inner product:

$$\langle\phi_1,\phi_2\rangle:=\int_{[G]}\overline{\phi_1}\phi_2(f_{-k_{\tau_0}}\otimes f_{k_{\tau_0}}\otimes\Upsilon^{\tau_0})(g)\,|\det(g)|^{-\nu}\,\mathrm{d}g.$$

We denote by $\underline{m}=(m_\tau)_{\tau\in\Sigma_F}:=(\frac{1}{2}(k_{1,\tau}+k_{2,\tau}+k_{3,\tau}))_{\tau\in\Sigma_F}$, $m_{3,\tau_0}=(k_{3,\tau_0}-m_{\tau_0})$ and, for $i=1,2,3$, we denote by $\underline{m}_i^{\tau_0}=(m_{i,\tau})_{\tau\neq\tau_0}=(m_\tau-k_{i,\tau})_{\tau\neq\tau_0}$.

Proposition 3.10. — *Let $\mathfrak{n}=\mathrm{mcm}(\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{n}_3)$. Assume that $\underline{k}_1$, $\underline{k}_2$ and $\underline{k}_3$ are unbalanced at τ_0 with dominant weight $\underline{k}_3$. Then for each $i\in\{1,2,3\}$ there exists $\phi_i\in H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k}_i,\nu_i))^{K_1^B(\mathfrak{n})}$ in $(\pi_i)_f$ such that*

$$\langle\phi_3,t(\phi_1,\phi_2)\rangle^2=\frac{(-1)^{\nu_3}C\cdot C(\phi_1,\phi_2,\phi_3)}{2^{4-2m_{3,\tau_0}}}\cdot\binom{k_{3,\tau_0}-2}{k_{2,\tau_0}+m_{3,\tau_0}-1}^2\cdot L\left(\frac{1-\nu_1-\nu_2-\nu_3}{2},\Pi_1\otimes\Pi_2\otimes\Pi_3\right),$$

where t is the trilinear product of Section 3.2, C is a non-zero constant independent of $(k_{1,\tau_0},k_{2,\tau_0},k_{3,\tau_0})$ and

$$C(\phi_1,\phi_2,\phi_3)=\frac{\langle\phi_1,\phi_1\rangle\langle\phi_2,\phi_2\rangle\langle\phi_3,\phi_3\rangle}{L(1,\Pi_1,\mathrm{Ad})L(1,\Pi_2,\mathrm{Ad})L(1,\Pi_3,\mathrm{Ad})}.$$

Proof. — Consider the morphism

$$t^{\tau_0}=\bigotimes_{\tau\neq\tau_0}t_\tau:\mathcal{P}^{\tau_0}(\underline{k}_3,\nu_3)\longrightarrow\mathcal{P}^{\tau_0}(\underline{k}_1,\nu_1)\otimes\mathcal{P}^{\tau_0}(\underline{k}_2,\nu_2).$$

Then by definition $t^{\tau_0}(\Upsilon^{\tau_0})=\Delta^{\tau_0}\in\mathcal{P}^{\tau_0}(\underline{k}_1,\nu_1)\otimes\mathcal{P}^{\tau_0}(\underline{k}_2,\nu_2)\otimes\mathcal{P}^{\tau_0}(\underline{k}_3,\nu_3)$, where:

$$\Delta^{\tau_0}:=\begin{vmatrix}x_3&y_3\\x_2&y_2\end{vmatrix}^{\underline{m}_1^{\tau_0}}\begin{vmatrix}x_3&y_3\\x_1&y_1\end{vmatrix}^{\underline{m}_2^{\tau_0}}\begin{vmatrix}x_1&y_1\\x_2&y_2\end{vmatrix}^{\underline{m}_3^{\tau_0}}.\tag{3.8}$$

If we put $v_\infty:=f_{-k_{3,\tau_0}}\otimes t_{\tau_0}(f_{k_{3,\tau_0}})\otimes\Delta^{\tau_0}\in\mathcal{D}(\underline{k}_1,\nu_1)\otimes\mathcal{D}(\underline{k}_2,\nu_2)\otimes\mathcal{D}(\underline{k}_3,\nu_3)$, this implies that

$$\langle\phi_3,t(\phi_1,\phi_2)\rangle=\int_{[G]}(\bar{\phi}_3\phi_1\phi_2)(f_{-k_{3,\tau_0}}\otimes t_{\tau_0}(f_{k_{3,\tau_0}})\otimes\Delta^{\tau_0})(g)\,|\det(g)|^{-\nu_3}\mathrm{d}g.$$

Let $i=1,2,3$. Consider $\widetilde{\phi}_i\in M_{(\underline{k}_i,-\nu_i)}(\Gamma_1(\mathfrak{n}_i),\chi_i^{-1})$ given by $\widetilde{\phi}_i(f):=\mathrm{sign}(f)^{\nu_i}\cdot\phi_i(f)\cdot(\epsilon_i\circ\det)^{-1}$. By Lemma 3.8 and the non-degeneracy of the inner product $\langle\cdot,\cdot\rangle$, we have that $\widetilde{\phi}_i \in \widetilde{\pi}_i^B$. It is clear by definition that $\langle\phi_3, t(\phi_1,\phi_2)\rangle = \langle\widetilde{\phi}_3, t(\widetilde{\phi}_1,\widetilde{\phi}_2)\rangle$. Hence, since $\widetilde{\overline{\phi}_3} = (-1)^{\nu_3}\overline{\widetilde{\phi}_3}$, one obtains

$$\begin{aligned}&\langle\phi_3, t(\phi_1,\phi_2)\rangle^2\\ &\quad= (-1)^{\nu_3}\iint_{[G]}(\bar{\phi}_3\phi_1\phi_2)(v_\infty)(g_1)(\widetilde{\bar{\phi}_3}\widetilde{\phi}_1\widetilde{\phi}_2)(\widetilde{v}_\infty)(g_2)|\det(g_1^{-1}g_2)|^{\nu_3}\,\mathrm{d}g_1\,\mathrm{d}g_2,\end{aligned}$$

where $\widetilde{v}_\infty \in \mathcal{D}(\underline{k}_1,-\nu_1)\otimes\mathcal{D}(\underline{k}_2,-\nu_2)\otimes\mathcal{D}(\underline{k}_3,-\nu_3)$ is defined analogously as v_∞. Notice that, again by Lemma 3.8, we can see $\varphi = (\bar{\phi}_3\phi_1\phi_2)(v_\infty)|\det|^{-\nu_3}$ as an element of π, and we can see $\widetilde{\varphi} = (\widetilde{\bar{\phi}_3}\widetilde{\phi}_1\widetilde{\phi}_2)(\widetilde{v}_\infty)|\det|^{\nu_3}$ as an element of $\widetilde{\pi}$. By Ichino's formula (Theorem. 3.7) we obtain

$$\begin{aligned}&\frac{\langle\phi_3, t(\phi_1,\phi_2)\rangle^2}{((\bar{\phi}_3\phi_1\phi_2)(v_\infty^0), \tau_{\mathfrak{n}}(\widetilde{\bar{\phi}_3}\widetilde{\phi}_1\widetilde{\phi}_2)(\widetilde{v}_\infty^0))}\\ &\qquad\qquad = \frac{(-1)^{\nu_3}}{2^3}\cdot\xi_F(2)^2\cdot\frac{L(1/2,\Pi)}{L(1,\Pi,\mathrm{Ad})}\cdot\prod_v I_v(\varphi_v\otimes\widetilde{\varphi}_v),\end{aligned}$$

where $\tau_{\mathfrak{n}} = (\tau_{\mathfrak{n}_1},\tau_{\mathfrak{n}_2},\tau_{\mathfrak{n}_3}) \in G(\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A})$ with $\tau_{\mathfrak{n}_i} = \left(\begin{smallmatrix}0 & -1\\ \varpi_{\mathfrak{n}_i} & 0\end{smallmatrix}\right)$,

$$\begin{aligned}v_\infty^0\otimes\widetilde{v}_\infty^0 := (f_{k_{1,\tau_0}}\otimes\widetilde{f}_{-k_{1,\tau_0}})\otimes(f_{k_{2,\tau_0}}\otimes\widetilde{f}_{-k_{2,\tau_0}})&\\ \otimes(f_{-k_{3,\tau_0}}\otimes\widetilde{f}_{k_{3,\tau_0}})\otimes\Upsilon_1^{\tau_0}\otimes\Upsilon_2^{\tau_0}\otimes\Upsilon_3^{\tau_0}&\in\pi_\infty\otimes\widetilde{\pi}_\infty,\end{aligned}$$

and the local terms $I_v(\varphi_v\otimes\widetilde{\varphi}_v) = \frac{L_v(1,\Pi_v,\mathrm{Ad})}{\xi_{F_v}(2)^2L_v(1/2,\Pi_v)}\cdot J_v$ with

$$J_v = \int_{F_v^\times\backslash B_v^\times}\frac{(\pi_{1,v}(b_v)\phi_{1,v},\widetilde{\phi}_{1,v})_v(\pi_{2,v}(b_v)\phi_{2,v},\widetilde{\phi}_{2,v})_v(\widetilde{\pi}_{3,v}(b_v)\bar{\phi}_{3,v},\widetilde{\bar{\phi}_{3,v}})_v\,\mathrm{d}b_v}{(\phi_{1,v},\tau_{\mathfrak{n}_1,v}\widetilde{\phi}_{1,v})_v(\phi_{2,v},\tau_{\mathfrak{n}_2,v}\widetilde{\phi}_{2,v})_v(\bar{\phi}_{3,v},\tau_{\mathfrak{n}_3,v}\widetilde{\bar{\phi}_{3,v}})_v},\qquad (v\nmid\infty)$$

$$J_{\tau_0} = \int_{F_{\tau_0}^\times\backslash B_{\tau_0}^\times}\frac{(b_{\tau_0}\widetilde{f}_{-k_{3,\tau_0}},f_{-k_{3,\tau_0}})_{\tau_0}((b_{\tau_0},b_{\tau_0})t_{\tau_0}(f_{k_{3,\tau_0}}),\widetilde{t_{\tau_0}(f_{k_{3,\tau_0}})})_{\tau_0}}{(f_{-k_{3,\tau_0}},\widetilde{f}_{k_{3,\tau_0}})_{\tau_0}(f_{k_{1,\tau_0}},\widetilde{f}_{-k_{1,\tau_0}})_{\tau_0}(f_{k_{2,\tau_0}},\widetilde{f}_{-k_{2,\tau_0}})_{\tau_0}}\,\mathrm{d}b_{\tau_0},$$

$$J_\tau = \int_{F_\tau^\times\backslash B_\tau^\times}\frac{((b_\tau,b_\tau,b_\tau)\Delta_\tau,\widetilde{\Delta}_\tau)_\tau}{(\Upsilon_{\tau,1})_\tau(\Upsilon_{\tau,2})_\tau(\Upsilon_{\tau,3})_\tau}\,\mathrm{d}b_\tau,\quad(\tau\neq\tau_0).$$

Here Δ_τ and $\widetilde{\Delta}_\tau$ are both equal to $\left|\begin{smallmatrix}x_3 & y_3\\ x_2 & y_2\end{smallmatrix}\right|^{m_{1,\tau}}\left|\begin{smallmatrix}x_3 & y_3\\ x_1 & y_1\end{smallmatrix}\right|^{m_{2,\tau}}\left|\begin{smallmatrix}x_1 & y_1\\ x_2 & y_2\end{smallmatrix}\right|^{m_{3,\tau}}$ as elements in $\mathcal{P}_\tau(k_{1,\tau},\nu_1)\otimes\mathcal{P}_\tau(k_{2,\tau},\nu_2)\otimes\mathcal{P}_\tau(k_{3,\tau},-\nu_3)$ and $\mathcal{P}_\tau(k_{1,\tau},-\nu_1)\otimes\mathcal{P}_\tau(k_{2,\tau},-\nu_2)\otimes\mathcal{P}_\tau(k_{3,\tau},\nu_3)$, respectively, and

$$\begin{aligned}t_{\tau_0}(f_{k_{3,\tau_0}})\otimes\widetilde{f}_{-k_{3,\tau_0}}&\in\mathcal{D}(\underline{k}_1,\nu_1)\otimes\mathcal{D}(\underline{k}_2,\nu_2)\otimes\mathcal{D}(\underline{k}_3,-\nu_3),\\ \widetilde{t_{\tau_0}(f_{k_{3,\tau_0}})}\otimes f_{-k_{3,\tau_0}}&\in\mathcal{D}(\underline{k}_1,-\nu_1)\otimes\mathcal{D}(\underline{k}_2,-\nu_2)\otimes\mathcal{D}(\underline{k}_3,\nu_3).\end{aligned}$$

By the $B_\tau^\times$-invariance of Δ_τ we have that

$$J_\tau = \frac{(\Delta_\tau, \widetilde{\Delta}_\tau)_\tau}{(\Upsilon_{\tau,1})_\tau (\Upsilon_{\tau,2})_\tau (\Upsilon_{\tau,3})_\tau}.$$

Since we can easily compute that $(\Upsilon_{\tau,i})_\tau = k_{i,\tau} + 1$, we obtain by [21, Lemma 4.12] (see also [21, Section 4.9])

$$I_\tau(\varphi_\tau \otimes \widetilde{\varphi}_\tau) = \frac{L_\tau(1, \Pi_\tau, \mathrm{Ad})}{\xi_{F_\tau}(2)^2 L_\tau(1/2, \Pi_\tau)} \cdot \frac{\frac{(m_\tau+1)! m_{1,\tau}! m_{2,\tau}! m_{3,\tau}!}{k_{1,\tau}! k_{2,\tau}! k_{3,\tau}!}}{(k_{1,\tau}+1)(k_{2,\tau}+1)(k_{3,\tau}+1)} = \pi^{-1}.$$

To compute J_{τ_0} notice that, by [36] and [34], the space

$$\mathrm{Hom}_{(G_{\tau_0}, O(2))}(\mathcal{D}(\underline{k}_1, \nu_1) \otimes \mathcal{D}(\underline{k}_2, \nu_2) \otimes \mathcal{D}(\underline{k}_3, -\nu_3), \mathbb{C})$$

is one dimensional, hence, we have that

$$\int_{F_{\tau_0}^\times \backslash B_{\tau_0}^\times} (b_{\tau_0} f_1, \widetilde{f}_1)_{\tau_0} (b_{\tau_0} f_2, \widetilde{f}_2)_{\tau_0} (b_{\tau_0} \widetilde{f}_3, f_3)_{\tau_0} \, \mathrm{d}b_{\tau_0} \\ = K \cdot (f_1 \otimes f_2, t_{\tau_0}(\widetilde{f}_3))_{\tau_0} \cdot (\widetilde{f}_1 \otimes \widetilde{f}_2, t_{\tau_0}(f_3))_{\tau_0},$$

for $f_1 \otimes f_2 \otimes \widetilde{f}_3 \in \mathcal{D}(\underline{k}_1, \nu_1) \otimes \mathcal{D}(\underline{k}_2, \nu_2) \otimes \mathcal{D}(\underline{k}_3, -\nu_3)$, $\widetilde{f}_1 \otimes \widetilde{f}_2 \otimes f_3 \in \mathcal{D}(\underline{k}_1, -\nu_1) \otimes \mathcal{D}(\underline{k}_2, -\nu_2) \otimes \mathcal{D}(\underline{k}_3, \nu_3)$, and some constant K (depending on k_{i,τ_0}). To compute K consider $f_1 \otimes f_2 \otimes \widetilde{f}_3 = f_{k_{1,\tau_0}} \otimes R^{m_{3,\tau_0}} f_{k_{2,\tau_0}} \otimes \widetilde{f}_{-k_{3,\tau_0}}$ and $\widetilde{f}_1 \otimes \widetilde{f}_2 \otimes f_3 = \widetilde{f}_{k_{1,\tau_0}} \otimes R^{m_{3,\tau_0}} \widetilde{f}_{k_{2,\tau_0}} \otimes f_{-k_{3,\tau_0}}$. By [21, Lemma 3.11] the left-hand-side is

$$\frac{\xi_{F_{\tau_0}}(2)^2 L_{\tau_0}(1/2, \Pi_{\tau_0})}{L_{\tau_0}(1, \Pi_{\tau_0}, \mathrm{Ad})} \cdot 2^{1-2m_{3,\tau_0}},$$

while the right-hand-side can be computed using the fact that $\langle f_{k_{i,\tau_0}}, \widetilde{f}_{-k_{i,\tau_0}} \rangle_{\tau_0} = 1$, $\langle Lf, \widetilde{f} \rangle_{\tau_0} = -\langle f, L\widetilde{f} \rangle_{\tau_0}$, $LR^j f_{k_{i,\tau_0}} = j(k_{i,\tau_0} + j - 1) R^{j-1} f_{k_{i,\tau_0}}$ (see equality (4.9)), and a formula for $t_{\tau_0}(\widetilde{f}_{-k_{3,\tau_0}})$ obtained by exchanging in (3.3), the $\widetilde{f}_{k_{3,\tau_0}}$ by $\widetilde{f}_{-k_{3,\tau_0}}$ and R^n by L^n. We obtain that

$$\begin{aligned} &\langle f_{k_{1,\tau_0}} \otimes R^{m_{3,\tau_0}} f_{k_{2,\tau_0}}, t_{\tau_0}(\widetilde{f}_{-k_{3,\tau_0}}) \rangle_{\tau_0} \\ &\quad = \binom{m-2}{k_{1,\tau_0} - 1} \langle f_{k_{1,\tau_0}} \otimes R^{m_{3,\tau_0}} f_{k_{2,\tau_0}}, \widetilde{f}_{-k_{1,\tau_0}} \otimes L^{m_{3,\tau_0}} \widetilde{f}_{-k_{2,\tau_0}} \rangle_{\tau_0} \\ &\quad = (-1)^{m_{3,\tau_0}} \frac{(m_{3,\tau_0})!(m-2)!}{(k_{1,\tau_0} - 1)!(k_{2,\tau_0} - 1)!}, \end{aligned}$$

and analogously for $\langle \widetilde{f}_{k_{1,\tau_0}} \otimes R^{m_{3,\tau_0}} \widetilde{f}_{k_{2,\tau_0}}, t_{\tau_0}(f_{-k_{3,\tau_0}})\rangle_{\tau_0}$. Moreover, we can compute similarly:

$$\begin{aligned}
&\langle t_{\tau_0}(f_{k_{3,\tau_0}}), t_{\tau_0}(\widetilde{f}_{-k_{3,\tau_0}})\rangle_{\tau_0}\\
&\quad= \sum_j C_j^2 \langle R^j(f_{k_{1,\tau_0}}), L^j(f_{-k_{1,\tau_0}})\rangle_{\tau_0} \langle R^{m_{3,\tau_0}-j}(f_{k_{2,\tau_0}}), L^{m_{3,\tau_0}-j}(f_{-k_{2,\tau_0}})\rangle_{\tau_0}\\
&\quad= (-1)^{m_{3,\tau_0}} \frac{(m-2)!(m_{3,\tau_0})!}{(k_{1,\tau_0}-1)!(k_{2,\tau_0}-1)!} \sum_j C_j,
\end{aligned}$$

where $C_j = \binom{m_{3,\tau_0}}{j}\binom{m-2}{k_{1,\tau_0}+j-1}$. Since $\sum_{j=0}^n \binom{A}{j}\binom{B}{n-j} = \binom{A+B}{n}$, we conclude

$$\begin{aligned}
&\langle t_{\tau_0}(f_{k_{3,\tau_0}}), t_{\tau_0}(\widetilde{f}_{-k_{3,\tau_0}})\rangle_{\tau_0}\\
&\qquad\qquad= (-1)^{m_{3,\tau_0}} \frac{(m-2)!(m_{3,\tau_0})!}{(k_{1,\tau_0}-1)!(k_{2,\tau_0}-1)!} \binom{k_{3,\tau_0}-2}{k_{2,\tau_0}+m_{3,\tau_0}-1}.
\end{aligned}$$

Putting all this together, we conclude that $I_{\tau_0}(\varphi_{\tau_0} \otimes \widetilde{\varphi}_{\tau_0}) = 2^{1-2m_{3,\tau_0}} \cdot \binom{k_{3,\tau_0}-2}{k_{2,\tau_0}+m_{3,\tau_0}-1}^2$.

Finally, we can choose the explicit test vectors provided in [21] to obtain that $I_v(\varphi_v \otimes \widetilde{\varphi}_v)$ is an explicit constant, which is equal to 1 if $v \nmid \mathfrak{n}$ ([21, Lemma 3.11]). We obtain that

$$\begin{aligned}
&\frac{\langle \phi_3, t(\phi_1,\phi_2)\rangle^2}{((\bar{\phi}_3\phi_1\phi_2)(v_\infty^0), \tau_{\mathfrak{n}}(\widetilde{\bar{\phi}_3\widetilde{\phi}_1\widetilde{\phi}_2})(\widetilde{v}_\infty^0))}\\
&\qquad= C\cdot \pi^{1-d}\cdot \frac{(-1)^{\nu_3}\cdot \xi_F(2)^2}{2^{4-2m_{3,\tau_0}}} \cdot \binom{k_{3,\tau_0}-2}{k_{2,\tau_0}+m_{3,\tau_0}-1}^2 \cdot \frac{L(1/2,\Pi)}{L(1,\Pi,\mathrm{Ad})},
\end{aligned}$$

for some constant C not depending on $k_{i,\tau}$.

The result follows from the fact that $L(\frac{1}{2},\Pi) = L(\frac{1-\nu_1-\nu_2-\nu_3}{2}, \Pi_1 \otimes \Pi_2 \otimes \Pi_3)$ and the denominator $((\bar{\phi}_3\phi_1\phi_2)(v_\infty^0), \tau_{\mathfrak{n}}(\widetilde{\bar{\phi}_3\widetilde{\phi}_1\widetilde{\phi}_2})(\widetilde{v}_\infty^0))$ is (up-to-constant) $\langle\phi_1,\phi_1\rangle\langle\phi_2,\phi_2\rangle\langle\phi_3,\phi_3\rangle$ by Lemma 3.8. $\square$

4. Modular forms for G'

In this section we introduce unitary Shimura curves which have a well behaved moduli interpretation. We define the sheaves which give raise to modular forms for these curves, and we interpret the triple product defined in Section 3 in more geometric terms.

4.1. Unitary Shimura curves

Let $\mathfrak{n}$ be an integral ideal of F prime to $\mathrm{disc}(B)$. We define the *unitary Shimura curve* over $\mathbb{C}$ of level $K_1'(\mathfrak{n})$ as:

$$X(\mathbb{C}) := G'(\mathbb{Q})_+\backslash(\mathfrak{H} \times G'(\mathbb{A}_f)/K_1'(\mathfrak{n})) = \bigsqcup_{t\in\mathrm{Pic}(E/F,\mathfrak{n})} \Gamma_{1,1}^t(\mathfrak{n})\backslash\mathfrak{H}, \tag{4.1}$$

the last decomposition comes from the fact that, by strong approximation, $\pi_T : G' \to T_E$ induces an isomorphism $G'(\mathbb{Q})_+\backslash G'(\mathbb{A}_f)/K_1'(\mathfrak{n}) \xrightarrow{\simeq} \mathrm{Pic}(E/F,\mathfrak{n})$.

To describe the moduli interpretation of the Shimura curve, we consider an Eichler order, $\mathcal{O}_{B,\mathfrak{m}} \subseteq \mathcal{O}_B$, of level $\mathfrak{m} \mid \mathfrak{n}$ such that $K_1^B(\mathfrak{n}) \subseteq \widehat{\mathcal{O}}_{B,\mathfrak{m}}^\times$. Let $\mathcal{O}_\mathfrak{m} := \mathcal{O}_{B,\mathfrak{m}} \otimes_{\mathcal{O}_F} \mathcal{O}_E \subset D$ and $V_\mathbb{Z} := \mathcal{O}_\mathfrak{m} \subset V$.

Lemma 4.1. — *Possibly enlarging $\mathfrak{m}$, there exists $\delta \in B$ such that $l \mapsto l^*$ stabilizes $\mathcal{O}_\mathfrak{m}$, $\mathcal{D}_{E/F}^{-1}\mathcal{O}_\mathfrak{m} = \{v \in V : \Theta(v,w) \in \mathbb{Z},$ for $w \in \mathcal{O}_\mathfrak{m}\}$ and Θ has integral values when restricted to $V_\mathbb{Z}$. Here $\mathcal{D}_{E/F}$ is the different of E/F.*

Proof. — As $l^* = \delta^{-1}\bar{l}\delta$, $\delta\mathcal{O}_{B,\mathfrak{m}}$ and $l \mapsto \bar{l}$ stabilizes $\mathcal{O}_\mathfrak{m}$, we only have to check that $\delta\mathcal{O}_\mathfrak{m}$ is a bilateral ideal. Consider the ideal $\mathfrak{I} = \{b \in B : \mathrm{Tr}_{B/F}(b\alpha) \in \mathcal{O}_F;$ for all $\alpha \in \mathcal{D}_{F/\mathbb{Q}}\mathcal{O}_{B,\mathfrak{m}}\}$ where $\mathcal{D}_{F/\mathbb{Q}}$ is the different of $F/\mathbb{Q}$. By [40, Lemme 4.7(1)] $\mathfrak{I}$ is a bilateral ideal of norm $\mathfrak{m}^{-1}\mathrm{disc}(B)^{-1}\mathcal{D}_{F/\mathbb{Q}}^{-2}$, hence, possibly enlarging $\mathfrak{m}$, we can assume that $\mathfrak{m}^{-1}\mathrm{disc}(B)^{-1}\mathcal{D}_{F/\mathbb{Q}}^{-2}$ is principal and generated by $d \in F_{<0}$. By strong approximation we have $\mathfrak{I} = \delta\mathcal{O}_\mathfrak{m}$. It is easy to check locally that δ can be chosen to satisfy $\delta^2 = d$, hence $\bar{\delta} = -\delta$. Since $\mathcal{D}_{E/\mathbb{Q}} = \mathcal{D}_{E/F}\mathcal{D}_{F/\mathbb{Q}}$, we obtain

$$\begin{aligned}\mathcal{D}_{E/F}^{-1}\mathcal{O}_\mathfrak{m} &= \{w \in D :\ \mathrm{Tr}_{D/E}(w\bar{v}\delta) \in \mathcal{D}_{E/F}^{-1}\mathcal{D}_{F/\mathbb{Q}}^{-1},\ \text{for all } v \in \mathcal{O}_\mathfrak{m}\} \\ &= \{w \in D :\ \Theta(d,w) \in \mathbb{Z},\ \text{for all } v \in \mathcal{O}_\mathfrak{m}\}.\end{aligned}$$

□

Let L/E be a finite extension such that $B \otimes_F L = \mathrm{M}_2(L)$. By [11, Section 2.3] the Riemann surface $X(\mathbb{C})$ has a model over L, X, which solves the following moduli problem: if R is a L-algebra then $X(R)$ corresponds to the set of the isomorphism classes of tuples (A,ι,θ,α) where:

(i) A is an abelian scheme over R of relative dimension $4d$.

(ii) $\iota : \mathcal{O}_\mathfrak{m} \to \mathrm{End}_R(A)$ gives an action of the ring $\mathcal{O}_\mathfrak{m}$ on A such that $\mathrm{Lie}(A)^{-,1}$ is of rank 1 and the action of $\mathcal{O}_F$ factors through $\mathcal{O}_F \subset E \subseteq L$.

(iii) θ is a $\mathcal{O}_\mathfrak{m}$-invariant homogeneous polarization of A whose Rosati involution sends $\iota(d)$ to $\iota(d^*)$.

(iv) A class α modulo $K_1'(\mathfrak{n})$ of $\mathcal{O}_\mathfrak{m}$-linear symplectic similitudes $\alpha : \widehat{T}(A) \xrightarrow{\simeq} \widehat{\mathcal{O}}_\mathfrak{m}$.

We denote by $\pi : A \to X$ the universal object and, for each $t \in \mathrm{Pic}(E/F,\mathfrak{n})$, let X^t be the corresponding connected component of X. We have the following description over $\mathbb{C}$.

$$A = \bigsqcup_{t\in\mathrm{Pic}(E/F,\mathfrak{n})} \Gamma^t_{1,1}(\mathfrak{n})\backslash\left(\mathfrak{H}\times(\mathbb{C}^2\otimes_{F,\tau_0}E)\times\mathrm{M}_2(\mathbb{C})^{\Sigma_F\smallsetminus\{\tau_0\}}\right)/D\cap t^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}}b_t^{-1},$$

where the action are $(z,v,M)\cdot d = (z, v + d\left(\begin{smallmatrix} z\\ 1\end{smallmatrix}\right), M + d)$ and $\gamma\cdot(z,v,M) = (\gamma z, (cz+d)^{-1}v, M\gamma^{-1})$ for $d \in D\cap t^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}}b_t^{-1}$, $\gamma \in \Gamma^t_{1,1}(\mathfrak{n})$ with $\tau_0(\gamma) = \left(\begin{smallmatrix} * & *\\ c & d\end{smallmatrix}\right)$ and $(z,v,M) \in \mathfrak{H}\times(\mathbb{C}^2\otimes_{F,\tau_0}E)\times\mathrm{M}_2(\mathbb{C})^{\Sigma_F\smallsetminus\{\tau_0\}}$. Its universal polarization is given by the restriction of the form Θ to $D\cap t^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}}b_t^{-1}$. Finally, observe that $\widehat{T}(A) = t^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}}b_t^{-1}$ and the class of $\mathcal{O}_{\mathfrak{m}}$-linear symplectic similitudes is given by $\alpha : \widehat{T}(A) \xrightarrow{\simeq} \widehat{\mathcal{O}}_{\mathfrak{m}},\ t^{-1}bb_t^{-1} \mapsto b$.

Remark 4.2. — Let $\mathfrak{c} \in \mathrm{Pic}(\mathcal{O}_F)$ and put $\mathcal{O}^{\mathfrak{c}}_{\mathfrak{m}} := b_{\mathfrak{c}}\widehat{\mathcal{O}}_{\mathfrak{m}}b_{\mathfrak{c}}^{-1}\cap D$. Let $X^{\mathfrak{c}}$ be the Shimura curve solving to the moduli problem of X but using $\mathcal{O}^{\mathfrak{c}}_{\mathfrak{m}}$ instead of $\mathcal{O}_{\mathfrak{m}}$. The irreducible components of $X^{\mathfrak{c}}$ are in bijection with $\mathrm{Pic}(E/F,\mathfrak{n})$ and $\Gamma^{\mathfrak{c}}_{1,1}(\mathfrak{n})\backslash\mathfrak{H}$ naturally appears as an irreducible component $X^{\mathfrak{c}}(\mathbb{C})$.

We give an alternative description of the points of $X^{\mathfrak{c}}$. Let (A,ι,θ,α) be a point in $X^{\mathfrak{c}}$, then there exists a unique isogenous pair (A_0,ι_0) with multiplication by $\mathcal{O}_{\mathfrak{m}}$ such that $\ker(A\to A_0) = \{P\in A:\ bP = 0,\ \text{for all } b\in b_{\mathfrak{c}}\widehat{\mathcal{O}}_{\mathfrak{m}}\cap D\}$. From θ, we obtain a homogeneous polarization $\theta_0 : A_0 \to A_0^\vee$. Moreover, the map $\alpha_0 : \widehat{T}(A_0) \xrightarrow{\varphi} \widehat{T}(A) \xrightarrow{\alpha} b_{\mathfrak{c}}\widehat{\mathcal{O}}_{\mathfrak{m}}b_{\mathfrak{c}}^{-1} \xrightarrow{\cdot\frac{b_{\mathfrak{c}}}{\deg\varphi}} b_{\mathfrak{c}}(\deg\varphi)^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}}$ has image in $\widehat{\mathcal{O}}_{\mathfrak{m}}$, and provides a symplectic isomorphism between $(\widehat{\mathcal{O}}_{\mathfrak{m}},\mathfrak{c}^{-1}\Theta)$ and $(\widehat{T}(A_0),\theta_0)$. Thus, $X^{\mathfrak{c}}$ classifies quadruples $(A_0,\iota_0,\theta_0,\alpha_0)$, where (A_0,ι_0,θ_0) is as for X and α_0 is a symplectic isomorphism between $(\widehat{\mathcal{O}}_{\mathfrak{m}},\mathfrak{c}^{-1}\Theta)$ and $(\widehat{T}(A_0),\theta_0)$.

Remark 4.3. — Working with the levels introduced in Section 2.2 instead of $K_1'(\mathfrak{n})$ we obtain analogous Shimura curves. For example we obtain the Shimura curves attached to $K_1'(\mathfrak{n},\mathfrak{q})$ for each ideal $\mathfrak{q}$ coprime with $\mathfrak{n}$, which satisfies a similar moduli problem as X with an extra $K_0'(\mathfrak{q})$-structure.

Remark 4.4. — As $p\nmid\mathrm{disc}(B)$ a class α of $\mathcal{O}_{\mathfrak{m}}$-linear symplectic similitudes modulo $K_1^{'}(\mathfrak{n})$ is decomposed as $\alpha = \alpha_p\times\alpha^p$ where $\alpha_p : T_p(A) \xrightarrow{\simeq} (\mathcal{O}_{\mathfrak{m}})_p \simeq \mathrm{M}_2(\mathcal{O}_E\otimes\mathbb{Z}_p)$ and $\alpha^p : \widehat{T}(A)^p \xrightarrow{\simeq} (\widehat{\mathcal{O}}_{\mathfrak{m}})^p$.

Θ induces a perfect pairing on $V_{\mathbb{Z}_p} = (\mathcal{O}_{\mathfrak{m}})_p = \mathrm{M}_2(\mathcal{O}_E\otimes\mathbb{Z}_p)$. As p splits in $\mathbb{Q}(\sqrt{\lambda})$ then $G'(\mathbb{Z}_p) \xrightarrow{\simeq} \mathrm{GL}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)\times\mathbb{Z}_p^\times$ via $b(t_1,t_2)\mapsto(bt_2,\det(b)t_1t_2)$ identifying $K_1^{'}(\mathfrak{n})_p$ with $K_1^B(\mathfrak{n})_p\times\mathbb{Z}_p^\times$.

The map α_p identifies $T_p(A)^{\pm}$ with $\mathrm{M}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$ and denote $\alpha_p^-: T_p(A)^- \xrightarrow{\simeq} \mathrm{M}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$. Reciprocally, a $\mathrm{M}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$-lineal isomorphism α_p^- gives rise to a symplectic similitude $\alpha_p=(\alpha_p^+,\alpha_p^-)$, where $\alpha_p^+: T_p(A)^+\to \mathrm{M}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$ is provided by the rule $\Theta(\alpha_p^+(v),\alpha_p^-(w)) = e_p(v,w)$ with e_p the corresponding perfect dual pairing on $T_p(A)$ (e_p is characterized by its image in $T_p(A)^+\times T_p(A)^-$ because, since the Rosati involution sends $\iota(\sqrt{\lambda})$ to $-\iota(\sqrt{\lambda})$, it vanishes at $T_p(A)^+\times T_p(A)^+$ and $T_p(A)^-\times T_p(A)^-$). The action of $(\gamma,n)\in \mathrm{GL}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)\times\mathbb{Z}_p^\times$ on α_p is given by $\alpha_p^-\mapsto\alpha_p^-\cdot\gamma$ and $\alpha_p^+\mapsto n\alpha_p^+\cdot\bar{\gamma}^{-1}$. Hence, to provide a α_p modulo $K_1'(\mathfrak{n})_p$ amounts to giving α_p^- modulo $K_1^B(\mathfrak{n})_p$, or equivalently, the point $P=(\alpha_p^-)^{-1}\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)(\mathrm{mod}\ \mathfrak{n})\in A[\mathfrak{n}\mathcal{O}_F\otimes\mathbb{Z}_p]^{-,1}$ that generates a subgroup isomorphic to $(\mathcal{O}_F\otimes\mathbb{Z}_p)/(\mathfrak{n}\mathcal{O}_F\otimes\mathbb{Z}_p)$. We have an analogous description for $K_0'(\mathfrak{n})$-structures.

4.2. Modular sheaves

We introduce the sheaves which give rise to the modular forms for G'. Let L_0/F be an extension such that $B\otimes_F L_0=\mathrm{M}_2(L_0)$, write $L=L_0(\sqrt{\lambda})\supset E$, and denote by X_L the base change to L of the unitary Shimura curve X. Using the universal abelian variety $\pi: A\to X_L$ we define the following coherent sheaves on X_L:

$$\omega := \left(\pi_*\Omega^1_{A/X_L}\right)^{+,2}\qquad \omega_- := \left(\left(R^1\pi_*\mathcal{O}_A\right)^{+,2}\right)^\vee\qquad \mathcal{H} := \left(\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}\right)^{+,2}.$$

Note that we have $\omega_-\simeq(\pi'_*\Omega^1_{A^\vee/X_L})^{+,2}$ and the sheaf $\mathcal{H}$ is endowed with a Gauss–Manin connection $\bigtriangledown:\mathcal{H}\to\mathcal{H}\otimes\Omega^1_{X_L}$. The natural $\mathcal{O}_D$-equivariant exact sequence:

$$0\longrightarrow\pi_*\Omega^1_{A/X_L}\longrightarrow\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}\longrightarrow R^1\pi_*\mathcal{O}_A\longrightarrow 0, \tag{4.2}$$

induces the Hodge exact sequence (see [15, Section 2.3.1]):

$$0\longrightarrow\omega\longrightarrow\mathcal{H}\longrightarrow\omega_-^\vee\longrightarrow 0. \tag{4.3}$$

If L contains the Galois closure of F then the natural decomposition $F\otimes_{\mathbb{Q}}L\cong L^{\Sigma_F}$ induces:

$$\omega=\bigoplus_{\tau\in\Sigma_F}\omega_\tau\qquad\qquad \mathcal{H}=\bigoplus_{\tau\in\Sigma_F}\mathcal{H}_\tau.$$

As the sheaves $(\Omega^1_{A/X_L})^{+,1}$ and $(\Omega^1_{A/X_L})^{+,2}$ are isomorphic, then condition (ii) of the moduli problem of X imply that ω_{τ_0} is locally free of rank 1, while ω_τ is of rank 2 for $\tau\neq\tau_0$. Moreover, since the Rosati involution maps

$\sqrt{\lambda}$ to $-\sqrt{\lambda}$, we deduce that ω_- is locally free of rank 1. Thus, $\omega_\tau = \mathcal{H}_\tau$ for each $\tau \neq \tau_0$, and we have the exact sequence

$$0 \longrightarrow \omega_{\tau_0} \longrightarrow \mathcal{H}_{\tau_0} \xrightarrow{\epsilon} \omega_-^\vee \longrightarrow 0. \tag{4.4}$$

Let $\underline{k} = (k_\tau) \in \mathbb{N}[\Sigma_F]$, we introduce the *modular sheaves* over X_L considered in this text:

$$\omega^{\underline{k}} := \omega_{\tau_0}^{\otimes k_{\tau_0}} \otimes \bigotimes_{\tau\neq\tau_0} \mathrm{Sym}^{k_\tau} \omega_\tau$$

$$\mathcal{H}^{\underline{k}} := \bigotimes_{\tau\in\Sigma_F} \mathrm{Sym}^{k_\tau} \mathcal{H}_\tau = \mathrm{Sym}^{k_{\tau_0}} \mathcal{H}_{\tau_0} \otimes \bigotimes_{\tau\neq\tau_0} \mathrm{Sym}^{k_\tau} \omega_\tau.$$

Definition 4.5. — *A* modular form *of weight $\underline{k}$ and coefficients in L for G' is a global section of $\omega^{\underline{k}}$, i.e. an element of* $\mathrm{H}^0(X_L, \omega^{\underline{k}})$.

4.3. Alternating pairings and Kodaira–Spencer

Notice that $\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}$ is a $\mathcal{O}_\mathfrak{m} \otimes_\mathbb{Z} \mathcal{O}_{X_L}$-module of rank one, hence if we fix a generator $\underline{w} \in \mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}$ we can define the unique symplectic $(\mathcal{O}_{B,\mathfrak{m}} \otimes_\mathbb{Z} \mathcal{O}_{X_L})$-linear involution

$$c : \mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L} \longrightarrow \mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L},$$

such that $c((b\otimes e)\underline{w}) = (b \otimes \bar{e})\underline{w}$ for all $b \in \mathcal{O}_{B,\mathfrak{m}}$ and $e \in \mathcal{O}_E$.

Since $\pi_*\Omega^1_{A/X_L} \simeq \mathrm{Lie}(A)^\vee$ and $R^1\pi_*\mathcal{O}_A \simeq \mathrm{Lie}(A^\vee)$, the polarization $\theta : \mathrm{Lie}(A) \to \mathrm{Lie}(A^\vee)$ provides a $\mathcal{O}_{X_L}$-linear morphism (see [15, Remarque 2.3.1]):

$$\Theta : \pi_*\Omega^1_{A/X_L} \times R^1\pi_*\mathcal{O}_A \longrightarrow \mathcal{O}_{X_L}$$

satisfying $\Theta(\lambda x, y) = \Theta(x, \lambda^* y)$ for $\lambda \in \mathcal{O}_\mathfrak{m}$. From (4.2) it is equivalent to give an alternating pairing Θ on $\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}$. The involution c and Θ, provide for each $\tau \in \Sigma_F$ an alternating pairing:

$$\bar{\varphi}_\tau : \left(\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}\right)^{+,2}_\tau \times \left(\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}\right)^{+,2}_\tau \longrightarrow \mathcal{O}_{X_L}$$

given by $\bar{\varphi}_\tau(u,v) := \Theta(u, \delta^{-1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)c(v))$. If $\tau \neq \tau_0$, this provides an isomorphism:

$$\varphi_\tau : \bigwedge^2 \left(\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}\right)^{+,2}_\tau = \bigwedge^2 \omega_\tau \xrightarrow{\simeq} \mathcal{O}_{X_L}. \tag{4.5}$$

If $\tau = \tau_0$, we obtain an isomorphism:

$$\varphi_{\tau_0} : \omega_{\tau_0} \xrightarrow{\simeq} \omega_- \tag{4.6}$$

given by $\bar{\varphi}_{\tau_0}(v,w) = \epsilon(v)(\varphi_{\tau_0}(w))$ for $w \in \omega_{\tau_0} \subset \mathcal{H}_{\tau_0}$ and $v \in \mathcal{H}_{\tau_0} = \left(\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_L}\right)^{+,2}_{\tau_0}$. Thus, the Kodaira–Spencer isomorphism (see [15, Lemme 2.3.4]) induces the isomorphism:

$$KS : \Omega^1_{X_L} \xrightarrow{\simeq} \omega_{\tau_0} \otimes \omega_- \xrightarrow{\varphi_{\tau_0}^{-1}} \omega_{\tau_0}^{\otimes 2}.$$

4.4. Katz Modular forms

Let R_0 be a L-algebra and $f \in H^0(X/R_0, \omega^{\underline{k}})$. Let R be a R_0-algebra, and (A,ι,θ,α) a tuple corresponding to a point of $X(\mathrm{Spec}(R))$. We denote by ω_A the R-module of the invariant differentials in $(\Omega^1_{A/R})^{+,2}$, i.e. the pull back to R of the sheaf ω introduced in Section 4.2.

Let $w = (f_0, (f_\tau, e_\tau)_{\tau\neq\tau_0})$ be a R-basis of ω_A then there exists $f(A,\iota,\theta,\alpha,w) \in \bigotimes_{\tau\neq\tau_0} \mathrm{Sym}^{k_\tau}\left(R^2\right)$ such that:

$$f(A,\iota,\theta,\alpha) = f(A,\iota,\theta,\alpha,w)((f_\tau,e_\tau)_{\tau\neq\tau_0}) \cdot f_0^{\otimes k_{\tau_0}}.$$

Thus a section $f \in H^0(X/R_0, \omega^{\underline{k}})$ is characterized as a rule that assigns to any R_0-algebra R and $(A,\iota,\theta,\alpha,w)$ over R a polynomial $f(A,\iota,\theta,\alpha,w) \in \bigotimes_{\tau\neq\tau_0} \mathrm{Sym}^{k_\tau}\left(R^2\right)$ such that

- (A1) $f(A,\iota,\theta,\alpha,w)$ depends only on the R-isomorphism class of (A,ι,θ,α);
- (A2) Formation of $f(A,\iota,\theta,\alpha,w)$ commutes with extensions $R \to R'$ of R_0-algebras;
- (A3) If $(t,\underline{g}) \in R^\times \times \mathrm{GL}_2(R)^{\Sigma_F\setminus\{\tau_0\}}$ and $w(t,\underline{g}) = (te_0, (f_\tau,e_\tau)g_\tau)$ then $f(A,\iota,\theta,\alpha,w(t,\underline{g})) = t^{-k_{\tau_0}} \cdot \left(\underline{g}^{-1} f(A,\iota,\theta,\alpha,w)\right)$.

Using the isomorphism φ_τ of (4.5) we obtain

$$\begin{aligned}\omega^{\underline{k}} &\simeq \omega_{\tau_0}^{k_{\tau_0}} \otimes \bigotimes_{\tau\neq\tau_0}\left(\left(\mathrm{Sym}^{k_\tau}\omega_\tau\right)^\vee \otimes \left(\bigwedge^2 \omega_\tau\right)^{\otimes k_\tau}\right)\\ &\overset{\varphi_\tau}{\simeq} \omega_{\tau_0}^{k_{\tau_0}} \otimes \bigotimes_{\tau\neq\tau_0}\left(\mathrm{Sym}^{k_\tau}\omega_\tau\right)^\vee.\end{aligned}$$

We obtain an alternative description of a section $f \in H^0(X/R_0, \omega^{\underline{k}})$ as a rule that assigns to a R_0-algebra R and $(A,\iota,\theta,\alpha,w)$ as above a linear form $f(A,\iota,\theta,\alpha,w) \in \bigotimes_{\tau\neq\tau_0} \mathrm{Sym}^{k_\tau}\left(R^2\right)^\vee$ with

$$f(A,\iota,\theta,\alpha) = f(A,\iota,\theta,\alpha,w)\left(\prod_{\tau\neq\tau_0}\begin{vmatrix} f_\tau & e_\tau \\ x_\tau & y_\tau\end{vmatrix}^{k_\tau} \varphi_\tau(f_\tau\wedge e_\tau)^{-k_\tau}\right) f_0^{\otimes k_{\tau_0}}.$$

This rule satisfy:

(B1) The element $f(A,\iota,\theta,\alpha,w)$ depends only on the R-isomorphism class of (A,ι,θ,α).

(B2) Formation of $f(A,\iota,\theta,\alpha,w)$ commutes with extensions $R\to R'$ of R_0-algebras.

(B3) If $(t,\underline{g})\in R^\times\times\mathrm{GL}_2(R)^{\Sigma_F\setminus\{\tau_0\}}$ then $f(A,\iota,\theta,\alpha,w(t,\underline{g}))=t^{-k_{\tau_0}}\cdot\left(\underline{g}^{-1}f(A,\iota,\theta,\alpha,w)\right)$.

4.5. Modular forms for G' and automorphic forms for G

Let $t\in\mathrm{Pic}(E/F,\mathfrak{n})$ and denote by $M_{\underline{k}}(\Gamma_{1,1}^t(\mathfrak{n}),\mathbb{C})$ the $\mathbb{C}$-vector space of holomorphic functions $f:\mathfrak{H}\to\bigotimes_{\tau\neq\tau_0}\mathcal{P}_\tau(k_\tau,k_\tau)^\vee$ such that $f(\gamma z)=(cz+d)^{k_{\tau_0}}\gamma f(z)$ for $\gamma\in\Gamma_{1,1}^t(\mathfrak{n})$ with $\tau_0(\gamma)=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)$. As $\det(\Gamma_{1,1}^t(\mathfrak{n}))=\{1\}$ then there is no action of the determinant. Over $\mathbb{C}$ the modular forms for G' are described by:

Lemma 4.6. — *We have a canonical isomorphism of $\mathbb{C}$-vector spaces:*

$$\mathrm{H}^0(X_{\mathbb{C}},\omega^{\underline{k}})=\bigoplus_{t\in\mathrm{Pic}(E/F,\mathfrak{n})}M_{\underline{k}}(\Gamma_{1,1}^t(\mathfrak{n}),\mathbb{C}).$$

Proof. — As in Section 4.1, over $\mathbb{C}$ we have $\mathrm{Lie}(A)=\left(\mathbb{C}^2\otimes_{F,\tau_0}E\right)\times\mathrm{M}_2(\mathbb{C})^{\Sigma_F\smallsetminus\{\tau_0\}}$ and $\mathrm{Lie}(A)^2=\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\mathrm{Lie}(A)=(\mathbb{C}\otimes_{F,\tau_0}E)\times(\mathbb{C}^2)^{\Sigma_F\smallsetminus\{\tau_0\}}$. Thus, $\omega_{A,\tau_0}=(\mathrm{Lie}(A)^{+,2}_{\tau_0})^\vee=\mathbb{C}\,\mathrm{d}x_{\tau_0}$ where $\mathrm{d}x_{\tau_0}:=\sqrt{\lambda}(\mathrm{d}x\otimes 1)+\mathrm{d}x\otimes\sqrt{\lambda}\in(\mathbb{C}\otimes_{F,\tau_0}E)^\vee$ and $\omega_{A,\tau}=\langle\mathrm{d}x_\tau,\mathrm{d}y_\tau\rangle\in(\mathbb{C}^2)^\vee$ for $\tau\neq\tau_0$.

If $t\in\mathrm{Pic}(E/F,\mathfrak{n})$ then a section $\varphi\in H^0(X_{\mathbb{C}},\omega^{\underline{k}})$ over X^t is given by $f(z)(\prod_{\tau\neq\tau_0}\left|\begin{smallmatrix}\mathrm{d}x_\tau&\mathrm{d}y_\tau\\X_\tau&Y_\tau\end{smallmatrix}\right|^{k_\tau})\,\mathrm{d}x_{\tau_0}^{k_{\tau_0}}$ where $f:\mathfrak{H}\to\bigotimes_{\tau\neq\tau_0}\mathcal{P}_\tau(k_\tau,k_\tau)^\vee$ is a holomorphic function. Since $\gamma^*\mathrm{d}x_{\tau_0}=(cz+d)^{-1}\mathrm{d}x_{\tau_0}$ and $\gamma^*(\mathrm{d}x_\tau,\mathrm{d}y_\tau)=(\mathrm{d}x_\tau,\mathrm{d}y_\tau)\tau(\gamma)^{-1}$ we deduce that $f(\gamma z)=(cz+d)^{k_{\tau_0}}\gamma f(z)$ for each $\gamma\in\Gamma_{1,1}^t(\mathfrak{n})$. □

Now we relate modular forms over $\mathbb{C}$ for G' and G. Let $\Delta:=(\mathcal{O}_F)_+^\times/U_{\mathfrak{n}}^2$ with $U_{\mathfrak{n}}=\{u\in\mathcal{O}_F^\times|u\equiv 1\bmod(\mathfrak{n})\}$. If $\nu\in\mathbb{Z}$ and $\underline{k}\in\Xi_\nu$, for $f\in M_{\underline{k}}(\Gamma_{1,1}^t(\mathfrak{n}),\mathbb{C})$ and $[s]\in\Delta$ we put

$$(s*_\nu f)(z):=s^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}}(cz+d)^{-k_{\tau_0}}\gamma_s^{-1}f(\gamma_s z).\tag{4.7}$$

here $\gamma_s\in\Gamma_1^t(\mathfrak{n})$ with $\det\gamma_s=s$ and $\tau_0\gamma_s=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)$. This gives an action of Δ on $M_{\underline{k}}(\Gamma_{1,1}^t(\mathfrak{n}),\mathbb{C})$.

Proposition 4.7. — *We have a natural isomorphism of $\mathbb{C}$-vector spaces*

$$\iota_{\underline{k},\nu}:H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k},\nu))^{K_1^B(\mathfrak{n})}\xrightarrow{\simeq}\bigoplus_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)}M_{\underline{k}}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}),\mathbb{C})^\Delta.$$

Proof. — By Remark 3.2, to give $\phi \in H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k},\nu))^{K_1^B(\mathfrak{n})}$ is equivalent to give a holomorphic function

$$f_\phi : \mathfrak{H} \times G(\mathbb{A}_f)/K_1^B(\mathfrak{n}) \longrightarrow \bigotimes_{\tau\neq\tau_0} \mathcal{P}_\tau(k_\tau,\nu)^\vee.$$

As $G(\mathbb{Q})^+\backslash G(\mathbb{A}_f)/K_1^B(\mathfrak{n}) \xrightarrow{\simeq} \mathbb{A}_{F,f}^\times/F_+^\times\widehat{\mathcal{O}}_F^\times = \mathrm{Pic}(\mathcal{O}_F)$ then f_ϕ is characterized by $f_\phi^{\mathfrak{c}} := f_\phi(\cdot, b_{\mathfrak{c}}) : \mathfrak{H} \to \bigotimes_{\tau\neq\tau_0}\mathcal{P}_\tau(k_\tau,\nu)^\vee$ with $[\mathfrak{c}] \in \mathrm{Pic}(\mathcal{O}_F)$. We can verify that $f_\phi^{\mathfrak{c}}(\gamma z) = \det\gamma^{-(\nu+k_{\tau_0})/2}(cz+d)^{k_{\tau_0}}\gamma f_\phi^{\mathfrak{c}}(\tau)$ if $\gamma \in \Gamma_1^{\mathfrak{c}}(\mathfrak{n})$. Since any $\gamma \in \Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}) \subseteq \Gamma_1^{\mathfrak{c}}(\mathfrak{n})$ has reduced norm 1 and $\Gamma_1^{\mathfrak{c}}(\mathfrak{n})/\Gamma_{11}^{\mathfrak{c}}(\mathfrak{n}) \simeq (\mathcal{O}_F)_+^\times/U_{\mathfrak{n}}^2 = \Delta$ then $f_\phi^{\mathfrak{c}} \in M_{\underline{k}}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}),\mathbb{C})^\Delta$. □

We define an action of Δ on $H^0(X^{\mathfrak{c}},\omega^{\underline{k}})$ using the description of Section 4.4. Let $f \in H^0(X^{\mathfrak{c}},\omega^{\underline{k}})$ and a tuple $(A,\iota,\theta,\alpha,w)$ over some ring R as in Section 4.4. For $[s] \in \Delta$ we put:

$$(s *_\nu f)(A,\iota,\theta,\alpha,w) := s^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} \cdot f(A,\iota,s^{-1}\theta,\gamma_s^{-1}\alpha,w).$$

This is well defined as, if $s = \eta^2 \in U_{\mathfrak{n}}^2$, we have

$$\begin{aligned} f(A,\iota,\theta,\alpha,w) &= f(A,\iota,s^{-1}\theta,\gamma_s^{-1}\alpha,\eta^{-1}w) \\ &= \tau_0(\eta)^{k_{\tau_0}} \prod_{\tau\neq\tau_0}(\tau(\eta)^{-k_\tau}) f(A,\iota,s^{-1}\theta,\gamma_s^{-1}\alpha,w) \\ &= \mathrm{N}_{F/\mathbb{Q}}(\eta)^{-\nu} s^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} f(A,\iota,s^{-1}\theta,\gamma_s^{-1}\alpha,w) \\ &= s^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} f(A,\iota,s^{-1}\theta,\gamma_s^{-1}\alpha,w). \end{aligned}$$

In Proposition 11.1 of the appendix we prove that both descriptions of the Δ-action coincide. Moreover, in Section 11, we describe the action of Hecke operators from the perspective of Katz modular forms due to the isomorphism $\iota_{\underline{k},\nu}$ of Proposition 4.7. Thus we obtain:

Corollary 4.8. — *We have the following decomposition compatible with Hecke operators:*

$$H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k},\nu))^{K_1^B(\mathfrak{n})} \xrightarrow{\simeq} \bigoplus_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)} H^0(X_{\mathbb{C}}^{\mathfrak{c},0},\omega^{\underline{k}})^\Delta,$$

where $X_{\mathbb{C}}^{\mathfrak{c},0}$ is the irreducible component of $X_{\mathbb{C}}^{\mathfrak{c}}$ corresponding to $1 \in \mathrm{Pic}(\mathrm{E/F},\mathfrak{n})$.

4.6. Connections and trilinear products

For $\underline{k}\in\mathbb{N}[\Sigma_F]$ and $m\in\mathbb{Z}$, we consider the sheaves:
$$\mathcal{H}_m^{\underline{k}}\coloneqq\omega_0^{k_{\tau_0}-m}\otimes\mathrm{Sym}^m\mathcal{H}_{\tau_0}\otimes\bigotimes_{\tau\neq\tau_0}\mathrm{Sym}^{k_\tau}\omega_\tau.$$
Then from (4.4) we obtain the exact sequence:
$$(4.8)\quad 0\longrightarrow\omega^{\underline{k}}\longrightarrow\mathcal{H}_m^{\underline{k}}$$
$$\xrightarrow{\epsilon}\omega^{\underline{k}-m\tau_0}\otimes\omega_-^{-1}\otimes\mathrm{Sym}^{m-1}(\mathcal{H}_{\tau_0})\overset{(4.6)}{\simeq}\mathcal{H}_{m-1}^{\underline{k}-2\tau_0}\longrightarrow 0.$$
By Griffiths transversality the Gauss–Manin connection (see [15]) induces a connection:
$$-\nabla_{\underline{k},m}:\mathcal{H}_m^{\underline{k}}\longrightarrow\omega^{\underline{k}-(m+1)\tau_0}\otimes\mathrm{Sym}^{m+1}(\mathcal{H}_{\tau_0})\otimes\Omega_X^1\xrightarrow{KS}\mathcal{H}_{m+1}^{\underline{k}+2\tau_0}.$$
Moreover, for each j we put $\nabla_{\underline{k}}^j\coloneqq\nabla_{\underline{k}+2(j-1)\tau_0,j-1}\circ\cdots\circ\nabla_{\underline{k},0}$.

As in Section 4.4, a global section f of $\mathcal{H}^{\underline{k}}$ is given by a rule that to each tuple (A,ι,θ,α) and $w^0=\{(f_\tau,e_\tau)\}_{\tau\neq\tau_0}$ a basis of $\bigoplus_{\tau\neq\tau_0}\omega_\tau$ assigns a linear form:
$$f(A,\iota,\theta,\alpha,w^0):\bigotimes_{\tau\neq\tau_0}\mathrm{Sym}^{k_\tau}\left(R^2\right)\longrightarrow\mathrm{Sym}^{k_{\tau_0}}(\mathcal{H}_{\tau_0}),$$
such that $f(A,\iota,\theta,\alpha)\coloneqq f(A,\iota,\theta,\alpha,w^0)(\prod_{\tau\neq\tau_0}\left\|\begin{smallmatrix}f_\tau & e_\tau\\ x_\tau & y_\tau\end{smallmatrix}\right\|^{k_\tau})$. Thus, if f is a section of $\omega^{\underline{k}}\subseteq\mathcal{H}^{\underline{k}}$ then for each $P\in\bigotimes_{\tau\neq\tau_0}\mathrm{Sym}^{k_\tau}\left(R^2\right)$ we have $f(A,\iota,\theta,\alpha,w^0)(P)=f(A,\iota,\theta,\alpha,w)(P)f_0^{\otimes k_{\tau_0}}$ with $w=(f_0,w^0)$.

Now let $\underline{k}_1\in\Xi_{\nu_1}$, $\underline{k}_2\in\Xi_{\nu_2}$ and $\underline{k}_3\in\Xi_{\nu_1+\nu_2}$ be unbalanced at τ_0 with dominant weight $\underline{k}_3$. Recall the notation $\underline{m}=(m_\tau)_{\tau\in\Sigma_F}\coloneqq((k_{1,\tau}+k_{2,\tau}+k_{3,\tau})/2)_{\tau\in\Sigma_F}$, $m_{3,\tau_0}=k_{3,\tau_0}-m_{\tau_0}$, and for $i=1,2,3$ we denote $\underline{m}_i^{\tau_0}=(m_{i,\tau})_{\tau\neq\tau_0}=(m_\tau-k_{i,\tau})_{\tau\neq\tau_0}$. The following result gives a geometric interpretation of the linear product t of (3.5) in terms of the isomorphism in Proposition 4.7.

Theorem 4.9. — *Let $\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)$. There exists a morphism $t_\mathfrak{c}:\omega^{\underline{k}_1}\times\omega^{\underline{k}_2}\to\omega^{\underline{k}_3}$ such that, if $f_i\in H^0(X^\mathfrak{c},\omega^{\underline{k}_i})$ for $i=1,2$ and $(A,\iota,\theta,\alpha)\in X^\mathfrak{c}$, then*
$$t_\mathfrak{c}(f_1,f_2)(A,\iota,\theta,\alpha)=\sum_{j=0}^{m_{3,\tau_0}}c_j\,\nabla_{\underline{k}_1}^j\,(f_1)\,\nabla_{\underline{k}_2}^{m_{3,\tau_0}-j}\,(f_2)(A,\iota,\theta,\alpha,w^0)(\Delta^{\tau_0})$$
where $\Delta^{\tau_0}\in\bigotimes_{\tau\neq\tau_0}\left(\mathrm{Sym}^{k_{1,\tau}}\left(R^2\right)\otimes\mathrm{Sym}^{k_{2,\tau}}\left(R^2\right)\otimes\mathrm{Sym}^{k_{\tau,3}}\omega_\tau\right)$ is given by $\Delta^{\tau_0}(x_1,y_1,x_2,y_2)=\prod_{\tau\neq\tau_0}\left|\begin{smallmatrix}f_\tau & e_\tau\\ x_{2,\tau} & y_{2,\tau}\end{smallmatrix}\right|^{m_{1,\tau}}\left|\begin{smallmatrix}f_\tau & e_\tau\\ x_{1,\tau} & y_{1,\tau}\end{smallmatrix}\right|^{m_{2,\tau}}\left|\begin{smallmatrix}x_{1,\tau} & y_{1,\tau}\\ x_{2,\tau} & y_{2,\tau}\end{smallmatrix}\right|^{m_{3,\tau}}$ and

$c_j = (-1)^j \binom{m_{3,\tau_0}}{j}\binom{m_{\tau_0}-2}{k_{1,\tau_0}+j-1}$. *Moreover, if* $\phi_i \in H^0(G(\mathbb{Q}), \mathcal{A}(\underline{k}_i, \nu_i))^{K_1^B(\mathfrak{n})}$ *for* $i = 1, 2$ *then:*

$$t_{\mathfrak{c}}(\iota_{\underline{k}_1,\nu_1}(\phi_1)_{\mathfrak{c}}, \iota_{\underline{k}_2,\nu_2}(\phi_2)_{\mathfrak{c}}) = \left(\frac{1}{2i}\right)^{m_{3,\tau_0}} \iota_{\underline{k}_3,\nu_1+\nu_2}(t(\phi_1,\phi_2))_{\mathfrak{c}}.$$

Proof. — By the definition we have that

$$\mathrm{Im}(t_{\mathfrak{c}}) \subseteq \omega^{\underline{k}_3 - m_{3,\tau_0}\tau_0} \otimes \mathrm{Sym}^{m_{3,\tau_0}}(\mathcal{H}_{\tau_0}),$$

thus, to prove that $t_{\mathfrak{c}}$ is well defined we need to check that $\mathrm{Im}(t_{\mathfrak{c}}) \subseteq \omega^{\underline{k}_3}$. From (4.8) it is enough to prove $\epsilon(t_{\mathfrak{c}}(f_1, f_2)) = 0$.

We will prove that, for each $\underline{k}$, $f \in H^0(X^{\mathfrak{c}}, \omega^{\underline{k}})$ and $j \in \mathbb{N}$, we have:

$$\epsilon \bigtriangledown_{\underline{k}}^j (f) = j(k_{\tau_0} + j - 1) \bigtriangledown_{\underline{k}}^{j-1} (f). \tag{4.9}$$

Indeed, assuming this

$$\begin{aligned}
&\epsilon(t_{\mathfrak{c}}(f_1, f_2)) \\
&\quad = \sum_{j=0}^{m_{3,\tau_0}} c_j \left(\epsilon \bigtriangledown_{\underline{k}_1}^j (f_1) \bigtriangledown_{\underline{k}_2}^{m_{3,\tau_0}-j} (f_2) + \bigtriangledown_{\underline{k}_1}^j (f_1) \epsilon \bigtriangledown_{\underline{k}_2}^{m_{3,\tau_0}-j} (f_2)\right) \\
&\quad = \sum_{n=0}^{m_{3,\tau_0}-1} \big(c_{n+1}(n+1)(k_{1,\tau_0} + n) \\
&\qquad\qquad + c_n(m_{3,\tau_0} - n)(k_{2,\tau_0} + m_{3,\tau_0} - n - 1)\big) \bigtriangledown_{\underline{k}_1}^n (f_1) \bigtriangledown_{\underline{k}_2}^{m_{3,\tau_0}-n-1} (f_2) \\
&\quad = 0.
\end{aligned}$$

To prove (4.9) it is enough to work locally, thus, let $U = \mathrm{Spec}(R)$ be an open of $X^{\mathfrak{c}}$ such that trivializes the sheaves $\mathcal{H}_{\tau_0}$ and ω. Let $u_1, u_2 \in \mathcal{H}_{\tau_0}$ be a basis such that $u_1 \in \omega_{\tau_0}$ is a basis and $\bar{\varphi}_{\tau_0}(u_1, u_2) = 1$, and let $\{f_\tau, e_\tau\}$ be an horizontal basis of ω_τ for $\tau \neq \tau_0$ (remark that it is possible as $\bigtriangledown(\omega_\tau) \subseteq \omega_\tau \otimes \Omega^1_U$). Let $D \in \mathrm{Der}(R)$ be the dual of $u_1 \otimes u_1 \in \omega_{\tau_0}^{\otimes 2} \overset{KS}{\simeq} \Omega^1_U$. Since the Kodaira–Spencer isomorphism is given by the composition

$$\omega_{\tau_0} \hookrightarrow \mathcal{H}_{\tau_0} \xrightarrow{\bigtriangledown} \mathcal{H}_{\tau_0} \otimes \Omega^1_U * \xrightarrow{(\varphi_{\tau_0}\circ\epsilon)\otimes\mathrm{id}} \omega_{\tau_0}^{-1} \otimes \Omega^1_U,$$

we have that $\bigtriangledown(D)(u_1) = (u_2 + au_1)$ for some $a \in R$. Moreover, from $0 = \bigtriangledown(D)(\bar{\varphi}_{\tau_0}(u_1, u_2)) = \bar{\varphi}_{\tau_0}(\bigtriangledown(D)u_1, u_2) + \bar{\varphi}_{\tau_0}(u_1, \bigtriangledown(D)u_2)$, we obtain $\bigtriangledown(D)(u_2) = (cu_1 - au_2)$, for some $c \in R$. By changing u_2 by $u_2 + au_1$, we can suppose that $a = 0$, thus $\bigtriangledown(u_1) = u_2u_1^2$ and $\bigtriangledown(u_2) = cu_1^3$.

For any $f = (\sum_{j=0}^{r} b_j u_1^{k_{\tau_0}-j} u_2^j) M \in \omega^{\underline{k}-r\tau_0} \otimes \mathrm{Sym}^r(\mathcal{H}_{\tau_0})$, where $M \in \bigotimes_{\tau\neq\tau_0} \omega_\tau^{k_\tau}$ is a monomial in w_τ^j, write $f(X) \coloneqq \sum_{j=0}^{r} b_j X^j \in R[X]$. Hence,

$$\begin{aligned}\nabla_{\underline{k},r}\, f &\\ &= \left(\sum_{j=0}^{r} (Db_j) u_1^{k_{\tau_0}-j+2} u_2^j + (k_{\tau_0}-j) b_j u_1^{k_{\tau_0}-j+1} u_2^{j+1} + jcb_j u_1^{k_{\tau_0}-j+3} u_2^{j-1}\right) M\end{aligned}$$

corresponds to $\nabla_{\underline{k},r}\,(f)\,(X) = Df(X) - (X^2-c)f'(X) + k_{\tau_0} X f(X)$. Since ϵ is given by derivation $f(X) \mapsto f'(X)$, we compute that $(\epsilon\circ\nabla_{\underline{k},r} - \nabla_{\underline{k}-2,r-1}\circ \epsilon) f(X) = k_{\tau_0} f(X)$. From this fact and a simple induction we deduce the equality (4.9).

Now we are going to prove the second statement of the theorem. Let $\rho : \mathfrak{H} \to \Gamma_{1,1}^t(\mathfrak{n})\backslash\mathfrak{H}$ be the projection to the connected component of $X_{\mathbb{C}}$ corresponding to $t \in \mathrm{Pic}(E/F,\mathfrak{n})$. Then, by the Riemann–Hilbert correspondence, we have:

$$\rho^*\mathcal{R}^1\pi_*\Omega^\bullet_{A/X_{\mathbb{C}}} = \rho^*\mathcal{R}^1\pi_*\mathbb{Z}\otimes_{\mathbb{C}}\mathcal{O}_{\mathfrak{H}} = \mathcal{O}_{\mathfrak{H}}\otimes_{\mathbb{C}}\mathrm{Hom}(D,\mathbb{C}),$$

here $\mathcal{O}_{\mathfrak{H}}$ is the sheaf of holomorphic functions of $\mathfrak{H}$ and each element in $\mathrm{Hom}(D,\mathbb{C})$ provides a horizontal section by inducing the corresponding linear form on $H_1(A,\mathbb{Z}) = D \cap \widehat{\mathcal{O}}_{\mathfrak{m}} b_t^{-1} \subset D$.

Then $\rho^*\omega_{\tau_0} = \mathcal{O}_{\mathfrak{H}}\,\mathrm{d}x_0 \subset \rho^*\mathcal{H}_{\tau_0} = \mathcal{O}_{\mathfrak{H}}\alpha + \mathcal{O}_{\mathfrak{H}}\beta$ and $\rho^*\mathcal{H}_\tau = \rho^*\omega_\tau = \mathcal{O}_{\mathfrak{H}}\,\mathrm{d}x_\tau^1 \oplus \mathcal{O}_{\mathfrak{H}}\,\mathrm{d}x_\tau^2$ for $\tau\neq\tau_0$. In the formulae above we have $\mathrm{d}x_0 = \alpha + z\beta$, $\alpha(\gamma\otimes e) = \widetilde{\tau}_0(e)\cdot d$ and $\beta(\gamma\otimes e) = \widetilde{\tau}_0(e)\cdot c$ for $e\in E$ and $\gamma\in B$ such that $\tau_0(\gamma) = \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$. Moreover if $\gamma \in D$, we have $\mathrm{d}x_\tau^1(\gamma) = c_\tau$ and $\mathrm{d}x_\tau^2(\gamma) = d_\tau$, where $\widetilde{\tau}(\gamma) = \left(\begin{smallmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau\end{smallmatrix}\right)$. Remark that α, β, $\mathrm{d}x_\tau^1$ and $\mathrm{d}x_\tau^2$ are horizontal, in particular we obtain $\nabla(\mathrm{d}x_0) = \beta\otimes\mathrm{d}z \in \rho^*\mathcal{H}_{\tau_0}\otimes\Omega^1_{\mathfrak{H}}$.

Observe that $(\mathrm{Lie}(A))^+_{\tau_0}$ is generated by the expressions $s(\gamma) \coloneqq \frac{1}{2}\tau_0(\gamma)\otimes 1 + \frac{1}{2\sqrt{\lambda}}\tau_0(\gamma)\otimes\sqrt{\lambda}$ for $\gamma\in B$. Since $\alpha(s(\gamma)) = d$ and $\beta(s(\gamma)) = c$ if $\tau_0(\gamma) = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ we obtain $\bar{\varphi}_{\tau_0}(\alpha,\beta) = \Theta(s\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right), \delta^{-1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)\overline{s\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0\end{smallmatrix}\right)}) = -1$ where $\overline{s(\gamma)} = \frac{1}{2}\tau_0(\gamma)\otimes 1 - \frac{1}{2\sqrt{\lambda}}\tau_0(\gamma)\otimes\sqrt{\lambda}$. We deduce from (4.6):

$$KS(\mathrm{d}x_0^2) = \epsilon\nabla(\mathrm{d}x_0)(\varphi_{\tau_0}\mathrm{d}x_0) = \epsilon(\beta)(\varphi_{\tau_0}\mathrm{d}x_0)\,\mathrm{d}z = \bar{\varphi}_{\tau_0}(\beta,\mathrm{d}x_0)\,\mathrm{d}z = \mathrm{d}z.$$

Now let $f \in M_{\underline{k}}(\Gamma_{1,1}^t(\mathfrak{n}),\mathbb{C})^\Delta$ and denote by s_f the section attached to f in Lemma 4.6, then:

$$\begin{aligned}\nabla_{\underline{k}}^j\, s_f(z) &\\ &= \left(\frac{1}{2i}\right)^j \frac{\phi\left(R^j f^\nu_{k_{\tau_0}} \otimes \left(\prod_{\tau\neq\tau_0} \left|\begin{smallmatrix} \mathrm{d}x_\tau^1 & \mathrm{d}x_\tau^2 \\ X_\tau & Y_\tau\end{smallmatrix}\right|^{k_\tau}\right)\right)}{f^\nu_{k_{\tau_0}+2j}(g_{\tau_0})} \mathrm{d}x_0^{k_{\tau_0}+2j} + \mathrm{d}\bar{x}_0 w,\end{aligned} \tag{4.10}$$

where $\mathrm{d}\bar{x}_0 = \alpha + \bar{z}\beta$ and some $w \in C^\infty(\mathfrak{H}) \otimes \mathrm{Sym}^{\underline{k}+2j-1}(\mathcal{H}^1)$. Indeed, since $\beta = (z-\bar{z})^{-1}(\mathrm{d}x_0 - \mathrm{d}\bar{x}_0) = (2iy)^{-1}(\mathrm{d}x_0 - \mathrm{d}\bar{x}_0)$ we obtain:

$$\begin{aligned}\nabla_{\underline{k}} s_f(z) = \frac{\partial}{\partial z} f(z) \left(\prod_{\tau \neq \tau_0} \left|\begin{smallmatrix} \mathrm{d}x_\tau^1 & \mathrm{d}x_\tau^2 \\ X_\tau & Y_\tau \end{smallmatrix}\right|^{k_\tau}\right) \mathrm{d}x_0^{k_1+2} \\ + k_{\tau_0} f(z) \left(\prod_{\tau \neq \tau_0} \left|\begin{smallmatrix} \mathrm{d}x_\tau^1 & \mathrm{d}x_\tau^2 \\ X_\tau & Y_\tau \end{smallmatrix}\right|^{k_\tau}\right) \mathrm{d}x_0^{k_{\tau_0}+1}\beta \\ = \left(\left(\frac{\partial}{\partial z} + \frac{k_{\tau_0}}{2iy}\right) f(z) \left(\prod_{\tau \neq \tau_0} \left|\begin{smallmatrix} \mathrm{d}x_\tau^1 & \mathrm{d}x_\tau^2 \\ X_\tau & Y_\tau \end{smallmatrix}\right|^{k_\tau}\right) \mathrm{d}x_0^{k_{\tau_0}+2}\right. \\ \left. - \frac{k_{\tau_0} f(z) \left(\prod_{\tau \neq \tau_0} \left|\begin{smallmatrix} \mathrm{d}x_\tau^1 & \mathrm{d}x_\tau^2 \\ X_\tau & Y_\tau \end{smallmatrix}\right|^{k_\tau}\right)}{2iy} \mathrm{d}x_0^{k_{\tau_0}+1} \mathrm{d}\bar{x}_0 \right).\end{aligned}$$

The claim follows from induction and the equality

$$\frac{\phi(Rf^\nu_{k_{\tau_0}} \otimes P)}{2if^\nu_{k_{\tau_0}+2}} = \left(\frac{\partial}{\partial z} + \frac{k_{\tau_0}}{2iy}\right) \frac{\phi(f^\nu_{k_{\tau_0}} \otimes P)}{f^\nu_{k_{\tau_0}}}.$$

Finally since $\mathrm{d}x_0$ and $\mathrm{d}\bar{x}_0$ are linearly independent, comparing formulas (4.10) and (3.3), we obtain the second claim of the theorem. □

Part B. p-adic families

We construct p-adic families of modular forms on the unitary Shimura curves X. From these families we construct families of autmorphic forms of B. Moreover, we construct local pieces of an adic eigenvariety.

5. Integral models and canonical groups

In this section we introduce the technical tools necessary to realize the p-adic variation of the modular sheaves introduced in Section 4.2.

5.1. Integral models and Hasse invariants

Let $\mathfrak{n}$ be an integral ideal of F prime to p and $\mathrm{disc}(B)$. From now on we will denote by X the unitary Shimura curve of level $K_1'(\mathfrak{n}, \prod_{\mathfrak{p} \neq \mathfrak{p}_0} \mathfrak{p})$ introduced in Remark 4.3. By [11, Section 5.3] X admits a canonical model over $\mathcal{O}_0$, representing the analogue moduli problem described in Section 4.1 but exchanging an E-algebra by an $\mathcal{O}_0$-algebra R. Namely, it classifies quadruples $(A, \iota, \theta, \alpha^{\mathfrak{p}_0})$ over R, where $\alpha^{\mathfrak{p}_0}$ is a class of $\mathcal{O}_D$-linear symplectic similitudes outside $\mathfrak{p}_0$. We denote this integral model by X_{int}, which has good reduction (see [11, Section 5.4]). Let $\mathfrak{X}$ be denote the formal scheme over $\mathrm{Spf}(\mathcal{O}_0)$ obtained as the completion of X_{int} along its special fiber which is denoted by $\overline{X}_{\mathrm{int}}$.

Let $\pi : \mathbf{A} \to X_{\mathrm{int}}$ be the universal abelian variety. Since we have added $K_0'(\mathfrak{p})$-structure for all $\mathfrak{p} \neq \mathfrak{p}_0$, $\mathbf{A}$ is endowed with a subgroup $C_{\mathfrak{p}} \subset \mathbf{A}[\mathfrak{p}]^{-,1}$ isomorphic to $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$ by Remark 4.4. The p-divisible group $\mathbf{A}[p^\infty]$ over X_{int} is decomposed as:

$$\mathbf{A}[p^\infty] = \mathbf{A}[\mathfrak{p}_0^\infty]^+ \oplus \left[\bigoplus_{\mathfrak{p}\neq\mathfrak{p}_0} \mathbf{A}[\mathfrak{p}^\infty]^+\right] \oplus \mathbf{A}[\mathfrak{p}_0^\infty]^- \oplus \left[\bigoplus_{\mathfrak{p}\neq\mathfrak{p}_0} \mathbf{A}[\mathfrak{p}^\infty]^-\right],$$

We are interested in the p-divisible groups $\mathcal{G}_0 := \mathbf{A}[\mathfrak{p}_0^\infty]^{-,1}$ and $\mathcal{G}_{\mathfrak{p}} := \mathbf{A}[\mathfrak{p}^\infty]^{-,1}$ if $\mathfrak{p} \neq \mathfrak{p}_0$, which are defined over X_{int} and endowed with actions of $\mathcal{O}_0$ and $\mathcal{O}_{\mathfrak{p}}$ respectively. The sheaves of invariant differentials of the corresponding Cartier dual p-divisible groups are denoted by $\omega_0 := \omega_{\mathcal{G}_0^D}$ and $\omega_{\mathfrak{p}} := \omega_{\mathcal{G}_{\mathfrak{p}}^D}$ if $\mathfrak{p} \neq \mathfrak{p}_0$. By Lemma 4.1, the universal polarization θ is an isomorphism over $\mathcal{O}_0$. Hence, this notations are justified because θ induces the following identifications:

$$\omega = \left(\pi_*\Omega^1_{\mathbf{A}/X_{\mathrm{int}}}\right)^{+,2} \overset{\theta^*}{\simeq} \left(\pi_*\Omega^1_{\mathbf{A}^\vee/X_{\mathrm{int}}}\right)^{-,1} = \omega_0 \oplus \bigoplus_{\mathfrak{p}\neq\mathfrak{p}_0} \omega_{\mathfrak{p}}. \tag{5.1}$$

The universal polarization provides a pairing (see Remark 4.4)

$$\Theta : \mathbf{A}[p^\infty]^+ \times \mathbf{A}[p^\infty]^- \longrightarrow \mathbb{G}_m[p^\infty].$$

Since p splits in $\mathbb{Q}(\sqrt{\lambda})$, we have that $\mathcal{O}_E \otimes \mathbb{Z}_p \simeq \mathcal{O}^2$, and the isomorphism that switches components induces an isomorphism $c : \mathbf{A}[p^\infty]^- \to \mathbf{A}[p^\infty]^+$. We obtain an isomorphism of p-divisible groups

$$\theta : \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_0^D; \qquad \theta(P)(Q) := \Theta\left(P, \delta^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c(Q)\right). \tag{5.2}$$

Hence, we have an isomorphism of sheaves of invariant differentials $\omega_0 \simeq \omega_{\mathcal{G}_0}$ compatible with (4.6).

Let $\widehat{\mathcal{O}}$ be a ring containing all the p-adic embeddings of $\mathcal{O}_F \hookrightarrow \mathcal{O}_{\mathbb{C}_p}$, then over $\widehat{\mathcal{O}}$ we have a decomposition $\omega_{\mathfrak{p}} = \bigoplus_{\tau\in\Sigma_{\mathfrak{p}}} \omega_{\mathfrak{p},\tau}$ with each $\omega_{\mathfrak{p},\tau}$ of rank 2, for $\mathfrak{p} \neq \mathfrak{p}_0$. The alternating pairing Θ provides, as in the complex setting, an isomorphism $\varphi_\tau : \bigwedge^2 \omega_{\mathfrak{p},\tau} \xrightarrow{\simeq} \mathcal{O}_{X_{\mathrm{int}}}$.

There is a dichotomy in X_{int} which says that any point in the generic fiber $\overline{X}_{\mathrm{int}}$ is ordinary or supersingular (with respect to $\mathcal{G}_0$), and there are finitely many supersingular points in $\overline{X}_{\mathrm{int}}$. From [25, Proposition 6.1] there exists $\mathrm{Ha} \in H^0(\overline{X}_{\mathrm{int}}, \omega_0^{p-1})$ that vanishes exactly at supersingular geometric points and these zeroes are simple. This is called the *Hasse invariant* and it is characterized as follows: For each open $\mathrm{Spec}(R) \subset \overline{X}_{\mathrm{int}}$ fix w a generator of $\omega_0 \mid_{\mathrm{Spec}(R)}$ and x a coordinate of $\mathcal{G}_0$ over R such that $w = (1 + a_1 x + a_2 x^2 \cdots)\,\mathrm{d}x$, then $[p](x) = a x^p + \cdots$ for some a and $\mathrm{Ha} \mid_{\mathrm{Spec}(R)} := a w^{p-1}$.

We denote by $\overline{\mathrm{Hdg}}$ the locally principal ideal of $\mathcal{O}_{\overline{X}_{\mathrm{int}}}$ described as follows: for each $U = \mathrm{Spec}(R) \subset \overline{X}_{\mathrm{int}}$ if $\omega_0 \mid_U = Rw$ and $\mathrm{Ha} \mid_U = H w^{\otimes(p-1)}$ then $\overline{\mathrm{Hdg}} \mid_U = HR \subseteq R$. Let Hdg the inverse image of $\overline{\mathrm{Hdg}}$ in $\mathcal{O}_{\mathfrak{X}}$, which is also a locally principal ideal. Note that Ha^{p^n} extends canonically to a section of $H^0(\mathfrak{X}, \omega_0^{p^n(p-1)} \otimes \mathbb{Z}/p^{n+1}\mathbb{Z})$, indeed, for any two extensions Ha_1 and Ha_2 of Ha we have $\mathrm{Ha}_1^{p^n} = \mathrm{Ha}_2^{p^n}$ modulo p^{n+1} by the binomial formula.

Remark 5.1. — From [8, Proposition 3.4] we obtain the existence of a $p-1$-root of the principal ideal Hdg. This ideal is denoted $\mathrm{Hdg}^{1/(p-1)}$.

Now we introduce some formal schemes in order to vary p-adically the modular sheaves and to produce p-adic families. For each integer $r \geqslant 1$ we denote by $\mathfrak{X}_r$ the formal scheme over $\mathfrak{X}$ which represents the functor (denoted by the same symbol $\mathfrak{X}_r$) that classifies for each p-adically complete $\widehat{\mathcal{O}}$-algebra R:

$$\mathfrak{X}_r(R) = \left\{ [(h,\eta)] \;\middle|\; \begin{array}{l} h \in \mathfrak{X}(R), \\ \eta \in H^0\big(\mathrm{Spf}(R), h^*(\omega_{\mathcal{G}}^{(1-p)p^{r+1}})\big), \\ \eta \cdot \mathrm{Ha}^{p^{r+1}} = p \bmod p^2 \end{array} \right\},$$

here the brackets means the class of the equivalence given by $(h,\eta) \equiv (h',\eta')$ if $h = h'$ and $\eta = \eta'(1+pu)$ for some $u \in R$. The formal scheme $\mathfrak{X}_r$ turns out to be the p-adic completion of the partial blow-up of $\mathfrak{X}$ at the zero locus of $\mathrm{Hdg}^{p^{r+1}}$ and p (see [3, Definition 3.1]).

5.2. Canonical subgroups

The theory of the canonical subgroup originally developed in the context of p-adic topology was generalized to the adic setting in [3]:

Proposition 5.2 ([3, Corollaire A.2]). — *There exists a canonical subgroup C_n of $\mathcal{G}_0[\mathfrak{p}_0^n]$ for $n \leqslant r$ over $\mathfrak{X}_r$. This is unique and satisfy the* *compatibility* $C_n[\mathfrak{p}_0^{n-1}] = C_{n-1}$. *Moreover, if we denote* $D_n := \mathcal{G}_0[\mathfrak{p}_0^n]/C_n$ *then* $\omega_{D_n} \simeq \omega_{\mathcal{G}_0[\mathfrak{p}^n]}/\mathrm{Hdg}^{\frac{p^n-1}{p-1}}$.

By [3, Proposition A.3], the cokernel of the map $\mathrm{dlog}_{\mathcal{G}_0^D[\mathfrak{p}_0^n]} : \mathcal{G}_0[\mathfrak{p}_0^n] \to \omega_{\mathcal{G}_0^D[\mathfrak{p}_0^n]} = \omega_0/p^n$ is killed by $\mathrm{Hdg}^{1/(p-1)}$. If we write $\Omega_0 \subseteq \omega_0$ for the subsheaf generated by the lifts of the image of the Hodge–Tate map and $\mathcal{I}_n := p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}$, we obtain a morphism

$$\mathrm{dlog}_0 : D_n(\mathfrak{X}_r) \longrightarrow \Omega_0 \otimes_{\mathcal{O}_{\mathfrak{X}_r}} (\mathcal{O}_{\mathfrak{X}_r}/\mathcal{I}_n) \subset \omega_0/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}. \tag{5.3}$$

In order to carry out the p-adic interpolation we work on covers of $\mathfrak{X}_r$. First notice that, by the moduli interpretation, the p-divisible group $\prod_{\mathfrak{p}\neq\mathfrak{p}_0} \mathcal{G}_\mathfrak{p} \to \mathfrak{X}_r$ is étale isomorphic to $\prod_{\mathfrak{p}\neq\mathfrak{p}_0}(F_\mathfrak{p}/\mathcal{O}_\mathfrak{p})^2$. Assume that $r \geqslant n$. Firstly, we add $\mathfrak{p}^n$-level corresponding to the primes $\mathfrak{p} \mid p$ such that $\mathfrak{p} \neq \mathfrak{p}_0$: We denote by $\mathfrak{X}_{r,n} \to \mathfrak{X}_r$ the formal scheme obtained by adding to the moduli interpretation a point of order $\mathfrak{p}^n$ for each $\mathfrak{p} \neq \mathfrak{p}_0$ whose multiples generate $\mathcal{G}_\mathfrak{p}[\mathfrak{p}]/C_\mathfrak{p}$ (see Remark 4.4). It is clear that the extension $\mathfrak{X}_{r,n} \to \mathfrak{X}_r$ is étale and its Galois group contains $\prod_{\mathfrak{p}\neq\mathfrak{p}_0}(\mathcal{O}_\mathfrak{p}/p^n\mathcal{O}_\mathfrak{p})^\times$ as a subgroup since F is unramified at p. Moreover, $\mathfrak{X}_{r,n}$ has also good reduction (see [11, Section 5.4]). Now we trivialize the subgroup D_n: Let $\mathcal{X}_{r,n}$ be the adic generic fiber of $\mathfrak{X}_{r,n}$. By [3, Corollaire A.2], the group scheme $D_n \to \mathcal{X}_{r,n}$ is also étale isomorphic to $p^{-n}\mathcal{O}_0/\mathcal{O}_0$. We denote by $\mathcal{IG}_{r,n}$ the adic space over $\mathcal{X}_{r,n}$ of the trivializations of D_n. Then the map $\mathcal{IG}_{r,n} \to \mathcal{X}_{r,n}$ is a finite étale with Galois group $(\mathcal{O}_0/p^n\mathcal{O}_0)^\times$. We denote by $\mathfrak{IG}_{r,n}$ the normalization $\mathcal{IG}_{r,n}$ in $\mathfrak{X}_{r,n}$ which is finite over $\mathfrak{X}_{r,n}$ and it is also endowed with an action of $(\mathcal{O}_0/p^n\mathcal{O}_0)^\times$. These constructions are captured by the following tower of formal schemes endowed with a natural action of $(\mathcal{O}/p^n\mathcal{O})^\times \simeq \prod_\mathfrak{p}(\mathcal{O}_\mathfrak{p}/p^n\mathcal{O}_\mathfrak{p})^\times$:

$$\mathfrak{IG}_{r,n} \longrightarrow \mathfrak{X}_{r,n} \longrightarrow \mathfrak{X}_r.$$

6. Overconvergent modular forms for G'

Following the approach introduced in [1], we deform the modular sheaves of G' which allow us to define overconvergent modular forms for G' and families of them. We also construct other overconvergent sheaves which will be useful to construct triple product p-adic L-functions.

6.1. Formal vector bundles

In this subsection we slightly modify the construction performed in [1, Section 2] which we briefly recall first. Let S be a formal scheme, $\mathcal{I}$ its (invertible) ideal of definition and $\mathcal{E}$ a locally free $\mathcal{O}_S$-module of rank n. We write $\overline{S}$ for the scheme with structural sheaf $\mathcal{O}_S/\mathcal{I}$ and put $\overline{\mathcal{E}}$ the corresponding $\mathcal{O}_{\overline{S}}$-module. We fix marked sections $s_1,\dots,s_m$ of $\overline{\mathcal{E}}$, namely, the sections $s_1,\dots,s_m$ define a direct sum decomposition $\overline{\mathcal{E}} = \mathcal{O}_{\overline{S}}^m \oplus Q$, where Q is a locally free $\mathcal{O}_{\overline{S}}$-module of rank $n-m$.

Let $S-\mathrm{Sch}$ be the category of formal S-schemes. There exists a formal scheme $\mathbb{V}(\mathcal{E})$ over S called the *formal vector bundle* attached to $\mathcal{E}$ which represents the functor, denoted by the same symbol, $S-\mathrm{Sch}\to\mathrm{Sets}$, and defined by $\mathbb{V}(\mathcal{E})(t:T\to S) := H^0(T,t^*(\mathcal{E})^\vee) = \mathrm{Hom}_{\mathcal{O}_T}(t^*(\mathcal{E}),\mathcal{O}_T)$. Crucial in [1] is the construction of the so called *formal vector bundles with marked sections* which is the formal scheme $\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)$ over $\mathbb{V}(\mathcal{E})$ that represents the following functor $S-\mathrm{Sch}\to\mathrm{Sets}$

$$\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)(t:T\longrightarrow S) = \left\{\rho\in H^0(T,t^*(\mathcal{E})^\vee)\mid \bar\rho(t^*(s_i))=1,\ i=1,\dots,m\right\},$$

here $\bar\rho$ is the reduction of ρ modulo $\mathcal{I}$. The construction of $\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)$ is as follows: the projection $\overline{\mathcal{E}}\to Q$ defines a quotient map $\bigoplus_k \mathrm{Sym}^k(\overline{\mathcal{E}})\to \bigoplus_k \mathrm{Sym}^k(Q)$ whose kernel is the ideal $(s_1,\dots,s_m)$ and, hence, defines a closed subscheme $C\subset\mathbb{V}(\overline{\mathcal{E}}) = \widehat{\mathrm{Sym}(\mathcal{E})}$, with corresponding ideal sheaf denoted by $\mathcal{J}$. Then $\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)$ is the $\mathcal{I}$-adic completion of the open formal subscheme of the blow-up of $\mathbb{V}(\mathcal{E})$ with respect to $\mathcal{J}$, defined by the requirement that the ideal generated by the inverse image of $\mathcal{J}$ coincides with the ideal generated by the inverse image of $\mathcal{I}$.

Given the fixed decomposition $\overline{\mathcal{E}} = Q\oplus\langle s_i\rangle_i$, let us consider now the subfunctor $\widehat{\mathbb{V}}_Q(\mathcal{E},s_1,\dots,s_m)$ that associates to any formal S-scheme $t:T\to S$ the subset of sections $\rho\in\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)(T)$ whose reduction $\bar\rho$ modulo $\mathcal{I}$ also satisfies $\bar\rho(t^*(m))=0$ for every $m\in Q$.

Lemma 6.1. — *The morphism $\widehat{\mathbb{V}}_Q(\mathcal{E},s_1,\dots,s_m)\to\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)$ is represented by a formal subscheme.*

Proof. — Since we have the direct sum decomposition $\overline{\mathcal{E}} = \mathcal{O}_{\overline{S}}^m\oplus Q$, we have the closed subscheme $\bigoplus_k\mathrm{Sym}^k(Q)\hookrightarrow\bigoplus_k\mathrm{Sym}^k(\overline{\mathcal{E}})$ and, hence, a closed subscheme $V\subset\mathbb{V}_0(\overline{\mathcal{E}},s_1,\dots,s_m)$. Let $\mathcal{J}$ be the corresponding ideal sheaf. Then we write $\widehat{\mathbb{V}}_Q(\mathcal{E},s_1,\dots,s_m)$ for the closed formal subscheme given by the inverse image of $\mathcal{J}$ in $\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)$.

Let $U=\mathrm{Spf}(R)$ be a formal affine open such that $\mathcal{I}$ is generated by $\alpha\in R$ and $\mathcal{E}|_U$ is free of rank n with basis $e_1,\dots,e_n$ such that $e_i\equiv s_i$ modulo $\mathcal{I}$, for $i=1,\dots,m$, and $Q=\langle e_{m+1},\dots,e_n\rangle$ modulo $\mathcal{I}$. Thus,

$$\mathbb{V}(\mathcal{E})|_U = \mathrm{Spf}(R\langle X_1,\dots,X_n\rangle)$$

and

$$\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)|_U = \mathrm{Spf}(R\langle Z_1, \ldots, Z_m, X_{m+1}, \ldots, X_n\rangle)$$

with the corresponding morphism $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \to \mathbb{V}(\mathcal{E})$ given by $X_i \mapsto 1 + \alpha Z_i$, for $i = 1, \ldots, m$ (see [1, Section 2]). Since the inverse image of $\mathcal{J}$ corresponds to $(\alpha, X_{m+1}, \ldots, X_n)$, we deduce that $\widehat{\mathbb{V}}_Q(\mathcal{E}, s_1, \ldots, s_m)|_U = \mathrm{Spf}(R\langle Z_1, \ldots, Z_m, T_{m+1}, \cdots T_n\rangle)$ with the corresponding morphism $\widehat{\mathbb{V}}_Q(\mathcal{E}, s_1, \ldots, s_m) \to \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ given by $X_j \mapsto \alpha Z_j$, for $j = m + 1, \ldots, n$.

Given a formal scheme $t : T \to U$, a section $\rho \in \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)(T)$ defined by the images $\rho^*(Z_i) = a_i$ and $\rho^*(X_j) = b_j$ provides the morphism $\rho \in \mathrm{Hom}_{\mathcal{O}_T}(t^*(\mathcal{E}), \mathcal{O}_T)$ satisfying $\rho(t^*(e_i)) = 1 + \alpha a_i$, for $i = 1, \ldots, m$, and $\rho(t^*(e_j)) = b_j$, for $j = m + 1, \ldots, n$. Thus, sections coming from $\widehat{\mathbb{V}}_Q(\mathcal{E}, s_1, \ldots, s_m)(T)$ will correspond to those ρ also satisfying $b_j = \alpha c_j$, for $j = m+1, \ldots, n$ and some $c_j \in R$. Hence, to sections such that $\bar{\rho}(t^*(m)) = 0$ for every $m \in Q$. $\square$

Remark 6.2. — Notice that this construction is also functorial with respect to $(\mathcal{E}, Q, s_1, \ldots, s_m)$. Indeed, given a morphism $\varphi : \mathcal{E}' \to \mathcal{E}$ of locally free $\mathcal{O}_S$-modules of finite rank and marked sections $s_1, \ldots, s_m \in \overline{\mathcal{E}}$, $s'_1, \ldots, s'_m \in \overline{\mathcal{E}}'$ such that $\bar{\varphi}(s'_i) = s_i$ and $\bar{\varphi}(Q') \subseteq Q$, we have the morphisms making the following diagram commutative

$$\begin{tikzcd}
\widehat{\mathbb{V}}_Q(\mathcal{E}, s_1, \ldots, s_m) \arrow[r] \arrow[d] & \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \arrow[r] \arrow[d] & \mathbb{V}(\mathcal{E}) \arrow[d] \\
\widehat{\mathbb{V}}_Q(\mathcal{E}', s'_1, \ldots, s'_m) \arrow[r] & \mathbb{V}_0(\mathcal{E}', s'_1, \ldots, s'_m) \arrow[r] & \mathbb{V}(\mathcal{E}').
\end{tikzcd}$$

Remark 6.3. — In fact, $\widehat{\mathbb{V}}_Q(\mathcal{E}, s_1, \ldots, s_m)$ depends on $Q \subset \overline{\mathcal{E}}$ and the image of s_i in $\overline{\mathcal{E}}/Q$. Indeed, since $\bar{\rho}(t^*(m_i)) = 0$ for $m_i \in Q$ and $t : T \to S$,

$$\begin{aligned}
&\widehat{\mathbb{V}}_Q(\mathcal{E}, (s_i + m_i)_i)(T) \\
&\qquad = \left\{\rho \in H^0(T, t^*(\mathcal{E})^\vee);\ \ \bar{\rho}(t^*(s_i + m_i)) = 1,\ \bar{\rho}(t^*(Q)) = 0\right\} \\
&\qquad = \widehat{\mathbb{V}}_Q(\mathcal{E}, (s_i)_i)(T).
\end{aligned}$$

6.2. Weight space for G'

We fix a decomposition $\mathcal{O}^\times \cong \mathcal{O}^0 \times H$ where H is the torsion subgroup of $\mathcal{O}^\times$ and $\mathcal{O}^0 \simeq 1 + p\mathcal{O}$ is a free $\mathbb{Z}_p$-module of rank d.

We put $\Lambda_F := \mathbb{Z}_p[\![\mathcal{O}^\times]\!]$ and $\Lambda_F^0 := \mathbb{Z}_p[\![\mathcal{O}^0]\!]$. Then remark that the choice of a basis $\{e_1, \dots, e_d\}$ of $\mathcal{O}^0$ furnishes an isomorphism $\Lambda_F^0 \cong \mathbb{Z}_p[\![T_1, \dots, T_d]\!]$ given by $1 + T_i = e_i$, for $i = 1, \dots, d$. Moreover, for each $n \in \mathbb{N}$ we consider the algebras:

$$\Lambda_n := \Lambda_F \left\langle \frac{T_1^{p^n}}{p}, \dots, \frac{T_d^{p^n}}{p} \right\rangle \qquad \Lambda_n^0 := \Lambda_F^0 \left\langle \frac{T_1^{p^n}}{p}, \dots, \frac{T_d^{p^n}}{p} \right\rangle.$$

The formal scheme $\mathfrak{W} = \mathrm{Spf}(\Lambda_F)$ is our formal *weight space* for G'. Thus, for each complete $\mathbb{Z}_p$-algebra R we have:

$$\mathfrak{W}(R) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^\times, R^\times).$$

We also consider the following formal schemes $\mathfrak{W}^0 = \mathrm{Spf}(\Lambda_F^0)$, $\mathfrak{W}_n := \mathrm{Spf}(\Lambda_n)$ and $\mathfrak{W}_n^0 := \mathrm{Spf}(\Lambda_n^0)$. By construction we have $\mathfrak{W} = \bigcup_n \mathfrak{W}_n$ and $\mathfrak{W}^0 = \bigcup_n \mathfrak{W}_n^0$. Moreover, we have:

$$\mathfrak{W}_n^0(\mathbb{C}_p) = \{k \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^0, \mathbb{C}_p^\times) : \ |k(e_i) - 1| \leqslant p^{-p^{-n}}, \ i = 1, \dots, d\}.$$

We denote by $\mathbf{k} : \mathcal{O}^\times \to \Lambda_F^\times$ the universal character of $\mathfrak{W}$, which decomposes as $\mathbf{k} = \mathbf{k}^0 \otimes \mathbf{k}_f$ where:

$$\mathbf{k}_f : H \longrightarrow \mathbb{Z}_p[H]^\times \qquad \mathbf{k}^0 : \mathcal{O}^0 \longrightarrow (\Lambda_F^0)^\times.$$

Let $\mathbf{k}_n^0 : \mathcal{O}^0 \to (\Lambda_n^0)^\times$ be $\mathbf{k}^0$ composed with $(\Lambda_F^0)^\times \subseteq (\Lambda_n^0)^\times$ and put $\mathbf{k}_n := \mathbf{k}_n^0 \otimes \mathbf{k}_f : \mathcal{O}^\times \to \Lambda_n^\times$.

Lemma 6.4. — *Let R be a p-adically complete Λ_n^0-algebra with maximal ideal m_R. Then $\mathbf{k}_n^0$ extends locally analytically to a character $\mathcal{O}^0(1 + p^n\lambda^{-1}\mathcal{O}_F \otimes R) \to R^\times$, for any $\lambda \in R$ such that $\lambda^{p-1} \in p^{p-2}m_R$. In particular, $\mathbf{k}_n^0$ is analytic on $1 + p^n\mathcal{O}$.*

Proof. — By definition $\mathbf{k}_n^0\left(\sum_i \gamma_i e_i\right) = \prod_{i=1}^d (T_i+1)^{\gamma_i} \in \mathbb{Z}_p[\![T_1, \dots, T_d]\!] \simeq \Lambda_F^0 \subseteq \Lambda_n^0$. Formally, we have that $(T_i + 1)^{\gamma_i} = \exp(\gamma_i \log(T_i + 1))$. For any $m = p^{n+s}m'$, $(m', p) = 1$ we have $p^n \frac{T_i^m}{m} = \frac{T_i^{m-(s+1)p^n}}{m'}\left(\frac{T_i^{p^n}}{p}\right)^{s+1} p \in p\Lambda_n^0$ and $m - (s+1)p^n = p^n(p^s m' - (s+1)) \geqslant 0$. Hence, $u_i := p^n \log(T_i + 1) \in p\Lambda_n^0$. We have that $\mathbf{k}_n^0(p^n\alpha) = \prod_{i=1}^{h_1} \exp(u_i\gamma_i)$ with $\alpha = \sum_i \gamma_i e_i$. This implies that, for any adic Λ_n^0-algebra R, the character $\mathbf{k}_n^0$ can be evaluated at the $\mathbb{Z}_p$-submodule $\mathcal{O}^0 \otimes_{\mathbb{Z}_p} \lambda^{-1}p^n R$, for any λ with valuation $\nu(\lambda) < \frac{p-2}{p-1}$. Note that, by means of the exponential map, we can identify $\mathcal{O}^0 \otimes_{\mathbb{Z}_p} \lambda^{-1}p^n R$ with $(1 + \lambda^{-1}p^n\mathcal{O} \otimes_{\mathbb{Z}_p} R) \subset (\mathcal{O} \otimes_{\mathbb{Z}_p} R)^\times$. We conclude that $\mathbf{k}_n^0$ can be evaluated at $\mathcal{O}^0 \cdot (1 + \lambda^{-1}p^n\mathcal{O} \otimes_{\mathbb{Z}_p} R) \subset (\mathcal{O} \otimes_{\mathbb{Z}_p} R)^\times$, and the result follows. $\square$

Let $\mathcal{O}^{\tau_0,0} := 1+p\mathcal{O}^{\tau_0}$ and recall the decomposition of rings $\mathcal{O} = \mathbb{Z}_p \times \mathcal{O}^{\tau_0}$. As above we introduce:

$$\begin{aligned} \Lambda_{\tau_0} &:= \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] & \Lambda_{\tau_0}^0 &:= \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] \simeq \mathbb{Z}_p[\![T]\!] \\ \Lambda^{\tau_0} &:= \mathbb{Z}_p[\![\mathcal{O}^{\tau_0 \times}]\!] & \Lambda^{\tau_0,0} &:= \mathbb{Z}_p[\![\mathcal{O}^{\tau_0,0}]\!] \simeq \mathbb{Z}_p[\![T_2,\dots,T_d]\!] \\ \Lambda_{\tau_0,n} &:= \Lambda_{\tau_0}\left\langle \frac{T^{p^n}}{p}\right\rangle & \Lambda_{\tau_0,n}^0 &:= \Lambda_{\tau_0}^0\left\langle \frac{T^{p^n}}{p}\right\rangle \\ \Lambda_n^{\tau_0} &:= \Lambda^{\tau_0}\left\langle \frac{T_2^{p^n}}{p},\dots,\frac{T_d^{p^n}}{p}\right\rangle & \Lambda_n^{\tau_0,0} &:= \Lambda^{\tau_0,0}\left\langle \frac{T_2^{p^n}}{p},\dots,\frac{T_d^{p^n}}{p}\right\rangle. \end{aligned}$$

Thus, we have decompositions $\Lambda_n^0 = \Lambda_{\tau_0,n}^0 \widehat{\otimes} \Lambda_n^{\tau_0,0}$ and $\Lambda_n = \Lambda_{\tau_0,n} \widehat{\otimes} \Lambda_n^{\tau_0}$. We denote by $\mathbf{k}_{\tau_0,n}^0 : (1+p\mathbb{Z}_p) \to \Lambda_{\tau_0,n}^0$, $\mathbf{k}_n^{\tau_0,0} : \mathcal{O}^{\tau_0,0} \to \Lambda_n^{\tau_0,0}$, $\mathbf{k}_{\tau_0,n} : \mathbb{Z}_p^\times \to \Lambda_{\tau_0,n}$ and $\mathbf{k}_n^{\tau_0} : \mathcal{O}^{\tau_0\times} \to \Lambda_n^{\tau_0}$ the universal characters. Then we have decompositions $\mathbf{k}_n^0 = \mathbf{k}_{\tau_0,n}^0 \otimes \mathbf{k}_n^{\tau_0,0}$ and $\mathbf{k}_n = \mathbf{k}_{\tau_0,n} \otimes \mathbf{k}_n^{\tau_0}$. Moreover,

$$\mathbf{k}_{\tau_0,n} \otimes \mathbf{k}_n^{\tau_0} = \mathbf{k}_n = \mathbf{k}_n^0 \otimes \mathbf{k}_f = \mathbf{k}_{\tau_0,n}^0 \otimes \mathbf{k}_n^{\tau_0,0} \otimes \mathbf{k}_f.$$

6.3. Overconvergent modular sheaves

We fix L a finite extension of $\mathbb{Q}_p$ containing the image of every p-adic embedding of F. We will work over the ring of integers of L. Let $r \geqslant n$. As in Section 5.2, we denote by $\mathcal{I}_n := p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}}$ considered now as an ideal of $\mathcal{O}_{\mathfrak{IG}_{r,n}}$, which is our ideal in order to perform the construction of Section 6.1. Let $\overline{\mathfrak{IG}_{\mathfrak{r},\mathfrak{n}}}$ be the reduction modulo $\mathcal{I}_n$, then from (5.3) we have

$$\mathrm{dlog}_0 : D_n(\mathfrak{IG}_{r,n}) \otimes_{\mathbb{Z}_p} (\mathcal{O}_{\mathfrak{IG}_{r,n}}/\mathcal{I}_n) \xrightarrow{\simeq} \Omega_0 \otimes_{\mathcal{O}_{\mathfrak{IG}_{r,n}}} (\mathcal{O}_{\mathfrak{IG}_{r,n}}/\mathcal{I}_n), \tag{6.1}$$

where Ω_0 is the $\mathcal{O}_{\mathfrak{IG}_{r,n}}$-submodule of ω_0 generated by all the lifts of $\mathrm{dlog}_0(D_n)$. By construction, there exist on $\mathfrak{IG}_{r,n}$ a universal canonical generator $P_{0,n}$ of D_n, and universal points $P_{\mathfrak{p},n}$ of order p^n in $\mathcal{G}_{\mathfrak{p}}[p^n]$. We put:

$$\Omega := \Omega_0 \oplus \bigoplus_{\mathfrak{p}\neq\mathfrak{p}_0} \omega_{\mathfrak{p}} = \Omega_0 \oplus \Omega^0 \tag{6.2}$$

where $\Omega^0 = \bigoplus_{\mathfrak{p}\neq\mathfrak{p}_0} \omega_{\mathfrak{p}}$, and we denote $\overline{\Omega}$ the associated $\mathcal{O}_{\overline{\mathfrak{IG}_{\mathfrak{r},\mathfrak{n}}}}$-module. Now we produce marked sections of $\overline{\Omega}$ in the sense of Section 6.1 as follows. Let $\mathfrak{p} \mid p$ and we consider two cases:

- if $\mathfrak{p} = \mathfrak{p}_0$ we denote by $s_0 \in \overline{\Omega}$ the image $\mathrm{dlog}_0(P_{0,n})$ using the isomorphism (6.1).

- if $\mathfrak{p} \neq \mathfrak{p}_0$ consider the decomposition $\omega_{\mathfrak{p}} = \bigoplus_{\tau\in\Sigma_{\mathfrak{p}}} \omega_{\mathfrak{p},\tau}$ over $\mathfrak{IG}_{r,n}$ and the dlog map:

$$\mathrm{dlog}_{\mathfrak{p}} : \mathcal{G}_{\mathfrak{p}}[p^n] \longrightarrow \omega_{\mathcal{G}_{\mathfrak{p}}[p^n]^D} = \omega_{\mathfrak{p}}/p^n\omega_{\mathfrak{p}} = \bigoplus_{\tau\in\Sigma_{\mathfrak{p}}} \omega_{\mathfrak{p},\tau}/p^n\omega_{\mathfrak{p},\tau}. \tag{6.3}$$

Then the image of $P_{\mathfrak{p},n}$ by $\mathrm{dlog}_{\mathfrak{p}}$ provides a set of sections $\{s_{\mathfrak{p},\tau}\}_{\tau\in\Sigma_{\mathfrak{p}}}$ of $\omega_{\mathfrak{p},\tau}/\mathcal{I}_n\omega_{\mathfrak{p},\tau} \subseteq \omega_{\mathfrak{p}}/\mathcal{I}_n\omega_{\mathfrak{p}}$.

The following lemma allows us to use the constructions recalled in Section 6.1:

Lemma 6.5. — *We have the following facts:*

- *The $\mathcal{O}_{\mathfrak{IG}_{r,n}}$-module Ω_0 is locally free of rank 1, and Ω is locally free of rank $2d-1$.*
- *The $\mathcal{O}_{\overline{\mathfrak{IG}_{r,n}}}$-module generated by $\{s_0\} \cup \bigcup_{\mathfrak{p}\neq\mathfrak{p}_0}\{s_{\mathfrak{p},\tau}\}_{\tau\in\Sigma_{\mathfrak{p}}}$ is a locally direct summand of $\overline{\Omega}$.*

Proof. — It follows directly from the isomorphism (6.1) and the fact that, since $\mathcal{G}_{\mathfrak{p}}$ is étale, $\mathrm{dlog}_{\mathfrak{p}}$ provides an isomorphism between $\omega_{\mathfrak{p}}/p^n\omega_{\mathfrak{p}}$ and $\mathcal{G}_{\mathfrak{p}}[p^n] \otimes_{\mathbb{Z}_p} (\mathcal{O}_{\mathfrak{IG}_{r,n}}/p^n\mathcal{O}_{\mathfrak{IG}_{r,n}})$. □

Applying the construction recalled in Section 6.1 to the locally free $\mathcal{O}_{\mathfrak{IG}_{r,n}}$-module Ω and the marked sections $\mathbf{s} := \{s_0\} \cup \bigcup_{\mathfrak{p}\neq\mathfrak{p}_0}\{s_{\mathfrak{p},\tau}\}_{\tau\in\Sigma_{\mathfrak{p}}}$ of $\overline{\Omega}$ we obtain the formal scheme $\mathbb{V}_0(\Omega,\mathbf{s})$ over $\mathfrak{IG}_{r,n}$. By construction we have the following tower of formal schemes:

$$\mathbb{V}_0(\Omega,\mathbf{s}) \longrightarrow \mathfrak{IG}_{r,n} \xrightarrow{g_n} \mathfrak{X}_r.$$

For any $\mathfrak{X}_r$-scheme T we have

$$\mathbb{V}_0(\Omega,\mathbf{s})(T) = \{(\rho,\varphi) \in \mathfrak{IG}_{r,n}(T) \times \Gamma(T,\rho^*\Omega^\vee) \,:\, \varphi(\rho^*s_i) \equiv 1 \bmod \mathcal{I}_n\}.$$

Let $\mathbf{s}^0 := \bigcup_{\mathfrak{p}\neq\mathfrak{p}_0}\{s_{\mathfrak{p},\tau}\}_{\tau\in\Sigma_{\mathfrak{p}}}$ then from (6.2) we have:

$$\mathbb{V}_0(\Omega,\mathbf{s}) = \mathbb{V}_0(\Omega_0,s_0) \times_{\mathfrak{IG}_{r,n}} \mathbb{V}_0(\Omega^0,\mathbf{s}^0).$$

As we are interested in locally analytic distributions (rather than functions) we perform the following construction. Let $t_0 \in \overline{\Omega_0}^\vee$ be such that $t_0(s_0) = 1$, $t_{\mathfrak{p},\tau} \in \overline{\omega}_{\mathfrak{p},\tau}$ any section such that $\varphi_\tau(s_{\mathfrak{p},\tau} \wedge t_{\mathfrak{p},\tau}) = 1$, and $Q_{\mathbf{s}^0} \subset \overline{\Omega^0}$ the direct summand generated by the sections in $\mathbf{s}^0$. We put $\mathbf{t}^0 := \bigcup_{\mathfrak{p}\neq\mathfrak{p}_0}\{t_{\mathfrak{p},\tau}\}_{\tau\in\Sigma_{\mathfrak{p}}}$ and we define:

$$\widehat{\mathbb{V}}(\Omega,\mathbf{s}) := \mathbb{V}_0(\Omega_0^\vee,t_0) \times_{\mathfrak{IG}_{r,n}} \widehat{\mathbb{V}}_{Q_{\mathbf{s}^0}}(\Omega^0,\mathbf{t}^0) \xrightarrow{f_0} \mathfrak{IG}_{r,n} \xrightarrow{g_n} \mathfrak{X}_r.$$

The $t_{\mathfrak{p},\tau}$ are defined modulo $Q_{\mathbf{s}^0}$, which is fine by Remark 6.3. By construction we have

$$\widehat{\mathbb{V}}(\Omega,\mathbf{s})(T) \quad = \quad \left\{\begin{matrix}(\rho,\varphi)\in\mathfrak{I}\mathfrak{G}_{r,n}(T)\times\Gamma(T,\rho^*(\Omega_0^\vee\oplus\Omega^0)^\vee);\\ \varphi(\rho^*t_0)\equiv\varphi(\rho^*t_{\mathfrak{p},\tau})\equiv 1,\ \varphi(\rho^*s_{\mathfrak{p},\tau})\equiv 0 \bmod \mathcal{I}_n\end{matrix}\right\}.$$

Recall that the morphism g_n is endowed with an action of $(\mathcal{O}/p^n\mathcal{O})^\times$. Then $\widehat{\mathbb{V}}(\Omega,\mathbf{s})/\mathfrak{X}_r$ is equipped with an action of $\mathcal{O}^\times(1+\mathcal{I}_n\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_a)\subseteq \mathrm{Res}_{\mathcal{O}/\mathbb{Z}_p}\mathbb{G}_m$. In fact, if $(\rho,\varphi)\in\widehat{\mathbb{V}}(\Omega,\mathbf{s})(T)$ and $\lambda(1+\gamma)\in\mathcal{O}^\times(1+\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{I}_n\mathcal{O}_T)$ then we write:

$$\lambda(1+\gamma)*(\rho,\varphi)=(\lambda\rho,\lambda(1+\gamma)*\varphi),$$

where $*$ denotes the extension of the natural action of $\mathcal{O}$ on $\Omega^\vee$ given by $\lambda*\varphi(w):=\varphi(\lambda w)$. This action is well defined since, for each $\tau\neq\tau_0$, we have $(\lambda_\tau(1+\gamma_\tau))*\varphi))\left((\lambda_\tau\rho)^*s_{\mathfrak{p},\tau}\right)\equiv\lambda_\tau\varphi(\rho^*\lambda_\tau s_{\mathfrak{p}_\tau})\equiv\lambda_\tau^2\varphi(\rho^*s_{\mathfrak{p}_\tau})\equiv 0 \bmod \mathcal{I}_n$, and, for each τ, we have

$$(\lambda_\tau(1+\gamma_\tau)*\varphi))\left((\lambda_i\rho)^*t_\tau\right)\equiv\lambda_i\varphi(\rho^*\lambda_\tau^{-1}t_\tau)=\varphi(\rho^*t_\tau) \bmod \mathcal{I}_n.$$

As $r\geqslant n$ by Lemma 6.4 (with $\lambda=\mathrm{Hdg}^{\frac{p^n}{p-1}}$) $\mathbf{k}_n^0$ extends to a locally analytic character

$$\mathbf{k}_n^0:\mathcal{O}^\times(1+\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{I}_n\mathcal{O}_{\mathfrak{X}_r}\otimes_{\mathbb{Z}_p}\Lambda_n^0)\longrightarrow\mathcal{O}_{\mathfrak{X}_r}\otimes_{\mathbb{Z}_p}\Lambda_n^0.$$

Definition 6.6. — *We consider the following sheaves over* $\mathfrak{X}_r\times\mathfrak{W}_n$*:*

$$\mathcal{F}_n:=\left((g_n\circ f_0)_*\mathcal{O}_{\hat{\mathbb{V}}(\Omega,\mathbf{s})}\widehat{\otimes}\Lambda_n\right)[\mathbf{k}_n^0]\qquad \Omega^{\mathbf{k}_f}:=\left(g_{1,*}(\mathcal{O}_{\mathfrak{I}\mathfrak{G}_1})\widehat{\otimes}\Lambda_n\right)[\mathbf{k}_f].$$

The overconvergent modular sheaf *over* $\mathfrak{X}_r\times\mathfrak{W}_n$ *is defined as* $\Omega^{\mathbf{k}_n}:=\Omega^{\mathbf{k}_n^0}\otimes_{\mathcal{O}_{\mathfrak{X}_r\times\mathfrak{W}_n}}\Omega^{\mathbf{k}_f}$*, where*

$$\Omega^{\mathbf{k}_n^0}:=\mathcal{F}_n^\vee=\mathcal{H}om_{\mathfrak{X}_r\times\mathfrak{W}_n}\left(\mathcal{F}_n,\mathcal{O}_{\mathfrak{X}_r\times\mathfrak{W}_n}\right).$$

Observe that $\mathcal{F}_n$ is the sheaf on $\mathfrak{X}_r\times\mathfrak{W}_n$ given by the sections s of $(g_n\circ f_0)_*\mathcal{O}_{\hat{\mathbb{V}}(\Omega,\mathbf{s})}\widehat{\otimes}\Lambda_n$ such $t*s=\mathbf{k}_n^0(t)\cdot s$, for all $t\in\mathcal{O}^\times(1+\mathcal{I}_r\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_a)$.

Definition 6.7. — *A section in* $\mathrm{H}^0(\mathfrak{X}_r\times\mathfrak{W}_n,\Omega^{\mathbf{k}_n})$ *is called a* family of locally analytic Overconvergent modular forms *for the group* G'*.*

Remark 6.8. — The present construction performs the p-adic variation of the modular sheaves for unitary Shimura curves over a d-dimensional weight space. Compare with [8] and [15] where the p-adic variation is essentially constructed over a curve inside our weight space.

Remark 6.9. — Considering the adic generic fiber of $\mathcal{F}_n$ we obtain a sheaf of Banach modules over $\mathcal{X}_r \times \mathcal{W}_n$ which, by construction, satisfies the properties stated in 1.1(i). The sheaves $\Omega^{\mathbf{k}_n}$ are introduced to construct of triple product p-adic L-functions.

6.4. Local description of $\mathcal{F}_n$

We consider the sheaf $\mathcal{F}'_n = ((f_0)_*\mathcal{O}_{\hat{\mathbb{V}}(\Omega,\mathbf{s})} \widehat{\otimes} \Lambda_n)[\mathbf{k}_n^0]$ over $\mathfrak{IG}_{r,n} \times \mathfrak{W}_n^0$, using notation of Section 6.3. Let $\rho : \mathrm{Spf}(R) \to \mathfrak{IG}_{r,n} \times \mathfrak{W}_n^0$ be a morphism of formal schemes over $\mathfrak{W}_n^0$ without p-torsion such that $\rho^*\Omega_0^\vee$ and $\rho^*\omega_{\mathfrak{p},\tau}$ are free R-modules of rank 1 and 2, respectively. Fix a basis $\rho^*\Omega_0^\vee = Re_0$ and $\rho^*\omega_{\mathfrak{p},\tau} = Rf_\tau \oplus Re_\tau$ for $\mathfrak{p} \neq \mathfrak{p}_0$ such that $f_\tau \equiv s_{\mathfrak{p},\tau}$, $e_0 \equiv t_0$, and $e_\tau \equiv t_{\mathfrak{p},\tau}$ modulo $\rho^*\mathcal{I}_n$. Moreover, we assume $\rho^*\mathcal{I}_n$ is generated by some $\alpha_n \in R$.

We denote by $e_0^\vee, e_\tau^\vee, f_\tau^\vee \in \rho^*(\Omega_0^\vee \oplus \Omega^0)^\vee$ the dual R-basis, then by definition, we have that

$$\widehat{\mathbb{V}}(\Omega,\mathbf{s})(\mathrm{Spf}(R))$$
$$= \left\{ \sum_{\tau\neq\tau_0} a_\tau f_\tau^\vee + \sum_{\tau\neq\tau_0} b_\tau e_\tau^\vee + b_0 e_0^\vee \ : \ b_0, b_\tau \in (1+\alpha_n R), a_\tau \in \alpha_n R \right\},$$

By Section 6.1 we have that $\rho^*\mathcal{O}_{\hat{\mathbb{V}}(\Omega,\mathbf{s})} = R\langle Y_{\tau_0}, (Z_\tau, Y_\tau)_{\tau\in\Sigma_\mathfrak{p},\mathfrak{p}\neq\mathfrak{p}_0}\rangle$, where $\sum_{\tau\neq\tau_0} a_\tau f_\tau^\vee + \sum_{\tau\neq\tau_0} b_\tau e_\tau^\vee + b_0 e_0^\vee$ as above corresponds to the point given by the ideal $(Y_\tau = \frac{b_\tau - 1}{\alpha_n},\ Z_\tau = \frac{a_\tau}{\alpha_n}) \subset \rho^*\mathcal{O}_{\hat{\mathbb{V}}(\Omega,\mathbf{s})}$.

Under the identification $\mathcal{O}_F \otimes_\mathbb{Z} R \simeq R^{\Sigma_F}$ the group $(1 + \mathcal{O}_F \otimes_\mathbb{Z} \mathcal{I}_n R)$ corresponds to $(1+\alpha_n R)^{\Sigma_F}$. A point $(1+\lambda\otimes x) \in (1+\mathcal{O}_F\otimes\mathcal{I}_n R)$ acts on $\phi \in \mathbb{V}_0(\Omega,\mathbf{s})(\mathrm{Spf}(R))$ by $(1+\lambda\otimes x) * \phi(w) = \phi(w) + x\phi(\lambda w)$. Then $(t_\tau) \in (1+\alpha_n R)^{\Sigma_F}$ acts on $f_\tau^\vee$ and $e_\tau^\vee$ by multiplication by t_τ, and on $e_0^\vee$ by multiplication by t_{τ_0}. Therefore $(t_\tau)_\tau$ acts as $Y_\tau \mapsto (t_\tau - 1)\alpha_n^{-1} + t_\tau Y_\tau$ for $\tau \in \Sigma_F$, and $Z_\tau \mapsto t_\tau Z_\tau$ for $\tau \neq \tau_0$.

Lemma 6.10. — *Let $P_{\alpha_n} \in R\langle x_\tau\rangle_{\tau\in\Sigma_F}$ such that $\mathbf{k}_n^0(1+\alpha_n z_\tau) = P_{\alpha_n}((z_\tau)_{\tau\in\Sigma_F})$. Then we have:*

$$\rho^*(\mathcal{F}'_n) = P_{\alpha_n}((Y_\tau)_{\tau\in\Sigma_F}) \cdot R\left\langle \frac{Z_\tau}{1+\alpha_n Y_\tau}\right\rangle_{\tau\in\Sigma_F\setminus\{\tau_0\}}.$$

Proof. — The inclusion $\supset$ is clear. Now we consider $f \in \rho^*(\mathcal{F}'_n) = R\langle Y_{\tau_0}, (Z_\tau, Y_\tau)_{\tau\in\Sigma_\mathfrak{p},\mathfrak{p}\neq\mathfrak{p}_0}\rangle[\mathbf{k}_n^0]$. Then $f/P_{\alpha_n}(Y_\tau) \in R\langle Z_\tau, Y_\tau\rangle^{(1+\alpha_n R)^{\Sigma_F}} = R\langle Y_{\tau_0}\rangle^{1+\alpha_n R}\widehat{\otimes}\widehat{\bigotimes}_{\tau\in\Sigma_\mathfrak{p},\mathfrak{p}\neq\mathfrak{p}_0} R\langle Z_\tau, Y_\tau\rangle^{1+\alpha_n R}$. As in [1, Lemmas 3.9 and 3.13], we have $R\langle Y_{\tau_0}\rangle^{1+\alpha_n R} = R$ and $R\langle Z_\tau, Y_\tau\rangle^{1+\alpha_n R} = R\langle Z_\tau/(1+\alpha_n Y_\tau)\rangle$, hence, the result follows. $\square$

Remark 6.11. — To obtain a description of $\mathcal{F}_n$ we need to consider the action of $\mathcal{O}^\times$ and descend using the morphism $g_n : \mathfrak{IG}_{r,n} \to \mathfrak{X}_r$ in the same way as in [1, Section 3.2.1]

6.5. Overconvergent modular forms à la Katz

Here we give a moduli description of the families of overconvergent modular forms introduced above.

6.5.1. Notations

Let R be a complete local $\widehat{\mathcal{O}}$-algebra, $k : \mathcal{O}^{\tau_0 \times} \to R$ be a character and $n \in \mathbb{N}$.

Definition 6.12. — *Let $C_n^k(\mathcal{O}^{\tau_0}, R)$ be the R-module of functions $f : \mathcal{O}^{\tau_0\times} \times \mathcal{O}^{\tau_0} \to R$ such that:*

- *$f(tx, ty) = k(t) \cdot f(x, y)$ for each $t \in \mathcal{O}^{\tau_0\times}$ and $(x, y) \in \mathcal{O}^{\tau_0\times} \times \mathcal{O}^{\tau_0}$;*
- *The function $y \mapsto f(1, y)$ is analytic on the disks $y_0 + p^n\mathcal{O}^{\tau_0}$, where y_0 belongs to a system of representatives of $\mathcal{O}^{\tau_0}/p^n\mathcal{O}^{\tau_0}$.*

The space of distributions *is defined by*

$$D_n^k(\mathcal{O}^{\tau_0}, R) := \mathrm{Hom}_R(C_n^k(\mathcal{O}^{\tau_0}, R), R).$$

Remark 6.13. — Note that $C_n^k(\mathcal{O}^{\tau_0}, R) \subseteq C_{n+1}^k(\mathcal{O}^{\tau_0}, R)$ and, if $k = \underline{k} \in \mathbb{N}[\Sigma_F \setminus \{\tau_0\}]$ is a classical weight, then $C_0^{\underline{k}}(\mathcal{O}^{\tau_0}, R)$ is the module of analytic functions and naturally contains $\mathrm{Sym}^{\underline{k}}(R^2)$. We obtain a natural projection $D_0^{\underline{k}}(\mathcal{O}^{\tau_0}, R) \to \mathrm{Sym}^{\underline{k}}(R^2)^\vee$.

The subgroup $K_0(p)^{\tau_0} \subset \mathrm{GL}_2(\mathcal{O}^{\tau_0})$ of upper triangular matrices modulo p acts on $C_n^k(\mathcal{O}^{\tau_0}, R)$ and $D_n^k(\mathcal{O}^{\tau_0}, R)$ by $(g * f)(x, y) = f((x, y)g)$ and $(g * \mu)(f) := \mu(g^{-1} * f)$, where $g \in K_0(p)^{\tau_0}$, $f \in C_n^k(\mathcal{O}^{\tau_0}, R)$ and $\mu \in D_n^k(\mathcal{O}^{\tau_0}, R)$. Since $y \mapsto f(1, y)$ is analytic on the disks $y_0 + p^n\mathcal{O}^{\tau_0}$, this action extends to an action of $K_0(p)^{\tau_0}(1 + p^n\mathrm{M}_2(R \otimes_{\mathbb{Z}_p} \mathcal{O}^{\tau_0}))$.

6.5.2. Locally analytic functions

For $r \geqslant n$ we describe the elements of $H^0(\mathfrak{X}_r \times \mathfrak{W}_n, \mathcal{F}_n)$ as rules, extending the classical interpretation given in Section 4.4.

Let R be a Λ_n-algebra and $P \in (\mathfrak{X}_r \times \mathfrak{W}_n)(\mathrm{Spf}(R))$ corresponding to the isomorphism class of $(A, \iota, \theta, \alpha^{\mathfrak{p}_0})$. We use the following notations introduced in Section 5 in the present situation: $C_n \subset \mathcal{G}_0$, D_n, $C_\mathfrak{p} \subset \mathcal{G}_\mathfrak{p}$ for $\mathfrak{p} \neq \mathfrak{p}_0$, $\mathcal{I}_n := p^n \mathrm{Hdg}(A)^{-\frac{p^n}{p-1}} \subset R$, $\mathrm{dlog}_0 : D_n(R) \to \Omega_0 \otimes_R (R/\mathcal{I}_n)$ and $\mathrm{dlog}_{\mathfrak{p},\tau} : \mathcal{G}_\mathfrak{p}[p^n] \to \omega_{\mathfrak{p},\tau} \otimes_R (R/\mathcal{I}_n)$.

Let $w = (f_0, \{(f_\tau, e_\tau)\}_\tau)$ such that f_0 is a basis of Ω_0 and if $\mathfrak{p} \neq \mathfrak{p}_0$ and $\tau \in \Sigma_\mathfrak{p}$ then $\{e_\tau, f_\tau\}$ is a basis of $\omega_{\mathfrak{p},\tau}$ and f_τ generate $\omega_{C_\mathfrak{p}^D}$. Suppose the reductions modulo $\mathcal{I}_n$ of f_0 and (f_τ, e_τ) lie in the image of dlog_0 and $\mathrm{dlog}_{\mathfrak{p},\tau}$, respectively. Then $p^{n-1}\mathrm{dlog}_{\mathfrak{p},\tau}^{-1}(f_\tau)$ generate $C_\mathfrak{p}$ for each $\tau \in \Sigma_\mathfrak{p}$.

For $s \in H^0(\mathfrak{X}_r \times \mathfrak{W}_n, \mathcal{F}_n)$ and $(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w)$ as above we assign the locally analytic function $s(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w) \in C_n^{\mathbf{k}_n^{\tau_0}, 0}(\mathcal{O}^{\tau_0}, R)$ defined as follows. Fix $(x, y) \in \mathcal{O}^{\tau_0 \times} \times \mathcal{O}^{\tau_0}$. As $A[\mathfrak{p}]^{-,1} = \langle p^{n-1}\mathrm{dlog}_{\mathfrak{p},\tau}^{-1} f_\tau, p^{n-1}\mathrm{dlog}_{\mathfrak{p},\tau}^{-1} e_\tau\rangle$, then $\langle p^{n-1}(\tau(y)\mathrm{dlog}_{\mathfrak{p},\tau}^{-1} f_\tau - \tau(x)\mathrm{dlog}_{\mathfrak{p},\tau}^{-1} e_\tau)_\tau\rangle$ generates $A[\mathfrak{p}]^{-,1}/C_\mathfrak{p} = \mathcal{G}_\mathfrak{p}[\mathfrak{p}]/C_\mathfrak{p}$. Thus, $(\mathrm{dlog}_0^{-1}(f_0), (\varphi_\tau(f_\tau \wedge e_\tau)^{-1}(\tau(y)\mathrm{dlog}_{\mathfrak{p},\tau}^{-1} f_\tau - \tau(x)\mathrm{dlog}_{\mathfrak{p},\tau}^{-1} e_\tau))_{\tau \neq \tau_0}) \in A[p^n]^{-,1}$ gives a point $P_{(x,y)}^w \in \mathfrak{IG}_{r,n}(R)$ lying above P. Finally we define:

$$s(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w)(x, y) := s\left(P_{(x,y)}^w, f_0 + \sum_{\tau \neq \tau_0} (\tau(x) f_\tau^\vee + \tau(y) e_\tau^\vee)\right).$$

Now we verify it is well defined. Firstly observe that

$$\left(P_{(x,y)}^w, f_0 + \sum_{\tau \neq \tau_0} \tau(x) f_\tau^\vee + \tau(y) e_\tau^\vee\right) \in \widehat{V}(\Omega, \mathbf{s})(R).$$

In fact, we have $s_0 = \mathrm{dlog}_0(P_{(x,y)}^w) = f_0$, $s_{\mathfrak{p},\tau} = \mathrm{dlog}_{\mathfrak{p},\tau}(P_{(x,y)}^w) = \varphi_\tau(f_\tau \wedge e_\tau)^{-1}(\tau(y) \cdot f_\tau - \tau(x) \cdot e_\tau)$ and $t_{\mathfrak{p},\tau} = \tau(r) \cdot f_\tau + \tau(s) \cdot e_\tau$ for any $r, s \in \mathcal{O}^{\tau_0}$ such that $rx + psy = 1$. Hence,

$$\left(f_0 + \sum_{\tau \neq \tau_0} \tau(x) f_\tau^\vee + \tau(y) e_\tau^\vee\right)(s_{\mathfrak{p},\tau}) \equiv 0;$$

$$\left(f_0 + \sum_{\tau \neq \tau_0} \tau(x) f_\tau^\vee + \tau(y) e_\tau^\vee\right)(t_i) \equiv 1 \ (i = 0, (\mathfrak{p}, \tau)).$$

Moreover, by the local description of Section 6.4, the function $y \mapsto s(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w)(1, y)$ is analytic over any open $y_0 + p^n\mathcal{O}^{\tau_0}$ (where $P_{(x,y)}^w$

is constant). Finally, if $t \in \mathcal{O}^{\tau_0\times}$ then $s(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)(tx,ty) =$

$$s\left(tP^w_{(x,y)}, f_0 + t\left(\sum_\tau \tau(x)f_\tau^\vee + \tau(y)e_\tau^\vee\right)\right)$$
$$= s\left(t * \left(P^w_{(x,y)}, f_0 + \sum_\tau \tau(x)f_\tau^\vee + \tau(y)e_\tau^\vee\right)\right)$$
$$= \mathbf{k}_n^{\tau_0,0}(t)\cdot s(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)(x,y).$$

If $t \in \mathbb{Z}_p^\times(1+\mathcal{I}_nR)$ and $g \in K_0(p)^{\tau_0}(1+\mathcal{I}_n\mathrm{M}_2(R))^{\Sigma_F\setminus\{\tau_0\}}$ then $(tf_0,\{(f_\tau,e_\tau)g\}_\tau)$ satisfies the same properties as w. If $t\in\mathbb{Z}_p^\times$ we have:

$$s(A,\iota,\theta,\alpha^{\mathfrak{p}_0},tw)(x,y) = s\left(t * \left(P^w_{(x,y)}, f_0 + \sum_\tau \tau(x)f_\tau^\vee + \tau(y)e_\tau^\vee\right)\right)$$
$$= \mathbf{k}^0_{n,\tau_0}(t)\cdot s(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)(x,y).$$

If $g\in K_0(p)^{\tau_0}$ we have $P^{wg}_{(x,y)g} = P^w_{(x,y)}$ and then:

$$s(A,\iota,\theta,\alpha^{\mathfrak{p}_0},wg)((x,y)g) = s\left(P^w_{(x,y)}, f_0 + \sum_\tau (\tau(x),\tau(y))\tau g\cdot \tau g^{-1}\begin{pmatrix} f_\tau^\vee \\ e_\tau^\vee\end{pmatrix}\right)$$
$$= s(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)(x,y).$$

These two last properties, and the fact that the action of $\mathbb{Z}_p^\times\times K_0(p)^{\tau_0}$ can be extended to an action of $R^\times\times K_0(p)^{\tau_0}(1+\mathcal{I}_n\mathrm{M}_2(R))$ analytically, allows us to extend the rule defined above to any basis w of Ω satisfying the above properties but not necessarily lying in the image of the dlog maps. Moreover, this rule characterizes the section s.

6.5.3. Distributions

Now let $\mu\in H^0(\mathfrak{X}_r\times\mathfrak{W}_n,\Omega^{\mathbf{k}_n})$. Using the construction above, we obtain a rule that assigns to each tuple $(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)$ as above, a distribution $\mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)\in D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},R)$. The rule $(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)\mapsto \mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)$ characterizes μ and satisfies:

(B1) $\mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)$ depends only on the R-isomorphism class of $(A,\iota,\theta,\alpha^{\mathfrak{p}_0})$.

(B2) The formation of $\mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)$ commutes with arbitrary extensions of scalars $R\to R'$ of Λ_n-algebras.

(B3-a) $\mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},a^{-1}w) = k_n^{\tau_0}(t)\cdot\mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)$, for all $t\in\mathbb{Z}_p^\times$.

(B3-b) $g*\mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},wg) = \mu(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)$, for all $g\in K_0(p)^{\tau_0}$.

Remark 6.14. — Note that this description fits with the classical setting of classical Katz modular forms explained in Section 4.4.

7. p-adic families and eigenvarieties for G

We construct $d+1$-dimensional families of Hecke eigenvalues of automorphic forms for G, using the spectral properties of the U_p-operator acting on families of overconvergent modular forms.

7.1. Weight space for G

The weight space for G', denoted by $\mathfrak{W}$, is formal spectrum of $\Lambda_F = \mathbb{Z}_p[\![\mathcal{O}^\times]\!]$ (see Section 6.2). For G we consider $\Lambda_F^G := \mathbb{Z}_p[\![\mathcal{O}^\times \times \mathbb{Z}_p^\times]\!] = \mathbb{Z}_p[H'] \widehat{\otimes} \Lambda_F^{G,0}$, where H' is an abelian finite group and $\Lambda_F^{G,0} = \Lambda_F^0 \widehat{\otimes} \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] = \mathbb{Z}_p[\![T_1, \dots, T_d, T]\!]$. We write also $\Lambda_n^G := \Lambda_F^G\langle \frac{T_1^{p^n}}{p}, \dots, \frac{T_d^{p^n}}{p}, \frac{T^{p^n}}{p}\rangle$.

Then we consider the formal schemes $\mathfrak{W}^G := \mathrm{Spf}(\Lambda_F^G)$ and $\mathfrak{W}_n^G := \mathrm{Spf}(\Lambda_n^G)$. The scheme $\mathfrak{W}^G$ is called the *weight space for G*. Similarly as in Section 6.2, we have the moduli interpretation:

$$\mathfrak{W}^G(R) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^\times \times \mathbb{Z}_p^\times, R).$$

The map $\mathcal{O}^\times \to \mathcal{O}^\times \times \mathbb{Z}_p^\times$ given by $t \mapsto (t^{-2}, N_{F/\mathbb{Q}}(t))$ induces a morphism of weight spaces:

$$k : \mathfrak{W}^G \longrightarrow \mathfrak{W}. \tag{7.1}$$

In terms of characters, if $(r, \nu) \in \mathfrak{W}^G(R)$ and $t \in \mathcal{O}^\times$ then $k(r,\nu)(t) = \nu(N(t)) \cdot r(t)^{-2}$.

Lemma 7.1. — *The morphism k extends to a well defined morphism $k : \mathfrak{W}_n^G \to \mathfrak{W}_n$.*

Proof. — By definition $\Lambda_F\langle \frac{T_1^{p^n}}{p}, \dots, \frac{T_d^{p^n}}{p}\rangle$ is the completion of the continuous Λ_F-algebra $\varphi : \Lambda_F \to \Lambda_F(\frac{T_1^{p^n}}{p}, \dots, \frac{T_d^{p^n}}{p}) \subseteq \Lambda_F[\frac{1}{p}]$ such that each $\frac{T_i^{p^n}}{p}$ is power bounded and satisfies the following universal property: for any non-archimedean topological continuous Λ_F-algebra $f : \Lambda_F \to B$ such that $f(p)$ is invertible in B and $f(T_i)f(p)^{-1}$ is power bounded in B, then there exists a unique continuous homomorphism $g : \Lambda_F(\frac{T_1^{p^n}}{p}, \dots, \frac{T_d^{p^n}}{p}) \to B$ with $f = g \circ \varphi$.

Each variable T_i corresponds to a $\mathbb{Z}_p$-generator α_i of Λ_F^0, and T corresponds to $\exp(p) \in 1 + p\mathbb{Z}_p$. Note that the corresponding algebra morphism $k^* : \Lambda_F \to \Lambda_F^G$ satisfies $k^*(T_i + 1) = (T_i + 1)^{-2}(T + 1)^{\beta_i}$, where

$\log_p(N_{F/\mathbb{Q}}\alpha_i) = p\beta_i$. Considering $F : \Lambda_F \xrightarrow{k^*} \Lambda_F^G \to \Lambda_F^G(\frac{T_1^{p^n}}{p}, \ldots, \frac{T_d^{p^n}}{p}, \frac{T^{p^n}}{p})$ we have

$$\begin{aligned} F(T_i^{p^n}) &= \left((T_i+1)^{-2}(T+1)^{\beta_i} - 1\right)^{p^n} \\ &= \left(\sum_{n\geqslant 1}\binom{\beta_i}{n}T^n + \sum_{m\geqslant 1}\binom{-2}{m}T_i^m \sum_{n\geqslant 0}\binom{\beta_i}{n}T^n\right)^{p^n} \\ &= p\left(\frac{T_i^{p^n}}{p}\left(\sum_{m\geqslant 1}\binom{-2}{m}T_i^{m-1}\right)^{p^n}\left(\sum_{n\geqslant 0}\binom{\beta_i}{n}T^n\right)^{p^n}\right. \\ &\qquad\qquad \left. + \frac{T^{p^n}}{p}\left(\sum_{n\geqslant 1}\binom{\beta_i}{n}T^{n-1}\right)^{p^n} + \lambda\right), \end{aligned}$$

for some $\lambda \in \Lambda_F^G$. This implies that $p^{-1}F(T_i^{p^n}) \in \Lambda_F^G(\frac{T_1^{p^n}}{p}, \ldots, \frac{T_d^{p^n}}{p}, \frac{T^{p^n}}{p})$ and it is power bounded. Thus, there exists a continuous morphism $k' : \Lambda_F(\frac{T_1^{p^n}}{p}, \ldots, \frac{T_d^{p^n}}{p}) \to \Lambda_F^G(\frac{T_1^{p^n}}{p}, \ldots, \frac{T_d^{p^n}}{p}, \frac{T^{p^n}}{p})$. such that $F = k' \circ \varphi$. The completion k' provides the morphism we are looking for. $\square$

7.2. Overconvergent descent from G' to G

Recall the Shimura curve $X^{\mathfrak{c}}$, for $\mathfrak{c} \in \mathrm{Pic}(\mathcal{O}_F)$, introduced in Remark 4.2. Repeating the constructions performed in Section 5 and Section 6 we obtain for $r \geqslant n$ a formal scheme $\mathfrak{X}_r^{\mathfrak{c}}$ and a sheaf $\Omega^{\mathbf{k}_n}$ over $\mathfrak{X}_r^{\mathfrak{c}} \times \mathfrak{W}_n$. The irreducible components of $X^{\mathfrak{c}}$ are in correspondence with $\mathrm{Pic}(E/F, \mathfrak{n})$, and similarly for $\mathfrak{X}_r^{\mathfrak{c}}$. We denote by $\mathfrak{X}_r^{\mathfrak{c},0}$ the irreducible component corresponding to $1 \in \mathrm{Pic}(E/F, \mathfrak{n})$. We consider the universal character of $\mathfrak{W}_n^G$

$$(\mathbf{r}_n, \boldsymbol{\nu}_n) : \mathcal{O}^\times \times \mathbb{Z}_p^\times \longrightarrow \Lambda_F^G \subset \Lambda_n^G,$$

and its image through $k : \mathfrak{W}_n^G \to \mathfrak{W}_n$ denoted $\mathbf{k}_n^G = (\mathbf{k}_n^{G,\tau_0}, \mathbf{k}_{n,\tau_0}^G) := k(\mathbf{r}_n, \boldsymbol{\nu}_n) : \mathcal{O}^\times \to \Lambda_n^G$. Let

$$M_{\mathbf{k}_n^G}^r(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}, p), \Lambda_n^G) := H^0(\mathfrak{X}_r^{\mathfrak{c},0} \times \mathfrak{W}_n^G, (\mathrm{id}, k)^*\Omega^{\mathbf{k}_n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In the same way as at the end of Section 4.5, we describe an action of Δ on $M_{\mathbf{k}_n^G}^r(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}, p), \Lambda_n^G)$. For $t = (t_0, t^0) \in \mathcal{O}^\times = \mathbb{Z}_p^\times \times \mathcal{O}^{\tau_0\times}$ we put $\mathbf{r}_n'(t) = \mathbf{r}_n(t^0)\boldsymbol{\nu}_n(t_0)\mathbf{r}_n(t_0^{-1})$. Now let $\mu \in M_{\mathbf{k}_n^G}^r(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}, p), \Lambda_n^G)$ and let $(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w)$ over some R be as in Section 6.5. Then for $s \in \Delta$ we put:

$$(s * \mu)(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w) := \mathbf{r}_n'(s) \cdot \mu(A, \iota, s^{-1}\theta, \gamma_s^{-1}\alpha^{\mathfrak{p}_0}, w),$$

where $\gamma_s \in K_1(\mathfrak{n}, p)$ and $\det(\gamma_s) = s$. This action is well defined: if $\eta \in U_{\mathfrak{n}}$ then $\eta : A \simeq A$ induces

$$\begin{aligned}\mu(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w) &= \mu(A, \iota, \eta^{-2}\theta, \eta^{-1}\alpha^{\mathfrak{p}_0}, \eta^{-1}w)\\ &= \mathbf{k}^G_{n,\tau_0}(\eta)\mathbf{k}^{G,\tau_0}_n(\eta^{-1}) \cdot \mu(A, \iota, \eta^{-2}\theta, \eta^{-1}\alpha^{\mathfrak{p}_0}, w)\\ &= \mathbf{r}'_n(\eta^2) \cdot \mu(A, \iota, \eta^{-2}\theta, \eta^{-1}\alpha^{\mathfrak{p}_0}, w),\end{aligned}$$

where the third equality follows from the fact that $N_{F/\mathbb{Q}}(\eta) = 1$. Finally, this action is compatible with the action introduced in Section 4.5 under specialization to classical weights.

In a compatible way with Corollary 4.8 we define:

Definition 7.2. — *The space of* families of locally analytic overconvergent automorphic forms for G with depth of overconvergence r *is:*

$$M^r_{(\mathbf{k}^G_n,\nu_n)}(\Gamma_1(\mathfrak{n}, p), \Lambda^G_n) := \bigoplus_{\mathfrak{c}\in \mathrm{Pic}(\mathcal{O}_F)} M^r_{\mathbf{k}^G_n}(\Gamma^{\mathfrak{c}}_{1,1}(\mathfrak{n}, p), \Lambda^G_n)^{\Delta}.$$

7.3. Hecke operators

We define Hecke operators acting on $\mu \in M^r_{(\mathbf{k}^G_n,\nu_n)}(\Gamma_1(\mathfrak{n}, p), \Lambda^G_n)$.

7.3.1. Hecke operators T_g, with $g \in G(\mathbb{A}^p_f)$

Given $(A, \iota, \theta, \alpha^{\mathfrak{p}_0}) \in \mathfrak{X}^{\mathfrak{c},0}_r(R)$, and the double coset

$$K^B_1\left(\mathfrak{n}, \prod_{\mathfrak{p}\neq\mathfrak{p}_0}\mathfrak{p}\right) g K^B_1\left(\mathfrak{n}, \prod_{\mathfrak{p}\neq\mathfrak{p}_0}\mathfrak{p}\right) = \bigsqcup_i g_i K^B_1\left(\mathfrak{n}, \prod_{\mathfrak{p}\neq\mathfrak{p}_0}\mathfrak{p}\right)$$

with $g_i \in G(\mathbb{A}^p_f)$ let $(A^{g_i}, \iota^{g_i}, \theta^{g_i}, (\alpha^{g_i})^{\mathfrak{p}_0}) \in \mathfrak{X}^{\mathfrak{c}_i\mathfrak{c},0}_r(R)$ as in Section 11, where $\mathfrak{c}_i = \det(g_i)$. For w a basis of ω_A, Proposition 11.3 suggests to define:

$$(T_g\mu)_{\mathfrak{c}}(A, \iota, \theta, \alpha^{\mathfrak{p}_0}, w) := \sum_i \mathbf{r}'_n(\det(\gamma_i)) \cdot \mu_{\mathfrak{c}'}(A^{g_i}, \iota^{g_i}, \det(\gamma_i)^{-1}\theta^{g_i}, k_i^{-1}(\alpha^{g_i})^{\mathfrak{p}_0}, w),$$

where $\mathfrak{c}'$ is the class of $\mathfrak{c}\det(g) = \mathfrak{c}\mathfrak{c}_i$ and $b_{\mathfrak{c}}g_i = \gamma_i^{-1}b_{\mathfrak{c}'}k_i$ for $\gamma_i \in G(\mathbb{Q})_+$, $k_i \in K^B_1(\mathfrak{n}, \prod_{\mathfrak{p}\neq\mathfrak{p}_0}\mathfrak{p})$, and the basis w in $\omega_{A^{g_i}}$ is the image of w through the isogeny $A \to A^{g_i}$, that is an isomorphism in tangent spaces. In the same way as in Section 11.2, we can verify that T_g is well defined.

7.3.2. Hecke operators $U_{\mathfrak{p}}$, where $\mathfrak{p} \mid p$ and $\mathfrak{p} \neq \mathfrak{p}_0$

We define $U_{\mathfrak{p}}$ for any $\mathfrak{p} \neq \mathfrak{p}_0$ dividing p by:

$$(U_{\mathfrak{p}}\mu)_{\mathfrak{c}}(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w) = \sum_{i\in\kappa_{\mathfrak{p}}} \left(\begin{pmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix} * \mu_{\mathfrak{c}}\right)\left(A_i,\iota_i,\varpi_{\mathfrak{p}}\theta_i,\alpha_i^{\mathfrak{p}_0},w\begin{pmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix}\right),$$

where $A_i = A/C_i$ with C_i defined as in Section 11.3. Following Remark 4.4, the morphism $\alpha_i^{\mathfrak{p}_0}$ is provided by the subgroup $A[\mathfrak{p}]^{-\cdot 1}/C_i$, and $w\left(\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix}\right)$ is the basis of ω_{A_i} given by w by means of the natural isogeny $A \to A_i$. Moreover, the action of $\left(\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix}\right)$ on $D_n^{\mathbf{k}_n^{G},\tau_0}(\mathcal{O}^{\tau_0},R)$ is given by:

$$\int_{\mathcal{O}^{\tau_0}\times\times\mathcal{O}^{\tau_0}} \phi\, \mathrm{d}\left(\begin{pmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix} * \mu\right) := \int_{\mathcal{O}^{\tau_0}\times\times\mathcal{O}^{\tau_0}} \left(\begin{pmatrix}1 & -i\\ 0 & \varpi_{\mathfrak{p}}\end{pmatrix} * \phi\right) \mathrm{d}\mu.$$

By Section 11.3 this definition is consistent with the classical definition up to constant.

7.3.3. Hecke operator $U_{\mathfrak{p}_0}$

We define $U_{\mathfrak{p}_0}$ by means of the formula

$$(U_{\mathfrak{p}_0}\mu)_{\mathfrak{c}}(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w) = \frac{1}{p}\sum_{i\in\mathbb{F}_p} \mu_{\mathfrak{c}}(A_i,\iota_i,\varpi_{\mathfrak{p}_0}\theta_i,\alpha_i^{\mathfrak{p}_0},w),$$

where the basis w is defined similarly as above, and $A_i = A/C_i$ is characterized by the fact that $C_i^{-,1} := C_i \cap A[\mathfrak{p}_0]^{-,1}$ has zero intersection with the canonical subgroup (see Section 11.3). By [8, Lemma 7.5], $(A_i,\iota_i,\varpi_{\mathfrak{p}_0}\theta_i,\alpha_i^{\mathfrak{p}_0}) \in \mathfrak{X}_r^{\mathfrak{c}}$, thus, $U_{\mathfrak{p}_0}$ defines a well defined operator in $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$.

Remark 7.3. — As noticed above, the specialization of the operators $U_{\mathfrak{p}}$ at classical weights and U of Section 3.3 coincides up to a constant. Indeed, by Section 11.3, for any classical $x = (\underline{k},\nu) \in \mathfrak{W}_n^G$ we have:

$$(U_{\mathfrak{p}}\mu)_x = \begin{cases} \varpi_{\mathfrak{p}}^{\frac{(\nu+2)\underline{1}+\underline{k}_{\mathfrak{p}}}{2}} \cdot U\mu_x, & \mathfrak{p}\neq\mathfrak{p}_0,\\ p^{\frac{\nu+k_{\tau_0}}{2}} \cdot U\mu_x, & \mathfrak{p}=\mathfrak{p}_0.\end{cases}$$

This implies that the pair of eigenvalues $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ of $U_{\mathfrak{p}}$ satisfy the Hecke polynomials $X^2 - a_{\mathfrak{p}}X + \varpi_{\mathfrak{p}}^{\underline{k}_{\mathfrak{p}}+\underline{1}}$ if $\mathfrak{p}\neq\mathfrak{p}_0$ and $X^2 - a_0X + p^{k_{\tau_0}-1}$ for $\mathfrak{p}=\mathfrak{p}_0$.

7.3.4. Hecke operator $V_{\mathfrak{p}_0}$

We analogously define $V_{\mathfrak{p}_0}$ as follows:

$$(V_{\mathfrak{p}_0}\mu)_{\mathfrak{c}}(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)=\mu_{\mathfrak{c}'}(A_\infty,\iota_\infty,\varpi_{\mathfrak{p}_0}\theta_\infty,\alpha_\infty^{\mathfrak{p}_0},w),$$

where the basis w is defined similarly as above, and $A_\infty=A/C_\infty$ is characterized by the fact that $C_\infty^{-,1}:=C_\infty\cap A[\mathfrak{p}_0]^{-,1}$ is the canonical subgroup. Since one can prove that $(A_\infty,\iota_\infty,\varpi_{\mathfrak{p}_0}\theta_\infty,\alpha_\infty^{\mathfrak{p}_0})\in\mathfrak{X}_{r-1}^{\mathfrak{c}}$, one obtains that, for $r\geqslant 1$, $V_{\mathfrak{p}_0}$ is well defined

$$V_{\mathfrak{p}_0}:M^{r-1}_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)\longrightarrow M^{r}_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G).$$

7.4. Banach space structure and compactness

Let $\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)$ and $y=(z,(r,\nu))\in(\mathfrak{X}_r^{\mathfrak{c},0}\times\mathfrak{W}_n^G)^{rig}$ with residue field L_y and consider the induced point in $\mathfrak{X}_r^{\mathfrak{c},0}\times\mathfrak{W}_n^G$ denoted again by y. The interpretation as Katz modular forms identifies

$$H^0(\mathrm{Spf}(\mathcal{O}_{L_y}),y^*(\mathrm{id},k)^*\Omega^{\mathbf{k}_n}))\otimes\mathbb{Q}_p\simeq D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{O}_{L_y})\otimes\mathbb{Q}_p,$$
$$k:=k(r,\nu)=(k_{\tau_0},k^{\tau_0}),$$

which is a Banach L_y-module. We denote by $|\cdot|_y$ the induced norm on $H^0(\mathrm{Spf}(\mathcal{O}_{L_y}),y^*(\mathrm{id},k)^*\Omega^{\mathbf{k}_n})\otimes\mathbb{Q}_p$, which, in fact, does not depend on the identification.

For any $\mu\in M^{r}_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$, we define the supremum norm

$$|\mu|=\sup\left\{|y^*\mu_{\mathfrak{c}}|_y;\ \ y\in(\mathfrak{X}_r^{\mathfrak{c},0}\times\mathfrak{W}_n^G)^{rig},\ \mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)\right\},$$
$$y^*\mu_{\mathfrak{c}}\in H^0(\mathrm{Spf}(\mathcal{O}_{L_y}),y^*(\mathrm{id},k)^*\Omega^{\mathbf{k}_n}))\otimes\mathbb{Q}_p.$$

Lemma 7.4. — *$|\cdot|$ is a norm on $M^{r}_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ which turns it into a potentially orthonormalizable $\Lambda_n^G\otimes\mathbb{Q}$-Banach module.*

Proof. — See [26, Lemma 2.2]. It is clear that $|\cdot|$ is an ultrametric norm. We have to prove that it is finite, complete and separated. Let $\{U_i=\mathrm{Spf}(A_i)\}_{i\in I}\subset\mathfrak{X}_r^{\mathfrak{c},0}\times\mathfrak{W}_n^G$ be a finite trivialization of Ω by affinoids. Thus, by Katz modular form interpretation $H^0(U_i,(\mathrm{id},k)^*\Omega^{\mathbf{k}_n})\overset{\sigma_i}{\simeq}D_n^{\mathbf{k}_n^{G,\tau_0}}(\mathcal{O}^{\tau_0},A_i)$. The supremum norm on $A_i\otimes\mathbb{Q}$ induces a finite norm $|\cdot|_i$ on $D_n^{\mathbf{k}_n^{G,\tau_0}}(\mathcal{O}^{\tau_0},A_i)\otimes\mathbb{Q}$. Moreover, we have that I is finite and $|\mu|=\max\{|\sigma_i\mu_{\mathfrak{c}}|_{U_i}|_i;\ i\in I,\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)\}$, thus, $|\cdot|$ is finite. Note that $|\cdot|_i$ is complete and separated on reduced affinoids. Since $\mathfrak{X}_r^{\mathfrak{c},0}\times\mathfrak{W}_n^G$ is reduced, $|\cdot|$ is separated. We deduce that $|\cdot|$ is also complete since every Cauchy sequence induce a Cauchy sequence in $D_n^{\mathbf{k}_n^{G,\tau_0}}(\mathcal{O}^{\tau_0}, A_i)\otimes\mathbb{Q}$ whose limits can be used, via identifications σ_i, to produce a section in $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$. which lies in the limit of the original Cauchy sequence.

That $|\cdot|$ is potentially orthonormalizable follows from [12, Section A], since L is discretely valued and for each classical weight (r,ν) we have $D_n^{\mathbf{k}_n^{G,\tau_0}}(\mathcal{O}^{\tau_0}, A_i)\simeq D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{O}_L))\,\widehat{\otimes}_{\mathcal{O}_L} A_i$, where $k:=k(r,\nu)$.

We conclude as the action of Δ is unitary and continuous. □

Proposition 7.5. — *The $\Lambda_n^G\otimes\mathbb{Q}$-lineal map*

$$U_p := \prod_{\mathfrak{p}|p} U_{\mathfrak{p}}$$

on $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ is compact.

Proof. — Using the notations of proof of 7.4 it is enough to prove compactness restricted to the affinoid $U_i = \mathrm{Spf}(A_i)$. Using notations from Section 6.5.1,

$$\begin{aligned} H^0(U_i,(\mathrm{id},k)^*\Omega^{\mathbf{k}_n})\otimes\mathbb{Q} &\simeq D_n^{\mathbf{k}_n^{G,\tau_0}}(\mathcal{O}^{\tau_0},A_i)\otimes\mathbb{Q}\\ &\simeq (A_i\otimes\mathbb{Q})\,\widehat{\otimes}_L\,\widehat{\bigotimes}_{\mathfrak{p}\neq\mathfrak{p}_0} D_n^{\mathbf{k}_{n,\mathfrak{p}}^G}(\mathcal{O}_{\mathfrak{p}},\mathcal{O}_L\otimes\Lambda_{n,\mathfrak{p}}^G)\otimes\mathbb{Q}. \end{aligned}$$

The operator U_p is the composition of the operator $U_{\mathfrak{p}_0}$ acting on $W_{\mathfrak{p}_0} := A_i\otimes\mathbb{Q}$ and, for each $\mathfrak{p}\neq\mathfrak{p}_0$, the operator $U_{\mathfrak{p}}$ acting on $W_{\mathfrak{p}} := D_n^{\mathbf{k}_{n,\mathfrak{p}}^G}(\mathcal{O}_{\mathfrak{p}},\mathcal{O}_L\otimes\Lambda_{n,\mathfrak{p}}^G)$ by $\int_{\mathcal{O}_{\mathfrak{p}}^\times\times\mathcal{O}_{\mathfrak{p}}}\phi\,\mathrm{d}U_{\mathfrak{p}}\mu := \sum_{i\in\kappa_{\mathfrak{p}}}\int_{\mathcal{O}_{\mathfrak{p}}^\times\times\mathcal{O}_{\mathfrak{p}}}\left(\begin{smallmatrix}1 & -i\\ 0 & \varpi_{\mathfrak{p}}\end{smallmatrix}\right)*\phi\,\mathrm{d}\mu$. It is enough to prove that each $U_{\mathfrak{p}}$ is compact in each $(\mathcal{O}_L\otimes\Lambda_{n,\mathfrak{p}}^G)$-Banach module $W_{\mathfrak{p}}$.

By [8, Lemma 7.5], the action of $U_{\mathfrak{p}_0}$ factors through the restriction $\mathfrak{X}_{r+1}^{\mathfrak{c},0}\to\mathfrak{X}_r^{\mathfrak{c},0}$. Similarly as in [27, Proposition 2.20], this implies that the action of $U_{\mathfrak{p}_0}$ on $W_{\mathfrak{p}_0}$ is compact.

Finally, by [38, Lemma 3.2.8] and [10, Corollary 2.9] the action of $U_{\mathfrak{p}}$ on $W_{\mathfrak{p}}$ is compact . □

7.5. Eigenvarieties

The following proposition allows the construction of the eigenvariety as stated in the introduction.

Proposition 7.6. — *The $\Lambda_n^G[1/p]$-module of families of overconvergent forms $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ is projective.*

Proof. — For each $\mathfrak{c}$, the neighborhood of the ordinary locus $\mathcal{X}_r^{\mathfrak{c}}$ is an affinoid, and we use Banach sheaves over them, hence, $M^r_{\mathbf{k}_n}(\Gamma^{\mathfrak{c}}_{1,1}(\mathfrak{n},p),\Lambda_n^G)^\Delta$ is projective. Thus $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ is a finite sum of projective spaces and then it is projective too. □

As $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ is a projective $\Lambda_n^G[1/p]$-module and the Hecke operator U_p acting on $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ is compact (see Proposition 7.5) then from [3, Section B.2.4] there exists the Fredholm determinant of U_p, which is denoted by $F_I(X) := \det(1 - XU_p \mid M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)) \in \Lambda_n^G[1/p]\{\{X\}\}$. We denote by $\mathcal{W}_n^G$ the corresponding adic weight space, and let $\mathcal{Z}_n \subset \mathbb{A}^1_{\mathcal{W}_n^G}$ be the spectral variety attached to $F_n(X)$ (see [3, Section B.3]). Then the natural map $\mathcal{Z}_n \to \mathcal{W}_n^G$ is locally finite and flat. By construction $\mathcal{Z}_n$ parametrizes reciprocals of the non-zero eigenvalues of U_p. In order to parametrize eigenvalues for the *all* Hecke operators we construct a finite cover of $\mathcal{Z}_n$ as follows.

The operator U_p is compact on $M^r_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ then using [3, Corollary B.1, Theorem B.2] we obtain a natural coherent sheaf denoted $\mathcal{M}_n$ over $\mathcal{Z}_n$ endowed with an action of the Hecke operators. The image of the Hecke operators in $\mathrm{End}_{\mathcal{O}_{\mathcal{Z}_n}}(\mathcal{M}_n)$ generates a coherent $\mathcal{O}_{\mathcal{Z}_n}$-algebra and the adic space attached to it is denoted $\mathcal{E}_n$, which by construction is equidimensional of dimension $d+1$ and endowed with a natural weight map $w : \mathcal{E}_n \to \mathcal{W}_n^G$ which is locally free and without torsion. This is the *adic eigenvariety* introduced in Theorem 1.1(ii) of the introduction.

7.6. Classicality

Let $\mathfrak{X}^{\mathfrak{c}}(\mathfrak{p}_0)$ be the Shimura curve obtained adding $\Gamma_0(\mathfrak{p}_0)$-level structure to the curve $\mathfrak{X}^{\mathfrak{c}}$ introduced in Section 4.2. Recall that, by the definition of $\mathfrak{X}^{\mathfrak{c}}$, the curve $\mathfrak{X}^{\mathfrak{c}}(\mathfrak{p}_0)$ has the full $\Gamma_0(p)$-level structure. Observe that its irreducible components are in bijection with $\mathrm{Pic}(E/F,\mathfrak{n})$ and let $\mathfrak{X}^{\mathfrak{c}}(\mathfrak{p}_0)^0$ be the irreducible component corresponding to 1. If $(\underline{k},\nu) \in \mathfrak{W}_n^G(\overline{\mathbb{Q}}_p)$ is a classical weight, i.e. $\underline{k} \in \Xi_\nu$, then the space of *classical forms of weight* $(\underline{k},\nu)$ *and level* $\Gamma_1(\mathfrak{n},p)$ is:

$$M_{(\underline{k},\nu)}\left(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p\right) = \bigoplus_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)} H^0\left(\mathcal{X}^{\mathfrak{c}}(\mathfrak{p}_0)^0,\omega^{\underline{k}}\right)^\Delta,$$

where $\omega^{\underline{k}}$ is defined in the same way as in Section 4.2. Observe that by Corollary 4.8 elements of $M_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)$ which are defined over number fields determine automorphic forms for G.

The space of *locally analytic overconvergent automorphic forms for G with depth of overconvergence r*, which we denote by $M^r_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)$, is defined as the specialization of the $\Lambda_n^G[1/p]$-module $M^r_{\mathbf{k}_n}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$ at $(\underline{k},\nu)$. Observe that a form $\mu\in M^r_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)$ is *overconvergent* in the sense that it is defined in some neighborhood of the ordinary locus and *locally analytic* in the sense that it is a section of a Banach sheaf. Remark the contrast with the coherent modular sheaves used to define classical modular forms.

The space of *overconvergent automorphic forms for G with depth of overconvergence r* is:

$$M^{r,\mathrm{alg}}_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)=\bigoplus_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)}H^0(\mathcal{X}_r^{\mathfrak{c},0},\omega^{\underline{k}})^{\Delta}.$$

Using the existence of the canonical subgroup in $\mathfrak{X}_r$ we obtain a canonical section of the natural morphism $\mathfrak{X}(\mathfrak{p}_0)^{\mathfrak{c}}\to\mathfrak{X}^{\mathfrak{c}}$ over $\mathfrak{X}_r$, this implies the following diagram:

$$\begin{tikzcd}
M^r_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p) \arrow[d, "\alpha"] & M_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p) \arrow[dl, hook, "\beta"] \\
M^{r,\mathrm{alg}}_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p). &
\end{tikzcd}$$

In the next result, if we work with respect to the morphism β, we consider the Hecke operator $U_{\mathfrak{p}_0}$ and, for $h\in\mathbb{R}_{\geqslant 0}$, the superscript h means the generalized eigenspace for $U_{\mathfrak{p}_0}$ of slope less than h. In the same way, if we work with respect to the morphism α, we consider Hecke operators $U_{\mathfrak{p}}$ for $\mathfrak{p}\neq\mathfrak{p}_0$ and we use superscripts $\underline{h}^0=(h_{\mathfrak{p}})_{\mathfrak{p}\neq\mathfrak{p}_0}\in\mathbb{R}_{\geqslant 0}^{\{\mathfrak{p}\neq\mathfrak{p}_0\}}$.

Proposition 7.7. — *Let $(h_0,\underline{h}^0)=(h_0,h_{\mathfrak{p}})_{\mathfrak{p}\neq\mathfrak{p}_0}\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}_{\geqslant 0}^{\{\mathfrak{p}\neq\mathfrak{p}_0\}}$.*

(i) If, for each $\mathfrak{p}\neq\mathfrak{p}_0$, we have $h_{\mathfrak{p}}<\min\{k_\tau+1:\tau\in\Sigma_{\mathfrak{p}}\}$ then α induces an isomorphism:

$$M^r_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)^{<\underline{h}^0}\xrightarrow{\sim}M^{r,\mathrm{alg}}_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)^{<\underline{h}^0}.$$

(ii) If $h_0<k_{\tau_0}-1$ then β induces an isomorphism:

$$M_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)^{<h_0}\xrightarrow{\sim}M^{r,\mathrm{alg}}_{(\underline{k},\nu)}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)^{<h_0}.$$

Proof. — The first statement is a consequence of the locally analytic BGG-resolution and our the description in Section 6.5. More precisely, we have seen in the proof of Proposition 7.5 that $U_{\mathfrak{p}}$ factors through res : $D^{\underline{k}_{\mathfrak{p}}}_{n-1}(\mathcal{O}_{\mathfrak{p}},\overline{\mathbb{Z}}_p)\otimes\mathbb{Q}_p\to D^{\underline{k}_{\mathfrak{p}}}_{n}(\mathcal{O}_{\mathfrak{p}},\overline{\mathbb{Z}}_p)\otimes\mathbb{Q}_p$, indeed since $\underline{k}_{\mathfrak{p}}$ is analytic $C^{\underline{k}_{\mathfrak{p}}}_{n,m}(\mathcal{O}_{\mathfrak{p}},\overline{\mathbb{Q}}_p)=C^{\underline{k}_{\mathfrak{p}}}_{m}(\mathcal{O}_{\mathfrak{p}},\overline{\mathbb{Q}}_p)$. Repeating this for each $\mathfrak{p}\neq\mathfrak{p}_0$ it is enough to consider analytic distributions instead of locally analytic ones.

Consider the last terms of the BGG-resolution (see [24] or [38]), which form a exact sequence:

$$\bigoplus_{\sigma\in\Sigma_F-\{\tau_0\}} D^{\underline{k}^{\tau_0}}_0(\mathcal{O}_{\mathfrak{p}},\overline{\mathbb{Z}}_p)\otimes\mathbb{Q}_p\xrightarrow{(\Theta^\vee_\sigma)_\sigma}\bigoplus_{\sigma\in\Sigma_F-\{\tau_0\}} D^{\underline{k}^{\tau_0}}_0(\mathcal{O}_{\mathfrak{p}},\overline{\mathbb{Z}}_p)\otimes\mathbb{Q}_p$$
$$\longrightarrow \mathrm{Sym}^{\underline{k}^{\tau_0}}(\overline{\mathbb{Q}}_p^2)\longrightarrow 0,$$

here $\Theta^\vee_\sigma$ is obtained from the morphism on the locally analytic functions given by applying $\frac{\mathrm{d}^{k_\sigma+1}}{\mathrm{d}y_\sigma^{k_\sigma+1}}$, i.e. taking $k_\sigma+1$-derivatives in the variable y_σ of the second coordinate. Remark that we have $\varpi_{\mathfrak{p}}^{k_\sigma+1}\Theta^\vee_\sigma\circ U_{\mathfrak{p}}=U_{\mathfrak{p}}\circ\Theta^\vee_\sigma$, if $\sigma\in\Sigma_{\mathfrak{p}}$, and $\Theta^\vee_\sigma\circ U_{\mathfrak{p}}=U_{\mathfrak{p}}\circ\Theta^\vee_\sigma$, if not. Thus, by the hypothesis on $\underline{h}^0$ we obtain an isomorphism

$$D^{\underline{k}^{\tau_0}}_0(\mathcal{O}^{\tau_0},\overline{\mathbb{Z}}_p)\otimes\mathbb{Q}_p^{<\underline{h}^0}\simeq\mathrm{Sym}^{\underline{k}^{\tau_0}}(\overline{\mathbb{Q}}_p^2)^{<\underline{h}^0}.$$

This directly implies the first claim by the Katz modular form interpretation. This argument is used in an analogous way in [2, Section 7.2 and Section 7.3].

The second statement belongs to a general type of results generalizing the geometric approach to classicity introduced in [26]. Our result is consequence, for example, of the main results of [7]. ☐

Part C. p-adic L-functions

We construct ordinary triple product p-adic L-functions. For this purpose, we introduce the space of (classical and dual) p-adic modular forms for G'. We also introduce the q-expansion of a p-adic modular form by using Serre–Tate coordinates.

8. p-adic modular forms

8.1. Definitions

Let $\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)$ and denote by $\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}$ the *ordinary locus* of $\mathfrak{X}^{\mathfrak{c}}$, namely, the (dense) open subscheme of $\mathfrak{X}^{\mathfrak{c}}$ obtained by removing the points specializing to supersingular points. Since it can also be defined as the open formal subscheme where Hdg is invertible, then $\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}\subset\mathfrak{X}^{\mathfrak{c}}_r$ for each $r>0$. Note that $\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}$ classifies quadruples $(A,\iota,\theta,\alpha^{\mathfrak{p}_0})$ where A has ordinary reduction. For each $\mathfrak{p}\neq\mathfrak{p}_0$, the $K_0'(\mathfrak{p})$-structure endows A with a subgroup $C_{\mathfrak{p}}\subset A[\mathfrak{p}]^{-,1}$. Moreover, as A is ordinary, the p-divisible group $\mathcal{G}_0:=\mathbf{A}[\mathfrak{p}_0^\infty]^{-,1}\to\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}$, attached to the universal abelian variety over $\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}$, lies in the exact sequence:

$$0\longrightarrow\mathbb{G}_m[p^\infty]\longrightarrow\mathcal{G}_0\longrightarrow\mathbb{Q}_p/\mathbb{Z}_p\longrightarrow 0.$$

Since Hdg is invertible, we have that $\Omega_0\mid_{\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}}=\omega_0\mid_{\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}}=\omega_{\mathcal{G}_0^D}\mid_{\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}}$ and then $\Omega\mid_{\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}}=\omega\mid_{\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}}$. Thus, it does not cause any confusion to put $\omega^{\mathbf{k}_n}:=\Omega^{\mathbf{k}_n}\mid_{\mathfrak{X}^{\mathfrak{c}}_{\mathrm{ord}}\times\mathfrak{W}_n}$.

Definition 8.1. — *The space of families of* p-adic modular forms *is:*

$$M^{p-\mathrm{adic}}_{(\mathbf{k}_n^G,\nu_n)}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G):=\bigoplus_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)}\mathrm{H}^0(\mathfrak{X}^{\mathfrak{c},0}_{\mathrm{ord}}\times\mathfrak{W}_n^G,\omega^{\mathbf{k}_n})^{\Delta}\otimes\mathbb{Q}.$$

8.2. q-expansions

Let W be the Witt vectors of $\overline{\mathbb{F}}_p$ and consider $\mathfrak{X}$ over W. Let $\mathcal{X}_{\mathrm{ord}}$ be the adic generic fiber of $\mathfrak{X}_{\mathrm{ord}}$. The p-divisible group $\mathcal{G}=\mathrm{A}[p^\infty]^{-,1}$ attached to the universal abelian variety $\mathbf{A}/\mathcal{X}_{\mathrm{ord}}$ lies in an exact sequence:

$$0\longrightarrow\mathbb{G}_m[p^\infty]\xrightarrow{\iota}\mathcal{G}\longrightarrow\mathbb{Q}_p/\mathbb{Z}_p\times\prod_{\mathfrak{p}\neq\mathfrak{p}_0}(F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})^2\longrightarrow 0. \tag{8.1}$$

The image of ι is the canonical subgroup of $\mathcal{G}$. As above, we have $\omega_{\mathcal{G}^D}=\omega_0\oplus\bigoplus_{\tau\in\Sigma_{\mathfrak{p}},\mathfrak{p}\neq\mathfrak{p}_0}\omega_{\mathfrak{p},\tau}$.

Definition 8.2. — *Let $\mathcal{X}(n)$ be the adic space representing the functor classifying $\mathcal{O}_F/p^n\mathcal{O}_F$-equivariant frames over $\mathcal{X}_{\mathrm{ord}}$ i.e. pairs of morphisms $(\imath,\pi)$ such that the following sequence is exact:*

$$0\longrightarrow\mathbb{G}_m[p^n]\xrightarrow{\imath}\mathcal{G}[p^n]\xrightarrow{\pi}p^{-n}\mathbb{Z}_p/\mathbb{Z}_p\times\prod_{\mathfrak{p}\neq\mathfrak{p}_0}(p^{-n}\mathcal{O}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})^2\longrightarrow 0,$$

and $(p^{-1},0)\in(p^{-n}\mathcal{O}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})^2$ generates $C_{\mathfrak{p}}$.

The space $\mathcal{X}(n)$ is finite étale over $\mathcal{X}_{\mathrm{ord}}$. Let $\mathfrak{X}(n)\to\mathfrak{X}_{\mathrm{ord}}$ be the normalization of $\mathfrak{X}_{\mathrm{ord}}$ in $\mathcal{X}(n)$ and consider the projective limit of $\mathrm{Spf}(W)$-schemes $\mathfrak{X}(\infty)=\varprojlim_n\mathfrak{X}(n)$, which is an affine formal scheme over W by [31, Section 2.3].

If R is a W-algebra we put $\mathcal{M}(\infty,R):=H^0(\mathfrak{X}(\infty),\mathcal{O}_{\mathfrak{X}(\infty)})\,\widehat{\otimes}\,R$. For $n\in\mathbb{N}$, the module of *universal convergent modular forms* (see [31]) is

$\mathcal{M}(\infty,\Lambda_n)$. Moreover, if $\mathcal{M}$ is a Λ_n-module we put

$$D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}) := \mathrm{Hom}_{\Lambda_n}(C_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\Lambda_n),\mathcal{M}).$$

In particular, we obtain $D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,\Lambda_n))$.

The universal object $(\mathbf{A},\iota,\theta,\alpha^{\mathfrak{p}_0})$ over $\mathfrak{X}(\infty)$ gives rise to a p-divisible group, denoted again by $\mathcal{G} = \mathbf{A}[p^\infty]^{-,1}$, equipped with the universal frame (8.1). We also have a canonical basis $\mathbf{f}_0 := \mathrm{dlog}_0(1)$ and $\mathbf{f}_\tau = \mathrm{dlog}_{\mathfrak{p},\tau}(1,0)$, $\mathbf{e}_\tau = \mathrm{dlog}_{\mathfrak{p},\tau}(0,1)$ for each $\mathfrak{p}\neq\mathfrak{p}_0$ and $\tau\in\Sigma_\mathfrak{p}$. Observe that $\varphi_\tau(\mathbf{f}_\tau\wedge\mathbf{e}_\tau)=1$ and f_τ generate $\omega_{C_\mathfrak{p}}$. Finally we put $\mathbf{w}=(\mathbf{f}_0,\{(\mathbf{f}_\tau,\mathbf{e}_\tau)\}_{\tau\neq\tau_0})$.

Definition 8.3. — *The* q-expansion *of* $\mu\in H^0(\mathfrak{X}_{\mathrm{ord}}\times\mathfrak{W}_n,\omega^{\mathbf{k}_n})$ *is the distribution* $\mu(q):=\mu(\mathbf{A},\iota,\theta,\alpha,\mathbf{w})\in D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,\Lambda_n))$ *(see Section 6.5 for the notations). Repeating the construction for the curves* $\mathfrak{X}^{\mathfrak{c},0}_{\mathrm{ord}}$ *we obtain a* q-expansion morphism*:*

$$M_{\mathbf{k}_n}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)\longrightarrow D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,\Lambda_n))^{\mathrm{Pic}(\mathcal{O}_F)}. \tag{8.2}$$

Now we introduce some coordinates and related notions which will be crucial to construct p-adic L-functions. In the same way as in [31], we have a classifying morphism:

$$c:\mathfrak{X}(\infty)\longrightarrow\mathbb{G}_m.$$

This is described as follows. Firstly observe that, using the classical Serre–Tate coordinates, $\mathbb{G}_m/W$ is identified with the moduli space of the extensions of $\mathbb{G}_m[p^\infty]$ by $\mathbb{Q}_p/\mathbb{Z}_p$. For example, let R be an Artinian ring with maximal ideal m_R and $m_R^{m+1}=0$, for $m>0$, then given such an exact sequence over R,

$$0\longrightarrow G_{m,R}[p^\infty]\simeq 1+m_R\stackrel{\imath}{\longrightarrow}E\stackrel{\pi_E}{\longrightarrow}\mathbb{Q}_p/\mathbb{Z}_p\longrightarrow 0, \tag{8.3}$$

we consider $s=p^m\pi_E^{-1}\left(\frac{1}{p^m}\right)-1\in m_R=\mathbb{G}_m(R)$, and this definition does not depend on m. Reciprocally, if $s\in\mathbb{G}_m(R)=m_R$, we consider the exact sequence (8.3) with $E=E_s:=(\mathbb{G}_{m,R}[p^\infty]\oplus\mathbb{Q}_p)/\langle((1+s)^z,-z),\ z\in\mathbb{Z}_p\rangle$, $\imath(a)=(a,0)$ and $\pi_s(a,b)=b$.

Thus, let $x\in\mathfrak{X}(\infty)$ corresponding to $(A,\iota,\theta,\alpha^{\mathfrak{p}_0})$, a morphism $\pi_\mathfrak{p}:\mathcal{G}_\mathfrak{p}[p^\infty]\stackrel{\simeq}{\to}(F_\mathfrak{p}/\mathcal{O}_\mathfrak{p})^2$ for each $\mathfrak{p}\neq\mathfrak{p}_0$, and an exact sequence $S_0: 0\to\mathbb{G}_m[p^\infty]\stackrel{\imath}{\to}\mathcal{G}_0[p^\infty]\stackrel{\pi_0}{\to}\mathbb{Q}_p/\mathbb{Z}_p\to 0$ where, as above, we use the notations $A[p^\infty]^{-,1}=\mathcal{G}=\mathcal{G}_0[p^\infty]\oplus\bigoplus_{\mathfrak{p}\neq\mathfrak{p}_0}\mathcal{G}_\mathfrak{p}[p^\infty]$. The morphism c maps x to the point classifying the extension S_0.

Proposition 8.4. — *There exists a morphism $\beta : \mathfrak{X}(\infty)\times_{\mathrm{Spf}(W)}\mathbb{G}_m \to \mathfrak{X}(\infty)$ such that the following diagram is commutative:*

$$\begin{tikzcd} \mathfrak{X}(\infty)\times_{\mathrm{Spf}(W)}\mathbb{G}_m \arrow[r, "\beta"] \arrow[d, "(c\times\mathrm{id})"'] & \mathfrak{X}(\infty) \arrow[d, "c"] \\ \mathbb{G}_m\times_{\mathrm{Spf}(W)}\mathbb{G}_m \arrow[r] & \mathbb{G}_m, \end{tikzcd}$$

where the bottom arrow is the formal group law. Moreover, if $x\in\mathfrak{X}(\infty)(\overline{\mathbb{F}}_p)$ is a closed point then c induces an isomorphism $c\mid_x\colon \mathfrak{X}(\infty)_x\to\mathbb{G}_m$ over W, here $\mathfrak{X}(\infty)_x$ is the formal completion at x.

Proof. — The proof can be found essentially in [31, Lemma 2.3.4, Proposition 2.3.5]. Notice that, since $F_{\mathfrak{p}_0}=\mathbb{Q}_p$, the Lubin–Tate p-divisible group is just $\mathbb{G}_m$ in this case. □

Let R be a W-algebra. Following [31, Definition 2.3.10] $f\in\mathcal{M}(\infty,R)$ is called *stable* if $\sum_{i=0}^{p-1} f(\beta(x,\xi_p^i))=0$ for each $\xi_p\in\mathbb{G}_m[p]$, and denote $\mathcal{M}(\infty,R)^{\heartsuit}\subseteq\mathcal{M}(\infty,R)$ the module of stable functions. Let $H^0(\mathfrak{X}_{\mathrm{ord}}\times\mathfrak{W}_n,\omega^{\mathbf{k}_n})^{\heartsuit}$ be the module of $\mu\in H^0(\mathfrak{X}_{\mathrm{ord}}\times\mathfrak{W}_n,\omega^{\mathbf{k}_n})$ such that $\mu(q)\in D_n^{\mathbf{k}_n^{\tau_0}}\left(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,\Lambda_n)^{\heartsuit}\right)$.

Repeating this construction but using $\mathcal{X}^{\mathfrak{c}}_{\mathrm{ord}}$, we define $M_{\mathbf{k}_n}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)^{\heartsuit}$ to be the p-adic modular forms μ whose sections $\mu_{\mathfrak{c}}$ lie in $H^0(\mathfrak{X}^{\mathfrak{c},0}_{\mathrm{ord}}\times\mathfrak{W}_n,\omega^{\mathbf{k}_n})^{\heartsuit}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, for each $\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)$.

8.3. Action of $U_{\mathfrak{p}_0}$

The previous description of $c:\mathfrak{X}(\infty)\to\mathbb{G}_m$ coordinates allows us to compute the $U_{\mathfrak{p}_0}$-operator in coordinates and to prove the following:

Lemma 8.5. — *Let $\mu\in M_{\mathbf{k}_n}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)$. Then*

$$\mu\in M_{\mathbf{k}_n}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n},p),\Lambda_n^G)^{\heartsuit}$$

if and only if $U_{\mathfrak{p}_0}\mu=0$.

Proof. — The above description of (8.3) implies that, for a big enough extension of R and given $s\in\mathbb{G}_m(R)$, we have

$$E_s[p]=\left\{\left(\xi_p^i(1+s)^{\frac{j}{p}},-\frac{j}{p}\right)_{i,j=0,\dots,p-1}\right\},$$

where ξ_p is a primitive p-root of unity. The canonical subgroup is $\mu_p = \{(\xi_p^j, 0)_{j=0,\dots,p-1}\}$ and the subgroups that intersect it trivially are $C_i = \{(\xi_p^{ij}(1+s)^{\frac{j}{p}}, -\frac{j}{p})_{j=0,\dots,p-1}\}$, for $i = 0,\dots,p-1$.

For each i we identify $E_s/C_i \simeq ((1+m_R)\oplus\mathbb{Q}_p)/\langle(\xi_p^{in}(1+s)^{\frac{n}{p}}, -\frac{n}{p}),\ n\in\mathbb{Z}_p\rangle$, and $\pi_i : E_s/C_i \to \mathbb{Q}_p/\mathbb{Z}_p$ is given by $\pi_i(a,b) = pb \mod \mathbb{Z}_p$. Thus, E_s/C_i corresponds to $\imath^{-1}(p^m\pi_i^{-1}(\frac{1}{p^m})) - 1 = \imath^{-1}(p^m(1,p^{-m-1})) - 1 = \imath^{-1}(1,p^{-1}) - 1 = (1+s)^{\frac{1}{p}}\xi_p^i - 1$. Then for $f\in\mathcal{O}_{\mathbb{G}_m} = W[\![q]\!]$ we have:

$$\begin{aligned} U_{\mathfrak{p}_0} f(q) &= \frac{1}{q_{\mathfrak{p}_0}} \sum_{i\in\mathcal{O}_F/\mathfrak{p}_0} f(\xi_p^i(1+q)^{1/p} - 1) \\ &= \frac{1}{q_{\mathfrak{p}_0}} \sum_{i\in\mathcal{O}_F/\mathfrak{p}_0} f(\widehat{\mathbb{G}}_m((1+q)^{1/p} - 1, \xi_p^i)), \end{aligned} \tag{8.4}$$

where $\widehat{\mathbb{G}}_m$ is the formal group law of the multiplicative group. The result follows directly from the above computation together with Proposition 8.4. □

Remark 8.6. — Notice that $W[\![q]\!]$ is topologically generated by $f_\alpha := (1+q)^\alpha$, with $\alpha\in\mathbb{Z}_p$. The above computation shows that $U_{\mathfrak{p}_0} f_\alpha = 0$, if $p\nmid\alpha$, and $U_{\mathfrak{p}_0} f_\alpha = f_{\alpha/p}$, if $p\mid\alpha$.

Remark 8.7. — In the same way we can describe the action of the operator $V_{\mathfrak{p}_0}$ in coordinates. In fact, remark that the quotient by the canonical subgroup is given by $G_s/\mu_p \simeq ((1+m_R)\oplus\mathbb{Q}_p)/\langle((1+s)^{np}, -n),\ n\in\mathbb{Z}_p\rangle$. Thus, G_s/μ_p corresponds to the point $\imath^{-1}(p^m\pi^{-1}(\frac{1}{p^m})) - 1 = \imath^{-1}(1,1) - 1 = (1+s)^p - 1$. Hence, $V_{\mathfrak{p}_0}$ acts on $f\in\mathcal{O}_{\mathbb{G}_m} = W[\![q]\!]$ by $V_{\mathfrak{p}_0} f(q) = f((1+q)^p - 1)$. Thus, $V_{\mathfrak{p}_0} f_\alpha = f_{\alpha p}$ for each $\alpha\in\mathbb{Z}_p$. One obtains the classical relation $U_{\mathfrak{p}_0}\circ V_{\mathfrak{p}_0} = \mathrm{Id}$.

9. Connections and Unit root splittings

9.1. Unit root splitting

Let $\mathcal{H}^1 := \mathcal{H}^1_{\mathcal{G}^D}$ denote the contravariant Dieudonné module attached to $\mathcal{G}^D \to \mathfrak{X}_{\mathrm{ord}}$, and remark that we have a decomposition $\mathcal{H}^1 = \mathcal{H}^1_0 \oplus \bigoplus_{\tau\in\Sigma_{\mathfrak{p}},\mathfrak{p}\neq\mathfrak{p}_0} \omega_{\mathfrak{p},\tau}$. Hence, the Hodge filtration $0\to\omega\to\mathcal{H}^1\to\omega_{\mathcal{G}}^\vee\to 0$, restricts to the exact sequence

$$0 \longrightarrow \omega_0 \longrightarrow \mathcal{H}^1_0 \xrightarrow{\epsilon} \omega_{\mathcal{G}}^\vee \longrightarrow 0. \tag{9.1}$$

As explained in Section 4.3, we have fixed an isomorphism $\omega_{\mathcal{G}} = \omega_{\mathcal{G}_0} \simeq \omega_0$ which, together with the Gauss–Manin connection $\triangledown : \mathcal{H}^1_0 \to \mathcal{H}^1_0 \otimes \Omega^1_{\mathfrak{X}_{\mathrm{ord}}}$, provides the Kodaira–Spencer isomorphism

$$KS : \omega_0 \hookrightarrow \mathcal{H}^1_0 \xrightarrow{\triangledown} \mathcal{H}^1_0 \otimes \Omega^1_{\mathfrak{X}_{\mathrm{ord}}} \xrightarrow{\epsilon} \omega^\vee_{\mathcal{G}_0} \otimes \Omega^1_{\mathfrak{X}_{\mathrm{ord}}},$$
$$KS : \Omega^1_{\mathfrak{X}_{\mathrm{ord}}} \simeq \omega_0 \otimes \omega_{\mathcal{G}_0} \simeq \omega_0^{\otimes 2}.$$

Lemma 9.1 ([31, Lemma 2.3.1]). — *There exists a unique morphism $\Phi : \mathfrak{X}_{\mathrm{ord}} \to \mathfrak{X}_{\mathrm{ord}}$ lifting the Frobenius morphism $x \mapsto x^p$ on the special fiber such that $\Phi^*\mathcal{G} \simeq \mathcal{G}/\mathcal{G}^0[p]$, where $\mathcal{G}^0$ is the formal part of $\mathcal{G}$. In particular, Φ induces an endomorphism Φ^* on $\mathcal{H}^1_{\mathcal{G}}$. Moreover, there exists a unique Φ^*-stable splitting $\mathcal{H}^1_{\mathcal{G}} = \omega_0 \oplus \mathcal{L}$ where $\mathcal{L}$ is an invertible quasi-coherent formal sheaf over $\mathfrak{X}_{\mathrm{ord}}$. In addition, $\mathcal{L}$ is horizontal with respect to the Gauss–Manin connection, that is, $\triangledown\mathcal{L} \subset \mathcal{L} \otimes \Omega^1_{\mathfrak{X}_{\mathrm{ord}}}$.*

Remark 9.2. — Since $\mathcal{G}^0 \simeq \mathbb{G}_m$, we have a decomposition $\mathcal{L} \simeq \mathcal{L}_0 \oplus \bigoplus_{\tau\in\Sigma_{\mathfrak{p}},\mathfrak{p}\neq\mathfrak{p}_0} \omega_{\mathfrak{p},\tau}$ and $\mathcal{L}_0$ defines a Unit Root splitting $\mathcal{H}^1_0 = \omega_0 \oplus \mathcal{L}_0$.

9.2. The overconvergent projection

As in Section 4.6, for $\underline{k}$ and $m \in \mathbb{Z}$ we consider the sheaves $\omega^{\underline{k}} \subseteq \mathcal{H}^{\underline{k}}_m := \omega_0^{k_{\tau_0}-m} \otimes \mathrm{Sym}^m \mathcal{H}^1_0 \otimes \bigotimes_{\tau\in\Sigma_{\mathfrak{p}},\mathfrak{p}\neq\mathfrak{p}_0} \mathrm{Sym}^{k_\tau} \omega_{\mathfrak{p},\tau}$. The elements of $H^0(\mathcal{X}_r, \mathcal{H}^{\underline{k}}_m)$ are called *nearly overconvergent modular forms*. In the same way as in [39, Proposition 3.2.4] using [39, Proposition 3.1.3] we have:

Proposition 9.3. — *The following morphism, which is induced by the Unit Root Splitting, is injective:*

$$\gamma_{\underline{k},m} : H^0(\mathcal{X}_r, \mathcal{H}^{\underline{k}}_m) \longrightarrow H^0(\mathcal{X}_{\mathrm{ord}}, \mathcal{H}^{\underline{k}}_m) \longrightarrow H^0(\mathcal{X}_{\mathrm{ord}}, \omega^{\underline{k}}).$$

As in Section 4.6, we have the morphism $\epsilon : \mathcal{H}^{\underline{k}}_m \to \mathcal{H}^{\underline{k}-2\tau_0}_{m-1}$, the Gauss–Manin connection induces morphisms $\triangledown_{\underline{k},m} : \mathcal{H}^{\underline{k}}_m \to \mathcal{H}^{\underline{k}+2\tau_0}_{m+1}$ and, for $j \in \mathbb{N}$, we put $\triangledown^j_{\underline{k}} := \triangledown_{\underline{k}+2(j-1)\tau_0,j-1} \circ \cdots \circ \triangledown_{\underline{k},0}$. The following result is analogous to [39, Lemma 3.3.4]:

Lemma 9.4. — *For each $f \in H^0(\mathcal{X}_r, \mathcal{H}^{\underline{k}}_m)$, where $2m < k_{\tau_0}$, there exist $g_j \in H^0(\mathcal{X}_r, \omega^{\underline{k}-2j\tau_0})$, for $j = 0, \dots, m$, such that we can write in a unique way:*

$$f = g_0 + \triangledown^1_{\underline{k}-2\tau_0}(g_1) + \triangledown^2_{\underline{k}-4\tau_0}(g_2) + \cdots + \triangledown^m_{\underline{k}-2m\tau_0}(g_m).$$

Proof. — We prove this result by induction on m. If $m=0$ the result is clear, then we suppose that $m>0$. Remark that $g_m := c^{-1}\epsilon^m f \in H^0(\mathcal{X}_r, \omega^{\underline{k}-2m\tau_0})$, where $c = \frac{m!(k_{\tau_0}-m-1)!}{(k_{\tau_0}-2m-1)!}$. From (4.9) we deduce that $\epsilon^m \bigtriangledown^m_{\underline{k}-2m\tau_0}(g_m) = c\cdot g_m = \epsilon^m f$. Then we obtain $\epsilon^m(f-\bigtriangledown^m_{\underline{k}-2m\tau_0} g_m) = 0$. Thus, we deduce $f - \bigtriangledown^m_{\underline{k}-2m\tau_0} g_m \in H^0(\mathcal{X}_r, \mathcal{H}^{\underline{k}}_{m-1})$, and the result follows from a simple induction. □

Definition 9.5. — *If $f \in H^0(\mathcal{X}_r, \mathcal{H}^{\underline{k}}_m)$ and $2m < k_{\tau_0}$, the* overconvergent projection *is $\mathcal{H}^r(f) := g_0$.*

The following result will be proved at the end of the Section :

Lemma 9.6. — *Let $e_{\mathrm{ord}} = \varprojlim_n U_p^{n!}$ be the standard ordinary operator, let $f \in H^0(\mathcal{X}_r, \mathcal{H}^{\underline{k}}_m)$ with $2m < k_{\tau_0}$, then we have $e_{\mathrm{ord}}(\gamma_{\underline{k},m}(f)) = e_{\mathrm{ord}}(\mathcal{H}^r(f))$.*

9.3. Connections on Formal Vector Bundles

Consider the formal vector bundle $f : \mathbb{V}(\mathcal{H}^1_0) \to \mathfrak{X}_{\mathrm{ord}}$. Then by [1, Section 2.4], the Gauss Manin connection extends to a connection:

$$\bigtriangledown : f_*\mathcal{O}_{\mathbb{V}(\mathcal{H}^1_0)} \longrightarrow f_*\mathcal{O}_{\mathbb{V}(\mathcal{H}^1_0)} \widehat{\otimes}\, \Omega^1_{\mathfrak{X}_{\mathrm{ord}}}.$$

The Unit Root Splitting gives $\mathbb{V}(\mathcal{H}^1_0) \simeq \mathbb{V}(\omega_0)\times\mathbb{V}(\mathcal{L}_0)$, hence, we can define the *Serre operator*

$$\Theta : f_*\mathcal{O}_{\mathbb{V}(\omega_0)} \hookrightarrow f_*\mathcal{O}_{\mathbb{V}(\mathcal{H}^1_0)} \xrightarrow{\bigtriangledown} f_*\mathcal{O}_{\mathbb{V}(\mathcal{H}^1_0)} \widehat{\otimes}\, \Omega^1_{\mathfrak{X}_{\mathrm{ord}}} \longrightarrow f_*\mathcal{O}_{\mathbb{V}(\omega_0)} \widehat{\otimes}\, \Omega^1_{\mathfrak{X}_{\mathrm{ord}}}$$

Lemma 9.7. — *Recall s_0 from Section 6.3. Then the Serre operator can be extended to $(g_n\circ f_0)_*\mathcal{O}_{\mathbb{V}_0(\omega_0,s_0)}$, i.e. there exists a morphism (also denoted by Θ) making the following diagram commutative*

$$\begin{tikzcd} f_*\mathcal{O}_{\mathbb{V}(\omega_0)} \arrow[r, "\Theta"] \arrow[d] & f_*\mathcal{O}_{\mathbb{V}(\omega_0)} \widehat{\otimes}\, \Omega^1_{\mathfrak{X}_{\mathrm{ord}}} \arrow[d] \\ (g_n\circ f_0)_*\mathcal{O}_{\mathbb{V}_0(\omega_0,s_0)} \arrow[r, "\Theta"] & (g_n\circ f_0)_*\mathcal{O}_{\mathbb{V}_0(\omega_0,s_0)} \widehat{\otimes}\, \Omega^1_{\mathfrak{X}_{\mathrm{ord}}}. \end{tikzcd}$$

Proof. — We check this locally. Let $\mathfrak{IG}_n$ be the formal scheme over $\mathfrak{X}_{\mathrm{ord}}$ defined analogously as $\mathfrak{IG}_{r,n}$ and $\rho^* : S = \mathrm{Spf}(R) \to \mathfrak{IG}_n$ be a neighborhood such that $\mathcal{H}^1_0 \mid_S = Rf_0 \oplus Re_0$, with $f_0 \in \omega_0$ congruent to s_0 modulo p^n, and $e_0 \in \mathcal{L}_0$. Le D be a derivation dual to a generator of $\Omega^1_{\mathfrak{X}_{\mathrm{ord}}}$ and write $\bigtriangledown(D)f_0 = a_0f_0 + b_0e_0$ and $\bigtriangledown(D)e_0 = c_0f_0 + d_0e_0$.

We claim that $a_0 \in p^n R$ or, equivalently, $\bigtriangledown(D)s_0 \in \bar{\mathcal{L}}_0$. We denote by σ the Frobenius morphism acting on W and we put $(\mathcal{G}_0^D)^\sigma = \mathcal{G}_0^D \times_{W,\sigma} W \to \mathcal{G}$. Let $\Sigma : (\mathcal{G}_0^D)^\sigma \to \mathcal{G}_0^D$ be the projection, $V : \mathcal{G}_0^\sigma \to \mathcal{G}_0$ be the Verschiebung morphism. As $\mathcal{H}_0^1 = \mathcal{H}_{\mathcal{G}_0}^1$ we can define $\varphi := D(V^D) \circ D(\Sigma)$ on $\mathcal{H}_0^1$. By [31, Lemma B.3.5] the image of $\mathrm{dlog}_0 : T_p(\mathcal{G}_0) \to \omega_0 \subset \mathcal{H}_0^1$ can be identified with the elements $\xi \in \mathcal{H}_0^1$ such that $\varphi\xi = p\xi$. By [31, Lemma B.3.6], we have that $p\varphi \bigtriangledown (D)\xi = \bigtriangledown(D)\varphi\xi = p \bigtriangledown (D)\xi$. Thus, $\varphi \bigtriangledown (D)\xi = \bigtriangledown(D)\xi$ and, again by [31, Lemma B.3.5], this implies that $\bigtriangledown(D)\xi \in \mathcal{L}_0$. Choosing ξ to be any pre-image of s_0, the claim follows.

Write $\rho^*\mathcal{O}_{\mathbb{V}(\mathcal{H}_0^1)} = R\langle X, Y\rangle$ with X and Y corresponding to f_0 and e_0, respectively, and remark that the map $R\langle X\rangle = \rho^*\mathcal{O}_{\mathbb{V}(\omega_0)} \to \rho^*\mathcal{O}_{\mathbb{V}(\omega_0,s_0)} = R\langle Z\rangle$ is given by $X \mapsto 1 + p^n Z$.

Finally, we define $\Theta(D) : \rho^*\mathcal{O}_{\mathbb{V}(\omega_0,s_0)} \to \rho^*\mathcal{O}_{\mathbb{V}(\omega_0,s_0)}$ by

$$\Theta(D)Q = DQ + \frac{a_0}{p^n}(1 + p^n Z)\frac{\partial}{\partial Z}Q.$$

Then, if $P(X) \in \rho^*\mathcal{O}_{\mathbb{V}(\omega_0)} \subset \rho^*\mathcal{O}_{\mathbb{V}(\mathcal{H}_0^1)}$, we have

$$\bigtriangledown(D)P = DP + (a_0 X + b_0 Y)\frac{\partial}{\partial X}P.$$

Thus, we have

$$\Theta(D)P = DP + a_0 X\frac{\partial}{\partial X}P,$$

which implies that $\Theta(D)$ satisfies the desired property. □

The Katz interpretation of the sheaves implies that $\omega^{\mathbf{k}_n^0} = \omega_0^{\mathbf{k}_{n,\tau_0}^0} \otimes \omega^{\mathbf{k}_n^{0,\tau_0}}$ where:

$$\omega^{\mathbf{k}_n^{0,\tau_0}} := \Big((g_n \circ f_0)_*\mathcal{O}_{\hat{\mathbb{V}}(\omega,\mathbf{s}^0)} \mathbin{\widehat{\otimes}} \Lambda_n\Big)[\mathbf{k}_n^{0,\tau_0}]^\vee,$$
$$\omega_0^{\mathbf{k}_{n,\tau_0}^0} := \big((g_n \circ f_0)_*\mathcal{O}_{\mathbb{V}_0(\omega_0,s_0)} \mathbin{\widehat{\otimes}} \Lambda_n\big)[\mathbf{k}_{n,\tau_0}^0].$$

Hence, the connection Θ of Lemma 9.7 gives rise to a connection $\Theta : \omega^{\mathbf{k}_n^0} \to \omega^{\mathbf{k}_n^0} \otimes \Omega^1_{\mathfrak{X}_{\mathrm{ord}}}$. On the other hand, recall $\omega^{\mathbf{k}_f} = (g_{1,*}(\mathcal{O}_{\mathfrak{IG}_1}) \mathbin{\widehat{\otimes}} \Lambda_n)[\mathbf{k}_f^{-1}]$ and remark that the covering $\mathfrak{IG}_1 \to \mathfrak{X}_{\mathrm{ord}}$ is étale, hence, the derivation on $\mathcal{O}_{\mathfrak{IG}_1}$ induces a connection $\bigtriangledown : \omega^{\mathbf{k}_f} \to \omega^{\mathbf{k}_f} \otimes \Omega^1_{\mathfrak{X}_{\mathrm{ord}}}$. Let $\mathbf{k}_n + 2\tau_0 : \mathcal{O}^\times \to \Lambda_n \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}$ be the character $(\mathbf{k}_n + 2\tau_0)(x) = (\tau_0 x)^2 \cdot \mathbf{k}_n(x)$. Then these connections induce the *Serre operator*:

$$\Theta : \omega^{\mathbf{k_n}} \longrightarrow \omega^{\mathbf{k_n}} \otimes \Omega^1_{\mathfrak{X}_{\mathrm{ord}}} \overset{KS}{\simeq} \omega^{\mathbf{k_n}} \otimes \omega_0^{\otimes 2} = \omega^{\mathbf{k}_n + 2\tau_0}.$$

Remark 9.8. — Let $U = \mathrm{Spf}(R)$ trivializing ω, and let $w = (f_0, (f_\tau, e_\tau)_{\tau\neq\tau_0})$ be a basis such that $\bigtriangledown(f_\tau) = \bigtriangledown(e_\tau) = 0$ and f_τ generate $\omega_{C_\mathfrak{p}}$. If $\mathbf{A}/R$ is the corresponding universal abelian variety, then a modular form is given by a distribution $\mu(\mathbf{A},\iota,\theta,\alpha^{\mathfrak{p}_0},w) \in D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},R)$. Thus, for each derivation D and $\phi \in C_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\Lambda_n)$, we have

$$\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}} \phi\,\mathrm{d}(\Theta(D)\mu)(\mathbf{A},\iota,\theta,\alpha^{\mathfrak{p}_0},w) = \Theta(D)\left(\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}} \phi\,\mathrm{d}\mu(\mathbf{A},\iota,\theta,\alpha^{\mathfrak{p}_0},w)\right).$$

Theorem 9.9. — *If $\mu \in H^0(\mathfrak{X}_{\mathrm{ord}}\times\mathfrak{W}_n,\omega^{\mathbf{k}_n})^{\heartsuit}$, there exists $M(\mu)(q) \in D_n^{\mathbf{k}_n^{\tau_0}}\left(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,\Lambda_n\,\widehat{\otimes}_W\,\Lambda_{\tau_0})\right)$ such that, if $k:\Lambda_{\tau_0}\to W$ is a classical weight, then $\Theta^k\mu \in H^0(\mathfrak{X}_{\mathrm{ord}}\times\mathfrak{W}_n,\omega^{\mathbf{k}_n+2k\tau_0})$ and:*

$$(\mathrm{id}\otimes k)(M(\mu)(q)) = (\Theta^k\mu)(q) \in D_n^{\mathbf{k}_n^{\tau_0}}\left(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,\Lambda_n)\right).$$

Proof. — Identifying $H^0(\mathbb{G}_m,\mathcal{O}_{\mathbb{G}_m}) \simeq W[\![q]\!] \simeq W[\![\mathbb{Z}_p]\!]$ the map β of Proposition 8.4 induces $\beta^*:\mathcal{M}(\infty,\Lambda_n)\to\mathcal{M}(\infty,\Lambda_n)\,\widehat{\otimes}_W\,W[\![\mathbb{Z}_p]\!]$. By [31, Lemma 2.1.6 and Remark 2.3.11], it restricts to $\beta^*:\mathcal{M}(\infty,\Lambda_n)^{\heartsuit}\to\mathcal{M}(\infty,\Lambda_n)^{\heartsuit}\,\widehat{\otimes}_W\,W[\![\mathbb{Z}_p^{\times}]\!] = \mathcal{M}(\infty,\Lambda_n\,\widehat{\otimes}_{\mathbb{Z}_p}\,\Lambda_{\tau_0})^{\heartsuit}$. We define $M(\mu)(q)$ by:

$$\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}} f(a,b)\,\mathrm{d}M(\mu)(q) := \beta^*\left(\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}} f(a,b)\,\mathrm{d}\mu(q)\right) \in \mathcal{M}(\infty,\Lambda_n\,\widehat{\otimes}_{\mathbb{Z}_p}\,\Lambda_{\tau_0})^{\heartsuit},$$

for each $f \in C_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\Lambda_n)$. To check the desired properties it is enough to consider points $x \in \mathfrak{X}(\infty)(\overline{\mathbb{F}}_p)$. By Proposition 8.4, the formal completion induces $\mathrm{res}_x:\mathcal{M}(\infty,\Lambda_n)\to\mathcal{M}(\infty,\Lambda_n)_x \simeq H^0(\mathbb{G}_m,\mathcal{O}_{\mathbb{G}_m})\,\widehat{\otimes}_W\,\Lambda_n \simeq W[\![q]\!]\,\widehat{\otimes}_W\,\Lambda_n$. We will verify that for each $f \in C_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0},\Lambda_n)$

$$\mathrm{res}_x\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}} f(a,b)\,\mathrm{d}\Theta\mu(q) = (q+1)\frac{\mathrm{d}}{\mathrm{d}q}\left(\mathrm{res}_x\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}} f(a,b)\,\mathrm{d}\mu(q)\right). \tag{9.2}$$

If we put $\mathbf{f_0} = \mathrm{dlog}_0(1)$ and $\eta_0 := \triangledown\left((q+1)\frac{\mathrm{d}}{\mathrm{d}q}\right)\mathbf{f}_0$ then we have

$$\triangledown\left((q+1)\frac{\mathrm{d}}{\mathrm{d}q}\right)\eta_0 = 0.$$

In fact, recall $\varphi = D(V^D)\circ D(\Sigma)$ acting on $\mathcal{H}_0^1$ as in the proof of Lemma 9.7. By [31, Lemmas B.3.5 and B.3.6], we have $\varphi\mathbf{f}_0 = p\mathbf{f}_0$ and $\triangledown((q+1)\frac{\mathrm{d}}{\mathrm{d}q})\varphi\xi = p\varphi\,\triangledown\,((q+1)\frac{\mathrm{d}}{\mathrm{d}q})\xi$, for all $\xi \in \mathcal{H}_0^1$. Hence, we deduce $\varphi\eta_0 = \eta_0$ and $p\varphi\,\triangledown\,((q+1)\frac{\mathrm{d}}{\mathrm{d}q})\eta_0 = \triangledown((q+1)\frac{\mathrm{d}}{\mathrm{d}q})\eta_0$. By [31, Lemma B.3.5], there exists basis $\{\alpha,\beta\}$ of $\mathcal{H}_0^1$ satisfying $\varphi\alpha = p\alpha$ and $\varphi\beta = \beta$. If we write $\triangledown((q+1)\frac{\mathrm{d}}{\mathrm{d}q})\eta_0 =$

$f_1\alpha + f_2\beta$ we have $f_1 = p^{2n}\varphi^n f_1$ and $f_2 = p^n\varphi^n f_2$, for all $n \in \mathbb{N}$, and, thus, $f_1 = f_2 = 0$ as required.

By [31, Theorem B.2.3], we have that $\mathbf{f}_0^2 = KS(\frac{\mathrm{d}q}{q+1})$. Thus, $\bigtriangledown\mathbf{f}_0 = \eta_0\mathbf{f}_0^2$ and $\bigtriangledown\eta_0 = 0$. Then $\Theta(f(q)\mathbf{f}_0) \in \omega_0$ is given by the projection of $(q+1)\frac{\mathrm{d}}{\mathrm{d}q}f(q)\mathbf{f}_0^3 + f(q)\eta_0\mathbf{f}_0^2 \in \mathcal{H}_0^1 = \omega_0 \oplus \mathcal{L}_0$ which is equal to $(q+1)\frac{\mathrm{d}}{\mathrm{d}q}f(q)\mathbf{f}_0^3$. This implies that, under the isomorphism

$$H^0(\mathfrak{X}_{\mathrm{ord}} \times \mathfrak{W}_n, \omega^{\mathbf{k}_n})_x \xrightarrow{\simeq} D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0}, \Lambda_n)\,\widehat{\otimes}_W\, W[\![q]\!];$$
$$\mu \longmapsto \mu(x, (\mathbf{f}_0, (\mathbf{f}_\tau, \mathbf{e}_\tau)_\tau)),$$

the Serre operator acts on $\mu \otimes f \in D_n^{\mathbf{k}_n^{\tau_0}}(\mathcal{O}^{\tau_0}, \Lambda_n)\,\widehat{\otimes}_W\, W[\![q]\!]$ by $\Theta(\mu \otimes f) = \mu \otimes (q+1)\frac{\mathrm{d}}{\mathrm{d}q}f$. Hence, the claim (9.2) follows.

Note that $\mathbb{Z}_p[\![q]\!] \simeq \mathbb{Z}_p[\![\mathbb{Z}_p]\!]$ is topologically generated by $f_\alpha(q) = (1+q)^\alpha$, with $\alpha \in \mathbb{Z}_p$, and $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ is topologically generated by f_α with $\alpha \in \mathbb{Z}_p^\times$. Hence, if we write $\Theta : \mathbb{G}_m = \mathbb{Z}_p[\![q]\!] \to \mathbb{G}_m = \mathbb{Z}_p[\![q]\!]$, $\Theta f = (q+1)\frac{\mathrm{d}}{\mathrm{d}q}f$, then for $\alpha \in \mathbb{Z}_p^\times$ we have $\Theta^k f_\alpha(q) = \alpha^k f_\alpha(q) = k^*((1+T)^\alpha)f_\alpha(q) = k^*(f_\alpha(\widehat{\mathbb{G}}_m(T,q)))$. This implies that $\mathrm{res}_x(\mathrm{id}\otimes k)(M(\mu)(q)) = \mathrm{res}_x(\Theta^k\mu)(q)$, by Proposition 8.4. Hence, the result follows. $\square$

Proof of Lemma 9.6. — By (9.2), the Serre operator Θ acts q-expansions as $\Theta f = (q+1)\frac{\mathrm{d}}{\mathrm{d}q}f$. By Remark 8.6, this implies that $U_{\mathfrak{p}_0}\Theta f = p\Theta U_{\mathfrak{p}_0} f$ (compare with [13, Lemma 2.7]). The result follows from Lemma 9.4, since $\gamma_{\underline{k},m}(\bigtriangledown_{\underline{k}}^1 g) = \Theta g$. $\square$

Remark 9.10. — The results of Sections 9.1, 9.2 and 9.3 are also valid using $\mathcal{X}_{\mathrm{ord}}^{\mathfrak{c}}$ instead of $\mathcal{X}_{\mathrm{ord}}$.

10. Triple product p-adic L-functions

In this section we put together the constructions performed in the text in order to produce triple product p-adic L-functions. Firstly, using the results from Section 9 we perform a p-adic interpolation of the trilinear products introduced in Section 3.2, which leads to triple product p-adic L-functions. These products are related to L-values via Proposition 3.10, which imply the interpolation property satisfied for these p-adic L-functions.

10.1. U_p action on distributions

Let R be a Λ_n-algebra and $k^{\tau_0} : \mathcal{O}^{\tau_0} \to R^\times$ be a character. Observe that $U_{\mathfrak{p}_0}$ acts naturally on $\mathcal{M}(\infty, R)$ through the moduli interpretation of the unitary Shimura curves. For $\mathfrak{p}\neq\mathfrak{p}_0$, the Hecke operator $U_{\mathfrak{p}}$ acts on $\mu\in D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R))$ by $U_{\mathfrak{p}}\mu=\sum_{i\in\kappa_{\mathfrak{p}}}\gamma_i((\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix})*\mu)$, where $(\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix})*\mu$ is the distribution given by $\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}}\phi\,\mathrm{d}((\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix})*\mu)=\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}}((\begin{smallmatrix}1 & -i\\ 0 & \varpi_{\mathfrak{p}}\end{smallmatrix})*\phi)\,\mathrm{d}\mu$ and $\gamma_i:\mathcal{M}(\infty,R)\to\mathcal{M}(\infty,R)$ corresponds to the (bijective) map that sends $(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)$ to $(A_i,\iota_i,\varpi_{\mathfrak{p}}\theta_i,\alpha_i^{\mathfrak{p}_0},w(\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix}))$ (see Section 7.3.2 for this notation). Let

$$U_p\coloneqq\prod_{\mathfrak{p}}U_{\mathfrak{p}}:D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R))\longrightarrow D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R)).$$

Let $\overline{D}_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},R)$ be the k^{τ_0}-homogeneous distributions of functions on $p\mathcal{O}^{\tau_0}\times\mathcal{O}^{\tau_0\times}$. If $\mathfrak{p}\neq\mathfrak{p}_0$, let $U_{\mathfrak{p}}$ act on $\varphi\in\mathrm{Hom}_R\left(\overline{D}_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},R),\mathcal{M}(\infty,R)\right)$ by $U_{\mathfrak{p}}\varphi(\mu)=\varphi(U_{\mathfrak{p}}^*\mu)$ where $U_{\mathfrak{p}}^*\mu=\sum_{i\in\kappa_{\mathfrak{p}}}\gamma_i(\left(\begin{smallmatrix}1 & -i\\ 0 & \varpi_{\mathfrak{p}}\end{smallmatrix}\right)*\mu)$ and

$$\int_{p\mathcal{O}^{\tau_0}\times\mathcal{O}^{\tau_0\times}}\phi\,\mathrm{d}\left(\begin{pmatrix}1 & -i\\ 0 & \varpi_{\mathfrak{p}}\end{pmatrix}*\mu\right)=\int_{p\mathcal{O}^{\tau_0}\times\mathcal{O}^{\tau_0\times}}\left(\begin{pmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix}*\phi\right)\mathrm{d}\mu.$$

Then, as before, we put:

$$U_p\coloneqq\prod_{\mathfrak{p}}U_{\mathfrak{p}}:\mathrm{Hom}_R(\overline{D}_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},R),\mathcal{M}(\infty,R))\longrightarrow\mathrm{Hom}_R(\overline{D}_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},R),\mathcal{M}(\infty,R)).$$

One checks that we have a well defined morphism of $R[U_p]$-modules:

$$D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R))\longrightarrow\mathrm{Hom}_R\left(\overline{D}_n^{k^{\tau_0}}\left(\mathcal{O}^{\tau_0},R\right),\mathcal{M}(\infty,R)\right)\tag{10.1}$$

which sends $\mu_1\in D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R))$ to the morphism:

$$\mu_2\longmapsto\int_{\mathcal{O}^{\tau_0\times}\times\mathcal{O}^{\tau_0}}\int_{p\mathcal{O}^{\tau_0}\times\mathcal{O}^{\tau_0\times}}k^{\tau_0}(xY-Xy)\,\mathrm{d}\mu_1(x,y)\,\mathrm{d}\mu_2(X,Y).$$

Definition 10.1. — *Considering the standard ordinary operator* $e_{\mathrm{ord}}\coloneqq\lim_n U_p^{n!}$ *we define:*

$$D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R))^{\mathrm{ord}}\coloneqq e_{\mathrm{ord}}\left(D_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R))\right),$$

$$\mathrm{Hom}_R\left(\overline{D}_n^{k^{\tau_0}}\left(\mathcal{O}^{\tau_0},R\right),\mathcal{M}(\infty,R)\right)^{\mathrm{ord}}\coloneqq e_{\mathrm{ord}}\left(\mathrm{Hom}_R\left(\overline{D}_n^{k^{\tau_0}}\left(\mathcal{O}^{\tau_0},R\right),\mathcal{M}(\infty,R)\right)\right),$$

the spaces where U_p is invertible.

Lemma 10.2. — *The morphism* (10.1) *induces an isomorphism of* $R[U_p]$-*modules:*

$$D_n^{\mathbf{k}^{\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{M}(\infty,R))^{\mathrm{ord}} \longrightarrow \mathrm{Hom}_R\left(\overline{D}_n^{\mathbf{k}^{\tau_0}}(\mathcal{O}^{\tau_0},R),\mathcal{M}(\infty,R)\right)^{\mathrm{ord}}.$$

Proof. — Since the morphism is continuous, it is enough to prove that the specialization of any classical weight $\underline{k}\in\mathbb{N}[\Sigma_F]$ is an isomorphism, since such classical weights form a dense set. Write $A=\mathcal{M}(\infty,R_{\underline{k}})$. The same arguments in the proof of Proposition 7.7 show that the ordinary parts of $D_{\overline{n}}^{\underline{k}^{\tau_0}}(\mathcal{O}^{\tau_0},A)$ and $\mathrm{Sym}^{\underline{k}^{\tau_0}}(A^2)^\vee$ are naturally isomorphic. Similarly, one can proof that the ordinary parts of $\mathrm{Hom}_{R_{\underline{k}}}(\overline{D}_{\overline{n}}^{\underline{k}^{\tau_0}}(\mathcal{O}^{\tau_0},R_{\underline{k}}),A)$ and $\mathrm{Sym}^{\underline{k}^{\tau_0}}(A^2)$ also agree. Thus, taking specialization at $\underline{k}$, we obtain the natural map between $\mathrm{Sym}^{\underline{k}^{\tau_0}}(A^2)^\vee$ and $\mathrm{Sym}^{\underline{k}^{\tau_0}}(A^2)$ which is an isomorphism. □

10.2. p-adic families of trilinear products

We fix integers $r\geqslant n_3\geqslant n_1,n_2\geqslant 1$. For $i=1,2,3$, we denote by $(\mathbf{r}_{n_i},\boldsymbol{\nu}_{n_i}):\mathcal{O}^\times\times\mathbb{Z}_p^\times\to\Lambda_{n_i}^G$ the universal characters of $\mathfrak{W}_{n_i}^G$ and $k:\mathfrak{W}_{n_i}^G\to\mathfrak{W}_{n_i}$ as introduced in (7.1). Then we put $\mathbf{k}_{n_3}^G:=k(\mathbf{r}_{n_3},\nu_{n_1}+\nu_{n_2})$ and $\mathbf{k}_{n_i}^G:=k(\mathbf{r}_{n_i},\boldsymbol{\nu}_{n_i})$ for $i=1,2$.

We put $\mathcal{R}:=\Lambda_{n_1}^G\widehat{\otimes}\Lambda_{n_2}^G\widehat{\otimes}_{\boldsymbol{\nu}_{n_3}=\nu_{n_1}+\boldsymbol{\nu}_{n_2}}\Lambda_{n_3}^G\simeq\Lambda_{n_1}^G\widehat{\otimes}\Lambda_{n_2}^G\widehat{\otimes}\Lambda_{n_3}$ and consider the characters:

$$\begin{aligned}\mathbf{m}_1^{\tau_0}&:=\mathbf{r}_{n_1}^{\tau_0}-\mathbf{r}_{n_3}^{\tau_0}-\mathbf{r}_{n_2}^{\tau_0}+\boldsymbol{\nu}_{n_2}\circ N:\mathcal{O}^{\tau_0\times}\longrightarrow\mathcal{R},\\ \mathbf{m}_2^{\tau_0}&:=\mathbf{r}_{n_2}^{\tau_0}-\mathbf{r}_{n_1}^{\tau_0}-\mathbf{r}_{n_3}^{\tau_0}+\boldsymbol{\nu}_{n_1}\circ N:\mathcal{O}^{\tau_0\times}\longrightarrow\mathcal{R},\\ \mathbf{m}_3^{\tau_0}&:=\mathbf{r}_{n_3}^{\tau_0}-\mathbf{r}_{n_1}^{\tau_0}-\mathbf{r}_{n_2}^{\tau_0}:\mathcal{O}^{\tau_0\times}\longrightarrow\mathcal{R},\\ \mathbf{m}_{3,\tau_0}&:=\mathbf{r}_{n_1,\tau_0}+\mathbf{r}_{n_2,\tau_0}-\mathbf{r}_{n_3,\tau_0}:\mathbb{Z}_p^\times\longrightarrow\mathcal{R},\end{aligned}$$

where $N:\mathcal{O}^{\tau_0\times}\to\mathbb{Z}_p^\times$ denotes the norm map. In the same way as in (3.8) we denote $\underline{\Delta}^{\tau_0}\in C_{n_1}^{\mathbf{k}_{n_1}^{G,\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{R})\otimes_{\mathcal{R}}C_{n_2}^{\mathbf{k}_{n_2}^{G,\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{R})\otimes_{\mathcal{R}}\overline{C}_{n_3}^{\mathbf{k}_{n_3}^{G,\tau_0}}(\mathcal{O}^{\tau_0},\mathcal{R})$ the function defined by:

$$\begin{aligned}&\underline{\Delta}^{\tau_0}((x_1,y_1),(x_2,y_2),(x_3,y_3))\\ &\qquad:=\mathbf{m}_1^{\tau_0}(x_3y_2-x_2y_3)\cdot\mathbf{m}_2^{\tau_0}(x_3y_1-x_1y_3)\cdot\mathbf{m}_3^{\tau_0}(x_1y_2-x_2y_1),\end{aligned}$$

where $\overline{C}_n^{k^{\tau_0}}(\mathcal{O}^{\tau_0},\cdot)$ denote the k^{τ_0}-homogeneous locally analytic functions on $p\mathcal{O}^{\tau_0}\times\mathcal{O}^{\tau_0\times}$, and the function is extended by 0 where $\mathbf{m}_3^{\tau_0}$ is not defined.

Let $\mu_1\in M^r_{\mathbf{k}_{n_1}^G}(\Gamma_1(\mathfrak{n},p),\Lambda_{n_1}^G)$ and $\mu_2\in M^r_{\mathbf{k}_{n_2}^G}(\Gamma_1(\mathfrak{n},p),\Lambda_{n_2}^G)$. For $i=1,2$ and $\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)$, let $\mu_{i,\mathfrak{c}}\in M^r_{\mathbf{k}_n^G}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n},p),\Lambda_{n_i}^G)^\Delta$ be the component (see Definition 7.2), and let $\mu_{i,\mathfrak{c}}(q) \in D_{n_i}^{\mathbf{k}_{n_i}^{G,\tau_0}}\left(\mathcal{O}^{\tau_0}, \mathcal{M}(\infty, \Lambda_{n_i}^G)\right)$ be its q-expansions (see Section 8.2). Using (8.4) and $U_{\mathfrak{p}_0} \circ V_{\mathfrak{p}_0} = \mathrm{Id}$, one checks that

$$\mu_{1,\mathfrak{c}}^{[p]}(q) := (1 - V_{\mathfrak{p}_0}U_{\mathfrak{p}_0})\, \mu_{1,\mathfrak{c}}(q) \in D_{n_1}^{\mathbf{k}_{n_1}^{G,\tau_0}}\left(\mathcal{O}^{\tau_0}, \mathcal{M}(\infty, \Lambda_{n_1}^G)^{\heartsuit}\right).$$

By Theorem 9.9 and Remark 9.10, we have

$$M(\mu_{1,\mathfrak{c}}^{[p]})(q) \in D_{n_1}^{\mathbf{k}_{n_1}^{G,\tau_0}}(\mathcal{O}^{\tau_0}, \mathcal{M}(\infty, \Lambda_{n_1}^G \otimes \Lambda_{\tau_0})^{\heartsuit})$$

and we put $\Theta^{\mathbf{m}_{3,\tau_0}}\mu_{1,\mathfrak{c}}^{[p]} := (\mathbf{m}_{3,\tau_0})^* M(\mu_{1,\mathfrak{c}}^{[p]}) \in D_{n_1}^{\mathbf{k}_{n_1}^{G,\tau_0}}(\mathcal{O}^{\tau_0}, \mathcal{M}(\infty, \mathcal{R}))$.

The *trilinear product* $t(\mu_{1,\mathfrak{c}}, \mu_{2,\mathfrak{c}}) \in \mathrm{Hom}_{\mathcal{R}}(\overline{D}_{n_3}^{\mathbf{k}_{n_3}^{G,\tau_0}}(\mathcal{O}^{\tau_0}, \mathcal{R}), \mathcal{M}(\infty, \mathcal{R}))$ is defined by:

$$\begin{aligned} &t(\mu_{1,\mathfrak{c}}, \mu_{2,\mathfrak{c}})(\mu) \\ &\quad := \int_{\mathfrak{L}_1^{\tau_0}} \int_{\mathfrak{L}_1^{\tau_0}} \int_{\mathfrak{L}_2^{\tau_0}} \underline{\Delta}^{\tau_0}(v_1, v_2, v_3)\, \mathrm{d}\left(\Theta^{\mathbf{m}_{3,\tau_0}}\mu_{1,\mathfrak{c}}^{[p]}\right)(v_1)\, \mathrm{d}\mu_{2,\mathfrak{c}}(v_2)\, \mathrm{d}\mu(v_3), \end{aligned}$$

where $\mathfrak{L}_1^{\tau_0} := \mathcal{O}^{\tau_0\times} \times \mathcal{O}^{\tau_0}$ and $\mathfrak{L}_2^{\tau_0} := p\mathcal{O}^{\tau_0} \times \mathcal{O}^{\tau_0\times}$. From Lemma 10.2 we obtain:

$$\begin{aligned} &(e_{\mathrm{ord}} t(\mu_{1,\mathfrak{c}}, \mu_{2,\mathfrak{c}}))_{\mathfrak{c}} \in M_{\mathbf{k}_{n_3}^G}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n}, p), \mathcal{R})^{\mathrm{ord}} \\ &\qquad\qquad := \bigoplus_{\mathfrak{c} \in \mathrm{Pic}(\mathcal{O}_F)} \left(D_{n_3}^{\mathbf{k}_{n_3}^{G,\tau_0}}(\mathcal{O}^{\tau_0}, \mathcal{M}(\infty, \mathcal{R})) \otimes \mathbb{Q}_p\right)^{\mathrm{ord}}. \end{aligned}$$

Note that the space $M_{\mathbf{k}_n^G}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n}, p), \mathcal{R})^{\mathrm{ord}}$ is endowed with the action of Hecke operators.

10.3. Construction

Let $\mu_1 \in M_{\mathbf{k}_{n_1}^G}^r(\Gamma_1(\mathfrak{n}, p), \Lambda_{n_1}^G)$ and $\mu_2 \in M_{\mathbf{k}_{n_2}^G}^r(\Gamma_1(\mathfrak{n}, p), \Lambda_{n_2}^G)$ be as before, and let us we take $\mu_3 \in M_{\mathbf{k}_{n_3}^G}^r(\Gamma_1(\mathfrak{n}, p), \Lambda_{n_3}^G)$ such that $U_{\mathfrak{p}}\mu_3 = \alpha_3^{\mathfrak{p}}\mu_3$, for some $\alpha_3^{\mathfrak{p}} \in (\Lambda_{n_3}^G)^\times$ and all $\mathfrak{p} \mid p$. Assume that there exists $\bar{\mu}_3 \in M_{\mathbf{k}_{n_3}^G}^r(\Gamma_1(\mathfrak{n}_0, p), \Lambda_{n_3}^G)$, for some $\mathfrak{n}_0 \mid \mathfrak{n}$, such that μ_3 is an element of

$$\begin{aligned} &M_{\mathbf{k}_{n_3}^G}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n}, p), \Lambda_{n_3}^G)^{\mathrm{ord}}[\bar{\mu}_3] \\ &\quad := \left\{\mu \in M_{\mathbf{k}_{n_3}^G}^{p-\mathrm{adic}}(\Gamma_1(\mathfrak{n}, p), \Lambda_{n_3}^G)^{\mathrm{ord}} :\ U_{\mathfrak{p}}\mu = \alpha_3^{\mathfrak{p}}\mu;\ T_\ell\mu = a_\ell\mu,\ \ell \nmid \mathfrak{n}\right\}, \end{aligned}$$

where ℓ are prime ideals of F, $T_\ell = T_g$ for any $g \in G(\mathbb{A}_f^p)$ of norm ℓ, and a_ℓ is the eigenvalue of $\bar{\mu}_3$. Let $\mathcal{R}' = \Lambda_{n_1}^G \mathbin{\widehat{\otimes}} \Lambda_{n_2}^G \mathbin{\widehat{\otimes}} \Lambda'_{n_3}$, where Λ'_{n_3} is the fraction field of Λ_{n_3}, thus, $\mathcal{R}'$ can be viewed as rational functions on $\mathfrak{W}_{n_1}^G \times \mathfrak{W}_{n_2}^G \times \mathfrak{W}_{n_3}$ with poles at finitely many weights in $\mathfrak{W}_{n_3}$.

In the rest we use the following notation: If $(x,y,z)\in\mathfrak{W}^G_{n_1}\times\mathfrak{W}^G_{n_2}\times\mathfrak{W}_{n_3}$, we denote by μ_x, μ_y, μ_z and $\bar\mu_z$ the specializations of the families μ_1 at x, μ_2 at y, and μ_3 and $\bar\mu_3$ at z. By Proposition 7.7, if $z\in\mathfrak{W}_{n_3}$ is a classical weight then the specialization μ_z is a classical modular form in $M_{k_z}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)^{\mathrm{ord}}$, where k_z is the specialization of $\mathbf{k}^G_{n_3}$ at z. Let us denote by $\bar\mu_z^*\in M_{k_z}(\Gamma_1(\mathfrak{n}_0,p),\overline{\mathbb{Q}}_p)$ the eigenvector for the adjoint of the $U_\mathfrak{p}$ operators associated with $\bar\mu_z$ as in Section 3.3. The following result is analogous to [13, Lemma 2.19].

Lemma 10.3. — *There exists $\mathcal{L}_p(\mu_1,\mu_2,\mu_3)\in\mathcal{R}'$ such that, for each classical point $(x,y,z)\in\mathfrak{W}^G_{n_1}\times\mathfrak{W}^G_{n_2}\times\mathfrak{W}_{n_3}$, we have:*

$$\mathcal{L}_p(\mu_1,\mu_2,\mu_3)(x,y,z)=\frac{\left\langle\mu_z^*,(e_{\mathrm{ord}}t(\mu_{1,\mathfrak{c}},\mu_{2,\mathfrak{c}}))_{\mathfrak{c}(x,y,z)}\right\rangle}{\langle\mu_z^*,\mu_z\rangle}$$

where $\langle\cdot,\cdot\rangle$ is the Petersson inner product defined in Section 3.4, and $\mu_z^*\in M_{k_z}(\Gamma_1(\mathfrak{n},p),\overline{\mathbb{Q}}_p)[\bar\mu_z^*]$ defines the dual basis of μ_z.

Proof. — Notice that $M^{p-\mathrm{adic}}_{\mathbf{k}^G_{n_3}}(\Gamma_1(\mathfrak{n},p),\Lambda'_{n_3})^{\mathrm{ord}}[\bar\mu_3]$ is a finite dimensional Λ'_{n_3}-module generated by the oldforms $\bar\mu_3^d$, for any $d\mid\mathcal{D}$ with $\mathfrak{n}=\mathfrak{n}_0\mathcal{D}$, where

$$\begin{aligned}(\bar\mu_3)^d_\mathfrak{c}&(A,\iota,\theta,\alpha^{\mathfrak{p}_0},w)\\&=\mathbf{r}'_n(\det(\gamma_d))\cdot(\bar\mu_3)_{\mathfrak{c}'}(A^{g_d},\iota^{g_d},\det(\gamma_d)^{-1}\theta^{g_d},k^{-1}(\alpha^{g_d})^{\mathfrak{p}_0},w),\end{aligned}$$

as in Equation (11.1) in Section 11.4 of the Appendix. The family $\bar\mu_3$ corresponds to an idempotent of the Hecke algebra, which induces a projection of $M^{p-\mathrm{adic}}_{\mathbf{k}^G_{n_3}}(\Gamma_1(\mathfrak{n},p),\mathcal{R})^{\mathrm{ord}}$ to $M^{p-\mathrm{adic}}_{\mathbf{k}^G_{n_3}}(\Gamma_1(\mathfrak{n},p),\mathcal{R}')^{\mathrm{ord}}[\bar\mu_3]$. The projection of $e_{\mathrm{ord}}t(\mu_{1,\mathfrak{c}},\mu_{2,\mathfrak{c}})$ to the line defined by μ_3 is a $\mathcal{R}'$-linear combination of the forms $\bar\mu_3^d$. It is therefore enough to show that, for all divisors d_1,d_2, we have $\langle\bar\mu_z^*,\bar\mu_z^{d_2}\rangle=\varrho(z)\cdot\langle\bar\mu_z^*,\bar\mu_z\rangle$, for some $\varrho\in\mathcal{R}\otimes\mathbb{Q}$. In fact, by Lemma 3.5, we have that

$$\varrho=\varrho\left((-\boldsymbol{\nu}_{n_1}-\nu_{n_2})(q_d)_x,\mathbf{a}_x\right)_{d|d_1d_2}\qquad\varrho(X_d,Y_d)_{d|d_1d_2}\in\overline{\mathbb{Q}}[X_d,Y_d]_{d|d_1d_2},$$

where $q_d:=N_{K/\mathbb{Q}}(d)$ and $\mathbf{a}\in\mathcal{R}$ is the eigenvalue for T_d. The result follows as q_d is prime to p. □

Definition 10.4. — *Let μ_1,μ_2,μ_3, where $\mu_i\in M^r_{\mathbf{k}^G_{n_i}}(\Gamma_1(\mathfrak{n},p),\Lambda^G_{n_i})$, be test vectors for three families of eigenvectors such that $U_\mathfrak{p}\mu_3=\alpha_3^\mathfrak{p}\mu_3$, for some $\alpha_3^\mathfrak{p}\in(\Lambda^G_{n_3})^\times$ and all $\mathfrak{p}\mid p$. The function $\mathcal{L}_p(\mu_1,\mu_2,\mu_3)\in\mathcal{R}'$ introduced in Lemma 10.3 is called the* triple product p-adic L-function *of μ_1,μ_2,μ_3.*

10.4. Interpolation property

Let $(\underline{r}_1,\nu_1)\in\mathfrak{W}^G_{n_1}$, $(\underline{r}_2,\nu_2)\in\mathfrak{W}^G_{n_2}$ and $\underline{r}\in\mathfrak{W}_{n_3}$ be classical weights and put $\underline{k}_1=k(\underline{r}_1,\nu_1),\underline{k}_2=k(\underline{r}_2,\nu_2)$ and $\underline{k}_3=k(\underline{r}_3,\nu_1+\nu_2)$ where k is the map (7.1). We suppose that $(\underline{k}_1,\underline{k}_2,\underline{k}_3)$ is unbalanced at τ_0 with dominant weight $\underline{k}_3$, and $k_{\tau,i}\in\mathbb{Z}_{>0}$ for each $\tau\in\Sigma_F$ and $i=1,2,3$.

We write $(x,y,z)\in\mathfrak{W}^G_{n_1}\times\mathfrak{W}^G_{n_2}\times\mathfrak{W}_{n_3}$ for the point corresponding to the triple $(\underline{k}_1,\nu_1)$, $(\underline{k}_2,\nu_2)$ and $(\underline{k}_3,\nu_1+\nu_2)$. As μ_3 is ordinary, then from Proposition 7.7 and Corollary 4.8 we deduce that its specialization at μ_z corresponds to an automorphic form of weight $(\underline{k}_3,\nu_1+\nu_2)$. If $\underline{k}_1$ and $\underline{k}_2$ are big enough, then the same is true for μ_x and μ_y, obtaining automorphic forms of weights $(\underline{k}_1,\nu_1)$ and $(\underline{k}_2,\nu_2)$, respectively. We denote by π_x, π_y and π_z the automorphic representations of $(B\otimes\mathbb{A}_F)^\times$ generated by these automorphic forms, and Π_x, Π_y and Π_z the corresponding cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$.

Assume that μ_i are eigenvectors for all the $U_\mathfrak{p}$ operators, namely $U_\mathfrak{p}\mu_i=\alpha_i^\mathfrak{p}\cdot\mu_i$, and write $\alpha_x^\mathfrak{p}$, $\alpha_y^\mathfrak{p}$ and $\alpha_z^\mathfrak{p}$ for the corresponding specializations at x, y and z. Moreover, we assume that $\bar\mu_x$ is the $\mathfrak{p}$-stabilization of the newform $\bar\mu_x^\circ$ for each $\mathfrak{p}\mid p$. Write $\beta_i^\mathfrak{p}$ for the other eigenvalue of $U_\mathfrak{p}$ as usual, and write $\beta_x^\mathfrak{p}$, $\beta_y^\mathfrak{p}$ and $\beta_z^\mathfrak{p}$ for the corresponding specializations.

Theorem 10.5. — *With the notations above we have:*

$$\mathcal{L}_p(\mu_1,\mu_2,\mu_3)(x,y,z)=K(\mu_x^\circ,\mu_y^\circ,\mu_z^\circ)\cdot\left(\prod_{\mathfrak{p}\mid p}\frac{\mathcal{E}_\mathfrak{p}(x,y,z)}{\mathcal{E}_{\mathfrak{p},1}(z)}\right)\cdot\frac{L\left(\frac{1-\nu_1-\nu_2-\nu_3}{2},\Pi_x\otimes\Pi_y\otimes\Pi_z\right)^{\frac12}}{\langle\bar\mu_z^\circ,\bar\mu_z^\circ\rangle}$$

here $K(\mu_1^\circ,\mu_2^\circ,\mu_3^\circ)$ is a non-zero constant,

$$\mathcal{E}_\mathfrak{p}(x,y,z)=\begin{cases}\left(1-\frac{\beta_x^\mathfrak{p}\beta_y^\mathfrak{p}\alpha_z^\mathfrak{p}}{\varpi_\mathfrak{p}^{\underline{m}_\mathfrak{p}+\underline{2}}}\right)\left(1-\frac{\alpha_x^\mathfrak{p}\beta_y^\mathfrak{p}\beta_z^\mathfrak{p}}{\varpi_\mathfrak{p}^{\underline{m}_\mathfrak{p}+\underline{2}}}\right)\left(1-\frac{\beta_x^\mathfrak{p}\alpha_y^\mathfrak{p}\beta_z^\mathfrak{p}}{\varpi_\mathfrak{p}^{\underline{m}_\mathfrak{p}+\underline{2}}}\right)\left(1-\frac{\beta_x^\mathfrak{p}\beta_y^\mathfrak{p}\beta_z^\mathfrak{p}}{\varpi_\mathfrak{p}^{\underline{m}_\mathfrak{p}+\underline{2}}}\right), & \mathfrak{p}\neq\mathfrak{p}_0,\\ \left(1-\frac{\alpha_x^{\mathfrak{p}_0}\alpha_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right)\left(1-\frac{\alpha_x^{\mathfrak{p}_0}\beta_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right)\left(1-\frac{\beta_x^{\mathfrak{p}_0}\alpha_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right)\left(1-\frac{\beta_x^{\mathfrak{p}_0}\beta_y^{\mathfrak{p}_0}\beta_z^{\mathfrak{p}_0}}{p^{m_0-1}}\right), & \mathfrak{p}=\mathfrak{p}_0,\end{cases}$$

$$\mathcal{E}_{\mathfrak{p},1}(z):=\begin{cases}(1-(\beta_z^\mathfrak{p})^2\varpi_\mathfrak{p}^{-\underline{k}_{3,\mathfrak{p}}-\underline{2}})\cdot(1-(\beta_z^\mathfrak{p})^2\varpi_\mathfrak{p}^{-\underline{k}_{3,\mathfrak{p}}-\underline{1}}), & \mathfrak{p}\neq\mathfrak{p}_0,\\ (1-(\beta_z^{\mathfrak{p}_0})^2p^{-k_{3,\tau_0}})\cdot(1-(\beta_z^{\mathfrak{p}_0})^2p^{1-k_{3,\tau_0}}), & \mathfrak{p}=\mathfrak{p}_0,\end{cases}$$

$m_0=\frac{k_{1,\tau_0}+k_{2,\tau_0}+k_{3,\tau_0}}{2}\geqslant 0$, *and* $\underline{m}_\mathfrak{p}=\frac{\underline{k}_{1,\mathfrak{p}}+\underline{k}_{2,\mathfrak{p}}+\underline{k}_{3,\mathfrak{p}}}{2}\in\mathbb{Z}[\Sigma_\mathfrak{p}]$.

Proof. — By construction we have:

$$\mathcal{L}_p(\mu_1,\mu_2,\mu_3)(x,y,z) = \frac{\left\langle \mu_z^*, e_{\mathrm{ord}}(\Theta^{m_{3,\tau_0}}\mu_{x,\mathfrak{c}}^{[p]}\cdot\mu_{y,\mathfrak{c}}(\underline{\Delta}^{\tau_0}_{(x,y,z)}))_{\mathfrak{c}}\right\rangle}{\langle \mu_z^*,\mu_z\rangle}. \tag{10.2}$$

Observe first that $\underline{\Delta}^{\tau_0}_{(x,y,z)}$ differs from Δ^{τ_0} of Equation (3.8). Indeed, $\underline{\Delta}^{\tau_0}_{(x,y,z)}((x_1,y_1),(x_2,y_2),(x_3,y_3))$ is extended by zero whether $(x_1y_2-x_2y_1)\notin \mathcal{O}^{\tau_0\times}$. Thus if we denote $\mu_{x,y}=\Theta^{m_{3,\tau_0}}\mu_{x,\mathfrak{c}}^{[p]}\cdot\mu_{y,\mathfrak{c}}$ then:

$$\begin{aligned}\varepsilon &:= \int \underline{\Delta}^{\tau_0}_{(x,y,z)}\,\mathrm{d}\mu_{x,y}\,\mathrm{d}\mu_z - \int \Delta^{\tau_0}\,\mathrm{d}\mu_{x,y}\,\mathrm{d}\mu_z\\ &= \int_D\int_{\mathfrak{L}_2^{\tau_0}} \Delta^{\tau_0}(v_1,v_2,v_3)\,\mathrm{d}\mu_{x,y}(v_1,v_2)\,\mathrm{d}\mu_z(v_3),\end{aligned}$$

where $D=\prod_{\mathfrak{p}\neq\mathfrak{p}_0}D_{\mathfrak{p}}$ and $D_{\mathfrak{p}}=\{((x_1,y_1),(x_2,y_2))\in(\mathcal{O}_{\mathfrak{p}}^\times\times\mathcal{O}_{\mathfrak{p}})^2:\ (x_1y_2-x_2y_1)\notin\mathcal{O}_{\mathfrak{p}}^\times\}=\bigcup_{i\in\kappa_{\mathfrak{p}}}(D_i\times D_i)$ with $D_i=\bigcup_{a\in\kappa_{\mathfrak{p}}^\times}(a+p\mathcal{O}_{\mathfrak{p}})\times(ai+p\mathcal{O}_{\mathfrak{p}})$.

Since μ_i are $U_{\mathfrak{p}}$-eigenvectors, a calculation similar to that of proof of Proposition 7.5 shows that the corresponding $\mathfrak{p}$-component

$$\varepsilon_{\mathfrak{p}} := \int_{D_{\mathfrak{p}}}\int_{p\mathcal{O}_{\mathfrak{p}}\times\mathcal{O}_{\mathfrak{p}}^\times}\Delta^{\tau_0}\,\mathrm{d}\mu_{x,y}\,\mathrm{d}\mu_z$$

satisfies

$$\begin{aligned}\varepsilon_{\mathfrak{p}} &= \frac{1}{\alpha_x^{\mathfrak{p}}\alpha_y^{\mathfrak{p}}}\sum_{i\in\kappa_{\mathfrak{p}}}\iint_{D_i^2}\int_{p\mathcal{O}_{\mathfrak{p}}\times\mathcal{O}_{\mathfrak{p}}^\times}\Delta^{\tau_0}\,\mathrm{d}U_{\mathfrak{p}}\mu_{x,y}\,\mathrm{d}\mu_z\\ &= \frac{1}{\alpha_x^{\mathfrak{p}}\alpha_y^{\mathfrak{p}}}\sum_{i\in\kappa_{\mathfrak{p}}}\gamma_i\iint_{(\mathcal{O}_{\mathfrak{p}}^\times\times\mathcal{O}_{\mathfrak{p}})^2}\int_{p\mathcal{O}_{\mathfrak{p}}\times\mathcal{O}_{\mathfrak{p}}^\times}\Delta^{\tau_0}(v_1\varpi_{\mathfrak{p}}g_i^{-1},v_2\varpi_{\mathfrak{p}}g_i^{-1},v_3)\\ &\qquad\qquad\mathrm{d}\mu_{x,y}(v_1,v_2)\,\mathrm{d}\mu_z(v_3)\\ &= \frac{\varpi_{\mathfrak{p}}^{m_{3,\mathfrak{p}}}}{\alpha_x^{\mathfrak{p}}\alpha_y^{\mathfrak{p}}}\sum_{i\in\kappa_{\mathfrak{p}}}\gamma_i\iint_{(\mathcal{O}_{\mathfrak{p}}^\times\times\mathcal{O}_{\mathfrak{p}})^2}\int_{p\mathcal{O}_{\mathfrak{p}}\times\mathcal{O}_{\mathfrak{p}}^\times}\Delta^{\tau_0}(v_1,v_2,v_3g_i)\,\mathrm{d}\mu_{x,y}(v_1,v_2)\,\mathrm{d}\mu_z(v_3)\\ &= \frac{\varpi_{\mathfrak{p}}^{m_{3,\mathfrak{p}}}}{\alpha_x^{\mathfrak{p}}\alpha_y^{\mathfrak{p}}}\iint_{(\mathcal{O}_{\mathfrak{p}}^\times\times\mathcal{O}_{\mathfrak{p}})^2}\int_{p\mathcal{O}_{\mathfrak{p}}\times\mathcal{O}_{\mathfrak{p}}^\times}\Delta^{\tau_0}(v_1,v_2,v_3)\,\mathrm{d}\mu_{x,y}(v_1,v_2)U_{\mathfrak{p}}^*\,\mathrm{d}\mu_z(v_3)\end{aligned}$$

since $\varpi_{\mathfrak{p}} g_j^{-1} * 1_{D_i} = 0$, if $i \neq j$, and $\varpi_{\mathfrak{p}} g_i^{-1} * 1_{D_i} = 1_{\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{O}_{\mathfrak{p}}}$. By Remark 7.3, $\alpha_x^{\mathfrak{p}} \beta_x^{\mathfrak{p}} = \varpi_{\mathfrak{p}}^{\underline{k}_{1,\mathfrak{p}} + \underline{1}}$, if $\mathfrak{p} \neq \mathfrak{p}_0$, and $\alpha_x^{\mathfrak{p}_0} \beta_x^{\mathfrak{p}_0} = p^{k_{1,\tau_0}-1}$. This implies that

$$\mathcal{L}_p(\mu_1,\mu_2,\mu_3)(x,y,z) = \left(\prod_{\mathfrak{p} \neq \mathfrak{p}_0} \left(1 - \beta_x^{\mathfrak{p}} \beta_y^{\mathfrak{p}} \alpha_z^{\mathfrak{p}} \varpi_{\mathfrak{p}}^{-\underline{m}_{\mathfrak{p}} - \underline{2}} \right) \right) \cdot \frac{\left\langle \mu_z^*, e_{\mathrm{ord}}(\Theta^{m_{3,\tau_0}} \mu_{x,\mathfrak{c}}^{[p]} \cdot \mu_{y,\mathfrak{c}}(\Delta^{\tau_0}))_{\mathfrak{c}} \right\rangle}{\langle \mu_z^*, \mu_z \rangle}.$$

As $\bigtriangledown_{\underline{k}_1}^{m_{3,\tau_0}} \mu_{x,\mathfrak{c}}^{[p]} \mu_{y,\mathfrak{c}}(\Delta^{\tau_0}) \in H^0(\mathcal{X}_r^{\mathfrak{c}}, \mathcal{H}_{m_{3,\tau_0}}^{\underline{k}_3})$ and $0 \leqslant 2m_{3,\tau_0} < k_{\tau_0,3}$, from Lemmas 9.4, 9.6 and Remark 9.10, we obtain $e_{\mathrm{ord}}(\Theta^{m_{3,\tau_0}} \mu_{x,\mathfrak{c}}^{[p]} \mu_{y,\mathfrak{c}}(\Delta^{\tau_0})) =$

$$e_{\mathrm{ord}}(\gamma_{\underline{k}_3, m_{3,\tau_0}} \bigtriangledown^{m_{3,\tau_0}} \mu_{x,\mathfrak{c}}^{[p]} \mu_{y,\mathfrak{c}}(\Delta^{\tau_0})) = e_{\mathrm{ord}}(\mathcal{H}^r(\bigtriangledown^{m_{3,\tau_0}} \mu_{x,\mathfrak{c}}^{[p]} \mu_{y,\mathfrak{c}}(\Delta^{\tau_0}))). \tag{10.3}$$

A laborious but straightforward computation shows that

$$\bigtriangledown^{m_{3,\tau_0}} \mu_{x,\mathfrak{c}}^{[p]} \mu_{y,\mathfrak{c}}(\Delta^{\tau_0}) = (-1)^{m_{3,\tau_0}} \binom{k_{3,\tau_0} - 2}{m_{3,\tau_0} + k_{2,\tau_0} - 1}^{-1} t(\mu_{x,\mathfrak{c}}^{[p]}, \mu_{y,\mathfrak{c}}) + \bigtriangledown \left(\sum_{i=0}^{m_{3,\tau_0}-1} a_i \bigtriangledown^i \mu_{x,\mathfrak{c}}^{[p]} \bigtriangledown^{m_{3,\tau_0}-1-i} \mu_{y,\mathfrak{c}}(\Delta^{\tau_0}) \right),$$

where $a_i = (-1)^{i+m_{3,\tau_0}+1} \binom{k_{3,\tau_0}-2}{m_{3,\tau_0}+k_{2,\tau_0}-1}^{-1} \left(\sum_{j=0}^{i} \binom{m_{3,\tau_0}}{j} \binom{m_{3,\tau_0}+k_{1,\tau_0}+k_{2,\tau_0}-2}{k_{1,\tau_0}+j-1}\right)$ and $t(\mu_{x,\mathfrak{c}}^{[p]}, \mu_{y,\mathfrak{c}})$ is defined as in Theorem 4.9. This relation implies:

$$\mathcal{H}^r(\bigtriangledown^{m_{3,\tau_0}} \mu_{x,\mathfrak{c}}^{[p]} \mu_{y,\mathfrak{c}}(\Delta^{\tau_0})) = (-1)^{m_{3,\tau_0}} \binom{k_{3,\tau_0} - 2}{m_{3,\tau_0} + k_{2,\tau_0} - 1}^{-1} \cdot t(\mu_{x,\mathfrak{c}}^{[p]}, \mu_{y,\mathfrak{c}}). \tag{10.4}$$

Since μ_z is ordinary, we obtain from (10.2), (10.3) and (10.4):

$$\mathcal{L}_p(\mu_1,\mu_2,\mu_3)(x,y,z) = \frac{(-1)^{m_{3,\tau_0}}}{\binom{k_{3,\tau_0}-2}{m_{3,\tau_0}+k_{2,\tau_0}-1}} \cdot \frac{\left\langle \mu_z^*, t(\mu_x^{[p]}, \mu_y) \right\rangle}{\langle \mu_z^*, \mu_z \rangle} \cdot \prod_{\mathfrak{p} \neq \mathfrak{p}_0} \left(1 - \beta_x^{\mathfrak{p}} \beta_y^{\mathfrak{p}} \alpha_z^{\mathfrak{p}} \varpi_{\mathfrak{p}}^{-\underline{m}_{\mathfrak{p}} - \underline{2}} \right).$$

Thus, the result follows by Theorem 4.9, Proposition 3.10, Remark 7.3 and Proposition 3.6. Notice that $K(\mu_x^\circ, \mu_y^\circ, \mu_z^\circ) = (C \cdot C(\mu_x^\circ, \mu_y^\circ, \mu_z^\circ))^{1/2} \cdot 2^{-2+m_{3,\tau_0}}$, where the constants C and $C(\mu_x^\circ, \mu_y^\circ, \mu_z^\circ)$ are as given in Proposition 3.10. $\square$

11. Δ-action and Hecke operators

We verify that the two actions of Δ introduced in Section 4.5 are compatible with the morphism in Lemma 4.6. Moreover, we describe the action of the Hecke operators on the space of quaternionic automorphic forms in terms of the associated moduli description of the unitary Shimura curves.

11.1. Compatibility of the Δ-action

Recall the space $M_{\underline{k}}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}),\mathbb{C})$ introduced in Section 4.5 endowed with the action of $\Delta=(\mathcal{O}_F)_+^\times/U_{\mathfrak{n}}^2$ given by (4.7). Fix $f\in M_{\underline{k}}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}),\mathbb{C})$.

By Lemma 4.6, f is interpreted as a section of $\omega^{\underline{k}}$ on a connected component of $X_{\mathbb{C}}^{\mathfrak{c}}$ and given by $f(z)(\prod_{\tau\neq\tau_0}|\begin{smallmatrix}\mathrm{d}x_\tau & \mathrm{d}y_\tau\\ x_\tau & Y_\tau\end{smallmatrix}|^{k_\tau})dx_{\tau_0}^{k_{\tau_0}}\in H^0(X_{\mathbb{C}}^{0,\mathfrak{c}},\omega^{\underline{k}})$, where $w=(\mathrm{d}x_{\tau_0},(\mathrm{d}x_\tau,\mathrm{d}y_\tau)_\tau)$ is a basis of ω_A.

Let $J_{\mathfrak{c}}:=\widehat{\mathcal{O}}_{\mathfrak{m}}b_{\mathfrak{c}}^{-1}\cap D$ and $V:=(\mathbb{C}^2\otimes_{F,\tau_0}E)\times\mathrm{M}_2(\mathbb{C})^{\Sigma_F\setminus\{\tau_0\}}$. We consider $\Lambda_z=v_z(J_{\mathfrak{c}})\subset V$ where $v_z(d):=(d(\begin{smallmatrix}z\\1\end{smallmatrix}),d)\in V$ for $d\in D$ and put $A_z:=V/\Lambda_z$. We consider the polarization $\theta_z:\Lambda_z\times\Lambda_z\to\mathbb{Z}$ given by $(v_z(\beta_1),v_z(\beta_1))\mapsto\Theta(\beta_1,\beta_2)$. Finally, let α_z be the isomorphism (see Remark 4.2) $\alpha_z:(\Lambda_z\otimes\widehat{\mathbb{Z}}\simeq\widehat{\mathcal{O}}_{\mathfrak{m}}b_{\mathfrak{c}}^{-1},\Theta_z)\to(\widehat{\mathcal{O}}_{\mathfrak{m}},\mathfrak{c}^{-1}\Theta)$ given by $\alpha_z(\beta)=\beta b_{\mathfrak{c}}$. Thus, as a Katz modular form we have $f(z)=f(A_z,\iota_z,\theta_z,\alpha_z,w)$.

Proposition 11.1. — *For $s\in\Delta$ write $k_s=b_{\mathfrak{c}}^{-1}\gamma_s^{-1}b_{\mathfrak{c}}$. Then, if $f\in M_{\underline{k}}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}),\mathbb{C})$, we have*

$$(s*f)(A_z,\iota_z,\theta_z,\alpha_z,w)=s^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}}\cdot f(A_z,\iota_z,s^{-1}\theta_z,k_s\alpha_z,w).$$

Remark 11.2. — Notice that $\gamma_s^{-1}\alpha_z$ provides an isomorphism between $(\widehat{T}A_z,\theta_z)$ and $(\widehat{\mathcal{O}}_{\mathfrak{m}},s\mathfrak{c}^{-1}\Theta)$, thus, an isomorphism between $(\widehat{T}A_z,s^{-1}\theta_z)$ and $(\widehat{\mathcal{O}}_{\mathfrak{m}},\mathfrak{c}^{-1}\Theta)$.

Proof. — Write $\tau_0(\gamma_s)=(\begin{smallmatrix}* & *\\ c & d\end{smallmatrix})$ and consider the map $\varphi_s:(A_{\gamma_s z},\iota_{\gamma_s z})\to(A_z,\iota_z)$ induced from the map on V given by $(v,M)\mapsto((cz+d)\cdot v,M\gamma_s)$. It is an isomorphism and sends $\theta_{\gamma_s z}$ to $s^{-1}\theta_z$, $\alpha_{\gamma_s z}$ to $(b_{\mathfrak{c}}^{-1}\gamma_s^{-1}b_{\mathfrak{c}})\alpha_z$ and w to $(cz+d)^{-1}w\gamma_s^{-1}$. This together with relation $f(z)=f(A_z,\iota_z,\theta_z,\alpha_z,w)$ and properties (B1) and (B3) from Section 4.4 conclude the proof. $\square$

11.2. Moduli description of Hecke operators

By Proposition 4.7, $f\in H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k},\nu))^{K_1^B(\mathfrak{n})}$ corresponds to $(f_{\mathfrak{c}})_{\mathfrak{c}}\in\bigoplus_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)}M_{\underline{k}}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}),\mathbb{C})^\Delta=\bigoplus_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)}H^0(X_{\mathbb{C}}^{\mathfrak{c},0},\omega^{\underline{k}})^\Delta$. Thus, for $g\in$

$G(\mathbb{A}_f)$, the last space inherits the action of the Hecke operator T_g attached to the double coset $K_1^B(\mathfrak{n})gK_1^B(\mathfrak{n}) = \bigsqcup_i g_i K_1^B(\mathfrak{n})$. Recall that the isomorphism of Proposition 4.7 depends on the choice of elements $b_{\mathfrak{c}} \in G(\mathbb{A}_f)$ whose norm is a representative of the class $\mathfrak{c}$.

Now we define a Hecke action in terms of the moduli problem. Let $\mathfrak{c} \in \mathrm{Pic}(\mathcal{O}_F)$, $\mathfrak{c}'$ be the class of $\mathfrak{c}\det(g)$. Write $b_{\mathfrak{c}} g_i = \gamma_i^{-1} b_{\mathfrak{c}'} k_i$ with $\gamma_i \in G(\mathbb{Q})_+$ and $k_i \in K_1^B(\mathfrak{n})$. Let $(A, \iota, \theta, \alpha) \in X^{\mathfrak{c}}(R)$ and $\bar{\alpha} : (\widehat{T}(A), \theta) \to (\widehat{\mathcal{O}}_{\mathfrak{m}}, \mathfrak{c}^{-1}\Theta)$ a representative of α. Let $n \in \mathbb{Z}$ be such that $ng_i \in \widehat{\mathcal{O}}_{\mathfrak{m}}$, thus, $\bar{\alpha}^{-1}(n^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}} g_i^{-1})$ provides a $\mathcal{O}_{\mathfrak{m}}$-submodule $C_i \subset A_{tor}$ isomorphic to $n^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}} g_i^{-1}/\widehat{\mathcal{O}}_{\mathfrak{m}}$. Then we consider $A^{g_i} := A/C_i$, the isogeny $\psi_{g_i} : (A^{g_i}, \iota^{g_i}) \to (A, \iota)$ and the polarization $\theta^{g_i} : A^{g_i} \to (A^{g_i})^\vee$ obtained from ψ_{g_i} and θ. If $\det(g_i) = \mathfrak{c}_i$, the abelian variety A^{g_i} has attached the class α_{g_i} modulo $K_{1,1}^B(\mathfrak{n})$ of the isomorphism $\bar{\alpha}^{g_i} : (\widehat{T}(A^{g_i}), \theta^{g_i}) \xrightarrow{\bar{\alpha}} (n^{-1}\widehat{\mathcal{O}}_{\mathfrak{m}} g_i^{-1}, \mathfrak{c}^{-1}\Theta) \xrightarrow{\cdot g_i} (\widehat{\mathcal{O}}_{\mathfrak{m}}, \mathfrak{c}_i^{-1}\mathfrak{c}^{-1}\Theta)$. Thus, we obtain a point $(A^{g_i}, \iota^{g_i}, \theta^{g_i}, \alpha^{g_i}) \in X^{\mathfrak{c}\mathfrak{c}_i}(R)$. We write $w = (dx_{\tau_0}, (dx_\tau, dy_\tau)_\tau) \in \omega_{A^{g_i}}$ for the pull-back of the basis $w \in \omega_A$ using $\psi_{g_i}^*$. We define:

$$(\widetilde{T}_g f)_{\mathfrak{c}}(A, \iota, \theta, \alpha, w) = \sum_i \det(\gamma_i)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} f_{\mathfrak{c}'}(A^{g_i}, \iota^{g_i}, \det(\gamma_i)^{-1}\theta^{g_i}, k_i^{-1}\alpha^{g_i}, w).$$

It is straightforward to verify it is well defined. For example, we clearly have isomorphisms of symplectic modules $k_i^{-1}\alpha^{g_i} : (\widehat{T}A^{g_i}, \det(\gamma_i)^{-1}\theta^{g_i}) \cong (\widehat{\mathcal{O}}_{\mathfrak{m}}, \det(\gamma_i)^{-1}\det(k_i)\mathfrak{c}^{-1}\mathfrak{c}_i^{-1}\Theta) = (\widehat{\mathcal{O}}_{\mathfrak{m}}, (\mathfrak{c}')^{-1}\Theta)$.

Independence of $\bar{\alpha}$. — If $\gamma \in K_{1,1}^B(\mathfrak{n})$, let $A_\gamma^{g_i}$ be constructed using $\gamma\bar{\alpha}$. Then $A_\gamma^{g_i} = A^{\gamma g_i}$. As $\gamma g_i = g_j k_0$ for some j and $k_0 \in K_1^B(\mathfrak{n})$, then

$$(A_\gamma^{g_i}, \iota_\gamma^{g_i}, \det(\gamma_i)^{-1}\theta_\gamma^{g_i}, k_i^{-1}(\gamma\bar{\alpha})^{g_i}) = (A^{g_j}, \iota^{g_j}, \det(\gamma_i)^{-1}\theta^{g_j}, k_i^{-1}k_0\bar{\alpha}^{g_j}),$$

since

$$(A^{g_j k_0}, \iota^{g_j k_0}, \theta^{g_j k_0}) = (A^{g_j}, \iota^{g_j}, \theta^{g_j})$$

and

$$(\gamma\bar{\alpha})^{g_i} = k_0\bar{\alpha}^{g_j}.$$

As $\gamma_j^{-1} b_{\mathfrak{c}'} k_j k_0 = (b_{\mathfrak{c}}\gamma b_{\mathfrak{c}}^{-1})\gamma_i^{-1} b_{\mathfrak{c}'} k_i$, then we can choose γ_i and γ_j such that $\det(\gamma_i) = \det(\gamma_j)$, and $k_i = k_j k_0$. Thus,

$$(A_\gamma^{g_i}, \iota_\gamma^{g_i}, \det(\gamma_i)^{-1}\theta_\gamma^{g_i}, k_i^{-1}(\gamma\bar{\alpha})^{g_i}) = (A^{g_j}, \iota^{g_j}, \det(\gamma_j)^{-1}\theta^{g_j}, k_j^{-1}\bar{\alpha}^{g_j})$$

and the claim follows.

Independent of the choice of γ_i. — If $b_{\mathfrak{c}} g_i = \overline{\gamma}_i^{-1} b_{\mathfrak{c}'} \overline{k}_i$ then $\beta = \overline{\gamma}_i \gamma_i^{-1} \in G(\mathbb{Q})_+ \cap b_{\mathfrak{c}'} K_1(\mathfrak{n}) b_{\mathfrak{c}'}^{-1} = \Gamma_1^{\mathfrak{c}'}(\mathfrak{n})$ and $\overline{k}_i = b_{\mathfrak{c}'}^{-1} \beta b_{\mathfrak{c}'} k_i$. If $s = \det(\beta)$ and $k_s = b_{\mathfrak{c}'}^{-1} \beta^{-1} b_{\mathfrak{c}'}$, then by Δ-invariance of $f_{\mathfrak{c}'}$:

$$s^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} f_{\mathfrak{c}'}(A^{g_i}, \iota^{g_i}, \det(\overline{\gamma}_i)^{-1}\theta^{g_i}, \overline{k}_i^{-1}\alpha^{g_i}, w) \\ = f_{\mathfrak{c}'}(A^{g_i}, \iota^{g_i}, \det(\gamma_i)^{-1}\theta^{g_i}, k_i^{-1}\alpha^{g_i}, w).$$

Only depends on $g_i K_1^B(\mathfrak{n})$. — If $k \in K_1^B(\mathfrak{n})$ then

$$(A^{g_i k}, \iota^{g_i k}, \theta^{g_i k}) = (A^{g_i}, \iota^{g_i}, \theta^{g_i})$$

and $\alpha^{g_i k} = k\alpha^{g_i}$.

Δ-invariant. — If $s \in \Delta$ and $k_s \in K_1^B(\mathfrak{n})$ with $\det(k_s) = s^{-1}$, by the Δ-invariance of $f_{\mathfrak{c}'}$ and Propositions 11.1 and 11.3], we have that

$$\begin{aligned} s * (\widetilde{T}_g f)_{\mathfrak{c}}(A, \iota, \theta, \alpha, w) &\\ &= s^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} \cdot (\widetilde{T}_g f)_{\mathfrak{c}}(A, \iota, s^{-1}\theta, k_s\alpha, w) \\ &= \sum_i (s\det(\gamma_i))^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} f_{\mathfrak{c}'}(A^{g_i}, \iota^{g_i}, s^{-1}\det(\gamma_i)^{-1}\theta^{g_i}, k_s k_i^{-1}\alpha^{g_i}, w) \\ &= (\widetilde{T}_g f)_{\mathfrak{c}}(A, \iota, \theta, \alpha, w). \end{aligned}$$

Proposition 11.3. — *We have that $T_g f = \widetilde{T}_g f$.*

Proof.

$$\begin{aligned} (T_g f)_{\mathfrak{c}}(A_z, \iota_z, \theta_z, \alpha_z, w) &\\ &= (T_g f)(z, b_{\mathfrak{c}}) = \sum_i f(z, b_{\mathfrak{c}} g_i) \\ &= \sum_i f(z, \gamma_i^{-1} b_{\mathfrak{c}'}) = \sum_i \det(\gamma_i)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} (cz+d)^{-k_{\tau_0}} \gamma_i^{-1} f(\gamma_i z, b_{\mathfrak{c}'}) \\ &= \sum_i \det(\gamma_i)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} (cz+d)^{-k_{\tau_0}} \gamma_i^{-1} f_{\mathfrak{c}'}(A_{\gamma_i z}, \iota_{\gamma_i z}, \theta_{\gamma_i z}, \alpha_{\gamma_i z}, w). \end{aligned}$$

Let $J_{\mathfrak{c}}^{g_i} := \widehat{\mathcal{O}}_{\mathfrak{m}} g_i^{-1} b_{\mathfrak{c}}^{-1} \cap D$ and $\Lambda_z^{\mathfrak{c}, g_i} := v_z(J_{\mathfrak{c}}^{g_i})$. We have an isomorphism $\psi_i : A_{\gamma_i z} = V/\Lambda_{\gamma_i z}^{\mathfrak{c}'} \to A_z^{g_i} = V/\Lambda_z^{\mathfrak{c}, g_i}$ given by

$$\psi_i(v, (M_\tau)_{\tau\neq\tau_0}) = ((cz+d)v, (M_\tau \widetilde{\tau}(\gamma_i))_{\tau\neq\tau_0}).$$

Notice that ψ_i is well defined since $\psi_i(v_{\gamma_i z}(\beta)) = v_z(\beta\gamma_i)$, for all $\beta \in J_{\mathfrak{c}'}$, and $\beta\gamma_i \in \widehat{\mathcal{O}}_{\mathfrak{m}} b_{\mathfrak{c}'}^{-1} \gamma_i \cap D = \widehat{\mathcal{O}}_{\mathfrak{m}} k_i g_i^{-1} b_{\mathfrak{c}}^{-1} \cap D = J_{\mathfrak{c}}^{g_i}$. Similarly as in the proof of Proposition 11.1, ψ_i sends $\theta_{\gamma_i z}$ to $\det(\gamma_i)^{-1}\theta_z^{g_i}$, $\alpha_{\gamma_i z}$ to

$$\begin{tikzcd} k_i^{-1}\alpha_z^{g_i} : \widehat{T}(A_z^{g_i}) \simeq \widehat{\mathcal{O}}_{\mathfrak{m}} g_i^{-1} b_{\mathfrak{c}}^{-1} \arrow[r, "\gamma_i^{-1}"] \arrow[dr, "b_{\mathfrak{c}}"'] & \widehat{\mathcal{O}}_{\mathfrak{m}} b_{\mathfrak{c}'}^{-1} \arrow[r, "b_{\mathfrak{c}'}"] & \widehat{\mathcal{O}}_{\mathfrak{m}}, \\ & \widehat{\mathcal{O}}_{\mathfrak{m}} g_i^{-1} \arrow[r, "g_i"] & \widehat{\mathcal{O}}_{\mathfrak{m}} \arrow[u, "k_i^{-1}"] \end{tikzcd}$$

and w to $(cz+d)^{-1}w\gamma_i^{-1} := ((cz+d)^{-1}\mathrm{d}x_{\tau_0}, (\mathrm{d}x_\tau, \mathrm{d}y_\tau)\widetilde{\tau}(\gamma_i)^{-1})$. Thus,

$$\begin{aligned}
&(T_g f)_{\mathfrak{c}}(A_z, \iota_z, \theta_z, \alpha_z, w) \\
&\quad \overset{(B1)}{=} \sum_i \det(\gamma_i)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} (cz+d)^{-k_{\tau_0}} \gamma_i^{-1} \\
&\qquad\qquad \times f_{\mathfrak{c}'}(A_z^{g_i}, \iota_z^{g_i}, \det(\gamma_i)^{-1}\theta_z^{g_i}, k_i^{-1}\alpha_z^{g_i}, (cz+d)^{-1}w\gamma_i^{-1}) \\
&\quad \overset{(B4)}{=} \sum_i \det(\gamma_i)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} f_{\mathfrak{c}'}(A_z^{g_i}, \iota_z^{g_i}, \det(\gamma_i)^{-1}\theta_z^{g_i}, k_i^{-1}\alpha_z^{g_i}, w). \qquad \square
\end{aligned}$$

11.3. The $U_{\mathfrak{p}}$-operator

Assume that f has Iwahori level at $\mathfrak{p} \mid p$. We denote by $U_{\mathfrak{p}}$ the Hecke operator T_g as defined above. We have a distinguished decomposition $K_1^B(\mathfrak{n},\mathfrak{p})\left(\begin{smallmatrix}\varpi_{\mathfrak{p}} & 0\\ 0 & 1\end{smallmatrix}\right)K_1^B(\mathfrak{n},\mathfrak{p}) = \bigsqcup_{i\in\kappa_{\mathfrak{p}}}\left(\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix}\right)K_1^B(\mathfrak{n},\mathfrak{p})$. For any $\mathfrak{c} \in \mathrm{Pic}(\mathcal{O}_F)$, assume that $\mathfrak{c}\mathfrak{p}$ lies in the class $\mathfrak{c}'$, this implies that $\mathfrak{c}\mathfrak{p} = \mathfrak{c}'(\gamma_{\mathfrak{p}})$, for some $\gamma_{\mathfrak{p}} \in \mathcal{O}_F$. Assuming that $\mathfrak{c}$ and $\mathfrak{c}'$ are coprime to $\mathfrak{p}$, $\gamma_{\mathfrak{p}}$ generates $\mathfrak{p}$ in $\mathcal{O}_{\mathfrak{p}}$. We have that $b_{\mathfrak{c}}\left(\begin{smallmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}\gamma_{\mathfrak{p}} & i\\ 0 & 1\end{smallmatrix}\right)b_{\mathfrak{c}'}k_i$ with $k_i \in K_1^B(\mathfrak{n},\mathfrak{p})$. By the previous computations,

$$\begin{aligned}
&(U_{\mathfrak{p}} f)_{\mathfrak{c}}(A, \iota, \theta, \alpha, w) \\
&\quad = \gamma_{\mathfrak{p}}^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}} \sum_{i\in\kappa_{\mathfrak{p}}} \begin{pmatrix}\gamma_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix} * f_{\mathfrak{c}'}\left(A_i, \iota_i, \gamma_{\mathfrak{p}}\theta_i, k_i^{-1}\alpha_i, w\begin{pmatrix}\gamma_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix}\right),
\end{aligned}$$

where $A_i = A/C_i$ (and $\iota_i, \theta_i, \alpha_i$ are also obtained from C_i) with

$$\begin{aligned}
C_i &\overset{\alpha}{\simeq} \widehat{\mathcal{O}}_{\mathfrak{m}}\begin{pmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix}^{-1}/\widehat{\mathcal{O}}_{\mathfrak{m}} \\
&= \left(\mathrm{M}_2(\mathcal{O}_{\mathfrak{p}})\begin{pmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix}^{-1}/\mathrm{M}_2(\mathcal{O}_{\mathfrak{p}})\right) \times \left(\mathrm{M}_2(\mathcal{O}_{\mathfrak{p}})\begin{pmatrix}\varpi_{\mathfrak{p}} & i\\ 0 & 1\end{pmatrix}^{-1}/\mathrm{M}_2(\mathcal{O}_{\mathfrak{p}})\right).
\end{aligned}$$

Thus C_i is characterized by $C_i^{-,1} := C_i \cap A[\mathfrak{p}]^{-,1}$. Moreover, $C_i^{-,1}$ corresponds to the cyclic subgroups of $A[\mathfrak{p}]^{-,1}$ not intersecting the canonical subgroup $C \subset A[\mathfrak{p}]^{-,1}$ that characterizes $\alpha_{\mathfrak{p}}$ by Remark 4.4.

11.4. Oldforms

Let $\mathfrak{n}=\mathfrak{n}_0\mathcal{D}$ and $d\mid\mathcal{D}$. Given a newform $\phi\in H^0(G(\mathbb{Q}),\mathcal{A}(\underline{k},\nu))^{K_1^B(\mathfrak{n}_0)}$ let $\phi^d\in H^0(G(F),\mathcal{A}(\underline{k},\nu))^{K_1^B(\mathfrak{n})}$ given by $\phi^d(f)(g):=\phi(f)(gg_d)$, where $g_d:=\left(\begin{smallmatrix}1&0\\0&\varpi_d\end{smallmatrix}\right)$ and $\varpi_d=\prod_{v\mid d}\varpi_v$.

We describe this oldform as in Section 4.4. Under Proposition 4.7, ϕ corresponds to a tuple $f=(f_{\mathfrak{c}})_{\mathfrak{c}\in\mathrm{Pic}(\mathcal{O}_F)}$ where $f_{\mathfrak{c}}\in M_{\underline{k}}(\Gamma_{1,1}^{\mathfrak{c}}(\mathfrak{n}_0),\mathbb{C})^{\Delta_{\mathfrak{n}_0}}=H^0(X_{\mathfrak{n}_0}^{\mathfrak{c},0},\omega^{\underline{k}})^{\Delta_{\mathfrak{n}_0}}$ and $\Delta_{\mathfrak{n}_0}=(\mathcal{O}_F)_+^\times/U_{\mathfrak{n}_0}^2$. If $b_{\mathfrak{c}}g_d=\gamma_d^{-1}b_{\mathfrak{c}'}k$ with $\gamma_d\in G(\mathbb{Q})_+$ and $k\in K_1^B(\mathfrak{n}_0)$, we have

$$\begin{aligned}f_{\mathfrak{c}}^d(A_z,\iota_z,\theta_z,\alpha_z,w)&=f^d(z,b_{\mathfrak{c}})=f(z,b_{\mathfrak{c}}g_d)\\&=\det(\gamma_d)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}}(cz+d)^{-k_{\tau_0}}\gamma_d^{-1}f(\gamma_dz,b_{\mathfrak{c}'})\\&=\frac{\det(\gamma_d)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}}}{(cz+d)^{k_{\tau_0}}}\gamma_d^{-1}f_{\mathfrak{c}'}(A_{\gamma_dz},\iota_{\gamma_dz},\theta_{\gamma_dz},\alpha_{\gamma_dz},w).\end{aligned}$$

We conclude by an analogous computation as above

$$\begin{aligned}&f_{\mathfrak{c}}^d(A,\iota,\theta,\alpha,w)\\&\quad=\det(\gamma_d)^{\frac{-\underline{k}+2k_{\tau_0}\tau_0+\nu\underline{1}}{2}}f_{\mathfrak{c}'}(A^{g_d},\iota^{g_d},\det(\gamma_d)^{-1}\theta^{g_d},k^{-1}\alpha^{g_d},w).\end{aligned}\tag{11.1}$$

BIBLIOGRAPHY

[1] F. Andreatta & A. Iovita, "Triple product p-adic L-functions associated to finite slope p-adic families of modular forms", *Duke Math. J.* **170** (2021), no. 9, p. 1989-2083.

[2] F. Andreatta, A. Iovita & V. Pilloni, "p-adic families of Siegel modular cuspforms", *Ann. Math. (2)* **181** (2015), no. 2, p. 623-697.

[3] ———, "Le halo spectral", *Ann. Sci. Éc. Norm. Supér. (4)* **51** (2018), no. 3, p. 603-655.

[4] D. Barrera, A. Cauchy, S. Molina & V. Rotger, "Kato's theorem for elliptic curves over totally real number fields", In progress.

[5] M. Bertolini, H. Darmon & V. Rotger, "Beilinson-Flach elements and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil L-series", *J. Algebr. Geom.* **24** (2015), p. 569-604.

[6] M. Bertolini, M. A. Seveso & R. Venerucci, "Diagonal classes and the Bloch–Kato conjecture", *Münster J. Math.* **13** (2020), no. 2, p. 317-352.

[7] S. Bijakowski, V. Pilloni & B. Stroh, "Classicité de formes modulaires surconvergentes", *Ann. Math. (2)* **183** (2016), no. 3, p. 975-1014.

[8] R. Brasca, "p-adic modular forms of non-integral weight over Shimura curves", *Compos. Math.* **149** (2013), no. 1, p. 32-62.

[9] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, 1997, xiv+574 pages.

[10] K. Buzzard, "Eigenvarieties", in *L-functions and Galois representations*, London Mathematical Society Lecture Note Series, vol. 320, Cambridge University Press, 2007, p. 59-120.

[11] H. Carayol, "Sur la mauvaise réduction des courbes de Shimura", *Compos. Math.* **59** (1986), no. 2, p. 151-230.

[12] R. F. Coleman, "p-adic Banach spaces and families of modular forms", *Invent. Math.* **127** (1997), no. 3, p. 417-479.

[13] H. Darmon & V. Rotger, "Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula", *Ann. Sci. Éc. Norm. Supér. (4)* **47** (2014), no. 4, p. 779-832.

[14] ———, "Diagonal cycles and Euler systems II: The Birch and Swinnerton–Dyer conjecture for Hasse–Weil–Artin L-functions", *J. Am. Math. Soc.* **30** (2017), no. 3, p. 601-672.

[15] Y. Ding, *Formes modulaires p-adiques sur les courbes de Shimura unitaires et compatibilité local-global*, Mémoires de la Société Mathématique de France. Nouvelle Série, vol. 155, Société Mathématique de France, 2017, viii+245 pages.

[16] Y. Fan, "Local expansion in Serre-Tate coordinates and p-adic iteration of Gauss–Manin connections", PhD Thesis, Università Degli Studi di Milano, 2018.

[17] M. Greenberg & M. A. Seveso, "Triple product p-adic L-functions for balanced weights", *Math. Ann.* **376** (2020), no. 1-2, p. 103-176.

[18] M. Harris & S. S. Kudla, "The central critical value of a triple product L-function", *Ann. Math. (2)* **133** (1991), no. 3, p. 605-672.

[19] ———, "On a conjecture of Jacquet", in *Contributions to automorphic forms, geometry, and number theory*, Johns Hopkins Univ. Press, 2004, p. 355-371.

[20] M. Harris & J. Tilouine, "p-adic measures and square roots of special values of triple product L-functions", *Math. Ann.* **320** (2001), no. 1, p. 127-147.

[21] M.-L. Hsieh, "Hida families and p-adic triple product L-functions", *Am. J. Math.* **143** (2021), no. 2, p. 411-532.

[22] A. Ichino, "Trilinear forms and the central values of triple product L-functions", *Duke Math. J.* **145** (2008), no. 2, p. 281-307.

[23] D. Jetchev, C. Skinner & X. Wan, "The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one", *Camb. J. Math.* **5** (2017), no. 3, p. 369-434.

[24] O. T. R. Jones, "An analogue of the BGG resolution for locally analytic principal series", *J. Number Theory* **131** (2011), no. 9, p. 1616-1640.

[25] P. L. Kassaei, "$\mathfrak{p}$-adic modular forms over Shimura curves over totally real fields", *Compos. Math.* **140** (2004), no. 2, p. 359-395.

[26] ———, "A gluing lemma and overconvergent modular forms", *Duke Math. J.* **132** (2006), no. 3, p. 509-529.

[27] ———, "Overconvergence and classicality: the case of curves", *J. Reine Angew. Math.* **631** (2009), p. 109-139.

[28] A. Kazi, "Twisted Triple Product p-adic L-function for Finite Slope Families of Hilbert Modular Forms", `https://arxiv.org/abs/2401.13230`, 2025.

[29] G. Kings, D. Loeffler & S. L. Zerbes, "Rankin–Eisenstein classes and explicit reciprocity laws", *Camb. J. Math.* **5** (2017), no. 1, p. 1-122.

[30] A. Lei, D. Loeffler & S. L. Zerbes, "Euler systems for Hilbert modular surfaces", *Forum Math. Sigma* **6** (2018), article no. e23 (67 pages).

[31] Y. Liu, S. Zhang & W. Zhang, "A p-adic Waldspurger formula", *Duke Math. J.* **167** (2018), no. 4, p. 743-833.

[32] D. Loeffler & S. L. Zerbes, "Rankin–Eisenstein classes in Coleman families", *Res. Math. Sci.* **3** (2016), article no. 29 (53 pages).

[33] ———, "On the Bloch–Kato conjecture for GSp(4)", `https://arxiv.org/abs/2003.05960`, 2024.

[34] H. Y. Loke, "Trilinear forms of $\mathfrak{gl}_2$", *Pac. J. Math.* **197** (2001), no. 1, p. 119-144.

[35] S. Molina, "Finite slope triple product p-adic L-functions over totally real number fields", *J. Number Theory* **223** (2021), p. 267-306.

[36] D. Prasad, "Trilinear forms for representations of GL(2) and local ϵ-factors", *Compos. Math.* **75** (1990), no. 1, p. 1-46.

[37] C. Skinner & E. Urban, "The Iwasawa main conjectures for GL_2", *Invent. Math.* **195** (2014), no. 1, p. 1-277.

[38] E. Urban, "Eigenvarieties for reductive groups", *Ann. Math. (2)* **174** (2011), no. 3, p. 1685-1784.

[39] ———, "Nearly overconvergent modular forms", in *Iwasawa theory 2012*, Contributions in Mathematical and Computational Sciences, vol. 7, Springer, 2014, p. 401-441.

[40] M.-F. Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics, vol. 800, Springer, 1980, vii+169 pages.

Manuscrit reçu le 8 mars 2022,
révisé le 3 janvier 2023,
accepté le 29 novembre 2024.

Daniel BARRERA SALAZAR
Universidad de Santiago de Chile
Alameda 3363, Estación Central,
Santiago (Chile)
daniel.barrera.s@usach.cl

Santiago MOLINA
Universitat de Lleida
Campus Universitari Igualada-UdL
Av. Pla de la Massa 8,
08700 Igualada (Spain)
santiago.molina@udl.cat