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THE GRUNWALD PROBLEM AND HOMOGENEOUS SPACES WITH NONSOLVABLE STABILISERS

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ABSTRACT. — We give an affirmative answer to the Grunwald problem for new families of nonsolvable finite groups G, away from the set of primes dividing $|G|$. Furthermore, we show that such G verify the condition (BM), that is, the Brauer–Manin obstruction to weak approximation is the only one for quotients of SL_n by G. These new families include extensions of groups satisfying (BM) by kernels which are products of symmetric groups $\mathfrak{S}_m$, with $m \neq 2, 6$, and alternating groups $\mathfrak{A}_5$. We also investigate (BM) for small groups by giving an explicit list of small order groups for which (BM) is unknown and we show that for many of them (BM) holds under Schinzel's hypothesis.

RÉSUMÉ. — Nous apportons une réponse positive au problème de Grunwald pour de nouvelles familles de groupes finis non résolubles G, en dehors de l'ensemble des places divisant $|G|$. De plus, nous montrons que ces groupes G vérifient la propriété (BM), c'est-à-dire que l'obstruction de Brauer–Manin à l'approximation faible est la seule pour les quotients de SL_n par G. Ces nouvelles familles sont formées d'extensions de groupes vérifiant (BM) par des produits de groupes symétriques $\mathfrak{S}_m$, pour $m \neq 2, 6$, et de copies du groupe alterné $\mathfrak{A}_5$. Nous étudions également (BM) pour des groupes de petit cardinal, en donnant une liste explicite de tels groupes pour lesquels (BM) est inconnue, et nous démontrons que pour plusieurs d'entre eux, (BM) est vérifiée si l'hypothèse de Schinzel est vraie.

1. Introduction

The Grunwald problem is a stronger version of the Inverse Galois Problem (IGP) originating in the classical problem of determining which division algebras admit a G-crossed product structure for a given finite group G,

see [29, 42, 48, 51, 57]. Further recent interest in the problem arose from its relation to the regular IGP [16, 34] and weak approximation [18, 28, 30].

The problem asks for the possible local behaviour of Galois extensions L/k with finite Galois group G, that is, the possibilities for their completions L_v/k_v at finitely many places v of k. It may be stated as follows. Given a finite set of places S of a number field k, and Galois field extensions $L^{(v)}/k_v$ for $v \in S$ with embeddings $\mathrm{Gal}(L^{(v)}/k_v) \to G$, determine if there exists a Galois extension L/k with $\mathrm{Gal}(L/k) \simeq G$ such that $L_v \cong L^{(v)}$ for all $v \in S$?

The Grunwald–Wang theorem [61] answers the problem affirmatively when G is abelian and S does not contain a place dividing 2. Nevertheless, in [60], a counterexample with $k = \mathbf{Q}$, $2 \in S$ and $G = \mathbf{Z}/8\mathbf{Z}$ is given by Wang. In fact, Grunwald problems are expected to be solvable whenever the places of S do not divide the order of G, a property also known as the *tame approximation property* for G over k [18]. The answer at places dividing $|G|$ is less clear, cf. [44, Section 3.1-3.2] and [49] for recent work on certain groups G.

A main approach to the IGP and the Grunwald problem originates in Noether's construction, where G is equipped with a free action on a rational space and the corresponding quotient is considered. Here we consider more generally a finite algebraic k-group G, choose an embedding $G \overset{\iota}{\hookrightarrow} \mathrm{SL}_{n,k}$, and let $X := \mathrm{SL}_{n,k}/G$ be the quotient. The solvability of all Grunwald problems for a constant group G is then equivalent to the *weak approximation* (WA) property on X [28], that is, to the set $X(k)$ of k-rational points being dense in $X(k_\Omega) := \prod_{v \in \Omega_k} X(k_v)$, where Ω_k is the set of places of k and the product is endowed with the product topology.

In all known instances where (WA) fails, the failure is explained by the *Brauer–Manin set*. It is well known that $X(k)$ is contained in a closed subset $X(k_\Omega)^{\mathrm{Brun}(X)} \subseteq X(k_\Omega)$ cut out by the Brauer–Manin obstruction, see Section 2.3. It may happen that the latter containment is strict, so that (WA) fails. In fact, it is conjectured by Colliot-Thélène in [9] (see also [13, Conjecture 14.1.2]) that $X(k)$ is dense in $X(k_\Omega)^{\mathrm{Brun}(X)}$ for smooth rationally connected varieties X, in particular for the quotient variety X defined above. When this holds, we say that *the Brauer–Manin obstruction to weak approximation is the only one on X* and that G *verifies (BM)*. This does not depend on n, nor on the embedding ι by the "no-name" lemma [12, Corollary 3.9] and [13, Proposition 13.3.11]. Furthermore, by a theorem of Lucchini Arteche [36, Corollary 6.3], when fixing a constant group G and letting X as above, if G verifies (BM) then $X(k)$ is dense

in $X(k_\Omega^S) := \prod_{v \in \Omega_k \setminus S} X(k_v)$ for any finite set $S \subseteq \Omega_k$ containing the prime ideals dividing $|G|$, that is, the tame approximation property holds for G. More generally, solving all Grunwald problems for G over k is equivalent to describing the closure of $X(k)$ in $X(k_\Omega)$, see [64, Section 3.3].

The property (BM) is known for many solvable groups such as finite abelian k-groups by Borovoi [3] and, in the constant finite abelian case, by Sansucs's reinterpretation [50] of Voskresenskiĭ's work on algebraic tori [59, Section 7.2, Theorem], see [64, Theorem 3.3.1]. It is also known for split extensions of k-groups that verify (BM) by a finite abelian k-group, by Harari [28], and for finite supersolvable k-groups by Harpaz and Wittenberg [30]. However, little is known about (BM) for nonsolvable groups. In fact the known examples of such groups are the symmetric groups $\mathfrak{S}_n$, the alternating group $\mathfrak{A}_5$ [37], the simple group of Lie type $\mathrm{PSL}_2(\mathbf{F}_7)$ [40] and, more generally, groups for which a generic polynomial exists, that is, when the quotient variety X is retract rational [32, Chapter 5].

In the present paper, we show that group extensions of a group verifying (BM) by certain nonsolvable kernels N, such as powers of $\mathfrak{A}_5$ and $\mathfrak{S}_n$, also verify (BM), yielding new families of nonsolvable groups verifying (BM) and having the tame approximation property.

THEOREM 1.1. — *Let k be a number field. Consider a short exact sequence of finite algebraic k-groups*

$$(1.1) \qquad 1 \longrightarrow N \longrightarrow E \longrightarrow Q \longrightarrow 1,$$

with $N(\bar{k}) \simeq \mathfrak{A}_5^{t_0} \times \prod_{i=1}^s \mathfrak{S}_{n_i}^{t_i}$ for $s, t_0, t_1, \ldots, t_s \in \mathbb{N} \cup \{0\}$ and $n_1, \ldots, n_s \in \mathbb{N} \setminus \{2, 6\}$. If Q verifies (BM) over k, then E also verifies (BM) over k.

In particular, if E is constant and S is a finite set of places of k not dividing the cardinal of G, then all Grunwald problems for E over S are solvable, that is, E has tame approximation over k.

In a parallel work, Harpaz and Wittenberg [31, (1) of Corollary 4.13] prove independently – with a different method – the theorem in case Q is supersolvable and $N(\bar{k}) \simeq \mathfrak{A}_5$ or $\mathfrak{S}_n$ for $n \in \mathbb{N} \setminus \{2, 6\}$. For supersolvable Q, their theorem [31, Theorem 4.5] actually allows N to be any group that satisfies the following condition:

(•)   For every homogeneous space X of $\mathrm{SL}_{n,k}$ and $\bar{x} \in X(\bar{k})$ whose stabiliser H over $\bar{x}$ satisfies $H(\bar{k}) \simeq N(\bar{k})$, the Brauer–Manin obstruction to weak approximation is the only one on X.

For the proof of Theorem 1.1, we do not require Q to be supersolvable but on the other hand assume Q verifies (BM) and reduce to the case

where (1.1) splits. In such case, we show that N can be chosen to be any group which satisfies (•) with the additional condition on the X's to have a rational point. For the purpose of proving this, we supply in Theorem 6.1 a dévissage method for (BM) which extends Harari [28, Théorème 1] and whose proof relies on his fibration method [28, Théorème 3]. In particular, this approach allows showing that (BM) holds for iterative semidirect products of groups verifying (BM) by abelian groups or by products of copies of $\mathfrak{A}_5$, $\mathfrak{S}_n$, $n \neq 2, 6$. Sufficient conditions to adding further groups to this list are provided in Corollary 6.2.

To allow powers of $\mathfrak{S}_n$ and $\mathfrak{A}_5$ in the kernel, we first consider more generally forms G of powers of constant k-groups N which are center-free and indecomposable, that is, which cannot be written as a product of two nontrivial groups. We show that such groups G are Weil restrictions along a finite étale algebra A/k of forms of the extension of N to A, see Theorem 3.1. This yields a description of homogeneous spaces of SL_r whose geometric stabilisers are powers of complete indecomposable center-free groups, see Theorem 4.1.

We then deduce that homogeneous spaces of SL_r with geometric stabiliser $\mathfrak{S}_n^t$, $n \neq 2, 6$, are stably rational, see Corollary 4.2. Similarly, we show that homogeneous spaces of SL_r with geometric stabiliser $\mathfrak{A}_5^t$ are retract rational, see Corollary 5.2, extending Maeda's theorem [37] on the stable rationality of the field of invariants of $\mathfrak{A}_5$. To carry this out, we use a "twisted" version of an argument of Buhler [32, Section 2.3, pp. 46–47] to produce a strongly versal G-torsor with a rational base for twisted forms G of $\mathfrak{A}_5^t$, $t \geqslant 1$.

Small groups

We apply Theorem 1.1 and previous known results in the literature to give an explicit list of the nonsolvable groups with cardinality at most 500 for which (BM) is known, see Proposition 6.4. The list of those "small" nonsolvable groups for which (BM) is unknown is given in Table 6.1. The smallest nonsolvable groups for which (BM) is unknown are central extensions of $\mathfrak{A}_5, \mathrm{Sym}_5$, and $\mathrm{PSL}_2(7)$ of orders 240 and 336. The smallest simple group for which (BM) is unknown is $\mathfrak{A}_6$. We also list the "small" (solvable) groups of cardinality at most 191 for which (BM) is unknown, see Proposition 7.4. Furthermore, under Schinzel's hypothesis (H) [52], we show that all but one, a semidirect product of $\mathbf{Z}/4\mathbf{Z}$ acting on the Heisenberg group He_3, verify (BM). We suspect that a similar reasoning could show that $\mathrm{He}_3 \rtimes \mathbf{Z}/4\mathbf{Z}$ verifies (BM) conditionally on Schinzel's hypothesis.

Property (BM) under Schinzel

To get (BM) under Schinzel's hypothesis, we prove the following theorem. Recall that a finite k-group Q is *supersolvable* if there exists a sequence $\{1\} = Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n = Q$ of normal k-subgroups of Q such that Q_i/Q_{i-1} is cyclic for all i.

THEOREM 1.2. — *Let k be a number field. Consider a short exact sequence of finite algebraic k-groups*

$$1 \longrightarrow N \longrightarrow E \longrightarrow Q \longrightarrow 1$$

such that the derived subgroup of $N(\bar{k})$ is $\mathbf{Z}/2\mathbf{Z}$. Assume that either Q is supersolvable, or the sequence splits and Q verifies (BM). If Schinzel's hypothesis (H) holds, then E verifies (BM).

Note that the class of supersolvable groups contains that of nilpotent groups. Further note that supersolvability differs from solvability by requiring that Q_{i-1} is normal in Q and not only in Q_i. As an example, the alternating group $\mathfrak{A}_4$ is solvable, but not supersolvable.

To prove the theorem, we show (BM) holds under the Schinzel's hypothesis (H) for homogeneous spaces of SL_r whose geometric stabiliser has derived subgroup $\mathbf{Z}/2\mathbf{Z}$, see Theorem 7.1. For this, we combine the fibration method [63, Chapitre 3, Corollaire 3.5] that relies on Schinzel's hypothesis (H), and a descent method for torsors under tori [30, Corollaire 2.2]. We then make use of Theorem 6.1 and the aforementioned theorem of Harpaz and Wittenberg [31, Theorem 4.5] to deduce Theorem 1.2 in Section 7.2.

Acknowledgement

2. Preliminaries

2.1. Notation

Throughout the paper, if k is a field, $\bar{k}$ denotes a fixed separable closure of k, and Γ_k the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$. A variety over k is a

separated k-scheme of finite type. If G is an algebraic group over a perfect field k, a homogeneous space of G is a k-variety X endowed with a left action of G such that the action of $G(\overline{k})$ on $X(\overline{k})$ is transitive.

For a connected scheme S and group S-schemes G and H, we say that H is an *S-form* of G if there exists an étale cover $T \to S$ such that the *base changes* G_T and H_T are T-isomorphic group schemes. When $S = \mathrm{Spec}(k)$, we say H is a k-form of G. Recall that S-forms of G are classified by the pointed set $\mathrm{H}^1_{\text{ét}}(S, \underline{\mathrm{Aut}}(G))$ [41, Chapter III, Section 4, p. 134] where $\underline{\mathrm{Aut}}(G)$ is the sheaf of automorphisms of G.

When $S' \to S$ is a finite locally free morphism of schemes and X' is a quasi-projective S'-scheme, we denote by $\mathrm{Res}_{S'/S}(X')$ the Weil restriction of X' along $S' \to S$, which is again an S-scheme [4, Section 7.6, Theorem 4].

2.2. Torsors

Let k be a field, G an affine algebraic k-group and $f : Y \to X$ a morphism of k-varieties. Assume a left (resp. right) action of G on Y is given, and let G act trivially on X. We say that f is a left (resp. right) torsor over X when f is a G-equivariant étale map and the morphism $G \times_X Y \to Y \times_X Y$ mapping (g, y) to $(g{\cdot}y, y)$ (resp. to $(y{\cdot}g, y)$) is an isomorphism. When the context is clear, we omit the left or right nature of the torsor.

Furthermore, when A is a k-algebra and G a group $\mathrm{Spec}(A)$-scheme, we shall denote by $\mathrm{H}^1(A, G)$ the pointed set $\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(A), G)$ defined in [13, Section 2.2.1]. When $A = k$, this corresponds to the nonabelian Galois cohomology defined in [54, Chapitre I, Section 5] and it classifies both right and left G-torsors over k. In this case, we let $\mathrm{Z}^1(k, G)$ denote the pointed set of 1-cocycles, that is, continous maps $(a_\gamma)_{\gamma \in \Gamma_k} : \Gamma_k \to E$ verifying $a_{\gamma\zeta} = a_\gamma \gamma(a_\zeta)$ for any $\gamma, \zeta \in \Gamma_k$, see [53, Chapitre I, Section 5.1].

When G is an affine algebraic group over a field k, $\sigma \in \mathrm{Z}^1(k, G)$ is a right k-torsor under G, and Y a quasi-projective variety endowed with a left G-action, we denote by $_\sigma Y$ the quotient of $\sigma \times_k Y$ by the action of G defined as $g{\cdot}(s, y) = (sg^{-1}, gy)$ (the existence of such a variety is for instance ensured by [56, Lemma 2.2.3]). The quotient $_\sigma Y$ is also known as the *contracted product* of σ and Y. In particular, for a k-group G and a right k-torsor $\sigma \in \mathrm{Z}^1(k, G)$ one gets the twisted group $_\sigma G$ when G is acting on itself on the left by $g{\cdot}h = ghg^{-1}$. Then, the twisted group $_\sigma G$ acts on $_\sigma Y$ in the following way: if (s, g) is the class of $\alpha \in {_\sigma G}(\overline{k})$ and (s', y) the class of $\beta \in {_\sigma Y}(\overline{k})$, then after choosing $h \in G(\overline{k})$ such that $s' = s{\cdot}h$, one may set $\alpha{\cdot}\beta$ as the class of $(s, (gh){\cdot}y)$ (also see [56, p. 20, Example 2]).

Let $f : Y \to X$ be a left G-torsor, for a linear algebraic k-group G. It is said to be *weakly versal* if, for every field extension M/k and every left G-torsor $t : T \to \mathrm{Spec}(M)$, there exists an M-point a of X such that t is the base change of f by a. If, moreover, a may be chosen in any nonempty Zariski open subset of X, then f is said to be *versal*. Following the terminology of [24, Section 1] and the definitions in [23, Section 1], we say that f is *strongly versal* if there exists a finite dimensional k-vector space V on which G acts on the left faithfully and a G-equivariant dominant rational map $V \dashrightarrow Y$. According to [23, Theorem 1.1], we have the following implications:

$$\text{strongly versal} \Longrightarrow \text{versal} \Longrightarrow \text{weakly versal}.$$

When $\underline{s}$ is a finite tuple of indeterminates, we say that a polynomial $f(\underline{s}, x) \in k(\underline{s})[x]$ is a *generic polynomial* over k for a finite group G if for every Galois extension E/M with group G and an overfield $M \supseteq k$, there exists $\underline{a} \in M^{|\underline{s}|}$ such that $f(\underline{a}, x)$ is well defined and E is a splitting field of $f(\underline{a}, x)$ over M. By [19, Theorem 1], the existence of a generic polynomial for G is equivalent to the existence of a weakly versal G-torsor whose base is a nonempty open subset of an affine space.

2.3. Local-global principles and rationality

Over number fields k, we consider weak approximation and the Brauer–Manin obstruction on a smooth k-variety X. Recall that X verifies *weak approximation* off a finite subset $S \subseteq \Omega_k$ whenever the diagonal embedding of $X(k)$ is dense in $\prod_{v \in \Omega_k \setminus S} X(k_v)$, the latter being endowed with the product topology. Let $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbf{G}_m)$ denote the Brauer group of X, and $\mathrm{Br}_{\mathrm{un}}(X)$ the Brauer group of any smooth compactification of X [13, Proposition 3.7.10]. Setting $X(k_\Omega) = \prod_{v \in \Omega_k} X(k_v)$, the *Brauer–Manin pairing* has been introduced by Manin [38] (see also [28, Section 1.3]) and is given by:

$$\mathrm{Br}_{\mathrm{un}}(X) \times X(k_\Omega) \longrightarrow \mathbf{Q}/\mathbf{Z}$$
$$((x_v), \alpha) \longmapsto \sum_{v \in \Omega_K} \mathrm{inv}_v \left(x_v^*(\alpha) \right),$$

where $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z}$ is the local invariant and x_v^* stands for the specialization morphism of Brauer groups $\mathrm{Br}(x_v) : \mathrm{Br}(X) \to \mathrm{Br}(k_v)$. Elements of $X(k_\Omega)$ orthogonal to $\mathrm{Br}_{\mathrm{un}}(X)$ form a closed subset of $X(k_\Omega)$ which contains $X(k)$ and which is called the Brauer–Manin set of X. We

shall say that *the Brauer–Manin obstruction to weak approximation is the only one on X* if $X(k)$ is dense in $X(k_\Omega)^{\mathrm{Brun}(X)}$. In this situation, we will write that X verifies (BM).

Furthermore, over an arbitrary field k, we describe the properties of varieties considered throughout the paper. Two k-varieties X and Y are said to be *stably k-birational*, or *stably k-birationally equivalent*, if there exist positive integers m and n such that $X \times \mathbf{P}_k^m$ and $Y \times \mathbf{P}_k^n$ are k-birational. The variety X is said to be *stably k-rational* if it is stably k-birational to $\mathbf{P}_k^n$ for some positive integer n. We also say that the variety X is *retract k-rational* if there exists a rational map $\mathbf{P}_k^n \dashrightarrow X$ with a rational section, for some positive integer n. When there is no confusion, the underlying field will be omitted in the preceding terminologies. Weak approximation and (BM) are stably k-birational invariants.

As a ubiquitous statement in this article, we recall the statement of the "no-name" lemma [12, Corollary 3.9]:

LEMMA 2.1 ("No-name" lemma). — *Let k be a field and G a finite k-group. For all positive integers r, s and all embeddings $G \hookrightarrow \mathrm{SL}_{r,k}$ and $G \hookrightarrow \mathrm{SL}_{s,k}$, the quotient varieties $\mathrm{SL}_{r,k}/G$ and $\mathrm{SL}_{s,k}/G$ are stably k-birational.*

Also, for k-groups we consider throughout this paper the following properties:

Notation 2.2. — When k is a field, consider the following properties on affine k-groups G:

   (1) there exists an embedding $G \hookrightarrow \mathrm{SL}_{n,k}$ such that the variety $\mathrm{SL}_{n,k}/G$ is stably k-rational;

   (2) there exists a strongly-versal G-torsor $Y \to X$ over k, where X is rational;

   (3) there exists an embedding $G \hookrightarrow \mathrm{SL}_{n,k}$ such that the variety $\mathrm{SL}_{n,k}/G$ is retract k-rational;

   (4) the field k is a number field and there exists an embedding $G \hookrightarrow \mathrm{SL}_{n,k}$ such that the Brauer–Manin obstruction to weak approximation is the only one for the variety $\mathrm{SL}_{n,k}/G$;

   (5) there exists a finite set of places S of k and an embedding $G \hookrightarrow \mathrm{SL}_{n,k}$ such that the variety $\mathrm{SL}_{n,k}/G$ verifies weak approximation off S.

One may notice that properties (1), (3), (4) and (5) of Notation 2.2 do not depend on the choice of the embedding, by Lemma 2.1 combined to the stably birational invariance of stable rationality, retract rationality, the Brauer–Manin obstruction to weak approximation and weak approximation

off a finite set of places [13, Proof of Proposition 13.2.3]. Furthermore, we have the following classical implications:

$$(1) \longrightarrow (2) \longrightarrow (3) \longrightarrow (4) \longrightarrow (5)$$

where the first implication is given by choosing $Y \to X$ as being $f \times \mathrm{id}_{\mathbf{P}_k^r}$ where $f : \mathrm{SL}_{n,k} \to \mathrm{SL}_{n,k}/G$ and $\mathbf{P}_k^r$ is chosen such that $\mathbf{P}_k^r \times \mathrm{SL}_{n,k}/G$ is rational. Moreover, the second implication is a consequence of the equivalence of (1) and (3) in [39, Proposition 4.2].

2.4. Preliminaries on group theory and almost complete stabilisers

Let C_n denote the cyclic group of order n, and $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$) the symmetric (resp. alternating) group of degree n.

Let us recall that a finite group G is *complete* if it is center-free with no nontrivial outer automorphism. Examples of complete groups are the $\mathfrak{S}_n$'s for $n \neq 2, 6$. Furthermore, we will say that a group G is *indecomposable* if it cannot be written as a product of nontrivial groups: this is for example the case of $\mathfrak{S}_n$ and $\mathfrak{A}_n$ when $n \geqslant 1$, since their normal subgroups have no direct factor.

Before going any further, let us state the following ubiquitous lemma on automorphisms of powers of indecomposable groups.

LEMMA 2.3 ([2, Theorem 3.1]). — *If N is a group and t a positive integer, then the following morphism is injective:*

$$\iota : \mathrm{Aut}(N)^t \rtimes \mathfrak{S}_t \longrightarrow \mathrm{Aut}(N^t)$$

$$((\varphi_i)_{i=1}^t, \sigma) \longmapsto \left[(n_1, \dots, n_t) \mapsto \prod_{i=1}^t \varphi_i(n_{\sigma^{-1}(i)}) \right],$$

where $\mathfrak{S}_t$ acts on $\mathrm{Aut}(N)^t$ by $\sigma \cdot (\varphi_i)_{i=1}^t = (\varphi_{\sigma^{-1}(i)})_{i=1}^t$ for any $\sigma \in \mathfrak{S}_t$ and $(\varphi_i)_{i=1}^t \in \mathrm{Aut}(N)^t$. Moreover, if N is indecomposable and center-free, then ι is an isomorphism.

Using the terminology of [31, Section 4.1.4], we shall say that a finite group G is *almost complete* if it is center-free and if the morphism $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ has a section. Examples of almost simple groups are the alternating groups $\mathfrak{A}_n$ for $n \notin \{2, 3, 6\}$, and all complete groups, such as the $\mathfrak{S}_n$'s for $n \notin \{2, 6\}$. As these groups are ubiquitous in this article, we summarise their basic properties in the following lemma.

LEMMA 2.4.

(1) A finite group G is almost complete if and only if any short exact sequence of profinite groups

$$1 \longrightarrow G \longrightarrow E \longrightarrow Q \longrightarrow 1$$

splits as a semi-direct product of profinite groups $E \simeq G \rtimes Q$.

(2) If G is a finite almost complete indecomposable group, then for any positive integer t, the group G^t is almost complete.

(3) If $H_1, \ldots, H_r$ are finite almost complete groups where no pair of the H_i have a common direct factor, then $\prod_{i=1}^{r} H_i$ is almost complete.

Proof. — Assertion (1) corresponds to [31, Lemma 4.12(1)]. To prove (2), first notice that G^t is center-free. Furthermore, since G is indecomposable, Lemma 2.3 ensures that $\mathrm{Aut}(G^t) \simeq \mathrm{Aut}(G)^t \rtimes \mathfrak{S}_t$ where $\mathfrak{S}_t$ acts on $\mathrm{Aut}(G)^t$ by permuting the coordinates. Denote by s a section of $a : \mathrm{Aut}(G) \to \mathrm{Out}(G)$. Then, the morphism $\mathrm{Aut}(G^t) \to \mathrm{Out}(G^t)$ corresponds to the morphism $\prod_{1 \leqslant i \leqslant t} a \times \mathrm{id}_{\mathfrak{S}_t} : \mathrm{Aut}(G)^t \rtimes \mathfrak{S}_t \to \mathrm{Out}(G)^t \rtimes \mathfrak{S}_t$ and a section is given by $\prod_{1 \leqslant i \leqslant t} s \times \mathrm{id}_{\mathfrak{S}_t}$, which prove (2).

As for (3), the group $\prod_{i=1}^{r} H_i$ is clearly center-free. Then [2, Theorem 2.2] ensures that $\mathrm{Aut}(\prod_{i=1}^{r} H_i) = \prod_{i=1}^{r} \mathrm{Aut}(H_i)$, so that the morphism

$$\mathrm{Aut}\left(\prod_{i=1}^{r} H_i\right) \longrightarrow \mathrm{Out}\left(\prod_{i=1}^{r} H_i\right),$$

which coincides with $\prod_{i=1}^{r}(\mathrm{Aut}(H_i) \to \mathrm{Out}(H_i))$, has a section since each of the H_i's is almost complete. $\square$

Combining Lemma 2.4(1) with a theorem of Pál–Schlank one gets:

PROPOSITION 2.5. — *Let X be a homogeneous space of $\mathrm{SL}_{n,k}$ and $\bar{x} \in X(\bar{k})$. Denote by G the stabiliser of $\bar{x}$. If the group $G(\bar{k})$ is finite and almost complete, then $X(k) \neq \emptyset$.*

Proof. — By Lemma 2.4(1) the exact sequence

$$1 \longrightarrow G(\bar{k}) \longrightarrow \pi_{\text{ét}}^1(X, \bar{x}) \longrightarrow \Gamma_k \longrightarrow 1,$$

splits. For homogeneous spaces of SL_n, the existence of such a section implies the existence of a rational point by [43, Theorem 7.6]. $\square$

Eventually, the following proposition gives a structural statement for homogeneous spaces whose stabilisers are products of almost complete groups with no pair having a common direct factor:

PROPOSITION 2.6. — *Let $H_1, \ldots, H_r$ be almost complete finite constant k-groups with no pair of the H_i's having a common direct factor, and m a positive integer. Suppose X is an $\mathrm{SL}_{m,k}$-homogeneous space whose geometric stabiliser is the direct product $\prod_{i=1}^{r} H_i$. Then X is stably birational to a product $\prod_{i=1}^{r} X_i$ where:*

(1) *there exists $n \in \mathbf{N}$ such that each X_i is a $\mathrm{SL}_{n,k}$-homogeneous space;*
(2) *for each $i \in \{1, \ldots, r\}$, and for any $\overline{x_i} \in X_i(\overline{k})$, if one denotes by G_i the stabiliser of $\overline{x_i}$, then $G_i(\overline{k}) \simeq H_i$.*

Proof. — By Proposition 2.5, one has $X(k) \neq \emptyset$. The choice of a rational point of X supplies a k-form G of the constant group $\prod_{i=1}^{r} H_i$ and an isomorphism of $\mathrm{SL}_{m,k}$-homogeneous spaces $X \simeq \mathrm{SL}_{m,k}/G$.

Furthermore, $\mathrm{Aut}(\prod_{i=1}^{r} H_i) = \prod_{i=1}^{r} \mathrm{Aut}(H_i)$ by [2, Theorem 2.2]. From this, the Galois action on $G(\overline{k})$ actually corresponds to a morphism $\Gamma_k \to \prod_{i=1}^{r} \mathrm{Aut}(H_i)$. For each $i \in \{1, \ldots, r\}$, the projection $\Gamma_k \to \mathrm{Aut}(H_i)$ of the Galois action on $\mathrm{Aut}(H_i)$ defines a k-form $\widetilde{H_i}$, so that $G \simeq \prod_{i=1}^{r} \widetilde{H_i}$. Now, choose a positive integer n and, for each $i \in \{1, \ldots, r\}$, an embedding $\widetilde{H_i} \hookrightarrow \mathrm{SL}_{n,k}$. Then, Lemma 2.1 ensures that $\mathrm{SL}_{m,k}/G$ is stably birational to $\prod_{i=1}^{r} \mathrm{SL}_{n,k}/\widetilde{H_i}$, so that after setting $X_i = \mathrm{SL}_{n,k}/\widetilde{H_i}$, one gets the required statement. $\qquad\square$

2.5. Nonabelian Shapiro's map

If Γ is a profinite group, a Γ-*group* is a discrete group E endowed with a continuous action $\Gamma \xrightarrow{\varphi} \mathrm{Aut}(E)$. We shall also say that φ yields a Γ-group, and write $g.e := \varphi(g)(e)$ for $(g, e) \in \Gamma \times E$ when there is no confusion as to what φ is. We then denote by $E \rtimes_\varphi \Gamma$ the semidirect product relative to φ and by $\mathrm{Sec}_\Gamma(E \rtimes_\varphi \Gamma)$ the set of continuous sections of $E \rtimes_\varphi \Gamma \to \Gamma$. We use the following well-known correspondence:

FACT 2.7 ([58, Lemma 7]). — *Let Γ be a profinite group and $\varphi : \Gamma \to \mathrm{Aut}(E)$ a Γ-group. Denote by $\mathrm{pr}_1 : E \rtimes_\varphi \Gamma \to E$ the first projection (which is not necessarily a morphism). Then the following map is bijective*

$$\mathrm{Sec}_\Gamma(E \rtimes_\varphi \Gamma) \longrightarrow \mathrm{Z}^1(\Gamma, E)$$

$$\sigma \longmapsto \mathrm{pr}_1 \circ \sigma.$$

For a closed subgroup $\Delta \leqslant \Gamma$ and a Δ-group E, we let

$$\mathrm{Ind}_\Delta^\Gamma E := \{f : \Gamma \longrightarrow E \mid f(dg) = df(g) \text{ for all } (d, g) \in \Delta \times \Gamma\}$$

denote the *induced* Γ-*group* endowed with the left action $(\gamma \cdot f)(g) = f(g\gamma)$ for $\gamma \in \Gamma$ and $f \in \mathrm{Ind}_\Delta^\Gamma E$. We may further identify $\mathrm{Ind}_\Delta^\Gamma E$ with $E^{\Delta\backslash\Gamma}$, where Γ acts on $\Delta\backslash\Gamma$ on the right, via the following lemma.

LEMMA 2.8. — *Let* Γ *be a profinite group,* Δ *a closed subgroup of* Γ *and* E *a* Δ-*group. Fix a set of representatives* $\{\varepsilon_i : i \in \Delta\backslash\Gamma\}$ *of* $\Delta\backslash\Gamma$ *and set*

$$\omega : \mathrm{Ind}_\Delta^\Gamma E \longrightarrow E^{\Delta\backslash\Gamma}$$
$$f \longmapsto (f(\varepsilon_i))_{i\in\Delta\backslash\Gamma}.$$

Furthermore for $i \in \Delta\backslash\Gamma$, $\gamma \in \Gamma$, *write* $\varepsilon_i\gamma = \delta_i(\gamma) \cdot \varepsilon_{i.\gamma}$ *for a unique* $\delta_i(\gamma) \in \Delta$. *Then:*

(1) *the map* ω *is an isomorphism of groups, which endows* $E^{\Delta\backslash\Gamma}$ *with a* Γ-*group structure defined as the composition of the* Γ-*group* $\Gamma \to \mathrm{Aut}(\mathrm{Ind}_\Delta^\Gamma E)$ *with*

$$\mathrm{Aut}(\omega) : \mathrm{Aut}(\mathrm{Ind}_\Delta^\Gamma E) \longrightarrow \mathrm{Aut}(E^{\Delta\backslash\Gamma})$$
$$\varphi \longmapsto \omega\varphi\omega^{-1};$$

(2) *furthermore, the action of* $\gamma \in \Gamma$ *on* $(e_i)_{i\in\Delta\backslash\Gamma} \in E^{\Delta\backslash\Gamma}$ *described in* (1) *is given by*

$$\gamma \cdot (e_i)_{i\in\Delta\backslash\Gamma} = (\delta_i(\gamma) \cdot e_{i.\gamma})_{i\in\Delta\backslash\Gamma};$$

(3) *consider the* Γ-*group structure on* $E^{\Delta\backslash\Gamma}$ *in* (1) *and let* 1 *denote the trivial coset in* $\Delta\backslash\Gamma$. *For* $a \in \mathrm{Z}^1(\Gamma, E^{\Delta\backslash\Gamma})$, *if we write* $a_\gamma = (e_i(\gamma))_{i\in\Delta\backslash\Gamma}$, *then:*

$$e_i(\gamma) = \delta_1(\varepsilon_i)^{-1} \cdot \left[e_1(\varepsilon_i)^{-1} e_1(\delta_i(\gamma)) \left(\delta_1(\delta_i(\gamma)) \cdot e_1(\varepsilon_{i.\gamma}) \right) \right];$$

(4) *if* Δ *acts trivially on* E, *then* ω *does not depend on the choice of the set of representatives of* $\Delta\backslash\Gamma$. *Furthermore, the conclusion of* (3) *may be rewritten as:*

$$e_i(\gamma) = e_1(\varepsilon_i)^{-1} e_1(\delta_i(\gamma)) e_1(\varepsilon_{i.\gamma}).$$

Proof. — The proof of (1) and (2) is straightforward, and (4) follows immediately from (3). Let us prove (3). Since a is a cocycle, $a_{\gamma\zeta} = a_\gamma \gamma(a_\zeta)$ for $\gamma, \zeta \in \Gamma$, so that (2) gives:

$$(2.1) \qquad\qquad e_i(\gamma\zeta) = e_i(\gamma)\big(\delta_i(\gamma) \cdot e_{i.\gamma}(\zeta)\big).$$

Noting that $i.\varepsilon_i^{-1} = 1$, and applying (2.1) with (i, γ, ζ) replaced by $(1, \varepsilon_i, \gamma)$, we get:

$$(2.2) \qquad e_1(\varepsilon_i\gamma) = e_{i.\varepsilon_i^{-1}}(\varepsilon_i\gamma) = e_1(\varepsilon_i)\big(\delta_1(\varepsilon_i) \cdot e_i(\gamma)\big).$$

Using $i.\varepsilon_i^{-1} = 1$ again, (2.1) with (i, γ, ζ) replaced by $(1, \delta_i(\gamma), \varepsilon_{i.\gamma})$ gives:

$$(2.3) \quad e_1\big(\delta_i(\gamma) \cdot \varepsilon_{i.\gamma}\big) = e_{i.\varepsilon_i^{-1}}\big(\delta_i(\gamma) \cdot \varepsilon_{i.\gamma}\big) = e_1(\delta_i(\gamma))\big(\delta_1(\delta_i(\gamma)) \cdot e_1(\varepsilon_{i.\gamma})\big).$$

But since $\varepsilon_i \gamma = \delta_i(\gamma) \cdot \varepsilon_{i.\gamma}$, we can equate (2.2) and (2.3), yielding (3). □

Returning to the general setting, note that the morphism of groups $\mathrm{Ind}_\Delta^\Gamma E \to E$ mapping $f : \Gamma \to E$ to $f(1)$ is Δ-equivariant, and hence induces *Shapiro's map* of pointed sets

$$\mathrm{sh} : \mathrm{Z}^1(\Gamma, \mathrm{Ind}_\Delta^\Gamma E) \longrightarrow \mathrm{Z}^1(\Delta, E)$$

which is known to be surjective by Shapiro's lemma [58, Proof of Proposition 8].

Moreover, if Δ acts trivially on E, then Shapiro's map coincides, via Fact 2.7, with a map:

$$(2.4) \qquad\qquad \mathrm{sh}' : \mathrm{Sec}_\Gamma(\mathrm{Ind}_\Delta^\Gamma E \rtimes \Gamma) \longrightarrow \mathrm{Hom}_{\mathrm{cont}}(\Delta, E).$$

In this case, if we use notations of Lemma 2.8 and identify $\mathrm{Ind}_\Delta^\Gamma E$ with $E^{\Delta \backslash \Gamma}$ via ω, the Shapiro map sh' can be described as follows:

$$(2.5) \qquad \begin{aligned} \mathrm{sh}' : \qquad\quad & \mathrm{Sec}_\Gamma(E^{\Delta \backslash \Gamma} \rtimes \Gamma) \longrightarrow \mathrm{Hom}_{\mathrm{cont}}(\Delta, E) \\ & \gamma \in \Gamma \mapsto ((e_i(\gamma))_{i \in \Delta \backslash \Gamma}, \gamma) \longmapsto \delta \mapsto e_1(\delta) \end{aligned}.$$

3. Structure of forms for powers of groups

The following theorem is the main result regarding powers of groups.

THEOREM 3.1. — *Let N be a finite constant k-group and t a positive integer. Assume further that N is indecomposable, with trivial center. If an algebraic k-group G is a k-form of N^t, then there exists an étale k-algebra A of degree t and a k-form $\widetilde{N_A}$ of $N_A := N \otimes_k A$ such that $G = \mathrm{Res}_{A/k}(\widetilde{N_A})$.*

Before giving a proof, let us recall how Weil restrictions are related to induced Γ_k-groups.

LEMMA 3.2 ([62, Theorem 1.3.1]). — *Let L/k be a finite separable extension of fields. Fix an embedding $\sigma_0 : L \hookrightarrow \bar{k}$ where $\bar{k}$ is a separable closure of k and set $\Gamma_L = \mathrm{Gal}(\bar{k}/L)$ which is a closed subgroup of Γ_k. Then for any algebraic group G over L, there exists a Γ_k-equivariant bijection*

$$(\mathrm{Res}_{L/k}G)(\bar{k}) \simeq \mathrm{Ind}_{\Gamma_L}^{\Gamma_k}(G(\bar{k}))$$

of Γ_k-groups, where Γ_L acts on the left on $G(\bar{k}) = \mathrm{Hom}_{\mathrm{Spec}(L)}(\mathrm{Spec}(\sigma_0), G)$ by composition, where $\mathrm{Spec}(\sigma_0)$ denotes the $\mathrm{Spec}(L)$-scheme $\mathrm{Spec}(\bar{k})$ corresponding to the embedding σ_0.

We also use the following technical lemma concerning Shapiro's map sh' from (2.4) when E is taken to be $\mathrm{Aut}(N)$. For this purpose, choosing a set of representatives of $\Delta\backslash\Gamma$, we identify $\mathrm{Aut}(\mathrm{Ind}_\Delta^\Gamma N)$ with $\mathrm{Aut}(N^{\Delta\backslash\Gamma})$ via the isomorphism $\mathrm{Aut}(\omega)$ from Lemma 2.8, and $\mathrm{Aut}(N)^{\Delta\backslash\Gamma}\rtimes\mathrm{Bij}(\Delta\backslash\Gamma)$ with a subgroup of $\mathrm{Aut}(N^{\Delta\backslash\Gamma})=\mathrm{Aut}(\mathrm{Ind}_\Delta^\Gamma N)$ via Lemma 2.3. Eventually, we denote by

$$(3.1) \qquad \begin{aligned} \rho:\Gamma &\longrightarrow \mathrm{Bij}(\Delta\backslash\Gamma) \\ \gamma &\longmapsto \rho(\gamma):i\mapsto i.\gamma^{-1} \end{aligned}$$

the right action of Γ on $\Delta\backslash\Gamma$.

LEMMA 3.3. — *Let N be a finite group and Γ a profinite group. Let Δ be a closed subgroup of Γ, assume that Δ acts trivially on $\mathrm{Aut}(N)$ and consider $\sigma\in\mathrm{Sec}_\Gamma(\mathrm{Ind}_\Delta^\Gamma\mathrm{Aut}(N)\rtimes\Gamma)$ and $\theta:=\mathrm{sh}'(\sigma)\in\mathrm{Hom}_{\mathrm{cont}}(\Delta,\mathrm{Aut}(N))$. Denote by $\mathrm{Ind}_\Delta^\Gamma N$ the Γ-group induced by θ and let $\beta:\Gamma\to\mathrm{Aut}(\mathrm{Ind}_\Delta^\Gamma N)$ be the associated Γ-action. Then β factors as:*

$$\Gamma \xrightarrow{\ \sigma\ } \mathrm{Ind}_\Delta^\Gamma\mathrm{Aut}(N)\rtimes\Gamma \xrightarrow{(\mathrm{id},\rho)} \mathrm{Ind}_\Delta^\Gamma\mathrm{Aut}(N)\rtimes\mathrm{Bij}(\Delta\backslash\Gamma)$$
$$\Big\downarrow{\psi}$$
$$\mathrm{Ind}_\Delta^\Gamma\mathrm{Aut}(N)\rtimes\mathrm{Bij}(\Delta\backslash\Gamma)$$

for some inner automorphism ψ of $\mathrm{Aut}(N)^{\Delta\backslash\Gamma}\rtimes\mathrm{Bij}(\Delta\backslash\Gamma)$.

Proof. — Let $\{\varepsilon_i:i\in\Delta\backslash\Gamma\}$ be a set of representatives for $\Delta\backslash\Gamma$, and $\delta_i(\gamma)\in\Delta$ such that $\varepsilon_i\gamma=\delta_i(\gamma)\varepsilon_{i.\gamma}$. Write $\sigma(\gamma)=((\varphi_i(\gamma))_{i\in\Delta\backslash\Gamma},\gamma)$ for $\gamma\in\Gamma$, so that $\varphi:\Gamma\to\mathrm{Aut}(N)^{\Delta\backslash\Gamma}$, $\gamma\mapsto(\varphi_i(\gamma))_{i\in\Delta\backslash\Gamma}$ lies in $\mathrm{Z}^1(\Gamma,\mathrm{Aut}(N)^{\Delta\backslash\Gamma})$ by Fact 2.7. Note that by Lemma 2.8(3):

$$(3.2) \qquad \varphi_i(\gamma)=\varphi_1(\varepsilon_i)^{-1}\circ\varphi_1(\delta_i(\gamma))\circ\varphi_1(\varepsilon_{i.\gamma}), \quad \text{for} \quad \gamma\in\Gamma, i\in\Delta\backslash\Gamma.$$

Recall that β induces a Γ-action on $N^{\Delta\backslash\Gamma}$ via $(\mathrm{Aut}(\omega)\circ\beta)(\gamma)=\omega\circ\beta(\gamma)\circ\omega^{-1}$. Moreover, by Lemma 2.8(2), we have

$$(3.3) \qquad \begin{aligned} \omega\circ\beta(\gamma)\circ\omega^{-1}:N^{\Delta\backslash\Gamma} &\longrightarrow N^{\Delta\backslash\Gamma} \\ (n_i)_{i\in\Delta\backslash\Gamma} &\longmapsto (\delta_i(\gamma).n_{i.\gamma})_{i\in\Delta\backslash\Gamma}. \end{aligned}$$

Since Δ acts on N through θ, and $\theta(\delta)=\varphi_1(\delta)$ by the explicit description (2.5) of sh', morphism (3.3) takes the form:

$$\begin{aligned} \omega\circ\beta(\gamma)\circ\omega^{-1}:N^{\Delta\backslash\Gamma} &\longrightarrow N^{\Delta\backslash\Gamma} \\ (n_i)_{i\in\Delta\backslash\Gamma} &\longmapsto (\varphi_1(\delta_i(\gamma))(n_{i.\gamma}))_{i\in\Delta\backslash\Gamma}. \end{aligned}$$

By definition of ι from Lemma 2.3 and by definition of ρ in (3.1), this is none but the image by ι of the automorphism

$$\left(\left(\varphi_1(\delta_i(\gamma))\right)_{i\in\Delta\backslash\Gamma}, \rho(\gamma)\right) \in \mathrm{Aut}(N)^{\Delta\backslash\Gamma} \rtimes \mathrm{Bij}(\Delta\backslash\Gamma).$$

However, by (3.2), this image is conjugate by the element $(\varphi_1(\varepsilon_i))_{i\in\Delta\backslash\Gamma} \in \mathrm{Aut}(N)^{\Delta\backslash\Gamma}$ to

$$((\mathrm{id}, \rho) \circ \sigma)(\gamma) = (\varphi_i(\gamma), \rho(\gamma))$$

which proves the statement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We may now combine Lemmas 3.2 and 3.3 to give a proof of Theorem 3.1.

Proof of Theorem 3.1. — Let G be a k-form of N^t. After choosing an isomorphism of groups $G(\bar{k}) \simeq N^t$, the action of Γ_k on $G(\bar{k})$ gives rise to a continuous action of Γ_k on N^t, that is, to a morphism $\varphi : \Gamma_k \to \mathrm{Aut}(N^t)$. By Lemma 2.3, one has $\mathrm{Aut}(N^t) \simeq \mathrm{Aut}(N)^t \rtimes \mathfrak{S}_t$, where $\mathfrak{S}_t$ acts on $\mathrm{Aut}(N)^t$ by permuting the coordinates. The composition $\psi : \Gamma_k \to \mathfrak{S}_t$ of φ with the projection $\mathrm{Aut}(N)^t \rtimes \mathfrak{S}_t \to \mathfrak{S}_t$ yields a right action of Γ_k on $\{1, \ldots, t\}$ defined by $l.\gamma := \psi(\gamma)^{-1}(l)$. Now pick a set of representatives $I \subseteq \{1, \ldots, t\}$ of the distinct orbits of $\{1, \ldots, t\}$ under the right action of Γ_k, so that $N^t = \prod_{i\in I} N^{i.\Gamma_k}$. Thus φ factors as

$$(3.4) \qquad \Gamma_k \xrightarrow{\prod_{i\in I}\alpha_i} \prod_{i\in I}\mathrm{Aut}(N^{i.\Gamma_k}) \longrightarrow \mathrm{Aut}(N^t)$$

where $\alpha_i : \Gamma_k \to \mathrm{Aut}(N^{i.\Gamma_k})$ denotes the action of Γ_k on the i-th factor $N^{i.\Gamma_k}$ of N^t. Let us denote by G_i the k-form of $N^{i.\Gamma_k}$ corresponding to $\alpha_i \in \mathrm{Z}^1(k, \mathrm{Aut}(N^{i.\Gamma_k}))$. By the factorisation (3.4), we thus have $G = \prod_{i\in I} G_i$.

Next, we fix $i \in I$ and rewrite G_i as a Weil restriction. For this purpose, we first use Lemma 2.3 to identify $\mathrm{Aut}(N^{i.\Gamma_k}) = \mathrm{Aut}(N)^{i.\Gamma_k} \rtimes \mathrm{Bij}(i.\Gamma_k)$. Furthermore, if we denote by $\rho_i : \Gamma_k \to \mathrm{Bij}(i.\Gamma_k)$ the right action described above, then α_i factors uniquely as

$$(3.5) \qquad \Gamma_k \xrightarrow{\sigma_i} \mathrm{Aut}(N)^{i.\Gamma_k} \rtimes \Gamma_k \longrightarrow \mathrm{Aut}(N)^{i.\Gamma_k} \rtimes \mathrm{Bij}(i.\Gamma_k)$$

for $\sigma_i \in \mathrm{Sec}_\Gamma(\mathrm{Aut}(N)^{i.\Gamma_k} \rtimes \Gamma_k)$. Letting Γ_i be the stabiliser of i under the right action of Γ_k on $i.\Gamma_k$, we may further identify:

$$i.\Gamma_k \simeq \Gamma_i\backslash\Gamma_k.$$

The Shapiro map (2.4) supplies a morphism $\theta_i := \mathrm{sh}'(\sigma_i) : \Gamma_i \to \mathrm{Aut}(N)$. Denote by $\mathrm{Ind}_{\Gamma_i}^{\Gamma_k}N$ the Γ_i-group induced by θ_i and $\beta_i : \Gamma_k \to \mathrm{Aut}(\mathrm{Ind}_{\Gamma_i}^{\Gamma_k}N)$

the corresponding action of Γ_k. Since $\alpha_i = (\mathrm{id}, \rho) \circ \sigma$ by (3.5), Lemma 3.3 yields an inner automorphism ψ_i of $\mathrm{Aut}(N^{i.\Gamma_k})$ such that

$$(3.6) \qquad \mathrm{Aut}(\omega_i) \circ \beta_i = \psi_i \circ \alpha_i.$$

Now, choosing $\chi_i \in \mathrm{Aut}(N^{i.\Gamma_k})$ such that ψ_i^{-1} is the conjugation map by χ_i, equation (3.6) yields the following commutative diagram:

$$
\begin{array}{ccc}
& \Gamma_k & \\
{}^{\beta_i}\swarrow & & \searrow^{\alpha_i} \\
\mathrm{Aut}(\mathrm{Ind}_{\Gamma_i}^{\Gamma_k} N) \xrightarrow[\mathrm{Aut}(\chi_i \circ \omega_i)]{\sim} & & \mathrm{Aut}(N^{i.\Gamma_k}).
\end{array}
$$

In other words, the map

$$(3.7) \qquad \chi_i \circ \omega_i : \mathrm{Ind}_{\Gamma_i}^{\Gamma_k} N \longrightarrow N^{i.\Gamma_k}$$

is a Γ_k-equivariant isomorphism of groups, where Γ_k acts on $\mathrm{Ind}_{\Gamma_i}^{\Gamma_k} N$ (resp. on $N^{i.\Gamma_k}$) via β_i (resp. α_i). Furthermore, recall that by definition of G_i, there is a Γ_k-equivariant isomorphism

$$(3.8) \qquad G_i(\bar{k}) \simeq N^{i.\Gamma_k}$$

where the action of Γ_k on the left-hand side (resp. on the right-hand side) is the Galois action on points (resp. is given by α_i). Also, if we set $M_i := \bar{k}^{\Gamma_i}$ and $\widetilde{N_{M_i}}$ the M_i-form of N associated to $\theta_i \in \mathrm{Z}^1(M_i, \mathrm{Aut}(N))$, then Lemma 3.2 supplies a Γ_k-equivariant isomorphism of groups

$$(3.9) \qquad \mathrm{Res}_{M_i/k}(\widetilde{N_{M_i}})(\bar{k}) \simeq \mathrm{Ind}_{\Gamma_i}^{\Gamma_k} N.$$

where the action of Γ_k on the left-hand side (resp. right-hand side) is the Galois action on points (resp. is given by β_i). Now, if we combine (3.8) and (3.9) with (3.7), the map $\chi_i \circ \omega_i$ may be rewritten as a Γ_k-equivariant isomorphism

$$G_i(\bar{k}) \simeq \mathrm{Res}_{M_i/k}(\widetilde{N_{M_i}})(\bar{k})$$

where the action of Γ_k on both sides is the Galois action on points. By Galois descent, this extends to an isomorphism of algebraic groups

$$G_i \simeq \mathrm{Res}_{M_i/k}(\widetilde{N_{M_i}}).$$

Thus, by setting $A := \prod_{i \in I} M_i$ and $\widetilde{N_A} := \prod_{i \in I} \widetilde{N_{M_i}}$, we eventually get

$$G \simeq \prod_{i \in I} \mathrm{Res}_{M_i/k}(\widetilde{N_{M_i}}) = \mathrm{Res}_{A/k}(\widetilde{N_A})$$

which proves the statement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We deduce the following description of homogeneous spaces for powers of groups.

Corollary 3.4. — *Let N be a finite constant k-group, t a positive integer and G a k-form of N^t. Assume further that N is indecomposable with trivial center. Then there exist an étale k-algebra A of degree t and a k-form $\widetilde{N_A}$ of N_A such that, for every k-embedding $G \hookrightarrow \mathrm{SL}_{r,k}$ and A-embedding $\widetilde{N_A} \hookrightarrow \mathrm{SL}_{s,A}$, the varieties $\mathrm{SL}_{r,k}/G$ and $\mathrm{Res}_{A/k}(\mathrm{SL}_{s,A}/\widetilde{N_A})$ are stably birationally equivalent.*

Proof. — By Theorem 3.1, there exists an étale k-algebra A of degree t and an A-form $\widetilde{N_A}$ of N_A endowed with an isomorphism $G \simeq \mathrm{Res}_{A/k}(\widetilde{N_A})$. Let $\widetilde{N_A}$ act faithfully on an affine space $\mathbf{A}_A^n$. By [15, Corollary A.5.4(3)], one then has that

$$\mathrm{Res}_{A/k}\left(\mathbf{A}_A^n/\widetilde{N_A}\right) \simeq \left(\mathrm{Res}_{A/k}\left(\mathbf{A}_A^n\right)\right)/\mathrm{Res}_{A/k}\left(\widetilde{N_A}\right).$$

Since $\mathrm{Res}_{A/k}\left(\mathbf{A}_A^n\right) \simeq \mathbf{A}_k^{nt}$ and $\mathrm{Res}_{A/k}(\widetilde{N_A}) \simeq G$, one deduces from the latter that $\mathrm{Res}_{A/k}\left(\mathbf{A}_A^n/\widetilde{N_A}\right) \simeq \mathbf{A}_k^{nt}/G$. The statement then follows from Lemma 2.1. $\qquad\square$

Denote by $\mathcal{P}_k$ any of the properties on affine k-groups listed in Notation 2.2.

Corollary 3.5. — *Let t be a positive integer and N a constant indecomposable finite k-group with trivial center. If every L-form of N_L verifies $\mathcal{P}_L$ for every separable field extension L/k with $[L : k] \leqslant t$, then any k-form of N^t verifies $\mathcal{P}_k$.*

Proof. — Let t be a positive integer and G a k-form of N^t. Let us now choose an étale k-algebra A of degree t and an A-form $\widetilde{N_A}$ of N_A as in Corollary 3.4. After choosing a k-embedding $G \hookrightarrow \mathrm{SL}_{r,k}$ and an A-embedding $\widetilde{N_A} \hookrightarrow \mathrm{SL}_{s,A}$, Corollary 3.4 ensures that $\mathrm{SL}_{r,k}/G$ and $\mathrm{Res}_{A/k}\left(\mathrm{SL}_{s,A}/\widetilde{N_A}\right)$ are stably birational. Since A is étale over k, it may be written as $A = \prod_{1 \leqslant i \leqslant n} M_i$ where the field extensions M_i/k, $i = 1, \ldots, r$ are separable of degree at most t. Moreover, $\mathrm{H}^1(A, \mathrm{Aut}(N)) = \prod_{1 \leqslant i \leqslant n} \mathrm{H}^1(M_i, \mathrm{Aut}(N))$ so that one may write $\widetilde{N_A} = \prod_{1 \leqslant i \leqslant n} \widetilde{N_{M_i}}$, where the M_i-group $\widetilde{N_{M_i}}$ is an M_i-form of N_{M_i} for $i = 1, \ldots, n$. From this, we infer that $\mathrm{SL}_{r,k}/G$ is stably birational to the variety

$$\prod_{1 \leqslant i \leqslant n} \mathrm{Res}_{M_i/k}\left(\mathrm{SL}_{s,M_i}/\widetilde{N_{M_i}}\right).$$

Furthermore, for each $i \in \{1, \ldots, n\}$, the variety $\mathrm{SL}_{s,M_i}/\widetilde{N_{M_i}}$ is assumed to satisfy $\mathcal{P}_{M_i}$. Since the Weil restriction of a variety which is retract rational, resp. rational, resp. satisfies that the Brauer–Manin obstruction to weak

approximation is the only one, resp. satisfies that weak approximation holds off a finite set of places, satisfies the same property (see [7, Theorem 1.1] for the Brauer–Manin obstruction to weak approximation, and [14, Example 4.2] for weak approximation), the statement of this proposition follows when $\mathcal{P}_k$ is one of the properties (1), (3), (4) and (5) in Notation 2.2.

Assume now that $\mathcal{P}_k$ is property (2) in Notation 2.2. For each $i \in \{1, \ldots, n\}$, choose a strongly-versal $\widetilde{N_{M_i}}$-torsor $Y_i \to X_i$ where X_i is M_i-rational. By definition of strong versality, let us pick an M_i-vector space V_i, a faithful action of $\widetilde{N_{M_i}}$ on V_i and an $\widetilde{N_{M_i}}$-equivariant rational dominant map $V_i \dashrightarrow Y_i$. After passing to Weil restrictions and products one gets a rational G-equivariant map $\prod_i \operatorname{Res}_{M_i/k}(V_i) \dashrightarrow \prod_i \operatorname{Res}_{M_i/k}(Y_i)$, which is dominant by [15, Corollary A.5.4(1)], and by [15, Corollary A.5.4(3)] we get a G-torsor

$$\pi : \prod_i \operatorname{Res}_{M_i/k}(Y_i) \longrightarrow \prod_i \operatorname{Res}_{M_i/k}(X_i)$$

whose base is rational. Since $\prod_i \operatorname{Res}_{M_i/k}(V_i)$ is a k-vector space on which G acts linearly and faithfully, π is strongly versal, which proves the statement. $\qquad\square$

4. Complete groups

The following theorem reduces our study of homogenous spaces whose geometric stabiliser is a power of a complete group to homogeneous spaces whose stabiliser is the group itself.

THEOREM 4.1. — *Let H be an indecomposable complete constant finite k-group and t a positive integer. Consider a homogeneous space X of $\operatorname{SL}_{n,k}$ and $\bar{x} \in X(\bar{k})$ whose stabiliser is denoted by G. If $G(\bar{k}) \simeq H^t$, then there exists an étale k-algebra A of degree t such that for any A-embedding $H_A \hookrightarrow \operatorname{SL}_{r,A}$, the varieties X and $\operatorname{Res}_{A/k}(\operatorname{SL}_{r,A}/H_A)$ are stably k-birational.*

Note that, under its assumptions, this theorem guarantees that in Corollary 3.4, the form $\widetilde{N_A}$ may be chosen to be constant. Let us start by giving an application of Theorem 4.1 when the given complete group is a symmetric group:

COROLLARY 4.2. — *Let k be a field, X a homogeneous space of $\operatorname{SL}_{r,k}$ and $\bar{x} \in X(\bar{k})$. Denote by G the stabiliser of $\bar{x}$. If there exist positive integers n, t with $n \notin \{2, 6\}$ and $G(\bar{k}) \simeq \mathfrak{S}_n^t$, then X is stably rational.*

Proof. — First note that the symmetric group $\mathfrak{S}_n$ is complete [35, Chapter IV, Section 13, p. 92]. Further, since it is indecomposable, Theorem 4.1 supplies an étale k-algebra A of degree t such that X is stably birational to $\operatorname{Res}_{A/k}(\operatorname{SL}_{r,A}/(\mathfrak{S}_n)_A)$. The fundamental theorem of symmetric polynomials, combined with Lemma 2.1, ensures that $\operatorname{SL}_{r,k}/(\mathfrak{S}_n)_k$ is stably k-rational, meaning that there exist nonnegative integers r and s and a birational k-map

$$\mathbf{P}_k^r \times_k \operatorname{SL}_{r,k}/(\mathfrak{S}_n)_k \dashrightarrow \mathbf{P}_k^s.$$

After tensoring by $\operatorname{Spec}(A)$ and passing to Weil restriction along A/k, we derive from the latter a birational k-map

$$\operatorname{Res}_{A/k}(\mathbf{P}_A^r) \times_k \operatorname{Res}_{A/k}(\operatorname{SL}_{r,A}/(\mathfrak{S}_n)_A) \dashrightarrow \operatorname{Res}_{A/k}(\mathbf{P}_A^s).$$

Since $\operatorname{Res}_{A/k}(\mathbf{A}_A^r)$ (resp. $\operatorname{Res}_{A/k}(\mathbf{A}_A^s)$), which is an affine space, is an open subset of $\operatorname{Res}_{A/k}(\mathbf{P}_A^r)$ (resp. $\operatorname{Res}_{A/k}(\mathbf{P}_A^s)$), the latter is k-rational. From this, we conclude that $\operatorname{Res}_{A/k}(\operatorname{SL}_{r,A}/(\mathfrak{S}_n)_A)$ is stably k-rational, which proves the statement.   □

For the proof of Theorem 4.1, we need the following lemma that classifies the stabilisers of a homogeneous space of SL_n in terms of the outer Galois action on one of them. We use the following terminology. Given an isomorphism of finite groups $A \xrightarrow{\varphi} B$ and a continuous action of a profinite group Γ on A, we shall call the action of $\sigma \in \Gamma$ on B, given by $b \mapsto \varphi(\sigma\varphi^{-1}(b))$, *the action induced by φ*. Given two continuous actions $\alpha : \Gamma \to \operatorname{Aut}(A)$ and $\beta : \Gamma \to \operatorname{Aut}(B)$ we say that φ is compatible with the outer actions of Γ on A and B if the following diagram is commutative:

$$
\begin{array}{ccc}
 & & \operatorname{Out}(A) \\
 & \nearrow & \downarrow \scriptstyle{\operatorname{Out}(\varphi)} \\
\Gamma & & \\
 & \searrow & \\
 & & \operatorname{Out}(B)
\end{array}
$$

where the upper diagonal map (resp. the lower diagonal map) is the composition of α (resp. β) with the quotient morphism $\operatorname{Aut}(A) \to \operatorname{Out}(A)$ (resp. $\operatorname{Aut}(B) \to \operatorname{Out}(B)$), and $\operatorname{Out}(\varphi)$ is induced by the isomorphism $\operatorname{Aut}(\varphi) : \operatorname{Aut}(A) \to \operatorname{Aut}(B)$.

LEMMA 4.3. — *Let X be a homogeneous space of $\operatorname{SL}_{N,k}$ and $x \in X(k)$ a rational point whose stabiliser is a finite group G over k.*

(1) *Let $b \in \operatorname{SL}_N(\bar{k})$ be such that $bx \in X(k)$. If H denotes the stabiliser of bx, then $H = bGb^{-1}$. Moreover, via the isomorphism $H(\bar{k}) \simeq$*

$G(\overline{k})$ mapping $h \in H(\overline{k})$ to $b^{-1}hb$, the induced action of $\sigma \in \Gamma_k$ on $G(\overline{k})$ is given by:

$$g \longmapsto b^{-1}\sigma(b)\sigma(g)\sigma(b^{-1})b \quad for \quad g \in G(\overline{k}), \sigma \in \Gamma_k.$$

(2) Conversely, let H be a finite group over k endowed with an isomorphism of finite groups $H(\overline{k}) \simeq G(\overline{k})$ compatible with the outer action of Γ_k. Assume further that $G(\overline{k})$ has trivial center. Then there exists $y \in X(k)$ whose stabiliser is H.

To give a proof of this lemma, we remind the following fact from Galois cohomology.

REMINDER 4.4. — *Consider a short exact sequence of discrete groups*

$$1 \longrightarrow H \longrightarrow E \longrightarrow Q \xrightarrow{\pi} 1$$

and Γ a profinite group. Let $\alpha, \beta : \Gamma \to E$ be two continuous morphisms coequalised by π, that is, morphisms α, β satisfying $\pi \circ \alpha = \pi \circ \beta$. Then the map $\sigma \in \Gamma \mapsto \beta(\sigma)\alpha(\sigma)^{-1}$ is a cocycle in $\mathrm{Z}^1(\Gamma, H)$ where the action of Γ on H is given by $\sigma \cdot h = \alpha(\sigma)h\alpha(\sigma)^{-1}$, for any $\sigma \in \Gamma$ and $h \in H$.

Proof. — By assumption, $h_\sigma := \beta(\sigma)\alpha(\sigma)^{-1}$ lies in H for $\sigma \in \Gamma$. Then, the tuple (h_σ) lies in $\mathrm{Z}^1(\Gamma, H)$ since, for all $\sigma, \tau \in \Gamma$, one has:

$$h_{\sigma\tau} = \beta(\sigma)\beta(\tau)\alpha(\tau)^{-1}\alpha(\sigma)^{-1} = \beta(\sigma)\alpha(\sigma)^{-1}\alpha(\sigma)\beta(\tau)\alpha(\tau^{-1})\alpha(\sigma)^{-1}$$

$$= h_\sigma\alpha(\sigma)h_\tau\alpha(\sigma)^{-1}. \qquad \square$$

Proof of Lemma 4.3. — For (1), note that the action induced by the isomorphism $\varphi : H(\overline{k}) \to G(\overline{k}), h \mapsto bhb^{-1}$ is given by:

$$\sigma \cdot g = \varphi\sigma\varphi^{-1}(g) = b^{-1}\sigma(bgb^{-1})b$$

$$= b^{-1}\sigma(b)\sigma(g)\sigma(b^{-1})b, \quad for \ g \in G(\overline{k}), \sigma \in \Gamma_k.$$

Let H be as in the statement of (2) and denote by $\varphi : H(\overline{k}) \to G(\overline{k})$ the given isomorphism compatible with the outer action of Γ_k. Denote by $\alpha : \Gamma_k \to \mathrm{Aut}(G(\overline{k}))$ the Galois action on $G(\overline{k})$ and $\beta : \Gamma_k \to \mathrm{Aut}(G(\overline{k}))$ the action induced by applying φ to the Galois action on $H(\overline{k})$. Since φ is compatible with the outer action of Γ_k, the maps α and β are coequalised by $\mathrm{Aut}(G(\overline{k})) \to \mathrm{Out}(G(\overline{k}))$. As $G(\overline{k})$ has trivial center, Reminder 4.4 applied to the short exact sequence

$$1 \longrightarrow G(\overline{k}) \xrightarrow{\iota} \mathrm{Aut}(G(\overline{k})) \longrightarrow \mathrm{Out}(G(\overline{k})) \longrightarrow 1$$

where ι maps g to the associated inner automorphism, implies that $\sigma \mapsto h_\sigma := \beta(\sigma)\alpha(\sigma)^{-1}$ lies in $\mathrm{Z}^1(k, G)$. Using the embedding $G(\overline{k}) \hookrightarrow \mathrm{SL}_N(\overline{k})$, this cocycles also lies in $\mathrm{Z}^1(k, \mathrm{SL}_N)$. Hilbert's theorem 90 hence supplies $b \in$

$\mathrm{SL}_N(\bar{k})$ such that $h_\sigma = b^{-1}\sigma(b)$ for all $\sigma \in \Gamma_k$. From the fact that $b^{-1}\sigma(b)$ is in the stabiliser $G(\bar{k})$ of $x \in X(k)$, we infer that $\sigma(bx) = bx$ for all $\sigma \in \Gamma_k$ so that $bx \in X(k)$.

To summarise, for $\sigma \in \Gamma_k$ and $g \in G(\bar{k})$, since $b^{-1}\sigma(b)$ acts on $G(\bar{k})$ via the inner automorphism $\iota(b^{-1}\sigma(b))$, we have:

$$\beta(\sigma) \circ \alpha(\sigma^{-1})(g) = b^{-1}\sigma(b)g\sigma(b^{-1})b$$

which boils down to

$$\beta(\sigma)(\sigma^{-1}(g)) = b^{-1}\sigma(b)g\sigma(b^{-1})b.$$

This being true for all g, one eventually gets that for all $\sigma \in \Gamma_k$ and $g \in G(\bar{k})$, the following holds:

$$\beta(\sigma)(g) = b^{-1}\sigma(b)\sigma(g)\sigma(b^{-1})b.$$

Thus, by (1), β coincides with the Galois action on the stabiliser of bx. By Galois descent, we get an isomorphism of H with the stabiliser of the rational point bx. $\qquad\square$

Proof of Theorem 4.1. — Since H is complete, assertion (2) of Lemma 2.4 ensures that H^t is almost complete. From Proposition 2.5, one then infers that X has a rational point. Thus, there exists a k-form G of H^t, an embedding $G \hookrightarrow \mathrm{SL}_{n,k}$ and an isomorphism of $\mathrm{SL}_{n,k}$-homogeneous spaces $X \simeq \mathrm{SL}_{n,k}/G$.

By Corollary 3.4, there exists an étale k-algebra A of degree t and an A-form $\widetilde{H_A}$ of H_A such that for any embedding $\widetilde{H_A} \hookrightarrow \mathrm{SL}_{r,A}$, the variety X is stably birational to $\mathrm{Res}_{A/k}(\mathrm{SL}_{r,A}/\widetilde{H_A})$. Now, since A is étale over k, it may be written as $A = \prod_{1 \leqslant i \leqslant d} M_i$ where for $i \in \{1, \ldots, d\}$, the field extension M_i/k is separable and finite. Furthermore, since $\mathrm{H}^1(A, \mathrm{Aut}(H)) = \prod_{1 \leqslant i \leqslant d} \mathrm{H}^1(M_i, \mathrm{Aut}(H))$, one may write $\widetilde{H_A} = \prod_{1 \leqslant i \leqslant d} \widetilde{H_{M_i}}$ where the M_i-group $\widetilde{H_{M_i}}$ is an M_i-form of H_{M_i} for $= 1, \ldots, d$. From this, one deduces that

$$(4.1) \qquad \mathrm{Res}_{A/k}(\mathrm{SL}_{r,A}/\widetilde{H_A}) = \prod_{i=1}^{d} \mathrm{Res}_{M_i/k}(\mathrm{SL}_{r,M_i}/\widetilde{H_{M_i}}).$$

For $i \in \{1, \ldots, d\}$, one may then choose an isomorphism

$$(4.2) \qquad\qquad \widetilde{H_{M_i}}(\bar{k}) \simeq H(\bar{k})$$

Since H is complete, its outer automorphisms are trivial, so that isomorphism (4.2) is compatible with the outer Galois action of Γ_k. Lemma 4.3(2)

may then be applied to ensure that H_{M_i} is embedded in SL_{r,M_i} and there exists an isomorphism of SL_{r,M_i}-homogeneous spaces

$$(4.3) \qquad \mathrm{SL}_{r,M_i}/\widetilde{H_{M_i}} \simeq \mathrm{SL}_{r,M_i}/H_{M_i}.$$

The combination of (4.1) and (4.3) then implies that

$$\mathrm{Res}_{A/k}(\mathrm{SL}_{r,A}/\widetilde{H_A}) \simeq \prod_{i=1}^{d} \mathrm{Res}_{M_i/k}(\mathrm{SL}_{r,M_i}/H_{M_i}).$$

The latter may be rewritten as $\mathrm{Res}_{A/k}(\mathrm{SL}_{r,A}/\widetilde{H_A}) \simeq \mathrm{Res}_{A/k}(\mathrm{SL}_{r,A}/H_A)$, to which X is thus stably birational. $\qquad\square$

5. Forms of powers of $\mathfrak{A}_5$

In this section, we assume that k is a field of characteristic 0 and we construct strongly-versal torsors for twisted forms of powers of $\mathfrak{A}_5$. Recall that $\mathfrak{A}_5$ is center-free and there is a commutative diagram

$$(5.1) \qquad \begin{array}{ccc} \mathrm{Aut}(\mathfrak{A}_5) & \longrightarrow & \mathrm{Out}(\mathfrak{A}_5) \\ \| & & \| \\ \mathfrak{S}_5 & \longrightarrow & C_2 \end{array}$$

where the bottom map is the signature morphism.

THEOREM 5.1. — *Suppose* char $k = 0$, *let* t *a positive integer, and* G *a k-form of* $\mathfrak{A}_5^t$. *Then there exists a strongly-versal G-torsor* $Y \to X$ *where* X *is k-rational of dimension $2t$.*

We start by stating the following application of Theorem 5.1:

COROLLARY 5.2. — *Let k be a field of characteristic zero, X a homogeneous space of* $\mathrm{SL}_{r,k}$ *and* $\bar{x} \in X(\bar{k})$. *Denote by N the stabiliser of $\bar{x}$. If there exists a positive integer t such that* $N(\bar{k}) \simeq \mathfrak{A}_5^t$, *then X is retract rational.*

Proof. — Since any transposition defines a section of (5.1), the group $\mathfrak{A}_5$ is almost complete by (1) of Lemma 2.4. Further, it is indecomposable so that $\mathfrak{A}_5^t$ is also almost complete by (2) of Lemma 2.4. Proposition 2.5 ensures that the variety X has a rational point, and the choice of such a point supplies a k-form G of $\mathfrak{A}_5^t$ and an isomorphism of $\mathrm{SL}_{r,k}$-homogeneous spaces $X \simeq \mathrm{SL}_{r,k}/G$. Now, Theorem 5.1 gives a versal left G-torsor $Y \to X$ where X is k-rational. From the equivalence of (1) and (3) in [39, Proposition 4.2], one thus gets that $\mathrm{SL}_{r,k}/G$ is retract-rational. $\qquad\square$

By Corollary 3.5, applied with $\mathcal{P}_k$ as being property (5) of Notation 2.2, the proof of Theorem 5.1 amounts to the following proposition. Denote the base change of $\mathfrak{A}_5$ to a field M by $\mathfrak{A}_{5,M}$.

PROPOSITION 5.3. — *Let M/k be a finite separable extension of fields and $(a_\sigma) \in \mathrm{Z}^1(M, \mathfrak{S}_5)$. Let us denote by $\widetilde{\mathfrak{A}_5}$ the M-form of $\mathfrak{A}_{5,M}$ associated to the 1-cocycle (a_σ) via the identification $\mathfrak{S}_5 = \mathrm{Aut}(\mathfrak{A}_5)$. Then there exists a strongly-versal left $\widetilde{\mathfrak{A}_5}$-torsor over an M-rational surface.*

The proof of Proposition 5.3 follows that of [32, Theorem 2.3.7], and requires the following lemmas. The proof of the first two lemmas may be skipped in a first reading.

The first lemma is a geometric reformulation of [19, Theorem 1, equivalence of (1) and (4)], and is about extending splitting fields of generic polynomials to weakly versal torsors.

LEMMA 5.4 ([19, Theorem 1]). — *Let G a finite constant group over k and $f(\underline{s}, x) \in k(\underline{s})[x]$ a generic polynomial for G. Denote by L a splitting field of $f(\underline{s}, x)$ over $k(\underline{s})$. Then, the G-extension $L/k(\underline{s})$ extends to a weakly versal left G-torsor $Y \to X$, where X is an open subset of $\mathbf{A}_k^{|s|}$.*

The proof of Lemma 5.4 given by DeMeyer and McKenzie in [19, Proof of Theorem 1] is ring-theoretic in flavour. We supply a more geometric proof in Appendix A.

The next lemma ensures that twisted linear actions remain linear:

LEMMA 5.5. — *Let G be a finite constant k-group, H a normal subgroup of G and $\sigma \in \mathrm{Z}^1(k, G)$. Consider a left G-variety Y and a G-equivariant morphism $Y \to X$, where G acts trivially on X. Assume further that $Y \to X$ factors as $Y \to Z \to X$, where Z is a left G/H-variety, $Y \to Z$ is H-equivariant, and $Z \to X$ is G/H-equivariant with G/H acting trivially on X. Then:*

(1) *The $_\sigma G$-equivariant morphism $_\sigma Y \to X$ factors canonically as $_\sigma Y \to {}_\tau Z \to X$, where τ is the 1-cocycle given by the composition of σ with the quotient morphism $G \to G/H$. Furthermore, $_\sigma Y \to {}_\tau Z$ is $\widetilde{H}$-equivariant, where $\widetilde{H}$ is the k-form of H defined as the image of σ by the map $\mathrm{Z}^1(k, G) \to \mathrm{Z}^1(k, \mathrm{Aut}(H))$ induced by the conjugation morphism $G \to \mathrm{Aut}(H)$. If moreover $Y \to X$, $Y \to Z$ and $Z \to X$ are respectively a G-torsor, an H-torsor, and a G/H-torsor, then any of the previous twisted maps is also a torsor.*

(2) *Let V be a k-vector space on which G acts faithfully on the left, and assume further that the factorisation $Y \to Z \to X$ is $V \to V/H \to$*

V/G. In the notation of (1), if $\sigma \in \mathrm{Z}^1(k, G)$, then $_\sigma V$ is a k-vector space on which $_\sigma G$ acts faithfully and linearly. In particular, the action of $\widetilde{H}$ on $_\sigma V$ is also faithful and linear, so that $_\sigma V \to {}_\sigma(V/H)$ is a versal $\widetilde{H}$-torsor over an irreducible open subscheme of $_\sigma(V/H)$.

Proof. — For the proof of (1), the canonical factorisation of $_\sigma Y \to X$ comes from the very definition of the contracted product. Since $\widetilde{H}$ is a subgroup of $_\sigma G$, it acts on $_\sigma Y$ and the map $_\sigma Y(\overline{k}) \to X(\overline{k})$ is $\widetilde{H}(\overline{k})$-equivariant compatibly with the action of Γ_k.

Let us now prove (2). First note that the twisted form $_\sigma V$ of the k-vector space V corresponds to a class in $\mathrm{H}^1(k, \mathrm{GL}(V))$. Since the latter classifies k-vector spaces which are geometrically isomorphic to V, it ensures that $_\sigma V$ is naturally endowed with a k-vector space structure. Furthermore, after twisting the action morphism $a : G \times_k V \to V$, one gets an action $b : {}_\sigma G \times_k {}_\sigma V \to {}_\sigma V$. Geometrically, b corresponds to a morphism $_\sigma G(\overline{k}) \to \mathrm{GL}(_\sigma V)(\overline{k})$ which is compatible with the Γ_k-action. It thus descends to a morphism $_\sigma G \to \mathrm{GL}(_\sigma V)$ which defines b. This proves that $_\sigma G$ acts linearly on $_\sigma V$. The last part of the statement is inferred by $\widetilde{H}$ being a subgroup of $_\sigma G$. □

The proof of Proposition 5.3 uses the following setting. By [32, Section 2.2, Proposition 2.3.8], the polynomial

$$f(s, u, x) = x^5 + sx^3 + u(x + 1) \in k(s, u)[x],$$

where s, u and x are indeterminates, is a generic polynomial of $\mathfrak{S}_5$ over $\mathbf{Q}$, hence over k. Denote by L a splitting field of f over $k(s, u)$. The genericity of f ensures that $L/k(s, u)$ is an $\mathfrak{S}_5$-extension. We thus have the following tower of field extensions

$$k(s, u) \subseteq k(s, u, \sqrt{\mathrm{disc} f}) \subseteq L$$

where L is an $\mathfrak{A}_5$-extension of $k(s, u, \sqrt{\mathrm{disc} f})$. By Lemma 5.4, this tower of fields can be extended to morphisms of schemes

$$Y \longrightarrow Z \longrightarrow X$$

where $Y \to X$ is a weakly versal left $\mathfrak{S}_5$-torsor, $Z = Y/\mathfrak{A}_5$ and X is an open subset of $\mathbf{A}_k^2$.

The following lemma first ensures that $Y \to X$ is a strongly-versal $\mathfrak{S}_5$-torsor.

LEMMA 5.6. — For any vector space V endowed with a faithful left linear action of $\mathfrak{S}_5$, there exists an $\mathfrak{S}_5$-equivariant dominant rational map $V \dashrightarrow Y$.

Proof. — Let t be the map $V \to V/\mathfrak{S}_5$. There exist an integral open subset U of $V/\mathfrak{S}_5$ such that, after setting $W = t^{-1}(U)$, the restriction of t to W is an $\mathfrak{S}_5$-torsor. Since $Y \to X$ is weakly versal, one may assume, after shrinking U, that there exist $\mathfrak{S}_5$-equivariant maps $r : W \to Y$ and $b : U \to X$ such that the following diagram is cartesian:

$$
\begin{array}{ccc}
W & \xrightarrow{\ r\ } & Y \\
\downarrow & & \downarrow \\
U & \xrightarrow{\ b\ } & X.
\end{array}
$$

It remains to show that the map b is dominant, which then implies that so is r. Indeed, otherwise the closure B of the image of b is of dimension 0 or 1. If it were of dimension 0, then b is constant. Since $U(k) \neq \emptyset$, this would mean that B is a rational point of X, so that the $\mathfrak{S}_5$-torsor $Y_B \to B$ corresponds to a Galois extension of fields with group $\mathfrak{S}_5$. Furthermore, the versality of the $\mathfrak{S}_5$-torsor $W \to U$ would imply that $Y_B \to B$ is versal so that the trivial $\mathfrak{S}_5$-torsor over k is a pullback of $Y_B \to B$. In particular, Y_B consists in a k-point, hence $Y_B \to B$ is the identity morphism of $\mathrm{Spec}(k)$, which can not be. Furthermore, if B were of dimension 1, then the curve $C := Y \times_X B$ would be dominated by W. It would thus be unirational, hence rational by the combination of [33, Lemma 2.3] and Lüroth's theorem [8, Section 4.6, Theorem 6.8]. But $C \to B$ being an $\mathfrak{S}_5$-torsor, this supplies an embedding

$$
\mathfrak{S}_5 \hookrightarrow \mathrm{Aut}(C/B) = \mathrm{Aut}(k(C)/k(B)) = \mathrm{PGL}_2(k(B)).
$$

Nevertheless, as $k(B)$ is of characteristic 0, there is no such embedding by [1, Introduction][1]. From this, one eventually deduces that $B = X$, so that b is dominant. $\qquad\square$

We can now give a proof of Proposition 5.3:

Proof of Proposition 5.3. — Let M/k be a finite extension, $(a_\sigma) \in \mathrm{Z}^1(M, \mathfrak{S}_5)$ and $\widetilde{\mathfrak{A}_5}$ the M-form of $\mathfrak{A}_{5,M}$ associated to the 1-cocycle (a_σ). Let V be an M-vector space on which $\mathfrak{S}_5$ acts faithfully on the left. Now, Lemma 5.6 supplies an $\mathfrak{S}_5$-equivariant dominant rational map $V \dashrightarrow Y$.

---

[1] See also [17, Corollary 1.3] for an explicit list of 1-parameter generic polynomials over fields of characteristic 0.

This yields a commutative diagram of rational maps:

$$
\begin{array}{ccc}
V & \dashrightarrow^{\alpha} & Y \\
{\scriptstyle \mathfrak{A}_5}\downarrow & & {\scriptstyle \mathfrak{A}_5}\downarrow \\
V/\mathfrak{A}_5 & \dashrightarrow^{\beta} & Z \\
{\scriptstyle C_2}\downarrow & & {\scriptstyle C_2}\downarrow \\
V/\mathfrak{S}_5 & \dashrightarrow & X
\end{array}
$$

where every horizontal map is dominant, α (resp. β) being $\mathfrak{S}_5$-equivariant (resp. $\mathfrak{A}_5$-equivariant) and each vertical map is a left torsor under the constant M-group written on its left. Using Lemma 5.5, one may twist such a diagram via the cocycle (a_σ), which ensures a commutative diagram of rational maps

(5.2)
$$
\begin{array}{ccc}
\widetilde{V} & \dashrightarrow^{\tilde{\alpha}} & \widetilde{Y} \\
{\scriptstyle \widetilde{\mathfrak{A}_5}}\downarrow & & {\scriptstyle \widetilde{\mathfrak{A}_5}}\downarrow \\
\widetilde{V}/\widetilde{\mathfrak{A}_5} & \dashrightarrow^{\tilde{\beta}} & \widetilde{Y}/\widetilde{\mathfrak{A}_5} \\
{\scriptstyle C_2}\downarrow & & {\scriptstyle C_2}\downarrow \\
V/\mathfrak{S}_5 & \dashrightarrow & X
\end{array}
$$

where each vertical map is a left torsor under the M-group written on its left. Moreover, any horizontal map of (5.2) is still dominant and $\tilde{\alpha}$ (resp. $\tilde{\beta}$) is $\widetilde{\mathfrak{S}_5}$-equivariant (resp. C_2-equivariant). Now, part (2) of Lemma 5.5 ensures that $\widetilde{V}$ is an M-vector space on which $\widetilde{\mathfrak{A}_5}$ acts faithfully and linearly. Thus, the dominance of $\tilde{\alpha}$ yields the strong versality of the $\widetilde{\mathfrak{A}_5}$-torsor $\widetilde{Y} \to \widetilde{Y}/\widetilde{\mathfrak{A}_5}$.

To see that $\widetilde{Z} := \widetilde{Y}/\widetilde{\mathfrak{A}_5}$ is M-rational, note that $\widetilde{Z}$ is none other than the twisted form of the C_2-torsor $Z \to X$ via the image $(\bar{a}_\sigma) \in \mathrm{Z}^1(M, C_2)$ of the 1-cocycle $(a_\sigma) \in \mathrm{Z}^1(M, \mathfrak{S}_5)$ induced by the quotient map $\mathfrak{S}_5 \to \mathfrak{S}_5/\mathfrak{A}_5 = C_2$. The 1-cocycle $(\bar{a}_\sigma)$ then corresponds to an extension $M(\sqrt{\alpha})/M$ for some $\alpha \in M$. By the very definition of twisted forms, the function field of $\widetilde{Z}$ is then the fixed subring of $M(s, u, \sqrt{\mathrm{disc} f}) \otimes_{M(s,u)} M(s, u, \sqrt{\alpha})$ under the action of C_2. If $\sqrt{\alpha} \in M$ the action of C_2 is trivial, so that

(5.3) $M(\widetilde{Z}) = M(s, u, \sqrt{\mathrm{disc} f}) \otimes_{M(s,u)} M(s, u, \sqrt{\alpha}) = M(s, u, \sqrt{\mathrm{disc} f}).$

Let us now compute $M(\widetilde{Z})$ when $\sqrt{\alpha} \notin M$. For this purpose, using the proof of [32, Theorem 2.3.7], we have

$\mathrm{disc}(f) = (108s^5 + 16s^4u - 900s^3u - 128s^2u^2 + 2000su^2 + 3125u^2 + 256u^3)u^2$

from which we deduce that $\mathrm{disc}(f)$ has odd degree. In particular, $\alpha \cdot \mathrm{disc}(f) \notin (M(s,u)^{\times})^2$ so that

(5.4)
$$
\begin{aligned}
M(\widetilde{Z}) &= \left(M(s,u,\sqrt{\mathrm{disc}f}) \otimes_{M(s,u)} M(s,u,\sqrt{\alpha}) \right)^{C_2} \\
&= M\left(s,u,\sqrt{\alpha \cdot \mathrm{disc}(f)} \right)
\end{aligned}
$$

by the following simple lemma:

LEMMA 5.7. — *If $k(\sqrt{\alpha})/k$ (resp. $k(\sqrt{\beta})/k$) is a quadratic extension on which C_2 acts by mapping $\sqrt{\alpha}$ (resp. $\sqrt{\beta}$) to $-\sqrt{\alpha}$ (resp. $-\sqrt{\beta}$), then under the diagonal action of C_2 on $k(\sqrt{\alpha}) \otimes_k k(\sqrt{\beta})$, the k-algebra of fixed elements $\left(k(\sqrt{\alpha}) \otimes_k k(\sqrt{\beta})\right)^{C_2}$ may be described as follows:*

$$
\left(k(\sqrt{\alpha}) \otimes_k k(\sqrt{\beta})\right)^{C_2} = \begin{cases} k(\sqrt{\alpha\beta}) & \text{if } \alpha\beta \notin (k^{\times})^2, \\ k \times k & \text{otherwise.} \end{cases}
$$

The simple proof of the lemma is postponed to the end of the section. Thus, by comparing (5.3) and (5.4), unconditionally on $\alpha \in M$ we have that:

$$
M(\widetilde{Z}) = M\left(s,u,\sqrt{\alpha . \mathrm{disc}(f)} \right).
$$

It is thus enough to show that $M\left(s,u,\sqrt{\alpha . \mathrm{disc}(f)} \right)$ is M-rational. Indeed, following the proof of [32, Theorem 2.3.7], one has:

$$
\begin{aligned}
\mathrm{disc}(f) &= (108s^5 + 16s^4 u - 900s^3 u - 128s^2 u^2 + 2000su^2 + 256u^3 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 3125u^2)u^2 \\
&= \frac{1}{5^5}\left((5^5 v + P)^2 - 4(9 - 20w)Q^2 \right)s^{10}
\end{aligned}
$$

where $P = 1000w^2 - 450w + 54$, $Q = (9 - 20w)(1 - 5w)$, $v = u^2/s^5$ and $w = u/s^2$. Hence,

$$
5\left(\frac{25\sqrt{\mathrm{disc}(f)}}{s^5 Q} \right)^2 = \left(\frac{5^5 v + P}{Q} \right)^2 - 4(9 - 20w)
$$

which, after multiplying by α, leads to

$$
5\left(\frac{25\sqrt{\alpha \cdot \mathrm{disc}(f)}}{s^5 Q} \right)^2 = \alpha\left(\frac{5^5 v + P}{Q} \right)^2 - 4\alpha(9 - 20w).
$$

Thus, after setting $A := \frac{25\sqrt{\alpha \cdot \mathrm{disc}(f)}}{s^5 Q}$ and $B := \frac{5^5 v + P}{Q}$, one gets

(5.5)
$$
5A^2 = \alpha B^2 - 4\alpha(9 - 20w).
$$

The definition of A and B ensures that $M(s, u, \sqrt{\alpha \cdot \mathrm{disc}(f)}) \supseteq M(A, B)$. Furthermore, the reverse inclusion holds since $w \in M(A, B)$ by (5.5), and hence $v \in M(A, B)$ by the definition of B. But then $s = w^2/v$ and $u = s^3 v/w$ are in $M(A, B)$. In particular $\sqrt{\alpha \cdot \mathrm{disc}(f)} \in M(A, B)$ by the very definition of A, and hence $M(s, u, \sqrt{\alpha \cdot \mathrm{disc}(f)}) = M(A, B)$ is M-rational. $\qquad\square$

Proof of Lemma 5.7. — If $\alpha\beta \notin (k^\times)^2$, then $k(\sqrt{\alpha\beta})$ is a k-subalgebra of dimension 2 of $k(\sqrt{\alpha}) \otimes_k k(\sqrt{\beta})$ fixed by C_2. Since $\left(k(\sqrt{\alpha}) \otimes_k k(\sqrt{\beta}) \right)^{C_2}$ is a k-algebra of dimension 2, this proves the statement in that case.

Now, if $\alpha\beta \in (k^\times)^2$, then $k(\sqrt{\beta}) = k(\sqrt{\alpha})$, from which one infers that $k(\sqrt{\alpha}) \otimes_k k(\sqrt{\beta}) = k(\sqrt{\alpha}) \times k(\sqrt{\alpha})$ on which C_2 acts diagonally, so that $\left(k(\sqrt{\alpha}) \otimes_k k(\sqrt{\beta}) \right)^{C_2} = k \times k$. $\qquad\square$

6. Application to Grunwald problems for some nonsolvable groups

When k is a number field, let us recall that the (BM) property for finite k-groups is a widely open question. In Section 6.1, we provide a criterion for an extension of finite k-groups to verify (BM). Afterwards, in Section 6.2, we combine the results of previous sections with Theorem 6.1 to infer a proof of Theorem 1.1, which supplies new families of nonsolvable groups verifying (BM). Eventually, Section 6.3 is dedicated to reviewing the nonsolvable groups with cardinality at most 500 for which (BM) holds.

6.1. Extensions of groups verifying (BM)

The proof of the criterion is similar to the proof of [28, Théorème 1], where Harari proves the case of a split exact sequence with abelian kernel. Our statement extends the latter:

THEOREM 6.1. — *Let k be a number field and consider a short exact sequence of finite algebraic k-groups:*

$$(\mathcal{S}) \qquad\qquad 1 \longrightarrow N \longrightarrow E \longrightarrow Q \longrightarrow 1.$$

Assume (BM) holds for Q and for every homogeneous space X of $\mathrm{SL}_{r,k}$ admitting a point $\bar{x} \in X(\bar{k})$ whose stabiliser S satisfies $S(\bar{k}) \simeq N(\bar{k})$. Assume further that one of the following conditions is satisfied:

(1) the set $X(K)$ is nonempty for every field extension K/k and every homogeneous space X of $\mathrm{SL}_{r,K}$ admitting a point $\bar{x} \in X(\overline{K})$ whose stabiliser S satisfies $S(\overline{K}) \simeq N(\overline{K})$;

(2) the exact sequence $(\mathcal{S})$ is split.

Then E verifies (BM).

Proof. — First choose an embedding of E (resp. Q) in SL_m (resp. SL_n) and if $(\mathcal{S})$ splits, fix a section $\sigma : Q \to E$ of $(\mathcal{S})$ and replace the embedding of Q by the embedding: $Q \overset{\sigma}{\hookrightarrow} E \hookrightarrow \mathrm{SL}_n$ to assume $m = n$ in this case. Consider the projection morphism $p : \mathrm{SL}_m \times \mathrm{SL}_n \to \mathrm{SL}_n$. The algebraic group E acts on the right on SL_n via Q, which induces a diagonal action of E on the right on $\mathrm{SL}_m \times \mathrm{SL}_n$. Then, p is a (right) E-equivariant morphism, so that it induces a morphism

$$f : (\mathrm{SL}_m \times \mathrm{SL}_n)/E \longrightarrow \mathrm{SL}_n/E = \mathrm{SL}_n/Q.$$

Let us apply the fibration method [28, Théorème 3] to f. For this purpose, we verify that f satisfies the following conditions:

(a) the base variety SL_n/Q verifies (BM);

(b) any fibre of f over a rational point verifies (BM);

(c) the generic fibre of f is unirational.

First note that (a) holds by assumption. For (b), fix a point x of SL_n/Q, denote by κ its residue field, and let us describe $X := f^{-1}(x)$. For this purpose, first note that SL_m acts on the left on $(\mathrm{SL}_m \times \mathrm{SL}_n)/E$ with $h \in \mathrm{SL}_m$ acting on the class of $(a,b) \in \mathrm{SL}_m \times \mathrm{SL}_n$ by $h \cdot (a,b) := (ha, b)$. Moreover, when letting SL_m act trivially on SL_n/Q, the morphism f is SL_m-equivariant. In particular, this induces a left action of $\mathrm{SL}_{m,\kappa}$ on X. Now, let $\overline{\kappa}$ be an algebraic closure of κ and choose $s \in \mathrm{SL}_n(\overline{\kappa})$ a representative of the class $x \in \mathrm{SL}_n(\overline{\kappa})/Q(\overline{\kappa})$. Then $X(\overline{\kappa})$ is the subset of $(\mathrm{SL}_m(\overline{\kappa}) \times \mathrm{SL}_n(\overline{\kappa}))/E(\overline{\kappa})$ consisting of classes of elements of the form (g, s) with $g \in \mathrm{SL}_m(\overline{\kappa})$. Since $h \in \mathrm{SL}_m(\overline{\kappa})$ maps the class of (g, s) to that of (hg, s), the action of $\mathrm{SL}_m(\overline{\kappa})$ on $X(\overline{\kappa})$ is transitive, so that X is a homogeneous space of $\mathrm{SL}_{m,\kappa}$.

Besides, the stabiliser of the class of $(1, s)$ consists of elements $h \in \mathrm{SL}_m(\overline{\kappa})$ for which there exists $e \in E(\overline{\kappa})$ such that $(h, s) = (e, se)$. As E acts on SL_n via Q, this equality holds if and only if $e \in N(\overline{\kappa})$, i.e. if and only if $h \in N(\overline{\kappa})$. From this, we deduce that the stabiliser of the class of $(1, s)$ is $N(\overline{\kappa})$, and hence $X(\overline{\kappa})$, endowed with its $\mathrm{SL}_m(\overline{\kappa})$-action, is isomorphic to $\mathrm{SL}_m(\overline{\kappa})/N(\overline{\kappa})$.

Thus, if x is a rational point of SL_n/Q, then $f^{-1}(x)$ is a homogeneous space of SL_m whose geometric stabilisers are isomorphic to $N(\overline{k})$. Moreover, such homogeneous space verify (BM) by assumption, so that (b) is verified.

To verify (c) holds, first assume that (1) holds. If x is the generic point of SL_n/Q, then we deduce from assumption (1) that the SL_m-homogeneous space $f^{-1}(x)$ (whose geometric stabiliser is isomorphic to $N(\bar{k})$) has a rational point. If y is such a rational point and S denotes its stabiliser, it induces an isomorphism $\mathrm{SL}_m/S \xrightarrow{\sim} f^{-1}(x)$ defined by mapping the class of $g \in \mathrm{SL}_m$ to $g \cdot y$. After composing with the quotient map $\mathrm{SL}_m \to \mathrm{SL}_m/S$, we obtain a dominant map $\mathrm{SL}_m \to f^{-1}(x)$, so that $f^{-1}(x)$ is unirational.

Now, assume that (2) holds. Since $(\mathcal{S})$ splits, the choice of embeddings we made in the beginning of the proof ensures that the diagonal embedding $\mathrm{SL}_n \to \mathrm{SL}_n \times \mathrm{SL}_n = \mathrm{SL}_m \times \mathrm{SL}_n$ is right E-equivariant. It thus induces a morphism $\mathrm{SL}_n/E \to (\mathrm{SL}_n \times \mathrm{SL}_n)/E$ which is a section of f. In particular, the generic fibre of f has a rational point. The previous paragraph thus implies that the generic fibre of f is unirational.

We may now apply [28, Théorème 3] to deduce that $(\mathrm{SL}_m \times \mathrm{SL}_n)/E$ verifies (BM), but it remains to verify that it is stably birational to SL_m/E. For this purpose, denote by $q : \mathrm{SL}_m \times \mathrm{SL}_n \to \mathrm{SL}_m$ the projection morphism, which is E-equivariant with respect to the right E-actions, so that it induces a quotient map $g : (\mathrm{SL}_m \times \mathrm{SL}_n)/E \to \mathrm{SL}_m/E$. First note that since E acts freely on $\mathrm{SL}_m \times \mathrm{SL}_n$, then the following diagram, whose horizontal maps are the canonical quotient maps, is cartesian:

$$
\begin{array}{ccc}
\mathrm{SL}_m \times \mathrm{SL}_n & \longrightarrow & (\mathrm{SL}_m \times \mathrm{SL}_n)/E \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\
\mathrm{SL}_m & \longrightarrow & \mathrm{SL}_m/E.
\end{array}
$$

Since the map $\mathrm{SL}_m \to \mathrm{SL}_m/E$ is fppf, we deduce from this diagram that g is a SL_m-torsor for the fppf topology, hence for the Zariski topology by [41, Proposition 4.9] (which is stated with GL_m although the proof is the same with SL_m). In other words, the SL_m-torsor g is locally trivial for the Zariski topology, yielding the claim. $\qquad\square$

COROLLARY 6.2. — *Let k be a number field and consider the exact sequence of finite k-groups*

$$1 \longrightarrow N \longrightarrow E \longrightarrow Q \longrightarrow 1.$$

Assume that Q verifies (BM), that $N(\bar{k})$ is almost complete, and that every k-form G of $N(\bar{k})$ verifies (BM). Then E verifies (BM).

Proof. — Since $N(\bar{k})$ is almost complete, Proposition 2.5 ensures that condition (1) of Theorem 6.1 is satisfied. In particular, for a homogeneous

space X of $\mathrm{SL}_{r,k}$ with geometric stabiliser $N(\bar{k})$, we have $X(k) \neq \emptyset$. For $x \in X(k)$, if G denotes the stabiliser of x, we thus have $X = \mathrm{SL}_{r,k}/G$ and $G(\bar{k}) \simeq N(\bar{k})$. Since G is a k-form of $N(\bar{k})$, the variety X verifies (BM). All assumptions of Theorem 6.1 are therefore satisfied, so that E verifies (BM). $\qquad\square$

6.2. Proof of Theorem 1.1

Let us start by the following theorem, which is a combination of Proposition 2.6 with Corollaries 4.2 and 5.2. It gives the geometric nature of homogeneous spaces of SL_r whose geometric stabilisers are products of alternating and symmetric groups.

THEOREM 6.3. — *Let k be a field, X a homogeneous space of $\mathrm{SL}_{r,k}$ and $\bar{x} \in X(\bar{k})$. Denote by N the stabiliser of $\bar{x}$.*

(1) *If there exist distinct positive integers $n_1, \ldots, n_s \notin \{2, 6\}$ and non-negative integers $t_1, \ldots, t_s \in \mathbb{N}$ such that*

$$N(\bar{k}) \simeq \prod_{i=1}^{s} \mathfrak{S}_{n_i}^{t_i},$$

then X is stably rational.

(2) *If k is of characteristic zero and if there exist distinct positive integers $n_1, \ldots, n_s \notin \{2, 6\}$ and $t_0, \ldots, t_s \in \mathbb{N}$ such that*

$$N(\bar{k}) \simeq \mathfrak{A}_5^{t_0} \times \prod_{i=1}^{s} \mathfrak{S}_{n_i}^{t_i},$$

then X is retract rational.

Proof. — We start by observing that for integers as in (2), the group $\mathfrak{A}_5^{t_0} \times \prod_{i=1}^{s} \mathfrak{S}_{n_i}^{t_i}$ is almost complete by (2) and (3) of Lemma 2.4, since, for each $n \notin \{2, 3\}$, the groups $\mathfrak{S}_n$ and $\mathfrak{A}_5$ are almost complete and indecomposable.

Let us first prove (1). By Proposition 2.6, there exist a positive integer n, and $\mathrm{SL}_{n,k}$-homogeneous spaces $X_1, \ldots, X_s$ with points $\bar{x_i} \in X_i(\bar{k})$ such that X is stably birational to $\prod_{i=1}^{s} X_i$ and the stabilisers $G_i(\bar{k})$ of $\bar{x_i}$ are isomorphic to $\mathfrak{S}_{n_i}^{t_i}$ for $i = 1, \ldots, s$. Since the n_i's are different from 2 and 6, Corollary 4.2 ensures that the X_i's are stably rational. Thus X is also stably rational.

To prove (2), Proposition 2.6 supplies a positive integer n and $\mathrm{SL}_{n,k}$-homogeneous spaces $X_0, \ldots, X_s$ such that X is stably birational to $\prod_{i=1}^{s} X_i$,

where the geometric stabilisers of X_0 (resp. X_i, for all $i \in \{1, \ldots, s\}$) is isomorphic to $\mathfrak{A}_5^{t_0}$ (resp. $\mathfrak{S}_{n_i}^{t_i}$). Thus, Corollary 5.2 ensures that X_0 is retract rational and Corollary 4.2 ensures that the X_i's are stably rational for each $i \in \{1, \ldots, s\}$, so that X is also retract rational. $\qquad\square$

When combining Theorems 6.1 and 6.3, we get a proof of Theorem 1.1:

Proof of Theorem 1.1. — Let us apply Corollary 6.2. By assumption, Q verifies (BM). Moreover, $N(\bar{k})$ is almost complete by (2) and (3) of Lemma 2.4, since the groups $\mathfrak{S}_n$ and $\mathfrak{A}_5$ are almost complete and indecomposable. Now if k is a number field, by (2) of Theorem 6.3, every homogeneous space of $\mathrm{SL}_{r,k}$, $r \geqslant 1$, with geometric stabiliser $N(\bar{k})$ is retract rational, so that it verifies weak approximation, hence it has (BM). This shows that the assumptions of Corollary 6.2 are verified, so that E verifies (BM). $\qquad\square$

6.3. The (BM) property for "small" nonsolvable groups

In this section, we combine our results with those in the literature to review the nonsolvable groups of order at most 500 for which (BM) is known. We encapsulate this in the following proposition.

PROPOSITION 6.4. — *Over any number field k, the (BM) property holds for every finite nonsolvable group whose cardinality is at most* 500, *except perhaps for those appearing in Table 6.1.*

To describe the groups in Tables 6.1 we use the following notations. For finite groups G, H and Q, we write $G = H \rtimes Q$ for a semidirect product of Q by H, and $G = H.Q$ when G is a non-split extension of Q by H. Let $D_n = C_n \rtimes C_2$ denote the dihedral group of order $2n$. Let Q_8 denote the quaternion group, that is, the unique non-split extension of C_2 by C_4. Let $\mathrm{GL}_2(\mathbf{F}_q)$, $\mathrm{SL}_2(\mathbf{F}_q)$, $\mathrm{PGL}_2(\mathbf{F}_q)$, and $\mathrm{PSL}_2(\mathbf{F}_q)$ denote respectively the 2×2 general, special, projective general, and projective special linear groups over $\mathbf{F}_q$.

The proof of Proposition 6.4 makes use of Theorem 1.1, Harari's theorem [28, Théorème 1] on split extensions of groups verifying (BM) by abelian groups, and a theorem of Plans on the field of invariants of double covers of symmetric groups [47, Theorem 11]:

TOOL A ([28, Théorème 1]). — *Given a split short exact sequence of finite groups*

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

with A abelian and G verifying (BM), the group E also verifies (BM).

Table 6.1. List of nonsolvable groups with cardinality at most 500 for which (BM) is unknown

Order	Magma ID	Nonsolvable groups for which (BM) is a priori unknown
240	240,89	$\mathrm{CSU}_2(\mathbf{F}_5) = C_2.\mathfrak{S}_5$
	240,93	$C_4.\mathfrak{A}_5$
336	336,114	$\mathrm{SL}_2(\mathbf{F}_7)$
	336,208	$\mathrm{PGL}_2(\mathbf{F}_7)$
360	360,118	$\mathfrak{A}_6$
480	480,218	$\mathrm{GL}_2(\mathbf{F}_5)$
	480,219	$C_2^2.\mathfrak{S}_5$
	480,221	$C_8.\mathfrak{A}_5$
	480,946	$C_4.\mathfrak{S}_5$
	480,947	$C_4.\mathfrak{S}_5$
	480,948	$C_4.\mathfrak{S}_5$
	480,949	$C_2 \times \mathrm{CSU}_2(\mathbf{F}_5)$
	480,953	$C_2^2.\mathfrak{S}_5$
	480,955	$C_2 \times (C_4.\mathfrak{A}_5)$
	480,957	$D_4.\mathfrak{A}_5$
	480,959	$Q_8.\mathfrak{A}_5$

TOOL B ([47, Theorem 11]). — *Let $n \geqslant 3$ be an odd integer and identify $\mathfrak{S}_{n-1}$ with the subgroup of $\mathfrak{S}_n$ fixing n. Consider a positive integer r and an exact sequence of groups:*

$$1 \longrightarrow C_2^r \longrightarrow H \overset{\pi}{\longrightarrow} \mathfrak{S}_n \longrightarrow 1$$

such that the center of H contains C_2^r. If we denote by E the inverse image of $\mathfrak{S}_{n-1}$ by π, then H verifies (BM) if and only if E verifies (BM).

TOOL C. — *Finite products of finite groups verifying (BM) also verify (BM).*

We also use the following results regarding the rationality of fields of invariants:

REMINDER 6.5. — *The following properties hold over $\mathbf{Q}$:*
- *All symmetric groups $\mathfrak{S}_n$ have stably rational fields of invariants.*
- *The alternating group $\mathfrak{A}_5$ has stably rational fields of invariants by [37].*
- *The group $\mathrm{GL}_2(\mathbf{F}_3)$ has stably rational fields of invariants by [46].*

- *The group* $\mathrm{SL}_2(\mathbf{F}_5)$ *has a generic polynomial by* [45].
- *The simple group* $\mathrm{GL}_3(\mathbf{F}_2) \simeq \mathrm{PSL}_2(7)$ *has stably rational fields of invariants by* [40, *Théorème 3*].
- *The quaternion group* Q_8 *has stably rational fields of invariants by* [25].

In particular, all these groups verify (BM).

Proof of Proposition 6.4. — To prove the statement, we run the following algorithm, both by hand and using Magma, over the nonsolvable groups of cardinality $\leqslant 500$. To go over the list by hand, we use the online database of Dokchister [21], where the groups are listed via their Magma ID in the "Small Groups Library" of Magma. Our list is ordered by increasing Magma ID.

Step 1. — Following this order, determine all the groups G for which (BM) follows from Tool A, Tool C and Reminder 6.5 by looking at all the short exact sequences in which (I) G fits in the middle, the kernel is abelian, and (BM) is known for the quotient by the reminder, or (II) G is the product of two groups for which (BM) is known, one factor by the reminder and the other either by Tool A or the reminder. The code in [5, "nonsolvable" code file] outputs the minimal nonsolvable groups for which (BM) remains unknown after ruling out the groups in (I), that is, those which are not extensions of a smaller group by an abelian kernel. For the smaller groups, (BM) follows from the reminder. One additional group $Q_8 \times \mathfrak{A}_5$ is then ruled out in (II) as the product of two groups for which (BM) is known by Tool C.

Step 2. — Out of the remaining 22 groups, we rule out two central extension H of $\mathfrak{S}_5$, with Magma IDs 240,90-91, using Tool B. For these, B yields an extension E for which (BM) is known by the reminder and Proposition 7.4. We then rule out two more groups which are the direct products of these groups with C_2.

Step 3. — Out of the remaining 18 groups, we use Theorem 1.1 to rule out two group extensions, $\mathfrak{A}_5 \rtimes C_8$ and $\mathfrak{A}_5 \rtimes Q_8$ with kernel $\mathfrak{A}_5$ and quotients satisfying (BM) by Tool A.

The remaining 16 groups appear in Table 6.1.   □

Remark 6.6. — Using Tool B, we further reduce the verification of (BM) for the following groups to verifying (BM) for solvable groups of smaller order. We shall see in Section 7 that (BM) holds for these groups under Schinzel's hypothesis H.

- The groups with Magma ID $240, 89$ and $480, 949$ satisfy (BM) if and only if the solvable group with Magma ID $48, 28$ verifies (BM);
- The group with Magma ID $480, 219$ satisfies (BM) if and only if the solvable group with Magma ID $96, 66$ verifies (BM);
- The group with Magma ID $480, 953$ satisfies (BM) if and only if the solvable group with Magma ID $96, 190$ verifies (BM).

7. Metabelian stabilisers and Grunwald problems for small solvable groups

Throughout this section k is a number field. In the first part of this section, we prove that the Brauer–Manin obstruction to weak approximation is the only one for homogeneous spaces of $\mathrm{SL}_{n,k}$ whose geometric stabiliser has derived subgroup C_2, under Schinzel's hypothesis (H), and hence deriving Theorem 1.2. In Section 7.3 we make use of Theorem 1.2 and the results of Harari [28] and Harpaz–Wittenberg [30] to list the finite groups of order at most 191 for which (BM) is unknown.

7.1. Homogeneous spaces with metabelian stabilisers

The following is the main theorem on homogeneous spaces whose derived geometric stabilisers is C_2:

THEOREM 7.1. — *Let k be a number field, X a homogeneous space of $\mathrm{SL}_{n,k}$ and $\bar{x} \in X(\bar{k})$. Assume that the stabiliser G of $\bar{x}$ is a finite algebraic $\bar{k}$-group, and assume further that $G' = C_2$. If Schinzel's hypothesis (H) holds, then the Brauer–Manin obstruction to weak approximation is the only one on X.*

Let us recall that a reduced variety X over a field k is said to be *split* if it contains an irreducible component which is geometrically irreducible [55]. If L/k is a field extension, one says that X is split by L if $X \otimes_k L$ is split. Equivalently, this means that there exists an irreducible component Y of X such that the algebraic closure of k in $k(Y)$ embeds in L.

Throughout the section we denote by $\Pi^1_{\text{ét}}(X, \bar{x})$, for any geometric point $\bar{x} \in X(\bar{k})$, the following short exact sequence of profinite groups (see [27, Théorème 6.1]):

$$1 \longrightarrow \pi^1_{\text{ét}}(X \otimes_k \bar{k}, \bar{x}) \longrightarrow \pi^1_{\text{ét}}(X, \bar{x}) \longrightarrow \Gamma_k \longrightarrow 1.$$

In addition, we recall that if S is a k-group of multiplicative type and $\widehat{S}$ its dual, then an S-torsor $f : Y \to X$ is called *universal* if the homomorphism $\widehat{S}(\bar{k}) \to \mathrm{Pic}(X_{\bar{k}})$, mapping a character $\chi : S_{\bar{k}} \to \mathbf{G}_{m,\bar{k}}$ to the image of f by the pushfoward map $\chi^* : \mathrm{H}^1(X_{\bar{k}}, S_{\bar{k}}) \to \mathrm{H}^1(X_{\bar{k}}, \mathbf{G}_m) = \mathrm{Pic}(X_{\bar{k}})$, is an isomorphism (see also [56, p. 25]).

Proof of Theorem 7.1. — Assume that $X(k_\Omega)^{\mathrm{Brun}} \neq \emptyset$. Since $\mathrm{SL}_{n,k}$ is semisimple, Rosenlicht's lemma ensures that $\bar{k}[X_{\bar{k}}]^\times = \bar{k}^\times$. Furthermore, by [30, Section 5.2], one has $\mathrm{Pic}(X_{\bar{k}}) = \mathrm{Hom}(G^{\mathrm{ab}}(\bar{k}), \bar{k}^\times)$ so that $\mathrm{Pic}(X_{\bar{k}})$ is finite. Thus, [56, Proposition 6.1.4] ensures that there exists a universal left D-torsor $f : Y \to X$ in the sense of [10, Section 2] and [56, Definition 2.3.3], where $D(\bar{k})$ is the Cartier dual of $\mathrm{Pic}(X_{\bar{k}})$, hence $D(\bar{k}) = G^{\mathrm{ab}}(\bar{k})$ so that D is a finite abelian group. In particular, D is an algebraic group of multiplicative type and from [11, (1.3.2) in Proposition 1.3] (see also [31, Lemma 3.8]) one infers that D fits in a short exact sequence

$$1 \longrightarrow D \longrightarrow T \longrightarrow Q \longrightarrow 1$$

where T is a torus and Q a quasi-trivial torus.

As in Section 2.2, let $_TY$ be the quotient of $T \times Y$ under the diagonal action of D defined for any point d in D by $(t,y) \mapsto (td^{-1}, d \cdot y)$, and denote by $_Tf : {}_TY \to X$ and $p : {}_TY \to Q$ the two projections. In order to prove that the Brauer–Manin obstruction to weak approximation is the only one for X, we are to apply the descent method for torsors under tori [30, Corollaire 2.2] to the left T-torsor $_Tf$. For this purpose, let us fix $\sigma \in \mathrm{Z}^1(k,T)$, and let us prove that the Brauer–Manin obstruction to weak approximation is the only one for the twisted variety $_\sigma({}_TY)$.

After twisting the morphism $_TY \to Q$ by σ, one gets a morphism $g : {}_\sigma({}_TY) \to {}_\sigma Q$. Furthermore, since Q is quasi-trivial, Shapiro's lemma combined with Hilbert's theorem 90 ensures that $_\sigma Q \simeq Q$, so that Q is an open subset of an affine space $\mathbf{A}_k^d$. One may then choose a smooth compactification Z of $_\sigma({}_TY)$ such that g extends to a morphism $\bar{g} : Z \to \mathbf{P}_k^d$. Since $_\sigma({}_TY)$ is an open subset of Z, it is enough to prove that the Brauer–Manin obstruction to weak approximation is the only one for Z. We prove the latter by applying the fibration method [63, Chapitre 3, Corollaire 3.5] to the fibration $\bar{g}$. Indeed, the fibres of g are twists of f by an element of $\mathrm{Z}^1(k,D)$, that is, they are universal D-torsors over X. But [30, Corollaire 5.4] ensures that such universal torsors are homogeneous spaces of $\mathrm{SL}_{n,K}$ with geometric stabiliser C_2. Thus, Lemma 7.2 below implies that the smooth fibres of g over rational points of $_\sigma Q$ have the weak approximation property, and Lemma 7.3 below guarantees that the fibres

of $\bar{g}$ over points of codimension one of $\mathbf{P}_k^d$ are split by a quadratic extension. In other words, the assumptions of [63, Chapitre 3, Corollaire 3.5] are satisfied, when applied to the fibration $\bar{g}$, yielding the statement. $\qquad\square$

LEMMA 7.2. — *Let V be an $\mathrm{SL}_{n,k}$-homogeneous space with geometric stabiliser C_2. Then V verifies weak approximation.*

Proof. — Let us first note that V verifies the Hasse principle. Indeed, assume that V has local points and denote by $\bar{v}$ a geometric point of V. Note that $\Pi_{\text{ét}}^1(V, \bar{v})$ is isomorphic to

$$1 \longrightarrow C_2 \longrightarrow \pi_{\text{ét}}^1(V, \bar{v}) \longrightarrow \Gamma_k \longrightarrow 1.$$

Since V has local points, the class $[\Pi_{\text{ét}}^1(V, \bar{v})] \in \mathrm{H}^2(k, C_2) = {}_2\mathrm{Br}(k)$ is mapped to the trivial class in $\mathrm{H}^2(k_v, C_2) = {}_2\mathrm{Br}(k_v)$ for each $v \in \Omega_k$. The Brauer–Hasse–Noether exact sequence thus implies that $[\Pi_{\text{ét}}^1(V, \bar{v})] \in \mathrm{H}^2(k, C_2)$ is trivial, so that $\Pi_{\text{ét}}^1(V, \bar{v})$ admits a section, which means that $V(k) \neq \emptyset$ by [43, Theorem 7.6].

Thus, if $V(k_\Omega) \neq \emptyset$, one has $V(k) \neq \emptyset$, so that $V \simeq \mathrm{SL}_{n,k}/C_2$. As $\mathbf{A}_k^2/C_2 \simeq \mathbf{A}_k^2$ is stably birational to V by Lemma 2.1, the variety V is then k-stably rational, hence verifies the weak approximation property. $\qquad\square$

LEMMA 7.3. — *Let R be a discrete valuation ring, K its fraction field and κ its residue field. Assume κ is of characteristic zero. Consider an $\mathrm{SL}_{n,K}$-homogeneous space V with geometric stabiliser C_2 and let $\mathscr{V}$ be a regular proper R-scheme. If the generic fibre of $\mathscr{V}$ contains V as an open dense subscheme, then $\mathscr{V} \otimes_R \kappa$ is split by a quadratic extension of κ.*

Proof. — First note that after replacing $\mathscr{V}$ by $\mathscr{V} \otimes_R \widehat{R}$, where $\widehat{R}$ denotes the completion of R, one may assume that R is complete, so that $R \simeq \kappa[\![t]\!]$ by [53, Chapitre II, Section 4, Théorème 2]. Denote by $\partial_R : \mathrm{H}^2(K, C_2) \to \mathrm{H}^1(\kappa, C_2)$ the Serre residue map [13, Section 1.4.1] and fix a geometric point $\bar{v}$ of V. Then the image of the class of $\Pi_{\text{ét}}^1(V, \bar{v})$ by ∂_R corresponds to a quadratic extension κ'/κ, and we are to prove that $\mathscr{V} \otimes_R \kappa$ is split by κ'/κ.

Since R is henselian, [41, Theorem II.3.10] ensures that $\mathrm{H}^1(R, C_2) = \mathrm{H}^1(\kappa, C_2)$ so that κ'/κ is the special fibre of an étale R-algebra R'. We replace $\mathscr{V}$ by $\mathscr{V} \otimes_R R'$ to assume $\kappa = \kappa'$ and $R = R'$ and claim that $\mathscr{V} \otimes_R \kappa$ is split. Since $\kappa' = \kappa$, the class of $\Pi_{\text{ét}}^1(V, \bar{v})$ has a trivial residue. Now, [22, Théorème (b)] supplies a field extension λ of κ such that κ is algebraically closed in λ and $\mathrm{cd}(\lambda) \leqslant 1$. Set $S = \lambda[\![t]\!]$, so that the R-algebra S is unramified. Furthermore, since κ is algebraically closed in λ, one may also assume, after tensoring $\mathscr{V}$ by S/R, that $R = S$, i.e. that $\mathrm{cd}(\kappa) = 1$.

Then, $[\Pi^1_{\text{ét}}(V, \bar{v})] \in \text{H}^2(K, C_2) = {}_2\text{Br}(K)$ having a trivial residue, [26, Proposition (2.1)] ensures that it comes from an element of $\text{Br}(R)$. But R being henselian, one infers from [13, Theorem 3.4.2(i)] that $\text{Br}(R) = \text{Br}(\kappa)$ where $\text{Br}(\kappa) = 0$ since $\text{cd}(\kappa) \leqslant 1$. Thus, the class of $\Pi^1_{\text{ét}}(V, \bar{v})$ in $\text{H}^2(K, C_2)$ is trivial, which means that $\Pi^1_{\text{ét}}(V, \bar{v})$ has a section. Hence [43, Theorem 7.6] ensures that $V(K) \neq \emptyset$, from which one gets that $\mathscr{V} \otimes_R \kappa$ is split, using [55, Lemma 1.1(b)]. $\qquad\square$

7.2. Proof of Theorem 1.2

We can now combine Theorem 7.1 with Theorem 6.1 and [31, Theorem 4.5] to supply a proof of Theorem 1.2.

Proof of Theorem 1.2. — Assuming Schinzel's hypothesis (H), Theorem 7.1 ensures that every homogeneous space of SL_r ($r \in \mathbb{N}$) whose geometric stabiliser has derived subgroup C_2 verifies (BM).

Let us first consider the case where Q is supersolvable and choose an embedding $E \hookrightarrow \text{SL}_r$. Then, we may apply [31, Theorem 4.5] to the homogeneous space SL_r/E and the embedding $N \subseteq E$. Indeed, if $\bar{x}$ is chosen to be the class of 1 in SL_r/E, then assumption (1) of [31, Theorem 4.5] is fulfilled since the outer Galois action of Γ_k on $E(\bar{k})$ factors through the Galois action Γ_k, which is trivial. Assumption (2) in *ibidem* holds since $E/N = Q$ is supersolvable. Finally, by the first paragraph of this proof, (BM) holds for every homogeneous space Y of SL_r with geometric stabiliser N as required in $(\star)$ of *ibidem*.

Now assume that the sequence of the statement is split and Q verifies (BM). We may then apply Theorem 6.1. For this purpose, first note that (2) holds since the short exact sequence of the statement splits. Furthermore, the assumptions of 6.1 preceding (1) and (2) are automatically satisfied since Q verifies (BM) and since every homogenous spaces with geometric stabiliser isomorphic to N verifies (BM) by the first paragraph. $\qquad\square$

7.3. The (BM) property for "small" solvable groups

Let us first state the main result of this subsection:

PROPOSITION 7.4. — *Let k be a number field. The (BM) property holds over k for every finite group of cardinality at most 191, except perhaps for those appearing in Tables 7.1 and 7.2. Furthermore, if one assumes that Schinzel's hypothesis (H) holds, then the (BM) property holds for every finite group appearing in Table 7.1.*

To describe the groups in Tables 7.1 and 7.2 we use the notations from Section 6.4. Further, let Dic_n denote the unique non-split extension $C_{2n}.C_2$ where C_2 acts on C_{2n} by inversion. For an odd prime p, let He_p denote the Heisenberg group, that is, the p-Sylow subgroup of $\mathrm{GL}_3(\mathbf{F}_p)$. Let 2^{1+2n} denote an extraspecial group whose center has order 2, the quotient being an elementary abelian group of rank $2n$.

Table 7.1. *List of finite groups with order less than* 191, *for which (BM) is known conditionally under Schinzel's hypothesis (H), but not unconditionally*

Order	Magma ID	Rewriting that allows to prove (BM)	Method used to prove (BM), using the rewriting
48	48,28	$\mathrm{CSU}_2(\mathbf{F}_3) = Q_8{\cdot}\mathfrak{S}_3$	Theorem 1.2
	48,33	$Q_8{\cdot}C_6$	Theorem 1.2
72	72,3	$Q_8 \rtimes C_9$	Theorem 1.2
96	96,3	$(C_2{\cdot}C_4^2) \rtimes C_3$	Theorem 1.2
	96,66	$Q_8 \rtimes \mathrm{Dic}_3$	Theorem 1.2
	96,67	$Q_8{\cdot}\mathrm{Dic}_3$	Theorem 1.2
	96,74	$Q_8{\cdot}C_{12}$	Theorem 1.2
	96,188	$C_2 \times \mathrm{CSU}_2(\mathbf{F}_3)$	Theorem 1.2 and Tool C
	96,190	$Q_8{\cdot}D_6$	Theorem 1.2
	96,191	$Q_8{\cdot}D_6$	Theorem 1.2
	96,192	$Q_8{\cdot}D_6$	Theorem 1.2
	96,193	$Q_8{\cdot}D_6$	Theorem 1.2
	96,200	$C_2 \times (Q_8{\cdot}C_6)$	Theorem 1.2 and Tool C
	96,201	$Q_8{\cdot}(C_2 \times C_6)$	Theorem 1.2
	96,202	$Q_8{\cdot}(C_2 \times C_6)$	Theorem 1.2
144	144,31	$Q_8{\cdot}D_9$	Theorem 1.2
	144,32	$Q_8 \rtimes D_9$	Theorem 1.2
	144,35	$C_2 \times (Q_8 \rtimes C_9)$	Theorem 1.2 and Tool C
	144,36	$Q_8{\cdot}C_{18}$	Theorem 1.2
	144,121	$C_3 \times \mathrm{CSU}_2(\mathbf{F}_3)$	Theorem 1.2 and Tool C
	144,124	$Q_8{\cdot}(C_3 \rtimes \mathfrak{S}_3)$	Theorem 1.2
	144,127	$Q_8{\cdot}(C_3 \times \mathfrak{S}_3)$	Theorem 1.2
	144,157	$C_3 \times (C_4{\cdot}\mathfrak{A}_4)$	Theorem 1.2 and Tool C
160	160,199	$2^{1+4} \rtimes C_5$	Theorem 1.2

Table 7.2. *The only group of order less than* 191 *for which (BM) is* a priori *unknown*

Order	Magma ID	Groups for which (BM) is *a priori* unknown
108	108,15	$\mathrm{He}_3 \rtimes C_4$

To prove Proposition 7.4, we append to Tools A, B and C a result of Harpaz–Wittenberg:

TOOL D ([30, Théorème B]). — *Any finite supersolvable group verifies (BM).*

We also use the fact that the solvable group $\mathrm{SL}_2(\mathbf{F}_3)$ verifies (BM). The stable rationality of its fields of invariants was proved by Rikuna in an unpublished paper. The retract rationality of such fields was later published by Burdick and Jonker [6, Theorem 3.6]:

REMINDER 7.5 ([6, Theorem 3.6]). — *The fields of invariants of* $\mathrm{SL}_2(\mathbf{F}_3)$ *are retract rational. In particular,* $\mathrm{SL}_2(\mathbf{F}_3)$ *satisfies (BM).*

Proof of Proposition 7.4. — The proof is derived from Tools A, C, D, Reminders 6.5, 7.5 and Theorem 1.2 as follows:

Step 1. — In our Magma code [5, "solvable" code file], we use the "Small Groups Library" of Magma to produce the list of finite groups of cardinality at most 191 for which neither Tool A nor Tool C nor Tool D might be used to prove (BM).

Step 2. — For any of the Magma IDs of this list, we look at all the short exact sequences in which the corresponding group fits in the middle, via an online database of Dokchister [20]. We then eliminate, by hand, all the groups among that list for which (BM) is known unconditionally, via a combination of Reminders 6.5 and 7.5 with Tools A and C. The remaining groups are listed in Tables 7.1 and 7.2.

Step 3. — Eventually, we determine, among the remaining groups, those for which (BM) modulo Schinzel's hypothesis (H) can be proved using Theorem 1.2. The remaining group is listed in Table 7.2.   □

Appendix A. Geometric proof of Lemma 5.4

We include here a geometric proof of the equivalence of (1) and (4) in [19, Theorem 1], that is, a proof of Lemma 5.4. We follow the notation of the lemma.

We start by setting $K := k(\underline{s})[x]/f(\underline{s}, x)$, $A := k\left[\underline{s}, \frac{1}{\mathrm{den}(f)\cdot\mathrm{disc}(f)}\right]$, $X :=$ $\mathrm{Spec}(A)$ and $Z := \mathrm{Spec}(A[x]/f(\underline{s}, x))$. First note that since f is monic in x, the natural morphism $\varphi : Z \to X$ is finite and flat. Besides, since X is an open subscheme of $\mathbf{A}_k^{|\underline{s}|}$ where $\mathrm{disc}(f)$ does not vanish, the fibres of f are étale. Thus f is flat with étale fibres, hence étale. Letting $\overline{k(\underline{s})}$ be a separable closure of $k(\underline{s})$, we denote by $\eta : \mathrm{Spec}(k(\underline{s})) \to X$ the generic point of X and $\bar\eta$ its composition with $\zeta : \mathrm{Spec}(\overline{k(\underline{s})}) \to \mathrm{Spec}(k(\underline{s}))$. From the following cartesian diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(A[x]/f(\underline{s}, x)) \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\
\mathrm{Spec}(k(\underline{s})) & \xrightarrow{\ \eta\ } & \mathrm{Spec}(A)
\end{array}
$$

one may then deduce a commutative diagram

$$
\begin{array}{ccc}
\pi_1^{\text{ét}}(\mathrm{Spec}(k(\underline{s})), \zeta) & & \\
& \searrow^{\alpha} & \\
\downarrow{\scriptstyle\pi_1^{\text{ét}}(\eta)} & & \mathrm{Sym}(Z_{\bar\eta}) \\
& \nearrow^{\beta} & \\
\pi_1^{\text{ét}}(X, \bar\eta) & &
\end{array}
$$

where α denotes the action of $\pi_1^{\text{ét}}(\mathrm{Spec}(k(\underline{s})), \zeta)$ on the fibre of ψ over $\mathrm{Spec}(\overline{k(\underline{s})})$ and β the action of $\pi_1^{\text{ét}}(X, \bar\eta)$ on the fibre $Z_{\bar\eta}$. Since the G-extension $L/k(\underline{s})$ is the Galois closure of $K/k(\underline{s})$, the image of α is a group that may be identified with G. The surjectivity of $\pi_1^{\text{ét}}(\eta)$, which comes from the normality of X [27, Exposé V, Proposition 8.2], thus ensures that the image of β is also G. Thus, there exists a Galois G-cover $Y \to X$ which factors through φ.

To prove that φ is weakly versal, let E/M be a G-extension of fields where M contains k. Since $f(\underline{s}, x)$ is a generic polynomial for G over k, there exists $a \in M^{|\underline{s}|}$ such that $f(a, x)$ is irreducible, $\mathrm{disc}(f)\cdot\mathrm{den}(f)$ does not vanish on a and such that E is a splitting field of $f(a, x)$ over M. Hence $a \in X(M)$, and we are to verify that $F := a \times_X Y$ is $\mathrm{Spec}(E)$. Choose $\overline{M}$ a separable closure of E, so that the morphisms $\mathrm{Spec}(\overline{M}) \to \mathrm{Spec}(E)$ and $\mathrm{Spec}(\overline{M}) \to \mathrm{Spec}(M)$ will both be denoted by ξ. Furthermore,

we set $\bar{a} := a \circ \xi$. We thus have a commutative diagram

$$
\begin{array}{ccccc}
\pi_1^{\text{ét}}(\operatorname{Spec}(M),\xi) & \xrightarrow{\pi_1^{\text{ét}}(a)} & \pi_1^{\text{ét}}(X,\bar{a}) & \xrightarrow{\sim} & \pi_1^{\text{ét}}(X,\bar{\eta}) \\
\downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow \\
& & G & \xrightarrow{\sim} & G \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Sym}(\operatorname{Spec}(\overline{M}[x]/f(a,x))) & = & \operatorname{Sym}(Z_{\bar{a}}) & \xrightarrow{\sim} & \operatorname{Sym}(Z_{\bar{\eta}})
\end{array}
$$

where the right part of the diagram is given by the choice of a path between $\bar{a}$ and $\bar{\eta}$ and the left part by functoriality of the étale fundamental group. But since the étale finite cover $Y \to X$ corresponds to the morphism $\pi_1^{\text{ét}}(X,\bar{\eta}) \to G$, the commutativity of the previous diagram ensures that the G-cover $F \to a$ corresponds to the morphism $\pi_1^{\text{ét}}(\operatorname{Spec}(M),\xi) \xrightarrow{\gamma} \operatorname{Im}(\gamma)$. Now, when identifying $\pi_1^{\text{ét}}(\operatorname{Spec}(M),\xi)$ with Γ_M, one may check that the image of γ is isomorphic to the group G. The Galois extension E/M thus corresponds to the subsequent morphism $\pi_1^{\text{ét}}(\operatorname{Spec}(M),\xi) \to G$, from which one deduces that $F = \operatorname{Spec}(E)$.

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