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FINITE ORBITS IN SURFACES WITH A DOUBLE ELLIPTIC FIBRATION AND TORSION VALUES OF SECTIONS

**by Pietro CORVAJA,
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Abstract. — We consider surfaces with two elliptic fibrations, each of them provided with a section. We study the orbits under the induced translation automorphisms proving that, under natural conditions, the finite orbits are confined to a curve. This goes in a similar direction of (and is motivated by) recent work by Cantat–Dujardin, although we use very different methods and obtain related but different results.

As a sample of application of similar arguments, we prove a new case of the Zilber–Pink conjecture, namely Theorem 1.5, for certain schemes over a 2-dimensional base, which was known to lead to substantial difficulties.

Most results rely, among other things, on recent theorems by Bakker and the second author of "Ax–Schanuel type"; we also relate a functional condition with a theorem of Shioda on unramified sections of the Legendre scheme. For one of our proofs, we also use recent height inequalities by Dimitrov–Gao–Habegger (or those by Yuan–Zhang).

Finally, in an appendix, we show that the Relative Manin–Mumford Conjecture over the complex number field is equivalent to its version over the field of algebraic numbers.

Résumé. — Nous considérons des surfaces munies de deux fibrations elliptiques, chacune avec une section. Les deux sections induisant des translations sur les fibres, nous nous intéressons aux orbites par cette action. Nous démontrons que, en dehors de cas exceptionnels classifiés, les orbites finies sont contenues dans une courbe algébrique. Alors que notre résultat s'insère dans la même veine d'un résultat semblable de Cantat–Dujardin, qui constitue la motivation principale de notre travail, les métodes que nous employons sont différentes et l'énoncé de finitude obtenu est aussi différent.

Notre approche mène aussi à la solution d'un cas de la conjecture de Zilber–Pink pour certains schémas abéliens sur une base de dimension deux, surmontant des difficultées bien connues.

Un outil essentiel pour la plus part de nos preuves consiste en un théorème récent de Bakker–Tsimerman de type « Ax–Schanuel »; nous utilisons aussi un théorème

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de Shioda sur les sections non-ramifiées du schéma de Legendre et une minoration pour la hauteur due à Dimitrov–Gao–Habegger et Yuan–Zhang.

Enfin, dans un appendice nous montrons que la conjecture de Manin–Mumford relative dans le cas complexe est équivalente à sa version dans le cas du corps des nombres algébriques.

1. Introduction

This paper originally arose through work of S. Cantat and R. Dujardin, who, in the paper [8], study *non elementary* subgroups of automorphisms of certain projective surfaces, and especially their finite orbits. In particular, they consider the discrete groups of the so-called Wehler surfaces, i.e. algebraic hypersurfaces of degree $(2,2,2)$ in $\mathbb{P}_1^3$; such surfaces, also called $K3$ surfaces of Markoff type, are endowed with three genus-1 fibrations. They prove various results which confine the finite orbits to a finite union of curves, under appropriate conditions. Their methods use equidistribution and the classification of measures invariant under certain groups.

Double elliptic fibration

Significant cases arise when the surface $\mathcal{X}$ is endowed with a double elliptic fibration; namely, we have two rational maps $\lambda, \mu : \mathcal{X} \to \mathbb{P}_1$, such that for each of them the general fiber is an elliptic curve on $\mathcal{X}$, with a zero section.[(1)]

We assume that the following hypotheses hold:

Hypotheses.

(H1) *The map $(\lambda, \mu) : \mathcal{X} \to \mathbb{P}_1^2$ is a finite morphism; in particular, the two elliptic fibrations are distinct.*

(H2) *None of the two schemes is isotrivial.*

(H3) *The surface $\mathcal{X}$ is smooth and projective.*

(H4) *We further suppose to have two sections σ, τ, resp. for $\mathcal{E}, \mathcal{F}$, none of which is torsion.*

(H5) *All these objects are defined over a number field K.*

(1) Note that allowing one of the base curves to be of positive genus would imply the other scheme to be isotrivial: indeed every map from a genus one curve to a higher genus curve is constant, which rules out the possibility of a base of genus $\geqslant 2$, and if the base curve had genus one, then only countably many fibers of the second fibration could admit a non-constant map to it.

The hypothesis on the finiteness of the map (λ, μ) is not restrictive, since it can be achieved, by changing the model, even starting from situations when λ or μ are just rational maps, or the morphism (λ, μ) contracts a curve.

We denote by E_a (resp. F_b) the fiber $\lambda^{-1}(a)$ (resp. $\mu^{-1}(b)$); note that the degree of the map $(\lambda, \mu) : \mathcal{X} \to \mathbb{P}_1 \times \mathbb{P}_1$ is the intersection product $E_a \cdot F_b$ between two generic fibers.

The sections σ, τ define birational transformations $\widetilde{\sigma}, \widetilde{\tau}$ of $\mathcal{X}$, e.g. putting

$$\widetilde{\sigma}(p) := p + \sigma(\lambda(p)), \quad p \in \mathcal{X}, \tag{1.1}$$

where the addition is meant on $E_{\lambda(p)}$, and similarly for τ in place of σ.

We denote by Γ the group generated by $\widetilde{\sigma}, \widetilde{\tau}$, and we are interested in the set of points of $\mathcal{X}$ having *finite* orbit through Γ. We stress that the elements of Γ are birational transformations so they can have isolated indeterminacy points. When we speak of the orbit of a point p under Γ we mean the set of elements of the form $\gamma(p)$, where $\gamma \in \Gamma$ is well defined at p. When $C \subset \mathcal{X}$ is an irreducible curve and γ a rational transformation, the strict transform $\gamma(C)$ is always well defined, since γ is well defined on a Zariski-open subset of C.

Goals

One of the goals of this paper is to prove in these cases stronger conclusions than in [8], with respect to the infinitude of orbits, using moreover a completely different method with respect to [8]. Specifically, we shall show that already the orbit of a point under a "small" part of the group is infinite, if the point lies outside a prescribed finite union of curves.

An example

A nice well-known example of the situation is provided by the quartic Fermat surface $\mathcal{X} \subset \mathbb{P}_3$ given by $x^4 + y^4 = z^4 + w^4$. (See e.g. the paper [11] for a study of arithmetic properties of that surface exploiting the presence of a double fibration.) Swinnerton-Dyer in [28] already constructed elliptic fibrations with sections, as follows: for a line on $\mathcal{X}$ (there are 48 of them), consider the pencil of planes through it, which intersects $\mathcal{X}$ in the line plus a cubic curve in the plane. By considering another line, its intersection with the cubic provides a section, which can be taken to be the zero section, so we have an elliptic family. Still another line gives another section.

Since $\mathcal{X}$ is a $K3$ surface, it is minimal and then it can be shown that the automorphisms which arise in this situation extend regularly to $\mathcal{X}$. However our method (with appropriate definitions) works even for birational automorphisms which cannot be extended to regular ones.

Two more examples, still involving $K3$ surfaces, derive from a work of N. Elkies [15] on the Euler equation $A^4 + B^4 + C^4 = D^4$ and a paper of E. Fuchs, M. Litman, J. Silverman and A. Tran [16] on $K3$ surfaces of Markoff type.

About the transformations

Each element of the group Γ is well-defined outside a finite set and might contract some (rational) curves; this is due to the fact that one cannot in general suppose that both elliptic fibrations are relatively minimal. In view of this possible phenomenon, we avoid points of indeterminacy by giving the definitions which follow.

For a point $p \in \mathcal{X}$ we set

$$(1.2)\quad O(p) = \left\{\widetilde{\sigma}^m\widetilde{\tau}^n(p)\ :\ m,n \in \mathbb{N} = \{0,1,\dots\},\ \widetilde{\sigma}^m\widetilde{\tau}^n \text{ is defined at } p\right\}.$$

Our attention is on *the set of p such that $O(p)$ is finite*; we note at once that this set is contained in $\mathcal{X}(\overline{K})$. We shall prove the following

Theorem 1.1. — *Under the assumptions* (H1), (H2), (H3), (H4), *and* (H5), *the points $p \in \mathcal{X}$ such that $O(p)$ is finite lie in a finite union of fibers F_x of the fibration μ.*

Under mild assumptions, Theorem 1.1 implies actually the finiteness of the set of points whose Γ-orbit is finite:

Corollary 1.2. — *Suppose moreover that the map $(\lambda,\mu) : \mathcal{X} \to \mathbb{P}_1^2$ is regular and finite. Then there are only finitely many points $p \in \mathcal{X}$ such that all sets $O(\widetilde{\sigma}^h(p))$, $h \in \mathbb{N}$, are finite.*

In particular, there are only finitely many points $p \in \mathcal{X}$ such that the set

$$\left\{\widetilde{\sigma}^m\widetilde{\tau}^n\widetilde{\sigma}^h(p), m, n, h \in \mathbb{Z}\right\}$$

is finite.

Actually, we shall deduce from Theorem 1.1 the finiteness of the points $p \in \mathcal{X}$ such that all the above set is finite while varying $m, n \in \mathbb{N}$ and $h \in \mathbb{Z}$.

Deduction of the Corollary 1.2 from the Theorem 1.1. — To prove the Corollary 1.2, note that if p is one of the points relevant in the corollary, by Theorem 1.1 all the $\widetilde{\sigma}^h(p)$, $h \in \mathbb{N}$, lie in a certain finite union $U = \bigcup_{x\in A} F_x$, where A is a finite subset of $\mathbb{P}_1$. Let δ be the degree of the j-function of the fibration μ and $h = \delta \cdot |A|$; then, since the fibrations are distinct, for any $x \in A$ the curves $F_x, \widetilde{\sigma}(F_x), \ldots, \widetilde{\sigma}^h(F_x)$ cannot all be of the shape F_y for $y \in A$.

Suppose that our set is infinite, then its intersection with some component F' of an F_a would be infinite, for suitable $a \in A$. Then for each $h \in \mathbb{N}$ the curve $\widetilde{\sigma}^h(F')$ would be inside U. But then some nonzero power $\widetilde{\sigma}^q$ would stabilise F'. On the other hand, by Assumption (H1) the function $\lambda : F' \to \mathbb{P}_1$ is generically surjective, implying that F' contains a point p' such that $\widetilde{\sigma}^q$ has infinite order at p'. This orbit lies in the intersection of F' with the λ-fiber at p', and this intersection is finite by assumption. The contradiction proves the assertion. □

Higher dimensions?

In this paper we consider only the case of surfaces, motivated by the mentioned work [8]. However, in principle the methods would apply also to analogous situations in higher dimensions, for instance to the case of n elliptic fibrations on a variety of dimension $n \geqslant 3$. Examples can be easily constructed, for instance by taking three 2-nets of elliptic curves in $\mathbb{P}_3$.

But in dimension $\geqslant 3$ there are other possibilities; for instance in dimension 3 we could consider both elliptic fibrations over a surface and fibrations over a curve by abelian surfaces (possibly in the birational sense). We believe that the method of this paper may be applied to such general context, using the recent results on heights of Dimitrov–Gao–Habegger [14] or Yuan–Zhang [29] in place of the Silverman–Tate bounds.

1.1. A kind of improvement

We can formulate the above situation in the following terms: we have a scheme over an open subset $\mathcal{X}_0$ of $\mathcal{X}$ with fibers given as products of three elliptic curves. Namely, above a point $x \in \mathcal{X}_0$ we associate the product of elliptic curves

$$E_{\lambda(x)} \times F_{\mu(\tilde{\sigma}(x))} \times E_{\lambda(\tilde{\tau}\circ\tilde{\sigma}(x)))}.$$

Here $\mathcal{X}_0 \subset \mathcal{X}$ is obtained by removing from $\mathcal{X}$ the subset where $\lambda, \mu \circ \widetilde{\sigma}$, or $\lambda \circ \widetilde{\tau} \circ \widetilde{\sigma}$ is not defined or some elliptic curve degenerates; hence $\mathcal{X}_0$ is the complement of finitely many curves and points in $\mathcal{X}$.

For simplicity, we denote by $\mathcal{E}_i \to \mathcal{X}_0$, $i = 1, 2, 3$, the corresponding elliptic schemes. We have an obvious section ξ given by

$$\xi(p) = \sigma(p) \times \tau(\widetilde{\sigma}(p)) \times \sigma((\widetilde{\tau} \circ \widetilde{\sigma})(p)),$$

where we simplified the notation by writing $\sigma(p)$ for $\sigma(\lambda(p))$, $\tau(p)$ for $\tau(\mu(p))$.

Using a recent result of Bakker with the second author, we shall prove a variation on Theorem 1.1, which is stronger in a sense.

We start by noting that $O(p)$ is certainly infinite if either $\sigma(p)$ or $\tau(p)$ has infinite order on the relevant elliptic curve. Thus we study the points p where both sections have finite order.

We suppose from now on that:

Hypothesis.

(H6) *The map (λ, μ) is regular.*

Definition 1.3. — *We define the subsets T_λ, T_μ of $\mathcal{X}$ by setting*

$$T_\lambda = \left\{x \in \mathcal{X}_0 \,:\, \sigma(\lambda(x)) \text{ has finite order on } E_{\lambda(x)}\right\}$$

and similarly for τ, μ.

The set is known to be a countable infinite union of fibers of curves; it is dense in $\mathcal{X}$, even in the complex topology; this can be deduced from Cantat's work [7] or via the non-constancy of the so-called Betti map; see for instance [9, 30]. Since by Hypothesis (H1) the map (λ, μ) is dominant, the intersection $T_\lambda \cap T_\mu$ is a countable Zariski-dense set. Similarly, the set of points $x \in T_\lambda$ such that $\widetilde{\sigma}(x) \in T_\mu$ is again Zariski-dense.

Theorem 1.4. — *The set*

$$\mathrm{Ex} := \{x \in \mathcal{X} \,:\, x \in T_\lambda,\ \widetilde{\sigma}(x) \in T_\mu,\ \widetilde{\tau} \circ \widetilde{\sigma}(x) \in T_\lambda\}\,. \tag{1.3}$$

is contained in a finite union of curves.

This result shows that adding the third constraint $\widetilde{\tau} \circ \widetilde{\sigma}(x) \in T_\lambda$ to the set $T_\lambda \cap T_\mu$ shrinks a Zariski-dense set into a degenerate one.

Theorems 1.1, 1.4 go in the same direction of the conclusions of [8], but with some significant differences. First, our groups can include also birational transformations, which are not everywhere regular; also we do not need to consider the whole orbits to deduce our conclusions. On the other hand, the paper [8], using completly different methods, explores more general fibrations and obtains a precise classification of the cases when the finiteness of orbits fails.

1.2. On the Betti map and Relative Manin–Mumford Conjecture on a two-dimensional base

While conceiving this work, we realized a few links with other problems.

1.2.1. Ax–Schanuel and the Betti map

The applications of known arithmetical methods led us to study new problems concerning the so-called Betti maps, related in turn to functional transcendence and theorems of Ax–Schanuel type; in particular we raised the question solved in the paper [2].

1.2.2. Relative Manin–Mumford

Some of the methods lead to cases of the Zilber–Pink conjecture for torsion points which until very recently weren't yet answered. Here we are thinking of the part of Zilber–Pink conjecture concerning the so-called *relative Manin–Mumford problem:* to confine the torsion values of a section of an abelian scheme to the appropriate closed subvariety. This had been dealt essentially for abelian schemes over a curve (see e.g. [10] for a recent result and a discussion of previous works). In [20] Ph. Habegger treated, via a somewhat ad hoc method, an instance of an elliptic scheme over a surface (see also [30, Theorem 3.13]). While the present paper was in preliminary form, Gao and Habegger [19] have announced a complete solution over varieties of arbitrary dimension. The methods of the present work, relying on the Bakker–Tsimerman results of Ax–Schanuel type, combined with crucial height estimates of Dimitrov–Gao–Habegger [14] (or Yuan–Zhang [29]; for simplicity, we shall use a version of the latter), allow a partially different proof of their results.

Here we include only a sample of this, which in fact was present in a previous version of this paper, predating Gao–Habegger announcement.

More specifically we shall prove the following:

Theorem 1.5. — *Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be non-isotrivial and pairwise non-isogenous elliptic schemes over a surface $\mathcal{X}$, and let $\sigma_1, \sigma_2, \sigma_3$ be non torsion sections, with respect to $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ respectively, all defined over $\overline{\mathbb{Q}}$. Then the points $x \in \mathcal{X}$ such that $\sigma_i(x)$ is torsion on $\mathcal{E}_i$ for $i = 1, 2, 3$ lie in the union of finitely many curves.*

The previous results could be obtained as special cases of this (the assumption on the three schemes not being isogenous is shown to hold in the context of Theorem 1.1; see Lemma 5.1). However, we have preferred to develop independent proofs, free of these tools, whose arguments could be useful for other tasks.

1.2.3. Relative Manin–Mumford over $\mathbb{C}$

While Gao–Habegger have announced their proof for varieties over $\overline{\mathbb{Q}}$, we realized that the transcendence methods invoked in this paper allow a specialization argument to formally deduce the general case over arbitrary complex bases. In the Appendix A, we give a proof of this deduction.[2]

1.2.4. Ramification of sections of the Legendre scheme

Our proofs rely on the Pila–Wilkie counting (see Section 2.2); as usual in the applications of this method, one has to deal eventually with the so-called *algebraic part* of the relevant varieties, proving that this is empty (or small). This leads to questions of *functional type*.

We prove the following result that we couldn't locate in the literature. Recall that an isogeny between elliptic curves is said to be cyclic if its kernel is a cyclic subgroup. Also, for points s_1, s_2 in an elliptic curve we say that they are linearly dependent if there exist integers a_1, a_2 not both zero such that $a_1 s_1 + a_2 s_2 = 0$; the same definition will be used for elliptic curves over function fields where points become sections.

Theorem 1.6. — *Let $\mathcal{E}$ be a non-isotrivial elliptic scheme over a complex algebraic curve B, with a non-torsion section σ. Consider an irreducible algebraic curve T inside $B \times B$ with the following property: for generic $(x, y) \in T$, there is a cyclic isogeny $\phi_{(x,y)} : E_x \to E_y$ such that the points $\phi_{(x,y)} \circ \sigma(x)$ and $\sigma(y)$ are linearly dependent in $E_y(\mathbb{C})$. Then ϕ is an isomorphism.*

The hypothesis that ϕ is cyclic prevents from taking for T the diagonal and for ϕ any multiplication-by-m-map, for any $m > 1$.

Remark. — We note that we can suppose that the degree of the isogeny $\phi_{(x,y)}$ is fixed, i.e. independent of the points $(x, y) \in T$. Also, the isogeny

(2) Ph. Habegger informed us that he and Gao also developed a reduction from $\overline{\mathbb{Q}}$ to $\mathbb{C}$, somewhat different from ours.

must vary algebraically on T; in other words, T is endowed with two isogenous elliptic schemes $\mathcal{E}_x$ and $\mathcal{E}_y$, whose fibers over the points $(x,y) \in T$ are E_x and E_y.

Also, the linear dependence relation between the points $\phi_{(x,y)} \circ \sigma(x)$ and $\sigma(y)$ can be phrased in terms of relations between the two sections: $T \ni (x,y) \mapsto \phi_{(x,y)}(\sigma(x)) \in E_y$ and $T \ni (x,y) \mapsto \sigma(y) \in E_y$. Namely, there exists non-zero integers n, m such that identically

$$n \cdot \phi_{(x,y)}(\sigma(x)) = m \cdot \sigma(y).$$

To prove Theorem 1.6 we found it necessary to invoke a strengthening of a result by Shioda on ramification of fields of definition for a non-torsion section of the Legendre scheme (see Subsection 2.1), recently obtained by the first and third authors in [12].

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We are also very grateful to an anonymous referee who read previous versions of this paper with great care, pointed out several inaccuracies and unclear points, and gave us precious suggestions to improve the presentation.

2. Ingredients

A fundamental ingredient for the present argument is the use of estimates of Pila–Wilkie (after work of Bombieri–Pila) for the distribution of rational points on *transcendental analytic subsets* (with certain properties). This has been applied by Pila and the third author to a new proof of the Manin–Mumford conjecture; see for instance the third author's book [30], where several applications of the same nature also appear, and the more recent book by Pila [26], devoted more specifically to the estimates.

Another ingredient is of *functional type*, and concerns results from differential Galois theory and monodromy. In the next subsection we comment on this. Another result of functional type, Lemma 5.2 due to Bakker–Tsimerman, will be stated and used later.

2.1. Monodromy of elliptic logarithms, Betti maps

We refer to the paper [12], by the first and third author for details of some proofs, and for other references.

Take the Legendre elliptic scheme $\mathcal{L}$ defined affinely by the Weierstrass equation

$$y^2 = x(x-1)(x-\lambda),$$

$\mathcal{L}_\lambda := \mathbb{C}/L_\lambda$, where L_λ is a lattice. It is classical that in the region $\max(|\lambda|, |1-\lambda|) < 1$ this lattice L_λ may be generated by the *periods* given by the hypergeometric function expansions (see Husemöller book [21], p. 184)

$$\omega_1(\lambda) = i\pi \sum_{n=0}^{\infty} \binom{1/2}{n}^2 (1-\lambda)^n, \qquad \omega_2(\lambda) = \pi \sum_{n=0}^{\infty} \binom{1/2}{n}^2 \lambda^n. \tag{2.1}$$

These functions may be analytically continued to (the universal cover of) $\mathbb{C} \setminus \{0,1\}$ and there is a monodromy action of the fundamental group $\pi_1(\mathbb{C} - \{0,1\})$ expressed by the matrices with rows $(1,2),(0,1)$ and $(1,0)$, $(2,1)$ corresponding to circles starting say at $1/2$ and encircling counter-clockwise 1 and 0 respectively. These matrices generate the free group denoted usually $\Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z})$.

These periods satisfy the Gauss differential equation corresponding to the operator

$$\mathcal{G} := 4\lambda(1-\lambda)D^2 + 4(1-2\lambda)D - I, \quad D = \mathrm{d}/\mathrm{d}\lambda.$$

Given an arbitrary non-isotrivial elliptic scheme $\mathcal{E} \to B$ over an affine curve B, we can enlarge if necessary the function field of B so that the 2-torsion sections become rational; this corresponds to taking an unramified cover $B' \to B$ and replacing the elliptic scheme $\mathcal{E} \to B$ by its pull back via the base change $B' \to B$. We can then reduce to the case where the two-torsion is rational; this provides a map $B \to \mathbb{C} - \{0,1\}$ expressing the elliptic scheme $\mathcal{E} \to B$ as a pull-back of the Legendre scheme $\mathcal{L} \to \mathbb{C} - \{0,1\}$. (See e.g. [12, Section 5.2]). The Lie algebras of the elliptic schemes $\mathcal{L} \to \mathbb{C} - \{0,1\}$ (resp. $\mathcal{E} \to B$) can be analytically identified with the products $(\mathbb{C} - \{0,1\}) \times \mathbb{C}$ (resp. $B \times \mathbb{C}$); they are endowed with projections to the total space of the elliptic schemes; on each fiber, this projection consists in the elliptic exponential map.

The base curve B is necessarily hyperbolic, hence its universal cover is provided by Poincaré's half-plane $\mathcal{H}$.

We then obtain the following commutative diagram, where the right-bottom 2×2 diagram is composed by algebraic varieties and morphisms:

$$\begin{array}{ccccc} \mathcal{H} \times \mathbb{C} & \longrightarrow & B \times \mathbb{C} & \longrightarrow & (\mathbb{C} - \{0,1\}) \times \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H} \times_B \mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & B & \longrightarrow & \mathbb{C} - \{0,1\} \end{array} \tag{2.2}$$

The periods can be viewed as locally well-defined sections of the projection $B \times \mathbb{C} \to B$ (middle vertical part of the above diagram), as well as globally well defined sections of the projection $\mathcal{H} \times \mathbb{C} \to \mathcal{H}$. The fundamental group $\pi_1(B)$ acts on the period-lattice; after choosing a base point $b_0 \in B$ and a basis of the period lattice ω_1, ω_2 in a neighborhhod of b_0, the monodromy action of the fundamental group provides a morphism $\pi_1(B, b_0) \to \Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z})$, whose image has finite index in Γ_2.

Now, if we have a section $\sigma : B \to \mathcal{E}$, we may associate to it an elliptic logarithm l, which is a locally well-defined function to $\mathbb{C}$ (ie. a section of the projection $B \times \mathbb{C} \to B$), which associates to each point $b \in B$ a complex number in the class of $\sigma(\pi(b))$ modulo $L_{\pi(b)}$. As just remarked, this function l is well-defined only locally, and is subject to a monodromy action under the fundamental group of $B(\mathbb{C})$. This function will also satisfy a linear differential equation with algebraic functions coefficients, actually the operator $\mathcal{G}$ applied to η produces a function in $\mathbb{C}(B)$. (See the papers [9, 12] for details.)

One may study the differential Galois theory associated to this situation, as done long ago by André and Bertrand independently, actually in much more general contexts, and for several sections (see Bertrand's paper [3] or André's [1]). Let us recall some results for the special case considered here.

The differential Galois group of the extension $\mathbb{C}(B)(\omega_1, \omega_2, \omega_1', \omega_2')/\mathbb{C}(B)$ generated by the periods and their derivatives over the function field of the curve B (equipped with a natural derivation extending $\mathrm{d}/\mathrm{d}\lambda$), embeds in $\mathrm{SL}_2(\mathbb{C})$ (representing its action on the periods), and is in fact equal to it, since for instance the monodromy group is well-known to be Zariski-dense in the differential Galois group. (In particular, the transcendence degree is $3 = \dim \mathrm{SL}_2$; note the relation over $\mathbb{C}(B)$ given by the Wronskian of the periods.)

As to the section, if G denotes the differential Galois group of the differential extension $\mathbb{C}(B)(\omega_1, \omega_2, \omega_1', \omega_2', l, l')/\mathbb{C}(B)$, then G is represented in SL_3 by matrices of the form

$$\begin{pmatrix} T & & a \\ & & b \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

where $T \in \mathrm{SL}_2$ and $a, b \in \mathbb{C}$ represent the coefficients of translations by the periods in the action on the logarithm. We let such matrices act on the right on row vectors. The algebraic group formed by the matrices of the above form is isomorphic to the semi-direct product $\mathbb{G}_a^2 \rtimes \mathrm{SL}_2$ and can be viewed as a group of affine transformations of the plane. André and Bertrand proved that, when the section is not torsion, the differential Galois group G is the maximal possible one, namely the full group of matrices of the form (2.3). So G has dimension 5, as the transcendence degree of the extension $\mathbb{C}(B)(\omega_1, \omega_2, \omega_1', \omega_2', l, l')/\mathbb{C}(B)$.

We summarize these results in the following proposition

Theorem 2.1. — *Let $\mathcal{E} \to B$ be a non-isotrivial elliptic scheme with a non-torsion section σ. Let $b_0 \in B$ be a point, ω_1, ω_2 be a base of the period lattice in a neighborhood of b_0, and l an elliptic logarithm for σ in the same neighborhood. Let $\pi_1(B, b_0) \to \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be the monodromy representation of the fundamental group of B acting on the periods. Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be the differential Galois group of the differential extension $\mathbb{C}(B)(\omega_1, \omega_2, \omega_1', \omega_2', l, l')/\mathbb{C}(B)$. Then $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is Zariski-dense in SL_2 and $G \subset \mathrm{SL}_3(\mathbb{C})$ is the group of the matrices of the form* (2.3).

In particular, the three functions ω_1, ω_2, l are algebraically independent over $\mathbb{C}(B)$.

Locally on the mentioned neighborhood of b_0 one can express the elliptic logarithm as a linear combination of the periods with real coefficients: namely we may write locally on B:

$$l(b) = \beta_1(b)\omega_1(b) + \beta_2(b)\omega_2(b),$$

for real-analytic functions β_1, β_2, locally well-defined on B.

In view of the last sentence in the above theorem, we can state the following very special case of a theorem of Manin [22].

Corollary 2.2. — *The Betti map for a non-torsion section of a non-isotrivial elliptic scheme is never constant.*

Consider now the monodromy group of this map; in other words, we must consider the monodromy of the elliptic logarithm *together* with that of the periods. This group $\widetilde{\Gamma}$ acts on the Betti map via affine transformations; it is naturally embedded into $\mathrm{SL}_3(\mathbb{Z}) \cap G$.

Theorem 2.1 from [12] asserts that:

Theorem 2.3. — *Let $\widetilde{\Gamma}$ be the monodromy group of the Betti map, as defined above. Then $\widetilde{\Gamma}$ has finite index in the affine group $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, namely in the group of integral points of the algebraic group $\mathbb{G}_A^2 \rtimes \mathrm{SL}_2$. In particular the kernel of its natural projection to the monodromy group $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of the periods is isomorphic to $\mathbb{Z}^2$.*

Equivalently, if we denote by B^* (resp. B_l) the minimal unramified topological covers of $B(\mathbb{C})$ where we can define the periods (resp. periods and l), then B_l is a Galois cover of B^* with Galois group isomorphic to $\mathbb{Z}^2$.

As proved in our paper [12], this result implies a theorem proved by Shioda in 1972, which we now recall.

Let σ be an algebraic section for the Legendre scheme, well-defined over some algebraic curve B, viewed as a cover of the λ-line. Such a section corresponds to a pair (ξ, ϕ) of algebraic functions of λ satisfying the Legendre equation: $\phi^2 = \xi(\xi - 1)(\xi - \lambda)$. Shioda's theorem concerns the field of definition $\mathbb{C}(\lambda, \xi, \phi)$, a finite extension of $\mathbb{C}(\lambda)$.

General results on elliptic curves prove that if the section is torsion, then this extension is *unramified* except possibly above $\lambda = 0, 1, \infty$. Shioda's theorem (proved by T. Shioda in 1972; it is stated as [12, Theorem 2.5] in our paper) asserts the converse:

Theorem 2.4. — *If $\mathbb{C}(\lambda, \xi, \phi)/\mathbb{C}(\lambda)$ is unramified outside $\{0, 1, \infty\}$ then (ξ, ϕ) is torsion.*

Already the case where we assume $\mathbb{C}(\lambda, \xi, \phi) = \mathbb{C}(\lambda)$ is not completely trivial. For the general case Shioda uses the Shioda–Tate formula and clever topological arguments. See the paper [12] for a sketch of this proof, and for a different proof using modular functions.

Bertrand's theorem 2.1 admits extensions, due to both Bertrand himself and Y. André, to several sections and possibly several elliptic schemes. We just quote the results we shall need:

Theorem 2.5. — *Let $\mathcal{E}_1, \ldots, \mathcal{E}_k \to B$ be pairwise non-isogenous elliptic schemes over a curve B and let $\sigma_1, \ldots, \sigma_k$ be rational sections. Let $\eta_1, \ldots, \eta_k$ be the corresponding elliptic logarithms in a neighborhood of a point $b_0 \in B$ and let $\omega_{1,i}, \omega_{2,i}$, for $i = 1, \ldots, k$, be bases of periods for the elliptic schemes. Then the $3k$ analytic functions $\eta_i, \omega_{1,i}, \omega_{2,i}$, $i = 1, \ldots, k$, are algebraically independent.*

Let $\mathcal{E} \to B$ be a non-isotrivial elliptic scheme and $\sigma_1, \dots, \sigma_k$ be linearly independent rational sections. Let as above $\eta_1, \dots, \eta_k$ be the corresponding elliptic logarithms in a neighborhood of a point $b_0 \in B$ and ω_1, ω_2 be a basis of the period lattice. Then the $k+2$ analytic functions $\omega_1, \omega_2, \eta_1, \dots, \eta_k$ are algebraically independent.

The above mentioned result of Bakker–Tsimerman can be viewed as an extension of the above statement to higher dimensional bases.

2.2. Distribution of rational points on analytic sets

We recall here a fundamental result about the distribution of rational points on real analytic sets, proved by Pila and Wilkie in [27]. Given a real analytic set $X \subset \mathbb{R}^n$, its *algebraic part* is the union of all connected semi-algebraic subsets of X of positive dimension. We refer to the paper of Bierston–Millman [4] for precise definitions. Denote by X^{alg} the algebraic part of X.

For a rational point $x = (a_1, \dots, a_n) \in \mathbb{Q}^n$, its multiplicative height is defined as the maximum of the height of the coordinates, while the height of a single rational number a/b, where a, b are coprime integers, is $\max(|a|, |b|)$.

For a subset $X \subset \mathbb{R}^n$ and a positive integer T we denote by $N(X, T)$ the number of rational points in X of (multiplicative) height $\leqslant T$:

$$N(X,T) = \sharp \{x \in X(\mathbb{Q}) \, : \, H(x) \leqslant T\} .$$

We shall consider families of analytic sets which are *definable* in the sense of the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$; for this notion we refer to [27, Section 1.7], and the book [30].

The result that we shall use is the following ([27, Theorems 1.8 and 1.9]):

THEOREM 2.6.

(1) *Let X be an $\mathbb{R}_{\mathrm{an,exp}}$-definable set and let $\epsilon > 0$. There is a constant $c(X, \epsilon)$ such that*

$$N\left(X - X^{\mathrm{alg}}, T\right) \leqslant c(X, \epsilon) T^{\epsilon}.$$

(2) *Let $Z \subset \mathbb{R}^m \times \mathbb{R}^n$ be an $\mathbb{R}_{\mathrm{an,exp}}$-definable set. For each $b \in \mathbb{R}^n$ denote by Z_b its fiber in Z. Then for every $\epsilon > 0$ there exists a positive number $c(Z, \epsilon)$ such that for every $b \in \mathbb{R}^n$*

$$N\left(Z_b - Z_b^{\mathrm{alg}}, T\right) \leqslant c(Z, \epsilon) T^{\epsilon}.$$

2.3. Height bounds

In the sequel the notation is as in Section 1. We fix a number field K where the variety, the fibrations, sections are defined and such that all the indeterminacy points for $\widetilde{\sigma}, \widetilde{\tau}$ are K-rational.

We recall the following lemma, which follows immediately from results proved first by Silverman and Tate, actually in much greater generality. We shall later need a stronger result recently proved by Yuan–Zhang in this context; a similar result, sufficient for our purposes, was obtained by Dimitrov–Gao–Habegger in [14].

We denote by $h(x)$ the additive Weil height of the algebraic number (or point) x (see e.g. the book [6]).

Lemma 2.7. — *There exist numbers $C_1 > 0$, $C_2 > 0$ such that:*

(1) *for each $x \in \mathbb{P}_1$ such that $\sigma(x)$ has finite order on E_x or $\tau(x)$ has finite order on F_x, we have $h(x) < C_1$.*

(2) *if N is the exact order of $\sigma(x)$ (resp. $\tau(x)$) then*

$$C_2^{-1}[\mathbb{Q}(x):\mathbb{Q}]^{1/2} \leqslant N \leqslant C_2 \cdot [\mathbb{Q}(x):\mathbb{Q}]^2.$$

Note that automatically these points are algebraic, since the sections are supposed to be non-torsion.

The first bound is a very special case of Silverman's specialization theorem (see [30, Proposition 3.2, p. 69], for a simple self-contained argument.) The upper bound for N derives from the upper bound $N \leqslant c[\mathbb{Q}(x):\mathbb{Q}]^2(1+h(x))$ due to S. David [13], combined with Silverman's bound for the height; a similar but somewhat weaker result is due to Masser; see [30, Appendix D (by Masser)] for a sketch of proof.

The lower bound for the torsion order in terms of the degree of the field of definition derives from the fact that multiplication by N on an elliptic curve is a morphism of degree N^2.

2.4. From height bounds to topological confinement

The boundedness of the height will be used in several respects, including the possibility of confining our relevant points to prescribed compact sets.

Lemma 2.8. — *Let V be a projective variety defined over a number field K, embedded in a projective space so that we have a height function coming from the embedding. Let H be a finite union of hypersurfaces defined over K. Let also T be a sufficiently large real number. Then* *there is a neighborhood H_T of H in the complex topology such that if $x \in (V - H)(\overline{\mathbb{Q}})$ satisfies $h(x) \leqslant T$ and $[K(x) : K] = d$ at least $d/2$ conjugates of x over K lie in $V - H_T$.*

Proof. — Let us cover V by finitely many affine open sets V_i on which H is defined by an equation $f_i = 0$, for a regular function $f_i \in K[V_i]$. We may work separately on each open set V_i. For every real $\delta > 0$ let us define the neighborhood H_δ by the inequality $|f_i| \leqslant \delta$. If n_i is the number of conjugates of x falling in the neighborhood H_δ then the height of $f_i(x)$ satisfies

$$h(f_i(x)) = h\left(\frac{1}{f_i(x)}\right) \geqslant \frac{n_i}{d \cdot [K : \mathbb{Q}]} \log(1/\delta)$$

(here we used that $f_i(x) \neq 0$). On the other hand $h(f_i(x)) \leqslant c_i h(x) \leqslant c_i T$, where c_i depends only on V, i, f_i. So we obtain that

$$n_i \leqslant \frac{c_i T d [K : \mathbb{Q}]}{\log 1/\delta}.$$

For sufficiently small δ we get the result. Actually if $\eta < 1$ is a number such that

$$\log(1/\delta)^{-1} \left(\sum_i c_i\right) [K : \mathbb{Q}] T < \eta \qquad \square$$

we obtain the result with ηd in place of $d/2$.

3. Proof of Theorem 1.1

We keep the notation of Theorem 1.1 and of Subsection 1.1. Recall we are interested in points $p \in \mathcal{X}_0$ whose orbit $O(p)$ is finite.

LEMMA 3.1. — *Let $p \in \mathcal{X}_0$ such that $O(p)$ is finite. Then p is an algebraic point.*

Proof. — By assumption the section σ takes a torsion value on $\lambda(p)$ while the section τ takes a torsion value on $\mu(p)$. Since both σ and τ, as well as the two elliptic fibrations, are defined over $\overline{\mathbb{Q}}$, the points of the bases where the section take torsion values are algebraic. So p lies in the intersection of a fibre of τ over an algebraic point with a fiber of λ over another algebraic point, so on the intersections of two algebraic curves defined over $\overline{\mathbb{Q}}$, so it is an algebraic point. $\square$

Let p be such a point and let us denote $a = \lambda(p) \in \overline{\mathbb{Q}}, b = \mu(p) \in \overline{\mathbb{Q}}$, so that $p \in E_a \cap F_b$.

By assumption $\tau(b)$ has finite order, say m, on F_b, and each of the points $\sigma(\lambda(\widetilde{\tau}^r(p)))$, $r = 0, 1, \dots, m-1$, has finite order, say n_r, on $E_{\lambda(\widetilde{\tau}^r(p))}$.

If we have a bound on m then b lies in a prescribed finite set, since τ is not a torsion section; hence p belongs to a finite union of fibers for μ. Therefore in what follows we may assume that m is larger than any prescribed number. For instance, we shall assume $m > k \cdot \deg \lambda_{F_b}$, where k is the number of critical values of λ.

The curve $F_b^0 := F_b \cap \mathcal{X}_0$ may be seen as the base curve for two elliptic schemes with sections. Namely, to $z \in F_b^0$ we associate the curves $E_{\lambda(z)}$, $E_{\lambda(z+\tau(b))}$ and the two respective sections given by

$$s_0(z) := \sigma(\lambda(z)) \quad \text{and} \quad s_1(z) := \sigma(\lambda(z + \tau(b))).$$

This yields a product scheme over F_b^0 with a section given by (s_0, s_1).

Note that none of these two sections is (identically) torsion. This holds by assumption, for otherwise σ itself would be identically torsion.

Before going ahead, note that under the action of $\mathrm{Gal}(\overline{K}/K)$ we may replace throughout b (and p) by any of its conjugates over K. Since b has bounded height (by Lemma 2.7), we may then suppose that b lies in a sufficiently large prescribed compact set Δ of $\mathbb{P}_1(\mathbb{C})$, not containing any of the finitely many points of bad reduction for μ or λ or $\lambda \circ \widetilde{\tau}$ (by Lemma 2.8).

We subdivide the proof in two parts. The first part will roughly represent a *uniform* version of the results in [24], and we shall follow that paper, indicating briefly the various steps and the necessary modifications. The main difference is that in the present case b must be allowed to vary, although it is confined in a prescribed compact set.

Proof of Theorem 1.1.

Case 1: We have $n_r > m^5$ for some r. — For b in the compact set Δ, we first remove from $F_b(\mathbb{C})$ a very small open neighbourhood of the set of bad reduction relative to λ and $\lambda \circ \widetilde{\tau}$. We may do this smoothly as b varies in Δ. Actually, we may use real-algebraic functions to express these neighbourhoods; such functions are *definable* (in the *o-minimal structure* $\mathbb{R}_{\mathrm{an,exp}}$) and will allow the application of Theorem 2.6.

Then we cover the complementary compact set in $F_b(\mathbb{C})$ with finitely many small compact simply-connected sets (again definable) such that for z inside any of them we may express analytically a pair of generating periods ω_{1i}, ω_{2i}, $i = 0, 1$, for the complex tori corresponding to the curves $E_{\lambda(z+i\tau(b))}$, and elliptic logarithms $\ell_i(z)$ of the section.

Note that all these functions depend also on b (a fact which we do not express explicitly for notational convenience), but since b lies in a compact set we may further suppose that these expressions hold uniformly and analytically for b in a small disk $\Delta_1 \subset \Delta$.

We may further write, for z varying on each of these finitely many small compact sets inside F_b,

$$\ell_i(z) = \beta_{1i}(z)\omega_{1i}(z) + \beta_{2i}(z)\omega_{2i}(z), \quad i = 0, 1, \tag{3.1}$$

for real analytic functions β_{1i}, β_{2i} (depending also analytically on $b \in \Delta_1$). Note that $s_i(z)$ is torsion (on $E_{\lambda(z+i\tau(b))}$) if and only if $\beta_{1i}(z)$ and $\beta_{2i}(z)$ are rational numbers (with common denominator equal to the order of $s_i(z)$).

Let us pick one of the compact sets in question, containing one value of $z \in F_b$, denoted ζ, with the property that both $s_0(\zeta), s_1(\zeta)$ are torsion (in the respective curve). This point ζ shall be of the shape $p + r\tau(b) = \widetilde{\tau}^r(p)$, for some r with $0 \leqslant r \leqslant m-1$. We are assuming that $n = n_r > m^5$.

By the second inequality in Lemma 2.7 we get $m^5 < n \leqslant C[K(\zeta) : K]^2$ so $[K(\zeta) : K] \geqslant C^{-1/2} n^{1/2}$. The other inequality of the same lemma provides $[K(b) : K] \leqslant C^2 m^2 \leqslant C^2 n^{2/5}$. Hence $[K(b,\zeta) : K(b)] \geqslant C' n^{1/2-2/5} = C' n^{1/10}$ for a suitable constant $C^{-5/2} = C' > 0$.

We can take then conjugate ζ_j, $j = 1, \dots, l_1 \geqslant C' n^{1/10}$ of ζ over $K(b)$, and in this way we obtain simultaneous torsion values $s_i(\zeta_j)$ on $E_{\lambda(z+i\tau(b))}$ $(i = 0, 1)$ for all $j = 1, \dots, l_1$.

Note that each ζ_j has uniformly bounded height by Lemma 2.7 and again by Lemma 2.8 we may assume that at least $c_1 l_1$ of these conjugates lie in a same compact set, denoted now $\Omega = \Omega_b \subset F_b(\mathbb{C})$, with the same properties as above, i.e. that the above functions are well-defined and analytic in Ω: here $c_1 > 0$ denotes a positive number depending only on the original data. Hence we may assume that Ω contains ζ_j for $j = 1, \dots, l$ where $l \geqslant c_2 n^{1/10}$.

Each ζ_j, $1 \leqslant j \leqslant l$, gives rise to a rational point $D(\zeta_j)$, of denominator $\geqslant n > m^5$, on the real-analytic surface described in $\mathbb{R}^4$ by $D(z) = D_b(z) := (\beta_{10}(z), \beta_{20}(z), \beta_{11}(z), \beta_{21}(z))$ as z varies in Ω. We denote by $Z_b \subset \mathbb{R}^4$ this surface.

We further note that each rational point $\rho \in \mathbb{Q}^4$ appears as a value $D(\zeta_j)$ for at most c_3 points ζ_j, where c_3 depends only on the opening data: this is because a function $\ell_i(z) - \rho_1\omega_{1i}(z) - \rho_2\omega_{2i}(z)$ has a uniformly bounded number of zeros in Ω for given ρ_1, ρ_2. In turn, this follows as in [23, Lemma 7.1], the only difference is that here b (which does not even appear in the notation) may vary; but locally the lemma holds uniformly, with the same proof. (Alternatively, one may use directly Gabrielov's theorem, see [4].)

We conclude that the real-analytic surface Z_b in $\mathbb{R}^4$ given by $\{B(z) : z \in \Omega\}$ contains at least l rational points of denominator $\leqslant n$, where again $l \geqslant c_4 n^{1/10}$. These points will have height[(3)] which is $\leqslant c_5 n$ where the implicit constant depends only on the involved compact sets.

In order to apply Pila–Wilkie's Theorem 2.6 to Z_b, we pause to prove that the algebraic part of Z_b is empty.

This follows from Theorem 2.5. We follow the exposition in [24]. If the algebraic part is non-empty there is a real-algebraic arc U inside Z_b; its pre-image under $(\beta_{10}, \beta_{20}, \beta_{11}, \beta_{12})$ in $\bigcup_{b\in\Delta} F_b^0 \subset \mathcal{X}^0$ is a real-analytic arc. Now the four functions β_{ij} restricted to this arc all depend algebraically on any of them which is nonconstant (if all are constant the thing is even easier). But then the two functions ℓ_j, when restricted to this arc, are algebraically dependent on the restrictions of the ω_{ij}, $i = 1, 2$, $j = 0, 1$ to this real-analytic arc. However all these functions are complex analytic (locally), so the dependence would hold identically on their domain, which violates Theorem 2.5.

At this point we apply Theorem 2.6, to obtain a bound for the number of these rational points in Z_b. More precisely, we may first decompose Δ as a finite union of small definable compact sets A_j so that the relevant functions are analytic in the union $\bigcup_{b\in A_j} \{b\} \times \Omega_b$, which is sent to $Z := A_j \times \mathbb{R}^4$. This Z is a definable family, where Z_b are the fibres.

From Theorem 2.6 we obtain that the transcendental part of Z_b (actually Z_b in this case) contains at most $c(Z, \epsilon)T^\epsilon$ rational points of height $\leqslant T$, for any $\epsilon > 0$.

At this point, taking $T = c_5 n^2$ and taking $\epsilon = 1/10$, say, we obtain a bound for n. This concludes the treatment of the first case.

We now go to the second case.

Case 2: We have $n_r \leqslant m^5$ for all r. — This means that the torsion orders of σ at the various $\lambda(\widetilde{\tau}^r(p))$ are not too much larger than the torsion order m of τ at $\mu(p) = b$.

The strategy will be similar to the one for the first case, but the rational points will be constructed differently, and the proof of emptyness of the algebraic part will also be different, not using Theorem 2.5 but a self-contained argument.

As previously, we select a compact domain Δ not containing the points of bad reduction for μ, and suppose that b lies in this set. Locally on Δ we have an analytic representation for the curves F_η, $\eta \in \Delta$ as quotients $\mathbb{C}/L_\eta$;

(3) We mean the naive height for a rational number, i.e. the maximum of the absolute values of numerator and denominator in a reduce fraction.

hence we may cover Δ with finitely many relatively open simply connected sets $A_l \subset \Delta$, such that for η in one of these A_l the lattice L_η corresponding to the fibre F_η is generated by the periods $\pi_1(\eta), \pi_2(\eta)$ varying analytically on A_l. We may also assume that these A_l are definable.

For our purposes we may work on each A_l at a time. Hence we omit the subscripts and denote by A one of these open sets containing b.

We denote by ξ a complex variable lying in a (compact) fundamental domain $\mathcal{D}(\eta) := \{t_1\pi_1(\eta) + t_2\pi_2(\eta) \,:\, 0 \leqslant t_1,\ t_2 \leqslant 1\}$ for the lattice L_η: note that these fundamental domains, for $\eta \in A$, form a definable family. Each point in F_η admits an elliptic logarithm in the domain $\mathcal{D}(\eta)$. Let us now remove from each fundamental domain small open disks with centers corresponding to points of bad reduction for λ restricted to the elliptic curve F_η. The remaining (compact) domain, denoted $\mathcal{D}^*(\eta)$, will continue to form a definable family for suitable parametrizations of the boundaries of these disks.

Note that, as above, by bounded height of b and $\lambda(p + r\tau(b))$ (obtained from Lemma 2.7), we may assume that our points admit elliptic logarithms lying in $\mathcal{D}^*(b)$. Let d_1 be the degree $[K(p) : K]$, and consider the conjugates p^g of p over K. By bounded height of b, we may assume that at least $c_1 d_1$ conjugates of b[4], for some $c_1 > 0$, will lie in a same A among the A_l. Indeed, the number of A_l is fixed, while $\geqslant c_2 d_1$ conjugates stay in the compact union of the A_l.

If for all of these conjugates there exist $m/2$ (say) values of r such that $\lambda(\widetilde{\tau}^r(b^g))$ lies in some of the small disks to be removed, then the sum $\sum_g \sum_{r_1 \neq r_2} |\lambda(p^g + r_1\tau(b^g)) - \lambda(p^g + r_2\tau(b^g))|^{-1}$, made over all conjugations g, will exceed any given multiple of $d_1 m^2$ (if the said disks are small enough), contradicting the uniform bound for the height of $\lambda(p + r\tau(b))$. (This argument is a refinement of that for Lemma 2.8.) Hence we may assume that for some conjugate of b, and for $\geqslant m/2$ values of r, the value $\lambda(p^g + r\tau(b^g))$ does not lie in any of the small "bad" disks to be removed.

We now express our definable analytic sets relevant in the application of the Theorem 2.6. For $\eta \in A$ and for $\xi \in \mathcal{D}(\eta)$, we write

$$\xi = t_1\pi_1(\eta) + t_2\pi_2(\eta); \tag{3.2}$$

we further write, as in (3.1), $\ell_0(z) = \beta_1(z)\omega_1(z) + \beta_2(z)\omega_2(z)$, where we now may omit the subscript "i" since we use only the section $s_0 = \sigma \circ \lambda$ and disregard s_1. Here z lies in F_η. (Recall however that these functions depend (locally analytically) also on η.)

(4) Some of these conjugates may coincide: we are taking conjugates of the field $K(p)$, which may be larger than $K(b)$.

Let finally $\mathbf{e} = \mathbf{e}_\eta : \mathcal{D}(\eta) \subset \mathbb{C} \to F_\eta$ denote the holomorphic elliptic-exponential.

Our analytic set Z is described in $A \times \mathbb{R}^4$ by the points $\eta \times (t_1, t_2, \beta_1(z), \beta_2(z))$, where $z = \mathbf{e}(t_1\pi_1(\eta) + t_2\pi_2(\eta)) \in F_\eta$ and where $0 \leqslant t_1, t_2 \leqslant 1$ and $\eta \in A$. We view this as a definable family of varieties Z_η, as η varies in A, and we shall be interested in a single Z_η, i.e. Z_b, where $b = \mu(p)$ (for our initial p or a well chosen conjugate of it).

For $r = 0, 1, \ldots, m-1$, let ξ_r denote a representative for $r\tau(b)$, lying in the domain $\mathcal{D}(b)$ and let $\widetilde{\xi}$ be a similar representative for p; by the above argument, we may assume that for a set $R \subset \{0, 1, \ldots, m-1\}$ with $|R| \gg m$, $\widetilde{\xi} + \xi_r$ lies in the double $2c\mathcal{D}^*(b)$ of the fundamental domain $\mathcal{D}^*(b)$, for $r \in R$. We have $\xi_r = \theta_{1r}\pi_1(b) + \theta_{2r}\pi_2(b)$, where θ_{1r}, θ_{2r} are rational numbers in $[0, 1]$ with denominator dividing m.[(5)] Also, letting $z_r = \mathbf{e}(\widetilde{\xi} + \xi_r)$, we have that $\beta_1(z_r), \beta_2(z_r)$ are rationals with denominator at most m^5, since we are in the Case 2.

These m rational points are pairwise distinct, so applying again Theorem 2.6 (with any fixed $\epsilon < 1/4$), we deduce that the algebraic part of Z_b is not empty, which means that Z_b contains an arc of real-algebraic curve in $\mathbb{R}^4$. On this arc the four functions $t_1, t_2, \beta_1, \beta_2$ become algebraically dependent.

Let ξ be the function defined by (3.2), restricted to this arc; then t_1, t_2 are algebraic functions of ξ. (Note that t_1, t_2 of course are determined as functions of ξ anyway, but if ξ is not restricted to the arc, these functions are not algebraic.) The same holds for $\beta_1(\mathbf{e}(\xi)), \beta_2(\mathbf{e}(\xi))$. In other words, we may view ξ as an elliptic logarithm (relative to F_b) of $z \in F_b$ and we have that $\ell_0(z)$ has "Betti cooordinates" which are algebraic functions of ξ, when we restrict ξ to a suitable real-analytic arc inside $\mathcal{D}(b)$. Recall that here $\ell_0(z)$ is the elliptic logarithm of $s_0(z)$ relative to the elliptic curve $E_{\lambda(z)}$.

Let us set $B_i(\xi) := \beta_i(\mathbf{e}(\xi))$, so $\ell_0(\mathbf{e}(\xi)) = B_1(\xi)\omega_1(\mathbf{e}(\xi)) + B_2(\xi)\omega_2(\mathbf{e}(\xi))$. When ξ runs on the said arc, the B_i become equal to certain algebraic functions of ξ, denoted $f_1(\xi), f_2(\xi)$. So the equality

$$\ell_0(\mathbf{e}(\xi)) = f_1(\xi)\omega_1(\mathbf{e}(\xi)) + f_2(\xi)\omega_2(\mathbf{e}(\xi)) \tag{3.3}$$

must hold on a set of real dimension $\geqslant 1$; since all involed functions are holomorphic, such equality must hold on a whole disk.[(6)]

To get a contradiction we use arguments based on monodromy. Consider the points $w \in F_b$ of bad reduction for E_λ, i.e. such that the fiber above

[(5)] We have $\theta_{ir} \equiv r\theta_{i1} \pmod{\mathbb{Z}}$ for $i = 1, 2$.

[(6)] Note however that (f_1, f_2) will not have to be equal to the Betti map of ℓ_0 on the whole disk.

$\lambda(w)$ is singular (possibly a multiple fiber); removing these points we obtain an affine curve F_b^* and locally well-defined holomorphic periods ω_1, ω_2 on it. The restriction of the elliptic exponential $\mathbf{e} : \mathbb{C} \to F_b$ gives rise to a topological cover with Galois group isomorhpic to $\mathbb{Z}^2$ of the form

$$\mathbf{e} : \mathbb{C}^0 \longrightarrow F_b^*, \quad \xi \longmapsto \mathbf{e}(\xi);$$

where $\mathbb{C}^0$ is the complement in $\mathbb{C}$ of a discrete set. Over $\mathbb{C}^0$ the expression (3.3) holds, where f_1, f_2 are *algebraic* functions of $\xi \in \mathbb{C}^0$. Their monodromy group is then finite. Then, the functions f_1, f_2 become globally well-defined on a finite cover $\mathcal{U} \to C^0$, giving rise to a Galois topological cover

$$\mathcal{U} \longrightarrow \mathbb{C}^0 \longrightarrow F_b^*$$

where the first arrow has finite degree, the second one denotes a cover with Galois group $\mathbb{Z}^2$. Letting H be the Galois group of the cover $\mathcal{U} \to F_b^*$, we obtain that the group H is given as an extension

$$0 \longrightarrow K \longrightarrow H \longrightarrow \mathbb{Z}^2$$

where K is a finite group.

Choosing a base point on $w_0 \in F_b^*$, the fundamental group of F_b^* acts on the periods ω_1, ω_2 and the logarithm l_0 via a representation $\pi_1(F_b^*, w_0) \to \mathrm{SL}_3(\mathbb{Z})$. By Theorem 2.1, its image is Zariski-dense in the five-dimensional affine group, isomorphic to $\mathbb{G}_a^2 \rtimes \mathrm{SL}_2$, formed by the matrices of the shape (2.3).

Choosing a base point $\xi_0 \in \mathbb{C}^0$ with $\mathbf{e}(\xi_0) = w_0$, we obtain a lift to $\mathbb{C}^0$ of the monodromy action of $\pi_1(F_b^*, b)$ to the lifting of periods and logarithm on $\mathbb{C}^0$. We also obtain a natural projection $\pi_1(F_b^*, b) \to \mathbb{Z}^2$. Chosing a base point u_0 above w_0 we obtain that the monodromy action of $\pi_1(F_b^*, b)$ on (f_1, f_2) (see eq. (3.3)) factors through H. Let us denote by $\bar{g}$ the projection of any point $g \in \pi_1(F_b^*, b)$ in the group H.

Choose now two elements $g_1, g_2 \in \pi_1(F_b^*, w_0)$ and consider the commutator $g_1 g_2 g_1^{-1} g_2^{-1}$. Since the Galois group of the cover $\mathbb{C}^0 \to F_b^*$ is finite, the image of this commutator in H lies in the finite group K. Hence

$$\left(\overline{g_1 g_2 g_1^{-1} g_2^{-1}}\right)^{|K|} \equiv 1$$

In terms of the matrix (2.3), this means that the last column must be ${}^t(0, 0, 1)$. However, the above relation does not hold identically on the whole mentioned five-dimensional algebraic group.

This contradiction concludes the proof of Theorem 1.1. □

4. Proof of Theorem 1.6

Let $\mathcal{E} \to B$ be the given non-isotrivial elliptic scheme and ω_1, ω_2 a basis for the period function on B, which are only locally well defined. Consider the universal cover $\mathcal{H} \to B$; the locally well defined period functions on B lift to well defined functions on $\mathcal{H}$, which we denote by $\widetilde{\omega}_1, \widetilde{\omega}_2 : \mathcal{H} \to \mathbb{C}$. For $z \in \mathcal{H}$, let $\tau(z) = \widetilde{\omega}_2(z)/\widetilde{\omega}_1(z)$ be the ratio of a given basis of periods [7].

The fundamental group $\pi_1(B)$ acts on $\mathcal{H}$ (after fixing a base point on B and a lift of it on $\mathcal{H}$). Given a non-torsion section $\sigma : B \to \mathcal{E}$ as in Theorem 1.6 and a determination $\ell : \mathcal{H} \to \mathbb{C}$ of its elliptic logarithm, Theorem 2.3 provides the existence of a subgroup $\Gamma_\sigma \subset \pi_1(B)$ fixing the periods and acting on the logarithm by translations by a two-dimensional lattice.

Consider now the algebraic curve $T \subset B \times B$ satisfying the hypothesis of Theorem 1.6. For coprime integers a, b, c, d with $ad - bc > 0$, we have a (connected) analytic curve $\mathcal{V} = \mathcal{V}_{a,b,c,d} \subset \mathcal{H} \times \mathcal{H}$ defined by the pairs of points in $\mathcal{H} \times \mathcal{H}$ that correspond to a given cyclic isogeny, i.e. $\mathcal{V} = \{(z_1, z_2) \in \mathcal{H} \times \mathcal{H} : \tau(z_2) = (a\tau(z_1) + b)/(c\tau(z_1) + d)\}$. If we fix $\delta := ad - bc$ to be the degree of our hypothetical isogeny, this curve $\mathcal{V}$ covers the curve T inside $B \times B$, corresponding to the cyclic isogenies of degree δ: this is because $\mathrm{SL}_2(\mathbb{Z})$ operates transitively on the primitive matrices of determinant δ (by primitive matrix, we mean a matrix with integral coefficients, not all divisible by a same prime number).

Note that $\mathcal{V}$ is sent to itself by the above mentioned subgroup Γ_σ (acting diagonally as $g(z_1, z_2) = (gz_1, gz_2)$). Indeed, Γ_σ fixes the periods, hence fixes τ.

The isogeny relating the elliptic curve over z_2 with that over z_1 is represented by multiplication by a $\xi = \xi(z_1, z_2) \in \mathbb{C}^*$ such that

$$\xi\widetilde{\omega}_1(z_2) = a\widetilde{\omega}_1(z_1) + b\widetilde{\omega}_2(z_1) \qquad \xi\widetilde{\omega}_2(z_2) = c\widetilde{\omega}_1(z_1) + d\widetilde{\omega}_2(z_1).$$

Note that $\xi(gz_1, gz_2) = \xi(z_1, z_2)$ for $g \in \Gamma_\sigma$, since Γ_σ fixes the periods.

Our assumption on the dependence of the sections means that for $(z_1, z_2) \in \mathcal{V}$, we have

$$n\xi\ell(z_2) = m\ell(z_1) + \eta(z_1, z_2), \tag{4.1}$$

where η is a certain period (for the curve corresponding to z_1) and m, n are nonzero integers. Then, acting on the last equation with Γ_σ, and using

[7] In the case of the Legendre scheme, we would have $\tau(z) = z$, where the local period $\omega_i(\lambda(z)) = \widetilde{\omega}_i(z)$

that Γ_σ sends $\mathcal{V}$ into itself, fixes ξ and acts on ℓ by translation by a full sub-lattice of $\widetilde{\omega}_1\mathbb{Z} \oplus \widetilde{\omega}_2\mathbb{Z}$, we get, for a suitable integer $h > 0$

$$n\xi(\ell(z_2) + h\widetilde{\omega}_i(z_2)) = m(\ell(z_1) + h\widetilde{\omega}_i(z_1)) + \eta(z_1, z_2), \quad i = 1, 2. \tag{4.2}$$

Hence, after subtracting (4.1) from (4.2) we obtain $nh\xi\widetilde{\omega}_i(z_2) = mh\widetilde{\omega}_i(z_1)$, and so $\tau(z_1) = \tau(z_2)$ which says that the elliptic curves over z_1 and over z_2, for $(z_1, z_2) \in T$, are indeed isomorphic. Since the isogeny ϕ is cyclic and the elliptic scheme is non-isotrivial, in particular without complex multiplication, the degree of the isogeny ϕ must be 1. □

It seems not easy, if at all possible, to formulate this argument directly in purely algebraic terms of Differential Galois Theory.

5. Proof of Theorem 1.4

We can view the context as a scheme over an open subset $\mathcal{X}_0$ of $\mathcal{X}$ with fibers given as products of three elliptic curves. Namely, above a point $x \in \mathcal{X}_0$ we associate the product of elliptic curves

$$E_{\lambda(x)} \times F_{\mu(\tilde{\sigma}(x))} \times E_{\lambda(\tilde{\tau}\circ\tilde{\sigma}(x)))}.$$

For simplicity of notation, we denote by $\mathcal{E}_i \to \mathcal{X}_0$, $i = 1, 2, 3$, the corresponding elliptic schemes; as before, $\mathcal{X}_0$ is the complement in $\mathcal{X}$ of a finite union of points and curves, namely the points where $\widetilde{\sigma}$ or $\widetilde{\tau} \circ \widetilde{\sigma}$ is not defined, plus the union of the singular fibers corresponding to the projections $\lambda, \mu \circ \widetilde{\sigma}, \lambda \circ \widetilde{\tau} \circ \widetilde{\sigma}$.

We have an obvious section ξ given by (with the above compressed notation) $\xi(p) = \sigma(p) \times \tau(\widetilde{\sigma}(p)) \times \sigma((\widetilde{\tau} \circ \widetilde{\sigma})(p))$.

We start by proving the following:

Lemma 5.1. — *The three elliptic schemes $\mathcal{E}_i \to \mathcal{X}_0$, $i = 1, 2, 3$, are pairwise non-isogenous.*

Proof. — Let us denote by $j_\lambda : \mathcal{X}_0 \to \mathbb{C}$ (resp. j_μ) the j-function corresponding to the λ-scheme on $\mathcal{X}_0$ (resp. to the μ-scheme). An isogeny between the schemes $\mathcal{E}_1$ and $\mathcal{E}_2$ would correspond to an algebraic relation of the form

$$P(j_\lambda, j_\mu \circ \widetilde{\sigma}) = 0$$

for a non-zero polynomial $P(X, Y) \in \mathbb{C}[X, Y]$. But now note that j_λ is trivially invariant by $\widetilde{\sigma}$, hence j_λ, j_μ would be in fact algebraically dependent, which is excluded since the two elliptic schemes defined by λ and μ are not isotrivial and the map (λ, μ), is dominant, which entails that the map (j_λ, j_μ) is also dominant.

An isogeny between $\mathcal{E}_1$ and $\mathcal{E}_3$ would imply an algebraic dependence relation as before between j_λ and $j_\lambda \circ \widetilde{\tau} \circ \widetilde{\sigma}$, which in turn would imply a dependence between j_λ and $j_\lambda \circ \widetilde{\tau}$ (again because $\lambda \circ \widetilde{\sigma} = \lambda$).

Now, if we restrict to a given generic fiber for λ, on which the function j_λ is constant, we obtain that $j_\lambda \circ \widetilde{\tau}$ would be also constant; this in turn implies that the automorphism $\widetilde{\tau}$ sends fibers of λ into fibers. This continues to be true for any iterates and so the fibers would all be isomorphic, therefore the first scheme would be isotrivial.

Finally, an isogeny between $\mathcal{E}_2$ and $\mathcal{E}_3$ would imply an algebraic dependence between $j_\mu \circ \widetilde{\sigma}$ and $j_\lambda \circ \widetilde{\tau} \circ \widetilde{\sigma}$, hence between j_μ and $j_\lambda \circ \widetilde{\tau}$. Noticing that j_μ is invariant under $\widetilde{\tau}$, we would obtain as in the first case a dependence between j_λ and j_μ, which is excluded. □

We now want to prove that the set where the above section ξ assumes torsion values is not Zariski-dense in $\mathcal{X}_0$. By definition, this set, denoted now Ex_0, is $\mathrm{Ex}_0 := \mathrm{Ex} \cap \mathcal{X}_0$.

We use the common procedure of counting, going back to [23].

We have bounded height for the $p \in \mathrm{Ex}_0$, by Lemma 2.7.

Also, if the section ξ has torsion order n at p, then the field of definition of p has degree $\geqslant c_1 n^{1/2}$ for some constant $c_1 > 0$ (by Lemma 2.7) and therefore there are at least such a number of distinct conjugates of p. Since we have bounded height $h(p)$, we may apply Theorem 2.8 and obtain that a positive proportion of these conjugates lies in a suitable compact set.

Now, in turn we can cover $\mathcal{X}_0$ with finitely many compact (bi)disks D (i.e. analytically isomorphic to products of two compact discs in $\mathbb{C}$) such that in each D we have three well defined pairs of periods ω_{i1}, ω_{i2} for the schemes $\mathcal{E}_i$, $i = 1, 2, 3$, and three well-defined elliptic logarithms ℓ_i for the sections, hence equations

$$\ell_i(z) = \beta_{i1}(z)\omega_{i1}(z) + \beta_{i2}(z)\omega_{i2}(z), \; z \in D, \quad i = 1, 2, 3, \tag{5.1}$$

where the Betti maps β_{i1}, β_{i2} take real values.

For $z \in D$, the sextuple of the $\beta_{ij}(z)$ describes a compact real definable set Z in $\mathbb{R}^6$.

Since the covering of $\mathcal{X}_0$ by the disks D is finite and may be chosen independently of p, we may also assume that D too contains at least $c_2 n^{1/2}$ conjugates of p, for some positive $c_2 > 0$.

Thus we obtain $c_2 n^{1/2}$ rational points in Z.

(AP) – The algebraic part. — Given that Z contains $\geqslant c_2 n^{1/2}$ rational points of height $\leqslant n$, for large values of n we deduce from Theorem 2.6 that there is an algebraic arc U in Z. We let U' be its inverse image in D; this U' will be a real-analytic set of dimension $\geqslant 1$.

If we restrict to U' the nine functions $\ell_i, \omega_{i1}, \omega_{i2}$, $i = 1, 2, 3$, the equations (2.1) will provide at least two independent algebraic relations among the nine functions (more precisely, each six-tuple of functions, namely for $i = 1, 2$, $i = 1, 3$ and $i = 2, 3$, will satisfy a nontrivial algebraic relation).

To exploit this fact, we apply a new result of Ax–Schanuel type, by Bakker–Tsimerman. We shall need only a special case, appearing as [2, Theorem 4.1], which we restate as follows:

Lemma 5.2. — *Let V be an algebraic variety, and for $i = 1, \dots, n$, let $\mathcal{E}_i \to V$, be non-isotrivial pairwise non-isogenous elliptic schemes, provided with sections $s_i : V \to \mathcal{E}_i$. For a simply-connected open set $D \subset V$ let, for $i = 1, \dots, n$, ℓ_i be an elliptic logarithm of s_i and $\omega_{i,1}, \omega_{i,2}$ a basis for the periods of $\mathcal{E}_i$, on D. Let $F = (\ell_i, \omega_{i,1}, \omega_{i,2})_{i=1\dots,n} : D \to \mathbb{C}^{3n}$ be the corresponding map. Let $T \subset \mathbb{C}^{3n}$ be a codimension*[8] *k algebraic subvariety and suppose that $F^{-1}(T)$ contains an irreducible analytic component R of codimension $< k$. Then $R \neq V$ and either two of the elliptic schemes become isogenous on R, or at least two of the sections become torsion on R, or an elliptic scheme becomes isotrivial on R.*

We apply this lemma with $n = 3$, $V = \mathcal{X}_0$, $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ as defined above and $k = 2$. By Lemma 5.1, the three schemes $\mathcal{E}_i$, $i = 1, 2, 3$ are pairwise non-isogenous, as requested in the above lemma.

We take for T the algebraic variety defined in $\mathbb{C}^9$ by the said two independent relations, while R will be a connected component of the analytic closure in D of U' (which is real-analytic, being the inverse image through an analytic map of a real-analytic arc).

Note also that the two independent algebraic relations among the said nine functions must continue to be true in R, and note that R has (complex) codimension < 2 since U has positive real-dimension.

We obtain from the lemma that one of the three following alternatives hold:

(i) Two of the elliptic schemes become isogenous on R.
(ii) Two of the sections become torsion on R.
(iii) One of the elliptic schemes becomes isotrivial on R.

[8] Here the codimension is meant in the ususal algebraic, equivalently complex-analytic, sense, whereas in the Pila–Wilkie estimates one considers real dimensions.

In each of the three cases we obtain the important consequence that R is actually algebraic, and we may assume it is an algebraic curve, because otherwise the conclusion would extend to the whole $\mathcal{X}$, which we have excluded since the beginning.

Each sextuple of functions on R given by $\ell_i, \omega_{i1}, \omega_{i2}$, i varying in any pair in $\{1, 2, 3\}$, generate over $\mathbb{C}$ a field of transcendence degree at most 5. This is because of the opening argument of the present section (AP). Now Theorem 2.5 implies that either one of the schemes becomes isotrivial on R, or, for any pair of them, either they are isogenous or one of the section is torsion.

(I). — Let us first work under the assumption that none of the three schemes becomes isotrivial on *R.*

This implies that none of the sections is torsion on R. Indeed, if for instance the first section were torsion, then, since σ is not torsion on $\mathcal{X}$, R would be a fiber of λ, hence the first scheme would be constant. Similarly, if the second section is torsion on R, this means that $\mu \circ \widetilde{\sigma}$ is constant on R, and the second scheme would be isotrivial. The same for the third one.

Hence we have a somewhat stronger conclusion with respect to that of Lemma 5.2, namely we can conclude that *any two of the schemes become isogenous on R, or one of the two is isotrivial on R.*

Recall also that we are assuming that the analytic set $Z \subset \mathbb{R}^6$ given by the Betti coordinates of the three sections (restricted to R) contains an algebraic arc.

Then, since (iii) and (ii) do not hold, (i) must hold so *two of the schemes* $\mathcal{E}_a$ *and* $\mathcal{E}_b$ *are isogenous* on R for a pair (a, b) with $1 \leqslant a < b \leqslant 3$. On the said algebraic arc, the four functions $\ell_a, \omega_{1a}, \omega_{2a}, \ell_b$ generate a field of transcendence degree at most 3. By Theorem 2.5 we conclude that the two sections are linearly dependent (if we view them as section for a same elliptic scheme, after applying the isogeny). Hence the two elliptic logarithms ℓ_a, ℓ_b are linearly dependent over R (up to the addition of periods). (Note that this is stronger than what is delivered by alternatives (i), (ii), (iii) above.)

Now, if the third scheme is not isogenous on R to $\mathcal{E}_a$ and $\mathcal{E}_b$, then by Theorem 2.5 we obtain that one of the sections is torsion and we apply the argument in *(I).*

Hence we may assume that:

(II). — Any two of the schemes are isogenous on R and the corresponding sections are linearly dependent.

Then we obtain that for $z \in R$ the curves $E_{\lambda(z)}$ and $F_{\mu(\tilde{\sigma}(z))}$ are isogenous, and so are the curves $E_{\lambda(z)}$ and $E_{\lambda(\tilde{\tau}\circ\tilde{\sigma})(z))}$. For notational convenience it is better (and equivalent) to say that for z on the curve $\widetilde{\sigma}(R)$ the curves $E_{\lambda(z)}$, $F_{\mu(z)}$ and $E_{\lambda(\tilde{\tau}(z))}$ are isogenous.

We shall apply this fact to the first and third sections, namely with $a = 1, b = 3$; note that in this case the elliptic schemes derive from the same scheme on $\mathcal{X}$, and with the same sections, but evaluated at different points, namely $\lambda(z), \lambda(\widetilde{\tau}(z))$.

We note that this result represents a problem of Unlikely Intersections in the pure function field case, namely without an analogue in the number-field case; so a fortiori we cannot say that this case comes from the number-field context.

To deal with this issue, we could argue again using the counting method and arguments of the type as in [23]. However we can conclude directly by means of Theorem 1.6, as follows.

In fact, when none of the schemes is isotrivial on R, Theorem 1.6 allows to improve Theorem 1.4 obtaining finiteness for the exceptional set.

If the second scheme is isotrivial then the last argument still works.

If two of them are isotrivial on a curve, then this curve must be a common component of a fiber (of $\lambda, \mu \circ \widetilde{\sigma}$ or $\lambda \circ \widetilde{\tau} \circ \widetilde{\sigma}$).

If the first or third scheme is isotrivial on R, we contend that the degree of the isogeny between the other two is bounded independently of R (note that indeed the other two must be isogenous since none of them can be isotrivial). Indeed, these isogenies correspond to certain (modular) algebraic relations between the corresponding j-invariants of the relevant elliptic curves (these relations appearing also in the proof of Theorem 1.6). These j-invariants are rational functions of bounded degree of the coordinates of the point $p \in \mathcal{X}_0$: in fact, the elliptic curves depend rationally on $p, \widetilde{\sigma}(p), (\widetilde{\tau} \circ \widetilde{\sigma})(p)$. If p lies either in a fiber of λ or in a fiber of $\lambda \circ \widetilde{\tau} \circ \widetilde{\sigma}$, these degrees are bounded independently of the fiber. Therefore the isogeny degree is bounded, and the finiteness of the exceptional curves follows. □

6. Proof of Theorem 1.5

We shall use an analogue of Silverman height bound Lemma 2.7, which was recently obtained by Dimitrov–Gao–Habegger in [14] and, independently, by Yuan–Zhang in [29] with somewhat different methods. We quote just a corollary of [29, Theorem 6.2.2]. Here is the statement:

Lemma 6.1. — *Let $A \to \mathcal{X}$ be an abelian scheme over an algebraic surface $\mathcal{X}$; let $\sigma_1, \sigma_2 : \mathcal{X} \to A$ be independent sections*[9]*. There exists a non-empty Zariski-open subset $\mathcal{X}_0 \subset \mathcal{X}$ such that the points of $\mathcal{X}_0$ where both sections take torsion values have uniformly bounded height.*

To derive this lemma from Theorem 6.2.2 we should in fact verify in our special situation that the *non-degeneracy condition* appearing in [29] holds. (This condition involves the so-called Betti map and appears also in the proofs of uniform versions of Mordell's conjecture in the paper [14].) The condition is rather technical to state and we omit it here. In our special case it follows from the results in the paper [10] where it is proved in particular that the Betti map is submersive for certain doubly elliptic schemes over a surface. We can refer to [10, Appendix 3]. Alternatively, note that if the rank of the Betti map is $\leqslant 2$, the fibers of the Betti map have positive dimension.

Since the torsion points for both sections are dense (this is easy to prove, and in any case can be assumed in the present situation) there would be infinitely many positive dimensional torsion subvarieties (actually curves) contradicting the main theorem of [10].

Let

$$\mathcal{T} = \{x \in \mathcal{X} : \sigma_i(x) \text{ is torsion on } \mathcal{E}_i \text{ for } i = 1, 2, 3\}$$

and assume it is infinite.

By Lemma 6.1, there is a non-empty Zariski-open subset $\mathcal{X}_0 \subset \mathcal{X}$ such that the points in $\mathcal{T}_0 := \mathcal{T} \cap \mathcal{X}_0$ have bounded height, hence in particular their degree over a number field of definition K tends to infinity. Let then $x \in \mathcal{T}_0$ be a point such that its degree $d = [K(x) : K]$ over K is sufficiently large.

By Lemma 2.8 we may find a given compact set $\Delta \subset \mathcal{X}_0(\mathbb{C})$, independent of x, such that at least $d/2$ conjugates of x lie in Δ.

We may now partition Δ in a large but finite number of compact subsets Δ_j, where we may assume that the periods related to the three schemes, and the logarithms of the sections, are well-defined in each Δ_j. We may assume that there is a compact Δ' independent of x, where the above functions are well-defined, and containing $> cd$ conjugates of x, where $c > 0$ is independent of x.

As before, we may write, for $\mathbf{z} \in \Delta'$,

$$\ell_i(\mathbf{z}) = \beta_{1i}(\mathbf{z})\omega_{1i}(\mathbf{z}) + \beta_{2i}(\mathbf{z})\omega_{2i}(\mathbf{z}), \quad i = 1, 2, 3, \tag{6.1}$$

where ω_{ij} are the periods and ℓ_i are the elliptic logarithms of the sections.

(9) I.e. independent in the $\mathbb{Z}$-module of rational sections.

For $\mathbf{z} = x \in \mathcal{T}_0$, the $\beta_{ij}(\mathbf{z})$ are rational numbers.

Again by bounded height, Lemma 2.7 (2) yields that the orders of torsion are bounded from above by $c' \cdot d^2$, for some positive constant c'. So the denominators of the said rationals are likewise upper bounded.

Comparing all the bounds we deduce that the image $Z \subset \mathbb{R}^6$ of the map $\mathbf{z} \mapsto (\beta_{ij}(\mathbf{z}))_{i=1,2,3,j=1,2}$ on Δ' contains $\geqslant cd$ rational points of height $\leqslant c'd^2$, for some positive constants c, c'. We then apply Theorem 2.6 to deduce that if d is large enough, the algebraic part of Z is non-empty, namely Z contains a real algebraic arc.

Let us consider its pre-image C' in Δ', which is a real-analytic set of dimension $\geqslant 1$. As in the previous proof, if we restrict to C' the nine functions $\ell_i, \omega_{i,1}, \omega_{i,2}$ for $i = 1, 2, 3$, the equations (6.1) will provide at least two independent algebraic equations among these functions (more precisely, each six-tuple of functions, namely for $i = 1, 2$, $i = 1, 3$ and $i = 2, 3$, will satisfy a nontrivial algebraic relation). We apply again Lemma 5.2 with $V = \mathcal{X}$, $n = 3$, $k = 2$ and $T \subset \mathbb{C}^9$ the algebraic variety defined by the algebraic relations among the $\ell_i, \omega_{i,1}, \omega_{i,2}$, $i = 1, 2, 3$; we obtain the existence of a complex-analytic curve $R \subset \mathcal{X}$ containing C' such that one of the following alternatives holds on R:

- two of the elliptic schemes become isogenous on R,
- or an elliptic scheme becomes isotrivial on R, or
- at least two of the sections become torsion on R.

In particular, we deduce that R is an algebraic curve.

Suppose that two of the schemes, say the first two of them, are isogenous but not isotrivial on R and that at most one section is torsion (so the second and third alternatives do not hold). Then consider the pair formed by the first and third schemes. Associated to these schemes, we have the four periods, the two logarithms and the corresponding Betti maps. As before, we deduce that these six functions are algebraically dependent on R. But by Theorem 2.5, this implies that these schemes are isogenous, hence all the three schemes become isogenous on R. (Then among the five analytic functions consisting on the two periods and the three logarithms, at most three of them are algebraically independent.) Then we use Pila's result on the André–Oort Conjecture for $\mathbf{C}^n$, i.e. [25, Theorem 1.1], in the special case of the variety $\mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$, where each line $\mathbb{A}^1(\mathbb{C})$ is viewed as $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$; actually we just need the functional part of the proof, asserting (in our case) that a non-special surface in $\mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$ contains only finitely many special curves.

We argue as follows. The three j-invariants (j_1, j_2, j_3) define a rational map $J : \mathcal{X} \to \mathbb{A}^3$, and the closure of the image is a surface in $\mathbb{A}^3$; this surface is non-special in the sense of the Shimura variety $\mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$ because the three elliptic schemes on $>\mathcal{X}$ are non isotrivial pair-wise non isogenous. Since the schemes become isogenous on R, the points of $J(R)$ with all three CM coordinates are Zariski-dense in $J(R)$; then $J(R)$ is a special curve and by the mentioned theorem of Pila there exist only finitely many such curves on $\mathcal{X}$.(10) So Theorem 1.5 is proved in this case.

Let us now consider the case when the first of the elliptic schemes becomes isotrivial on R. So R is a component of a curve of the shape $j_1(x) = c$, where j_1 is the j-invariant of the first scheme, viewed as a rational function on the base $\mathcal{X}$. Therefore R has bounded degree (with respect to a given embedding of $\mathcal{X}$).

The six functions $\ell_i, \omega_{i,1}, \omega_{i,2}$ for $i = 2, 3$, become algebraically dependent on C', hence on R. Again by Theorem 2.5, we deduce that either two of the schemes are isogenous or one of the section σ_2 or σ_3 is torsion on R. But this also leads to at most finitely many curves R because if the isogeny degree or the torsion order surpass a certain limit then the degree of the relevant curve grows.

Let us finally consider the case when none of the schemes is isotrivial on R and σ_1, σ_2 become torsion on R.

We can then apply the main theorem of [10] which provides the finiteness of such curves, as wanted. □

Appendix A. Relative Manin Mumford

A.1. The conjecture over $\overline{\mathbb{Q}}$ and $\mathbb{C}$

For a relative abelian scheme $\pi : A \to S$, we say that a subscheme $V \subset A$ is a torsion variety (or simply, torsion) if $V \subset A[N]$ for some positive integer N. If V is reduced, this condition can be checked on closed points, and thus this is equivalent to asking that for all $v \in V$, the point $v \in A_{\pi(v)}$ is torsion. For a subscheme $Y \subset A$, we denote by Y_{tor} the union of all torsion subvarieties of Y.

If $B \subset A$ is an abelian subvariety, then for any positive integer N, we say that any irreducible component of $B + A[N]$ is a torsion coset.

(10) The recourse to this case of André–Oort could be avoided, for instance on using the piece of Pila's proof dealing with the algebraic part of the relevant definable set.

The following is known as the *Relative Manin–Mumford conjecture (RMM).*

Conjecture A.1 (RMM). — *Let $\pi_0 : A_0 \to S_0$ be a relative Abelian scheme, where S_0 is an irreducible complex variety. Let $Y_0 \subset A_0$ be a subvariety with a Zariski-dense set of torsion points, such that $\dim Y_0 + \dim S_0 < \dim A_0$. Then either Y_0 doesn't dominate S_0, or else it is contained in a proper torsion coset.*

We note that RMM has been recently announced by Gao–Habbeger over $\overline{\mathbb{Q}}$. We wish to explain how to generalize this to $\mathbb{C}$. Our specific theorem doesn't rely on their methods but instead is a formal implication:

Theorem A.2. — *The RMM over $\mathbb{C}$ is a consequence of the RMM over $\overline{\mathbb{Q}}$.*

To prove this theorem, we shall borrow ideas from the recent work [5] on Ax–Schanuel, though we do not use their specific theorems.

We will in fact consider the following generalized version, with the case of $r = 0$ being the RMM:

Conjecture A.3. — *Let $\pi : A \to S$ be an Abelian scheme over an irreducible complex variety S, and ley $Y \subset A$ be an irreducible variety that dominates S. Let $Y_{\mathrm{tor}}^{\geqslant r}$ denote the set of irreducible components of Y_{tor} of dimension at least r. Suppose that $r + \dim A > \dim Y + \dim S$. If $Y_{\mathrm{tor}}^{\geqslant r}$ is Zariski dense in Y, then Y is contained in a proper torsion coset.*

Note that this is a special case of the Zilber–Pink conjecture, and is in fact equivalent to ZP restricted to a special case of weakly special subvarieties (those with no abelian part).

We will prove the following implication:

Theorem A.4. — *Conjecture A.3 is a consequence of the RMM over $\overline{\mathbb{Q}}$.*

We begin by showing that the general case of Conjecture A.1 follows from the case where all the varieties in question are defined over $\overline{\mathbb{Q}}$. This idea is already present in [10].

Lemma A.5. — *Conjecture A.3 over $\mathbb{C}$ follows from the $\overline{\mathbb{Q}}$ case.*

Proof. — To prove the reduction, we spread out to $\overline{\mathbb{Q}}$-varieties $\pi' : A' \to S' \to T$ with $Y' \subset A'$ whose generic point over T gives π. Now suppose that the theorem is false. Then there exists a Zariski-dense set of varieties s_i inside $Y_{\mathrm{tor}}^{\geqslant r}$. Now these spread out to get a set of varieties $S_i \subset Y$ whose fibers over the generic points of T are the s_i, and such that $S_i \to T$ is dominant and quasi-finite. The dimensions of S', A', S_i' are all $\dim T$ higher than what they were for their un-primed analogues. Thus replacing r by $r + \dim T$ and applying the $\overline{\mathbb{Q}}$ result yields that Y' is contained in a proper torsion coset of A', and taking the fiber over the genetic point of T yields the same for Y. □

Henceforth we focus on proving Theorem A.4 for A, S, Y defined over $\overline{\mathbb{Q}}$. Without loss of generality we may assume that A is principally polarized, as the problem is invariant under etale base change and isogenies.

A.2. A finite-dimensional family of leaves

Let $L = R^1\pi_*\mathbb{Z}$. Then L is a local system over S^{an} underlying a variation of Hodge structures of weight 1. Now corresponding to the principal polarization on A there is a mixed variation of hodge structures E over A of weights $0, 1$ which fits into an exact sequence

$$0 \longrightarrow \pi^*L^\vee \longrightarrow E \longrightarrow \mathbb{Z}(0) \longrightarrow 0.$$

We may construct E as follows: note that since A is principally polarized we have an isomorphism $A^\vee \cong A$. Next, we note that as sheaves on A we have $\mathcal{E}xt^1(\mathbb{Z}(0), \pi^*L^\vee) \cong A^\vee \times_S A \cong A \times_S A$. We define E to be the global extension corresponding to the diagonal section.

Within $E_\mathcal{O}$ there is a sub-bundle F^0E which contains $\pi^*F^0L^\vee$ and also a lift of $\mathbb{C}(0) \otimes_\mathbb{C} \mathcal{O}$. For a point $a \in A$, if we let e be a lift of a generator of $\mathbb{Z}(0)_a$ to $E_{a,\mathbb{Z}}$, then $e + l \in F^0E$ for $l \in \pi^*L^\vee$ implies that l maps to a under the natural universal covering map $L^{\vee,\mathrm{an}} \to A^{\mathrm{an}}$. We let $E_1 \subset E$ denote the closed subvariety of elements which map to $1 \in \mathbb{Z}(0)$.

Now let $\phi : E \to A$ denote the map from the total space of E and consider the splitting $\sigma : \phi^*TA \to TE$ given by the connection, locally cutting out the locus of a flat vector. Now we consider

$$\mathcal{F}_E := \big(\sigma(\phi^*(TA)) \cap TF^0E\big)\big|E_1.$$

In other words, $\mathcal{F}_E$ is the tangent directions in E_1 along which some vector stays flat, and stays in F^0. By the description above, this means that once we fix the vector it determines its image in A. In other words, $\mathcal{F}_E$ is integrable and yields a foliation of E_1 whose leaves are locally covering spaces over S. We shall be interested in the images of these leaves in A.

In local co-ordinates, let $\ell \subset L^\vee$ be a flat section, which is to say a locally constant section of the first relative homology of A over S. Then

ℓ has an image in A and this image is also the image of the leaf in $\mathcal{F}_E$ corresponding to $e + l$.

Importantly for us, for each N all the components of $A[N]$ are images of leaves corresponding to the vectors in $\frac{1}{N}E_{1,\mathbb{Z}}$.

We call an image in A of a leaf from $\mathcal{F}_E$ a *homology leaf.*

A.3. The foliation is unlikely everywhere

LEMMA A.6. — *There is a Zariski-open subset $U \subset Y$ such that all points of U have a homology leaf through them which intersects Y in dimension at least r.*

Proof. — We first claim that the set of points Y_*in Y which have a homology leaf through them which intersects Y in dimension at least r is constructible.

First, given a complex manifold M and a complex subvariety $Z \subset M$, Z has dimension at least d at a point $z \in Z$ iff for all $r \geqslant 1$ the length of the scheme $Z_r := Z \cap \operatorname{Spec}(O_{M,z}/m_{M,z}^{r+1})$ is at least $\binom{r+d}{r}$. One implication is by dimension theory, and the other follows from Noether normalization.

Next, consider the diagonal $\Delta \subset E_1^2$ and the r'th infinitesimal neighborhood Δ_r defined by the vanishing of the $r+1$'st power of its defining ideal. Then the fiber of Δ_r over a point $x \in E_1$ is naturally identified with the r'^{th} neighborhood of x in E_1. Now since $\mathcal{F}_E$ is integrable, we can solve it formally to order r and obtain a subscheme $\mathcal{F}_r \subset \Delta_r$ whose fiber at each point $x \in E_1$ is the homology leaf through x to order r.

Let $Z_r := \mathcal{F}_r \cap \phi^{-1}(Y)$, and let $T_r \subset E_1(\mathbb{C})$ denote the set of points at which the fiber of Z_r has length at least $\binom{r+d}{r}$. Since length of coherent sheaves is upper-semicontinuous, T_r is a closed subvariety. Therefore, $T := \bigcap_r T_r$ is also a closed subvariety. By our criterion for dimensionality, we see that $Y_* = \phi(T)$, and is therefore a constructible set by Chevalleys theorem.

Now Y_* contains $Y_{\text{tor}}^{\geqslant r}$, which by assumption is Zariski-dense in Y. Any Zariski-dense constructible set must contain a Zariski open subset, and thus the result follows. □

A.4. Unlikely Intersections in Mixed Shimura Varieties

Let $\mathbb{A}_g$ denote the universal principally polarized Abelian variety of dimension g. This sits over $\mathcal{A}_g$ (the moduli space of principally polarized abelian varieties). Now we have the following convenient lemma from [17, Proposition 1.1]

Lemma A.7. — *Let $W \subset \mathbb{A}_g$ denote a weakly special subvariety. Then W lies above a weakly special $W' \subset \mathcal{A}_g$. Moreover, the restriction of the universal abelian scheme to W' is isogenous to a direct sum of 3 abelian schemes $\mathbb{A}_g \mid W' \sim A \oplus B \oplus C$ such that C is isotrivial, and $W = A + (0, b, c)$ where $b \in B(\overline{W'})$ is a torsion section and $c \in C(W')$ is a constant section.*

Now, we apply the mixed Ax–Schanuel theorem of [18]. Let W be the weakly special closure of the image of A in $\mathbb{A}_g$ with image $W' \subset \mathcal{A}_g$, and note that the Hodge leaves are pullbacks of algebraic sets from the universal cover of $\mathbb{A}_g$. Then [18, Theorem 1.1] tells us that at every point $y \in Y$ and Hodge leaf L which intersects Y in dimension at least r, there exists a weakly special $V \subset W$ whose pre-image in A is a subvariety $R \supset Y \cap L$ with image $R' \subset S$, such that

$$\dim Y \cap R + \dim R' \geqslant \dim R + r. \tag{A.1}$$

Let $V' \subset \mathcal{A}_g$ denote the image of V. By assumption we have that $r + \dim A > \dim Y + \dim S$. Since $\dim Y \cap R \leqslant \dim Y$, it follows that

$$\dim A - \dim S + \dim R' > \dim R.$$

now

$$\dim A = \dim W - \dim W', \dim R - \dim R' \leqslant \dim V - \dim V'$$

and thus we see that

$$\dim V - \dim V' < \dim W - \dim W'.$$

Hence the fibers of V over V' are not a full g-dimensional abelian variety. Thus by Lemma A.7 that $A \mid R'$ must decompose as $B + C + D$ where D is isotrivial and $R \subset B + c + d$ where c is torsion, such that B a proper subvariety of $A \mid R'$.

Now, since this decomposition occurs at every point of $y \in U$, and since there are only countably many decompositions of an abelian variety, as well as countably many families of weakly special subvarieties, it follows that there is a global decomposition $A = B + C + D$ whose restrictions to the R' give the decomposition above, where the R' are all pulled back from an (algebraic) family of weakly special varieties V'. Since we only obtain torsion points of C, torsion points can't move continuosly, and Y is not contained in a proper weakly special of A, it follows that $C = 0$. That means there exists a global decomposition $A = B + D$ where D is isotrivial along the R' fibers, and each intersection $Y \cap L$ lies over a constant section of $D \mid R'$.

Proof of Theorem A.4. — We now complete the proof of Theorem A.4. Let $\psi : S \to T$ denote the quotient of S whose fibers are the R', so that D is the basechange of $E \to T$ along ψ. Consider $Z := \overline{\psi(Y)}$. Note that torsion of Z must be Zariski dense in Z and thus by the RMM over $\overline{\mathbb{Q}}$ it follows that

$$\dim Z + \dim T \geqslant \dim E. \tag{A.2}$$

Now let $e \in E$ be the image of a generic $y \in U$. By the description above it follows that R lies entirely within the preimage of e, and thus that $Y \cap R \subset Y_e$. Thus we obtain

$$\begin{aligned}\dim Y \cap R - \dim R + \dim R' &\leqslant \dim Y_e - \dim(B/S)\\ &= \dim Y - \dim Z - \dim(B/S)\\ &\leqslant \dim Y - \dim E + \dim T - \dim(B/S)\\ &= \dim Y - \dim(A/S)\\ &< r.\end{aligned}$$

contradicting (A.1). This completes the proof of Theorem A.4. □

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