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ON THE COHERENCE OF THE L^2 SUBSHEAF FOR A SINGULAR HERMITIAN METRIC WHOSE DETERMINANT HAS ANALYTIC SINGULARITIES

by Yongpan ZOU (*)

ABSTRACT. — We study the sheaf of the locally square integrable holomorphic section of a vector bundle with semi-positive curved singular Hermitian metric. We confirm the coherence when its induced determinant metric has analytic singularities.

RÉSUMÉ. — Nous étudions le faisceau des sections holomorphes localement à carré intégrable d'un fibré vectoriel avec une métrique hermitienne singulière à courbure semi-positive. Nous confirmons la cohérence lorsque la métrique déterminante induite présente des singularités analytiques.

1. Introduction

The multiplier ideal sheaves for singular Hermitian metric on holomorphic line bundle play an important role in complex geometry and algebraic geometry. A classical result of Nadel says that the multiplier ideal sheaf associated with a plurisubharmonic function is coherent, see [3, Proposition 5.7]. There is a natural higher rank analog of the multiplier ideal sheaf for the singular metric of a holomorphic vector bundle.

DEFINITION 1.1 (cf. [2]). — *Let (E, h) be a rank r holomorphic vector bundle with singular metric over the complex manifold X . Then we define the locally square integrable sheaf of $\mathcal{O}(E)$ as follows, we denote it by $\mathcal{E}(E, h)$,*

$$\mathcal{E}(E, h)_x := \{s \in \mathcal{O}(E)_x : |s|_h^2 \text{ is locally integrable around } x \in X\}.$$

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The fundamental question is when this sheaf is coherent. This problem has been confirmed in some cases, see [2, 6, 7, 8, 9, 10, 11] and so on. Since the usual proof makes use of Hörmander's weighted L^2 -estimate of $\bar{\partial}$ -equation, some strong positive condition like (singular) Nakano positivity may be necessary (cf. see Example 1.4). In [10, Conjecture 1.1], T. Inayama asks whether $\mathcal{E}(E, h)$ is coherent under the assumption that h is only singular Griffiths semi-positive, i.e., semi-positive curved. He partially answers this question by assuming the unbounded locus of $\det h$ is discrete.

The determinant line bundle $\det E := \wedge^r E$ of the vector bundle plays an essential role in this problem, we denote the induced metric on $\det E$ by $\det h$. In this note, we give another partial answer to Inayama's conjecture. The following theorem is our main result.

THEOREM 1.2 (Corollary 4.5). — *Let (E, h) be a holomorphic vector bundle over an n -dimensional complex manifold with a singular Griffiths semi-positive Hermitian metric h . If the weight of induced metric $\det h$ on the determinant line bundle $\det E$ has analytic singularities, then $\mathcal{E}(E, h)$ is coherent.*

Example 1.3. — We introduce one important example of G. Hosono. For holomorphic vector bundle E , If there exist some global sections $s_1, s_2, \dots, s_N \in H^0(X, E)$ generically generate E , the following morphism of bundle

$$p: X \times \mathbb{C}^N \longrightarrow E$$

$$(z, (a_1, \dots, a_N)) \longmapsto \sum_i a_i s_i(z)$$

is surjective on a Zariski open subset of X . The induced quotient metric h of E from the standard metric on \mathbb{C}^N is semi-positive curved. As the calculation in [6, Lemma 4.3], the determinant metric has analytic singularities. Thus $\mathcal{E}(E, h)$ is coherent by the above Theorem, for details see [6, Theorem 1.1].

Example 1.4. — We give two important examples of the sheaf of locally square integrable sections. The first comes from M. Iwai's paper [11, Theorem 1.2]. Let (E, h) be a holomorphic vector bundle on X with a singular Hermitian metric. Iwai proves that $\mathcal{E}(E, h)$ is coherent under the following three conditions:

- (1) There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.
- (2) the metric $he^{-\zeta}$ is a positively curved singular Hermitian metric on E for some continuous function ζ on X .

- (3) There exists a real number C such that $\sqrt{-1}\Theta_{E,h} - C\omega \otimes \text{Id}_E \geq 0$ on $X \setminus Z$ in the sense of Nakano.

According to [11, Lemma 2.3], we can replace the second condition with the so-called L^2 -adapted condition. Recall a local holomorphic frame v_1, \dots, v_r of vector bundle (E, h) is L^2 -adapted if for every measurable function f_1, \dots, f_r , the section $\sum f_i v_i$ is locally square integrable if and only if $f_i v_i$ is locally square integrable for every $i = 1, \dots, r$.

In [15, Corollary B], Schnell and Yang prove a similar result. Let X be a complex manifold and let D be an arbitrary divisor on X . Let \mathcal{V} be a polarized variation of rational Hodge structures (VHS) on $X \setminus D$. M. Saito's mixed Hodge module theory shows that \mathcal{V} uniquely corresponds to a polarizable Hodge module \mathcal{M} on X with strict support. Let E be the lowest nonzero piece in the Hodge filtration of \mathcal{V} and let $(\mathcal{M}, F_\bullet \mathcal{M})$ be the filtered D_X -module underlying \mathcal{M} . Saito also shows that E extends to the lowest nonzero piece of $F_\bullet \mathcal{M}$, which is a torsion-free sheaf on X . The Hodge metric h on E extends to a singular Hermitian metric. Let $j : X \setminus D \hookrightarrow X$ be the open embedding. Schnell–Yang proved an interesting fact as follows. Let \mathcal{F} be the subsheaf of $j_* E$ consisting of sections of E which are locally L^2 near D with respect to the Hodge metric on E and the standard Lebesgue measure, then \mathcal{F} is coherent.

It is worth noting that Iwai's result implies Schnell–Yang's result. Indeed, since E is the lowest nonzero piece in the Hodge filtration, by W. Schmid's curvature calculation [14, Lemma 7.18] it is Nakano semi-positive. On the other hand, according to [16, Proposition 2.6], the vector bundle $j_* E$ has one L^2 -adapted holomorphic frame with respect to the extended singular Hermitian metric. Therefore all three conditions above are satisfied.

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2. Preliminaries

In this section, we introduce some basic definitions and results in complex geometry. Unless otherwise mentioned, X denotes a complex manifold

of dimension n , and Δ denotes a polydisc in \mathbb{C}^n . The basic reference is [4]. Firstly, let us recall the concept of the Chern connection and the curvature form of a vector bundle. Let (E, h) be a holomorphic vector bundle on X . Corresponding to this smooth metric h , there exists the unique Chern connection $D = D_{(E, h)}$, which can be split in a unique way as a sum of a $(1, 0)$ and a $(0, 1)$ connection, i.e., $D = D'_{(E, h)} + D''_{(E, h)}$. Furthermore, the $(0, 1)$ part of the Chern connection $D''_{(E, h)} = \bar{\partial}$. The curvature form is defined to be $\Theta_{E, h} := D^2_{(E, h)}$. On a coordinate patch $\Omega \subset X$ with complex coordinate (z_1, \dots, z_n) , denote by (e_1, \dots, e_r) an orthonormal frame of vector bundle E with rank r . Set

$$\sqrt{-1}\Theta_{E, h} = \sqrt{-1} \sum_{\substack{1 \leq j, k \leq n, \\ 1 \leq \lambda, \mu \leq r}} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu, \quad c_{jk\mu\lambda} = \bar{c}_{jk\lambda\mu}.$$

Corresponding to $\sqrt{-1}\Theta_{E, h}$, there is a Hermitian form $\theta_{E, h}$ on $TX \otimes E$ defined by

$$\theta_{E, h}(\phi, \phi) = \sum_{jk\lambda\mu} c_{jk\lambda\mu}(x) \phi_{j\lambda} \bar{\phi}_{k\mu}, \quad \phi \in T_x X \otimes E_x.$$

DEFINITION 2.1. — *A holomorphic vector bundle (E, h) is said to be:*

- (1) Nakano positive (resp. Nakano semi-positive) if for every nonzero tensor $\phi \in TX \otimes E$, we have

$$\theta_{E, h}(\phi, \phi) > 0 \quad (\text{resp. } \geq 0).$$

- (2) Griffiths positive (resp. Griffiths semi-positive) if for every nonzero decomposable tensor $\xi \otimes e \in TX \otimes E$, we have

$$\theta_{E, h}(\xi \otimes e, \xi \otimes e) > 0 \quad (\text{resp. } \geq 0).$$

It is clear that Nakano positivity implies Griffiths positivity and that both concepts coincide if $r = 1$. In the case of a line bundle, E is merely said to be positive (resp. semi-positive).

The Nakano positivity has a close relationship with the solvability of bundle valued $\bar{\partial}$ -equation. The next theorem is fundamental.

THEOREM 2.2 ([3, Theorem 5.1]). — *Let X be a complete Kähler manifold with a Kähler metric ω which is not necessarily complete. Let (E, h) be a Hermitian vector bundle of rank r over X , and assume that the curvature operator $B := [i\Theta_{E, h}, \Lambda_\omega]$ is semi-positive definite everywhere on $\wedge^{n, q} T_X^* \otimes E$, for some $q \geq 1$. Then for any form $g \in L^2(X, \wedge^{n, q} T_X^* \otimes E)$ satisfying*

$\bar{\partial}g = 0$ and $\int_X \langle B^{-1}g, g \rangle dV_\omega < +\infty$, there exists $f \in L^2(X, \wedge^{n, q-1} T_X^* \otimes E)$ such that $\bar{\partial}f = g$ and

$$\int_X |f|^2 dV_\omega \leq \int_X \langle B^{-1}g, g \rangle dV_\omega.$$

In general, the Griffiths positive condition is not enough to solve the $\bar{\partial}$ -equation of vector bundle, but Demailly–Skoda have the following interesting result.

THEOREM 2.3 ([5]). — *Let (E, h) be a holomorphic vector bundle with smooth Hermitian metric h , if (E, h) is Griffiths semi-positive, then $(E \otimes \det E, h \otimes \det h)$ is Nakano semi-positive.*

Next, we come to the singular category, we first introduce positivity notions for singular Hermitian metrics. Let H_r be the space of semi-positive, possibly unbounded Hermitian forms on \mathbb{C}^r . A singular Hermitian metric h on vector bundle E is a measurable map from X to H_r such that $h(x)$ is finite and positive definite almost everywhere. In particular, we have $0 < \det h < +\infty$ almost everywhere.

DEFINITION 2.4. — *Let (E, h) be a singular Hermitian metric on X , then (E, h) is said to be:*

- (1) *Griffiths semi-negative (or semi-negative curved) if $\log |s|_h^2$ (or $|s|_h^2$) is plurisubharmonic (psh, for short) for any local holomorphic section s of E .*
- (2) *Griffiths semi-positive (or semi-positive curved) if the dual metric h^* on E^* is Griffiths semi-negative.*

The plurisubharmonic function is one of the essential concepts in complex geometry. A quasi-plurisubharmonic (quasi-psh, for short) function is a function v which is locally equal to the sum of a psh function and a smooth function. The following regularization lemma is very useful.

LEMMA 2.5 ([1, 13]).

- (1) *Suppose X is a polydisc (even a domain) in \mathbb{C}^n , and suppose h is a singular Hermitian metric on E which is semi-negative (resp. semi-positive) curved. Then there exists a sequence of smooth Hermitian metrics $\{h_\nu\}$ whose corresponding curvature tensor is Griffiths negative (resp. positive), and decreasing (resp. increasing) pointwise to h on any smaller polydisc (relatively compact subset).*
- (2) *Suppose a singular Hermitian metric h is semi-positive curved. Then $(-\log \det h) \in L^1_{loc}(X, \mathbb{R})$ and is a psh function.*

Sometimes it is more natural to consider the sheaf $K_X \otimes \mathcal{E}(E, h)$ instead of square integrable sheaf $\mathcal{E}(E, h)$ as the former has nice functorial property.

LEMMA 2.6 ([16, Proposition 4.3]). — *Let $\pi : X' \rightarrow X$ be a proper modification of complex manifolds, and (E, h) be the vector bundle on X with possible singular metric, then*

$$\pi_*(K_{X'} \otimes \mathcal{E}(\pi^*E, \pi^*h)) = K_X \otimes \mathcal{E}(E, h).$$

DEFINITION 2.7. — *A quasi-psh function u will be said to have analytic singularities if u can be written locally as*

$$u = \alpha \log(|f_1|^2 + \cdots + |f_N|^2) + v$$

where α is a positive real number, v is a locally bounded function and all f_i are holomorphic function. Moreover, if the coefficient α is a positive rational number, then we say u has algebraic singularities.

3. Singular Hermitian metrics with algebraic singularities

To investigate the coherence, we need to solve one special $\bar{\partial}$ equation first. The following theorem and its proof are the slight modifications of [10, Inayama's Theorem 3.1]. We present it here for the reader's convenience. Some simplified notations: $\det^m E := (\det E)^{\otimes m}$, $\det^m h := (\det h)^m$.

THEOREM 3.1. — *Let (X, ω) be a Stein manifold with a Kähler metric ω . It is known that, for every positive constant k , we can find a smooth strictly psh function ϕ with $\sqrt{-1}\partial\bar{\partial}\phi \geq k\omega$, so let us fix some $k > 0$ and ϕ with this property. Let (E, h) be the trivial holomorphic vector bundle with a Griffiths semi-positive singular Hermitian metric h . There exists a naturally induced metric $h \otimes \det^m h$ on the vector bundle $E \otimes \det^m E$ for any $m \in \mathbb{N}$. For any $\bar{\partial}$ -closed $E \otimes \det^m E$ valued (n, q) -form u with finite L^2 norm with respect to $h \otimes \det^m h$, there exists a $E \otimes \det^m E$ valued $(n, q-1)$ -form α such that $\bar{\partial}\alpha = u$ and*

$$\int_X |\alpha|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega \leq \frac{1}{qC} \int_X |u|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega$$

for some constant C .

Proof. — By the assumption of Stein, we may regard X as a submanifold of \mathbb{C}^N for some positive integer N . We denote by $i : X \hookrightarrow \mathbb{C}^N$ the inclusion of X and according to Siu's result in [17], there exists an open neighborhood M of X in \mathbb{C}^N and a holomorphic retraction $p : M \rightarrow X$ such that

$p \circ i = \text{Id}_X$. Now (p^*E, p^*h) is the trivial vector bundle with Griffiths semi-positive metric p^*h on M . From the Lemma 2.5, i.e., [13, Proposition 6.2], one can obtain a sequence of smooth Hermitian metrics $\{g_\nu\}$ with Griffiths semi-positive curvature increasing to p^*h on any relative compact subset in M . Taking an exhaustion $\{X_j\}_{j=1}^\infty$ of X , where each X_j is a relative compact Stein sub-domain in X , satisfying each X_j is the relatively compact subset of X_{j+1} and $\bigcup X_j = X$. Set $\{h_\nu := i^*g_\nu\}$ and so $\{h_\nu\}$ is an approximation sequence with Griffiths semi-positive curvature increasing to h on any relatively compact subset of X . Since each h_ν is Griffiths semi-positive, due to Demailly–Skoda’s Theorem 2.3, $h_\nu \otimes \det h_\nu$ is Nakano semi-positive. The curvature of $(E \otimes \det^m E, h_\nu \otimes \det^m h_\nu \cdot e^{-\phi})$ can be calculated as

$$\begin{aligned} \sqrt{-1}\Theta_{h_\nu \otimes \det^m h_\nu \cdot e^{-\phi}} &= \sqrt{-1}\Theta_{h_\nu \otimes \det^m h_\nu} + \sqrt{-1}\partial\bar{\partial}\phi \otimes \text{Id}_{E \otimes \det^m E} \\ &= \sqrt{-1}\Theta_{h_\nu \otimes \det h_\nu} \otimes \text{Id}_{\det^{m-1} E} \\ &\quad + (m-1)\sqrt{-1}\partial\bar{\partial}(-\log \det h_\nu) \otimes \text{Id}_{E \otimes \det E} \\ &\quad + \sqrt{-1}\partial\bar{\partial}\phi \otimes \text{Id}_{E \otimes \det^m E} \\ &\geq C\omega \otimes \text{Id}_{E \otimes \det^m E}. \end{aligned}$$

The last inequality is in the sense of Nakano and C is independent of ν . Thus for any $E \otimes \det^m E$ valued (n, q) -form u with finite norm, we have

$$\langle [\sqrt{-1}\Theta_{h_\nu \otimes \det^m h_\nu \cdot e^{-\phi}}, \Lambda_\omega]u, u \rangle \geq qC|u|_{h_\nu \otimes \det^m h_\nu \cdot e^{-\phi}}^2$$

Now we fix one sub-domain X_j . By using Theorem 2.2 we get a solution α_ν of the $\bar{\partial}$ -equation satisfying

$$\begin{aligned} \int_{X_j} |\alpha_\nu|_{\omega, h_\nu \otimes \det^m h_\nu}^2 e^{-\phi} dV_\omega &\leq \frac{1}{qC} \int_{X_j} |u|_{\omega, h_\nu \otimes \det^m h_\nu}^2 e^{-\phi} dV_\omega \\ &\leq \frac{1}{qC} \int_{X_j} |u|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega \\ &\leq \frac{1}{qC} \int_X |u|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega < +\infty \end{aligned}$$

for sufficiently large ν . Fix sufficiently large ν_0 . We have that for $\nu \geq \nu_0$

$$\begin{aligned} \int_{X_j} |\alpha_\nu|_{\omega, h_{\nu_0} \otimes \det^m h_{\nu_0}}^2 e^{-\phi} dV_\omega &\leq \int_{X_j} |\alpha_\nu|_{\omega, h_\nu \otimes \det^m h_\nu}^2 e^{-\phi} dV_\omega \\ &\leq \frac{1}{qC} \int_X |u|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega < +\infty. \end{aligned}$$

Then $\{\alpha_\nu\}_{\nu \geq \nu_0}$ forms a bounded sequence on X_j with respect to the norm

$$\int_{X_j} |\cdot|_{\omega, h_{\nu_0} \otimes \det^m h_{\nu_0}}^2 e^{-\phi} dV_\omega.$$

We can get a weakly convergent subsequence $\{\alpha_{\nu_0,k}\}_k$. Thus, the weak limit α_j satisfies

$$\int_{X_j} |\alpha_j|_{\omega, h_{\nu_0} \otimes \det^m h_{\nu_0}}^2 e^{-\phi} dV_\omega \leq \frac{1}{qC} \int_X |u|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega < +\infty.$$

Next, we fix $\nu_1 > \nu_0$. Repeating the above argument, we can choose a weakly convergent subsequence $\{\alpha_{\nu_1,k}\}_k \subset \{\alpha_{\nu_0,k}\}_k$ with respect to $\int_{X_j} |\cdot|_{\omega, h_{\nu_1} \otimes \det^m h_{\nu_1}}^2 e^{-\phi} dV_\omega$. Then by taking a sequence $\{\nu_n\}_n$ increasing to $+\infty$ and a diagonal sequence, we obtain a weakly convergent sequence $\{\alpha_{\nu_k,k}\}_k$ with respect to $\int_{X_j} |\cdot|_{\omega, h_{\nu_\ell} \otimes \det^m h_{\nu_\ell}}^2 e^{-\phi} dV_\omega$ for all ℓ . Hence, α_j satisfies

$$\int_{X_j} |\alpha_j|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega \leq \frac{1}{qC} \int_X |u|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega$$

thanks to the monotone convergence theorem. Since the right-hand side of the above inequality is independent of j , by using the exactly same argument, we can get an $E \otimes \det^m E$ -valued $(n, q-1)$ -form α satisfying $\bar{\partial}\alpha = u$ and

$$\int_X |\alpha|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega \leq \frac{1}{qC} \int_X |u|_{\omega, h \otimes \det^m h}^2 e^{-\phi} dV_\omega,$$

which completes the proof of Theorem 3.1. \square

Remark 3.2. — From now on, since the coherence is the local property, we assume $X := \Delta := \Delta^n$ to be the polydisc in \mathbb{C}^n . If possible, we can shrink Δ to meet our requirements. Let ω be the standard Kähler metric on Δ . With Theorem 3.1 at hand, one can prove the coherence of $\mathcal{E}(E \otimes \det^m E, h \otimes \det^m h)$. To be specific, one can prove that it is generated by the square integrable sections of $E \otimes \det^m E$. The proof is the same as the procedure in [3, Proposition 5.7].

THEOREM 3.3. — *Given a holomorphic vector bundle E with singular Griffiths semi-positive metric h . If the weight of Hermitian metric $\det h$ on the determinant line bundle $\det E$ has algebraic singularities, then the sheaf $\mathcal{E}(E, h)$ is coherent. Note that (E, h) always be Griffiths semi-positive.*

Proof. — We divide the proof into several steps.

Step 1. — We first simplify the metric by the proper modification. The sheaf $\mathcal{E}(E, h)$ is coherent if and only if $K_X \otimes \mathcal{E}(E, h)$ is coherent. We thus consider the coherence of $K_X \otimes \mathcal{E}(E, h)$ since it has the nice functorial property under the proper modification. Indeed, if $\pi : X' \rightarrow X$ be a proper modification, then due to previous Lemma 2.6, we have

$$\pi_*(K_{X'} \otimes \mathcal{E}(\pi^*E, \pi^*h)) = K_X \otimes \mathcal{E}(E, h).$$

According to Grauert's coherence theorem, if $K_X \otimes \mathcal{E}(\pi^*E, \pi^*h)$ is coherent, then its direct image $K_X \otimes \mathcal{E}(E, h)$ is coherent too. Note that (π^*E, π^*h) is also singular Griffiths semi-positive, see for example [12, Lemma 2.3.2]. By the assumption, the weight of induced metric $\det h$ on the determinant line bundle $\det E$ has algebraic singularities, i.e., locally can be written as

$$-\log \det h = \gamma \log \left(\sum_j |h_j|^2 \right) + \xi.$$

Here γ is a positive rational number and ξ is a bounded function. Therefore after blow-up or modification, we can assume that

$$-\log \det h = \sum_j \gamma_j \log |h_j|^2 + \xi.$$

All γ_j are positive rational number, we can write $\gamma_j = \frac{n_j}{m}$, where $n_j, m \in \mathbb{Z}$ and $m \neq 0$. So we have

$$(3.1) \quad -m \log \det h = \sum_j n_j \log |h_j|^2 + m\xi$$

be the local weight of $\det^m h$, the metric of line bundle $\det^m E$. Now locally the metric

$$(3.2) \quad \det^m h = \frac{1}{\prod_j |h_j|^{2n_j}} e^{-m\xi}.$$

Let $H = \prod h_j^{n_j}$ be the corresponding holomorphic function. Locally $|H|^2 \det^m h = e^{-m\xi}$ is positive and bounded function.

Step 2. — Since $\mathcal{E}(E \otimes \det^m E, h \otimes \det^m h)$ and $\mathcal{E}(\det^m E, \det^m h)$ are coherent, hence locally generated by the square integrable sections. As we have said, since the coherence is a local property, we may assume that $X := \Delta := \Delta^n$ is a small polydisc in \mathbb{C}^n . Let $H_{2, h \otimes \det^m h}^0(\Delta, E \otimes \det^m E)$ be the space of holomorphic section s of $E \otimes \det^m E$ on Δ such that

$$\int_{\Delta} |s|_{h \otimes \det^m h}^2 dV_{\omega} < +\infty.$$

We consider the evaluation map

$$ev : H_{2, h \otimes \det^m h}^0(\Delta, E \otimes \det^m E) \times \mathcal{O}_{\Delta} \longrightarrow \mathcal{O}(E \otimes \det^m E).$$

We denote by $\Gamma_{E \otimes \det^m E}$ the image of ev . Every coherent \mathcal{O}_Δ -sheaf enjoys the Noether property, it is obvious that $\Gamma_{E \otimes \det^m E}$ is coherent, and by the Remark 3.2 we have the equality

$$(3.3) \quad (\Gamma_{E \otimes \det^m E})_x = \mathcal{E}(E \otimes \det^m E, h \otimes \det^m h)_x.$$

Similarly, for the line bundle $(\det^m E, \det^m h)$, we denote by $\Gamma_{\det^m E}$ the image of square integrable sections under the evaluation map, and according to the coherence of $\mathcal{E}(\det^m E, \det^m h)$, we have

$$(3.4) \quad (\Gamma_{\det^m E})_x = \mathcal{E}(\det^m E, \det^m h)_x.$$

Step 3. — We introduce the algebraic criterion that yields the coherence of $\mathcal{E}(E, h)$. For the vector bundle (E, h) , we denote by e the image of square integrable sections under the evaluation map as above. We want to show that $e_x = \mathcal{E}(E, h)_x$ for any point $x \in \Delta$. Since e is coherent and $e_x \subseteq \mathcal{E}(E, h)_x$, one just need to check

$$(3.5) \quad e_x + \mathcal{E}(E, h)_x \cap m_x^k \cdot E_{(x)} = \mathcal{E}(E, h)_x$$

for any positive integer k , here $E_{(x)} := \varinjlim_{x \in U} H^0(U, E)$. Indeed, if this is the case, by the Artin–Rees lemma, there exists a positive integer l such that

$$\mathcal{E}(E, h)_x \cap m_x^k \cdot E_{(x)} = m_x^{k-l} \cdot (\mathcal{E}(E, h)_x \cap m_x^l \cdot E_{(x)})$$

holds for any $k > l$. Therefore according to the above equality (3.5), one have

$$\mathcal{E}(E, h)_x = e_x + \mathcal{E}(E, h)_x \cap m_x^k \cdot E_{(x)} \subset e_x + m_x \cdot \mathcal{E}(E, h)_x \subset \mathcal{E}(E, h)_x.$$

By Nakayama’s lemma, one can obtain $e_x = \mathcal{E}(E, h)_x$, which is the desired result. Now we begin to prove the equality (3.5), one inclusion is trivial, we just need to check that

$$(3.6) \quad e_x + \mathcal{E}(E, h)_x \cap m_x^k \cdot E_{(x)} \supseteq \mathcal{E}(E, h)_x.$$

Step 4. — In this step, we begin to prove the inequality (3.6), the key point is the choice of section g of $\det^m E$. Let $f_x \in \mathcal{E}(E, h)_x$, according to our assumption

$$\det^m h = \frac{1}{\prod_j |h_j|^{2n_j}} \cdot e^{-m\xi}$$

on Δ . So we can choose $g = H = \prod_j h_j^{n_j} \in H_{2, \det^m h}^0(\Delta, \det^m E)$ and

$$(3.7) \quad |g|_{\det^m h}^2 = |g|^2 \cdot \det^m h = e^{-m\xi} \in (C_1, C_2).$$

Here C_1, C_2 are two strictly positive bounded real numbers. In this situation, one has

$$(3.8) \quad \int_{\Delta'} |f|_h^2 \cdot |g|_{\det^m h}^2 dV_\omega \leq C_2 \int_{\Delta'} |f|_h^2 dV_\omega < +\infty.$$

Here Δ' is the very small neighborhood of x . Therefore we have $f_x \cdot g_x \in \mathcal{E}(E \otimes \det^m E, h \otimes \det^m h)_x$. According to equality (3.3), $(\Gamma_{E \otimes \det^m E})_x = \mathcal{E}(E \otimes \det^m E, h \otimes \det^m h)_x$, there exist a global section $r \in H_{2, h \otimes \det^m h}^0(\Delta, E \otimes \det^m E)$ such that

$$r_x = f_x \cdot g_x.$$

Step 5. — We solve the $\bar{\partial}$ equation with the psh weight φ_k , this weight forces the germ of the section to belong to some high-order maximal ideal m_x at the fixed point x . We choose a sufficient small neighbor U of x and a cut-off function ρ such that $\text{supp } \rho \subseteq U$ and $\rho = 1$ on a small neighbor of x .

Now we want to solve the $\bar{\partial}$ -equation $\bar{\partial}(\rho r) = \bar{\partial}u$ with the weighted L^2 -estimate. We first define two weighted psh functions

$$\begin{aligned} \varphi_k &= (n + k + k') \log |z - x|^2 + |z|^2; \\ \varphi_{k,\delta} &= (n + k + k') \log(|z - x|^2 + \delta^2) + |z|^2. \end{aligned}$$

Here k' represent the order of g at the point x , i.e., $g_x \in m_x^{k'}$ but $g_x \notin m_x^{k'+1}$. The previous Theorem 3.1 shows that there exists an $E \otimes \det^m E$ valued $(0, 0)$ form u such that $\bar{\partial}u = \bar{\partial}(\rho r)$ and

$$\int_{\Delta} |u|_{h \otimes \det^m h}^2 e^{-\varphi_{k,\delta}} dV_\omega \leq \int_{\Delta} |\bar{\partial}(\rho r)|_{h \otimes \det^m h}^2 e^{-\varphi_{k,\delta}} dV_\omega < +\infty.$$

Taking some subsequence and taking the limit when $\delta \rightarrow 0$, we can get

$$\int_{\Delta} |u|_{h \otimes \det^m h}^2 e^{-\varphi_k} dV_\omega \leq \int_{\Delta} |\bar{\partial}(\rho r)|_{h \otimes \det^m h}^2 e^{-\varphi_k} dV_\omega < +\infty.$$

By the definition of φ_k , we know that $u_x \in m_x^{k+k'+1} \cdot (E \otimes \det^m E)_{(x)}$, this is actually not obvious. Indeed, by [11, Lemma 2.3], if we write $u = \sum u_i \delta_i$, here δ_i are the holomorphic frame of vector bundle $E \otimes \det^m E$, we then have each $u_i \in m_x^{k+k'+1}$. Since $\bar{\partial}(\rho \cdot r - u) = 0$, this means $(\rho \cdot r - u)$ is holomorphic section of $E \otimes \det^m E$. Then $(\frac{\rho \cdot r}{g} - \frac{u}{g})$ is a meromorphic section of E on the whole Δ and holomorphic on $\Delta \setminus V(g)$, where $V(g) := \{z \in \Delta : g(z) = 0\}$. On one hand, due to the fact

$$\left(\frac{\rho \cdot r}{g} \right)_x = \frac{r_x}{g_x} = f_x$$

and ρ is a cut off function near x with small enough support, the first term $\frac{\rho \cdot r}{g}$ is smooth on the whole Δ and satisfying

$$\int_{\Delta} \left| \frac{\rho \cdot r}{g} \right|_h^2 dV_{\omega} < +\infty.$$

On the other hand, if we have $\int_{\Delta} \left| \frac{u}{g} \right|_h^2 dV_{\omega} < +\infty$, then $(\frac{\rho \cdot r}{g} - \frac{u}{g})$ is square integrable with respect to metric h , hence it can be extended to the whole Δ as a holomorphic section of E . Grant this for the time being, let $s = \frac{\rho \cdot r}{g} - \frac{u}{g}$, take the germ at x , we have $s_x = f_x - \frac{u_x}{g_x}$ and

$$f_x = s_x + \frac{u_x}{g_x} \in e_x + \mathcal{E}(E, h)_x \cap m_x^k \cdot E_{(x)}.$$

This is to say, $f_x \subseteq e_x + \mathcal{E}(E, h)_x \cap m_x^k \cdot E_{(x)}$ and therefore equality (3.5) is obtained. We now prove the desired result that $\mathcal{E}(E, h)$ is coherent.

Step 6. — The last step is to show that $\int_{\Delta} \left| \frac{u}{g} \right|_h^2 dV_{\omega} < +\infty$. But this is easy by the choose of the g , see (3.7), indeed, we have

$$\begin{aligned} \int_{\Delta} \left| \frac{u}{g} \right|_h^2 dV_{\omega} &= \int_{\Delta} \frac{|u|_{h \otimes \det^m h}^2}{|g|_{\det^m h}^2} dV_{\omega} \\ &\leq \frac{1}{C_1} \int_{\Delta} |u|_{h \otimes \det^m h}^2 dV_{\omega} < +\infty \end{aligned}$$

as desired. □

Remark 3.4. — Upon reviewing the proof of Theorem 3.3, it becomes evident that the next three conditions are sufficient to derive the conclusion, (here h_1 is not necessarily induced from h):

- (1) The vector bundle $(E \otimes \det^m E, h \otimes h_1)$ has the L^2 -estimate of the associated $\bar{\partial}$ -equation and therefore the sheaf $\mathcal{E}(E \otimes \det^m E, h \otimes \det^m h)$ is coherent.
- (2) The determinant bundle $(\det^m E, h_1)$ is singular Griffiths semi-positive. Thus $\mathcal{E}(\det^m E, h_1)$ is coherent, this is a classical result. Note that m can be any positive integer.
- (3) The weight of metric h_1 has algebraic singularities with integer coefficients, i.e., locally $-\log h_1 = \sum_j n_j \log |f_j|^2 + \xi$ just like (3.1).

4. Singular Hermitian metrics with analytic singularities

We have established the coherence of $\mathcal{E}(E, h)$ under the assumption that the weight $\psi = -\log \det h$ has algebraic singularities. It appears that the conclusion should hold even when ψ has analytic singularities. We aim to prove this extension now.

THEOREM 4.1. — Let (Δ, ω) be a small polydisk with a standard metric ω , and ϕ be a smooth strictly plurisubharmonic function on Δ . Let (E, h) be the trivial holomorphic vector bundle with a Griffiths semi-positive singular Hermitian metric h . There exists a naturally induced metric $h \otimes (\det h)^m$ on the vector bundle $E \otimes \det^m E$ for any $m \in \mathbb{N}$. If there is a psh function θ such that $\sqrt{-1}\partial\bar{\partial}\theta - m\sqrt{-1}\partial\bar{\partial}(-\log(\det h)) \geq 0$, then for any $\bar{\partial}$ -closed $E \otimes \det^m E$ valued (n, q) -form u satisfying

$$\int_{\Delta} |u|_{\omega, h \otimes e^{-\theta}}^2 \cdot e^{-\phi} dV_{\omega} < +\infty.$$

We can find a $E \otimes \det^m E$ valued $(n, q-1)$ -form α such that $\bar{\partial}\alpha = u$ and

$$\int_{\Delta} |\alpha|_{\omega, h \otimes e^{-\theta}}^2 \cdot e^{-\phi} dV_{\omega} \leq C \int_{\Delta} |u|_{\omega, h \otimes e^{-\theta}}^2 e^{-\phi} dV_{\omega}$$

for some constant C .

Remark 4.2. — Here we make $e^{-\theta}$ the non-trivial metric on the trivial line bundle $\det^m E$. This theorem tells us that the vector bundle $(E \otimes \det^m E, h \otimes e^{-\theta})$ has the L^2 -estimate of the associated $\bar{\partial}$ -equation and therefore the sheaf $\mathcal{E}(E \otimes \det^m E, h \otimes e^{-\theta})$ is coherent.

The proof of Theorem 4.1. — Like the proof in the previous Theorem 3.1, we first regularize the metric h and the psh function θ on the smaller polydisk as Lemma 2.5. The approximation sequence of smooth metrics h_{ν} increasing to h and the approximation sequence of smooth psh functions θ_{ν} decreasing to θ . We then calculate the curvature of trivial vector bundle $(E \otimes \det^m E, h_{\nu} \otimes e^{-\theta_{\nu}} \cdot e^{-\phi})$. Firstly we write $h_{\nu} \otimes e^{-\theta_{\nu}} = h_{\nu} \otimes \det^m h_{\nu} \cdot (\det^m h_{\nu})^{-1} \cdot e^{-\theta_{\nu}}$.

Due to Demailly-Skoda theorem, $h_{\nu} \otimes \det^m h_{\nu}$ is Nakano semi-positive. At the same time, we can arrange the sequence smooth psh function θ_{ν} such that

$$\sqrt{-1}\partial\bar{\partial}\theta_{\nu} - m\sqrt{-1}\partial\bar{\partial}(-\log(\det h_{\nu})) \geq -C\omega,$$

here C are some positive constant independent to ν . Hence the curvature of vector bundle $(E \otimes \det^m E, h_{\nu} \otimes e^{-\theta_{\nu}} \cdot e^{-\phi})$ can be calculated as follow

$$\begin{aligned} \sqrt{-1}\Theta_{h_{\nu} \otimes e^{-\theta_{\nu}} \cdot e^{-\phi}} &= \sqrt{-1}\Theta_{h_{\nu} \otimes \det^m h_{\nu}} \\ &\quad + \sqrt{-1}\partial\bar{\partial}(\theta_{\nu} + \log(\det h_{\nu})^m) \otimes \text{Id}_{E \otimes \det^m E} \\ &\quad + \sqrt{-1}\partial\bar{\partial}\phi \otimes \text{Id}_{E \otimes \det^m E}. \end{aligned}$$

Now if we choose the very positive psh function ϕ , we can make the curvature of vector bundle $(E \otimes \det^m E, h_{\nu} \otimes e^{-\theta_{\nu}} \cdot e^{-\phi})$ is strictly Nakano

positive, i.e.,

$$\begin{aligned} & \sqrt{-1}\Theta_{h_\nu \otimes \det^m h_\nu} + \sqrt{-1}\partial\bar{\partial}(\theta_\nu + \log(\det h_\nu)^m) \otimes \text{Id}_{E \otimes \det^m E} \\ & \quad + \sqrt{-1}\partial\bar{\partial}\phi \otimes \text{Id}_{E \otimes \det^m E} \geq \varepsilon\omega \otimes \text{Id}_{E \otimes \det^m E} \end{aligned}$$

Thus for any $E \otimes \det^m E$ valued (n, q) -form u , we have

$$\langle [\sqrt{-1}\Theta_{h_\nu \otimes e^{-\theta_\nu} \cdot e^{-\phi}}, \Lambda_\omega]u, u \rangle \geq q\varepsilon|u|^2.$$

The rest proof is the same as the previous Theorem 3.1 and we omit it. \square

COROLLARY 4.3. — *With the same condition in the Theorem 4.1, the sheaf $\mathcal{E}(E \otimes \det^m E, h \otimes e^{-\theta})$ is coherent.*

Remark 4.4. — Similar to Remark 3.4, once the next three conditions are satisfied, then from the proof of Theorem 3.3, we know $\mathcal{E}(E, h)$ is coherent.

- (1) The vector bundle $(E \otimes \det^m E, h \otimes e^{-\theta})$ has the L^2 -estimate of the associated $\bar{\partial}$ -equation and therefore the sheaf $\mathcal{E}(E \otimes \det^m E, h \otimes e^{-\theta})$ is coherent.
- (2) The determinant bundle $(\det^m E, e^{-\theta})$ is singular Griffiths semi-positive. Thus $\mathcal{E}(\det^m E, e^{-\theta})$ is coherent.
- (3) If the weight θ has algebraic singularities, and the coefficient is positive integer, i.e., $\theta = \alpha \log(|f_1|^2 + \cdots + |f_N|^2) + v$, for $\alpha \in \mathbb{Z}^+$ and holomorphic functions f_i and bounded function v .

COROLLARY 4.5. — *Let (E, h) be a holomorphic vector bundle over an n -dimensional complex manifold with a singular Griffiths semi-positive Hermitian metric h . If the weight of metric $\det h$ has analytic singularities, then $\mathcal{E}(E, h)$ is coherent.*

Proof. — Locally $(-\log \det h)$ can be written as

$$-\log \det h = \alpha \log(|f_1|^2 + \cdots + |f_N|^2) + v$$

where α is a positive real number, v is a locally bounded function and all f_i are holomorphic functions. We can choose a positive integer β such that $\beta \geq \alpha$. It is obviously that $\theta := \beta \log(|f_1|^2 + \cdots + |f_N|^2) + v$ is plurisubharmonic function and satisfying $\sqrt{-1}\partial\bar{\partial}\theta - \sqrt{-1}\partial\bar{\partial}(-\log \det h) \geq 0$. By Theorem 4.1 and Corollary 4.3, the sheaf $\mathcal{E}(E \otimes \det E, h \otimes e^{-\beta})$ is coherent. According to the above Remark 4.4, we can obtain the desired result, i.e., the sheaf $\mathcal{E}(E, h)$ is coherent. \square

Remark 4.6. — While attempting to establish the general case using Demailly's approximation theorems, we encounter a specific difficulty. We want to approximate $\det h$ with a sequence of psh functions f_i with analytic singularities. According to Theorem 4.1, one may need the next result:

$\sqrt{-1}\partial\bar{\partial}f_i - \sqrt{-1}\bar{\partial}\det h$ are uniformed bounded below, i.e., $\sqrt{-1}\partial\bar{\partial}f_i - \sqrt{-1}\bar{\partial}\det h \geq -C\omega$, where C is a constant independent to i . Relying solely on Demailly's approximation theorem, we find this task challenging. Demailly's approximation shows that if $\sqrt{-1}\partial\bar{\partial}\det h \geq \gamma$ for some smooth $(1, 1)$ -forms γ , then we have $\sqrt{-1}\partial\bar{\partial}f_i \geq -\gamma - \delta\omega$ for some constant δ . This is not enough for our purpose since this can not induce $\sqrt{-1}\partial\bar{\partial}f_i - \sqrt{-1}\bar{\partial}\det h \geq -C\omega$.

Remark 4.7. — Let E be a holomorphic vector bundle of rank r with a singular metric over the complex manifold X . We can thus have a dual projective bundle $\mathbb{P} := P(E^*)$ whose point can be identified with the hyperplane of E . Let $\pi : \mathbb{P}(E) \rightarrow X$ be the natural projection, then there is a tautological hyperplane sub-bundle S of π^*E over $\mathbb{P}(E)$ such that $S_{[\xi]} = \xi^{-1}(0) \subset E_x$ for all $\xi \in E_x^* \setminus \{0\}$. The quotient line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) := \pi^*E/S$ is the so-called tautological line bundle and we have the exact sequence of vector bundle over $\mathbb{P}(E)$,

$$0 \longrightarrow S \longrightarrow \pi^*E \xrightarrow{\iota} \mathcal{O}_{\mathbb{P}(E)}(1) \longrightarrow 0.$$

We know there have the isomorphism $\pi_*\mathcal{O}_{\mathbb{P}(E)}(k) \simeq \mathcal{O}(S^k E)$ and particularly have $\pi_*\mathcal{O}_{\mathbb{P}(E)}(1) \simeq \mathcal{O}(E)$. The isomorphism can be obtained by the next composition

$$\phi_U : H^0(U, E) \xrightarrow{\pi^*} H^0(\pi^{-1}(U), \pi^*E) \xrightarrow{\iota} H^0(\pi^{-1}(U), \mathcal{O}_{\mathbb{P}(E)}(1)).$$

The last morphism ι comes from the above exact sequence. We know the morphism $\iota : \pi^*E \rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ endows $\mathcal{O}_{\mathbb{P}(E)}(1)$ a quotient metric h_1 . If (E, h) is Griffiths semi-positive singular Hermitian metric, then (π^*E, π^*h) also be Griffiths semi-positive, and the quotient line bundle $(\mathcal{O}_{\mathbb{P}(E)}(1), h_1)$ is Griffiths semi-positive too. We consider the next commutative diagram

$$\begin{array}{ccc} \mathcal{O}(E) & \xrightarrow[\simeq]{\phi} & \mathcal{O}_{\mathbb{P}(E)}(1) \\ \uparrow \cup & & \cup \uparrow \\ \mathcal{E}(E, h) & \xrightarrow{\phi} & \text{Ima}(\mathcal{E}(E, h)). \end{array}$$

We try to find out the relationship between $\text{Ima}(\mathcal{E}(E, h))$ with $\mathcal{E}(\mathcal{O}_{\mathbb{P}(E)}(1), h_1)$, i.e., the square integrable section of $\mathcal{O}_{\mathbb{P}(E)}(1)$ with respect to h_1 . If there are coincident, then we can prove the coherence of $\mathcal{E}(E, h)$ since it is well known that $\pi_*\mathcal{E}(\mathcal{O}_{\mathbb{P}(E)}(1), h_1)$ is coherent.

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